Universidade de Lisboa

Faculdade de Ciências
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Inflation driven by 3-form fields

Mestrado em Física
Especialização em Astrofísica e Cosmologia

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Dissertação orientada pelo Doutor Nelson Nunes

2015

Abstract

Throughout this thesis I study primordial inflation, which was a period of accelerated expansion in the early universe, through models of tensorial fields, more specifically. 3-form fields.

I explore the dynamics of two coupled 3-form fields in the standard four dimensional case and compare it to the uncoupled model already addressed in the literature. I focus on the role of the coupling, in contrast with the uncoupled case, and see how it affects inflationary scenarios for two different forms of the potential. I follow to deduce the evolution equations and the equations of motion which I numerically solve. I propose a general form for the Lagrangian of $\mathcal{N}$ coupled 3-form fields and provide the respective equations of motion.

I present a single 3-form field model in a five dimensional braneworld, called the Randall-Sundrum II model, in which our universe in confined to a four dimensional 3-brane embedded in a five dimensional AdS bulk. The braneworld effects modify the evolution equations so I focus on the main differences from the five dimensional and the standard four dimensional case. Once again, I calculate the equations of motion and follow to rewrite these in the form of a system of first order differential equations which I numerically solve. I present inflationary solutions for different forms of the potential and study the dynamics, the critical points and their stability, and show the influence of the fifth dimension on these quantities. I follow to calculate the speed of sound for this model and present the evolution of scalar and tensorial perturbations by perturbing the 3-form and the metric.

Finally, by calculating the cosmological parameters, tensor to scalar ratio and spectral index, I show how my inflationary setting fits the recent Planck data and see how is it sensible to the value of the brane tension where in a particular case I find a lower bond for it.

Keywords: Inflation; 3-form inflation; 3-form cosmology; Randall-Sundrum II; Inflation in Randall-Sundrum II; N-Forms

Resumo

Inflação primordial foi um curto período de expansão acelerada no universo primitivo (10^{-36}s a 10^{-33}s). É uma teoria que foi introduzida por Alan Guth com o objectivo de dar resposta a alguns problemas do modelo standard da Cosmologia.

Nesta tese estudo a inflação a partir de campos tensoriais, em particular, campos 3-forma. Começarei por lembrar ao leitor alguns conceitos básicos e necessários de cálculo tensorial. Apresento resumidamente o modelo padrão da cosmologia e explico o que é a inflação primordial, porque é necessária e como é maioritariamente estudada na literatura a partir de um campo escalar a interagir com o seu potencial.

Escrevo os modelos standard para uma e para múltiplas 3-formas, já estudadas na literatura. Apresento um modelo para inflação conduzida por dois campos 3-forma acopladas onde calculo as equações de evolução, equações do movimento, tensor energia momento e alguns constrangimentos. Introduzindo quatro variáveis adimensionais úteis, reescrevo as equações do movimentos para os campos, com estas novas variáveis, sob a forma de um sistema de equações diferenciais de primeira ordem o qual resolvo numericamente. Calculo os parâmetros de slow roll e impondo as condições para termos inflação (condições de slow roll) consigo obter as condições iniciais para o sistema dinâmico. Estudo a dinâmica, os pontos críticos e estabilidade destes pontos para dois tipos de potenciais, quadrático e exponencial. Foco-me no papel do acoplamento em comparação com o caso desacoplado, já estudado na literatura, mostrando a sua influência na modificação das equações e na dinâmica inflacionária. Em particular, mostro que um acoplamento da ordem de $\sim 10^{-4}$ tem a influência de estender a duração da inflação entre 20 a 30 e-folds. Proponho uma forma geral para o Lagrangiano de $\mathcal{N}$ campos 3-forma acoplados introduzindo um novo termo que contém os acoplamentos onde imponho algumas restrições de forma a não termos repetições nos potenciais e mostro as respectivas equações do movimento.

Familiarizo o leitor com o modelo extra-dimensional Randall-Sundrum II, proposto por Lisa Randall e Raman Sundrum em 1999, onde o nosso universo está confinado a uma 3-brane de 4 dimensões embebido numa quinta dimensão (bulk) cuja geometria é Anti de Sitter. Apresento um modelo inflacionário conduzido por um campo 3-forma, confinado à brana, onde deduzo as equações do movimento e respectivos constrangimentos. Uma vez mais, introduzo variáveis adimensionais e reescrevo as equações do movimento sob a forma de um sistema dinâmico. Impondo as condições de slow roll para inflacção, estudo a dinâmica, os pontos críticos do sistema e sua estabilidade para diferentes formas do potencial. Em particular foco-me na influência da quinta dimensão em comparação com o caso padrão a quatro dimensões. Calculo a evolução das perturbações escalares e tensoriais perturbando a métrica e a 3-forma. Usando a forma da velocidade do som, deduzo os parâmetros cosmológicos, mais especificamente, a razão entre a amplitudes das perturbações tensorias-escalares e os índices espectrais n_s e n_T. Por fim comparo as previsões cosmológicas do modelo considerado com os recentes resultados do satélite Planck e observo como são sensíveis à quinta dimensão quando se altera o valor da tensão da brana. Encontro um limite inferior para a tensão da brana para um caso particular do potencial.

Palavras-chave: Inflação; Inflação 3-forma; Cosmologia 3-forma; Randall-Sundrum II; Inflação em Randall-Sundrum II; N-formas

Acknowledgements

I would like to express my enormous gratitude to my supervisor Nelson Nunes for always being available to help me throughout this last two years of my master course, for opening my eyes into Cosmology and for during all the obstacles and difficulties of this work keeping a calm and positive attitude that without it, it would be really difficult to keep my head up, finish my course and stand where I am today.

I would also like to thank my family, specially to my older sister Filipa and my father.

Contents

1 Introduction

Primordial inflation (4) was a period of accelerated expansion in the early universe and provides solutions for some cosmological problems such as the flatness and horizon problems and also explains the emergence of the primordial density fluctuations essential for the formation of the large scale structure observed today [1, 2]. Inflation is mostly studied considering a self interacting scalar field and has been widely studied in the literature (see Refs. [3, 4] for reviews). However, there is no study excluding the possibility of the energy source of the inflationary expansion to be of a non-scalar nature. It is, therefore, important to understand the nature of higher spin fields and how robust they are in order to fully test their applications in cosmology. Inflation considering higher spinor fields has been investigated in the past and these models are also important due to their connection to string theory scenarios[5, 6, 7]. Vector inflation has been studied in Ref. [8], however, for inflation to proceed, the vector needs a nonminimal coupling and seems to feature some instabilities. Inflation with a 2-form field resembles much the vector inflation with the same problems [9, 10].

A 3-form has been shown to present viable solutions, not only for inflation [11, 12, 14], but also for describing the dark energy sector [15]. Inflation driven by two uncoupled 3-form fields has also been studied and does presents interesting results [16].

In the Randall-Sundrum II model (6), proposed in 1999 [17], our universe is confined to a four dimensional 3-brane, where the standard model particles reside, embedded in a five dimensional slice of an anti-de Sitter (AdS) space-time, the bulk. The presence of the bulk modifies the evolution equations [18], more specifically, the Friedmann equation leads to a non-standard expansion law of the universe at high energies, while reproducing the standard four dimensional cosmology at low energies. One particular feature of the RSII model is that the tensor modes are enhanced due to the presence of the five dimensional bulk [20, 19]. Chaotic inflation on the brane has been investigated in Ref. [21] and it was shown that the inflationary predictions are modified from those in the four dimensional standard cosmology. Quintessential inflation from brane worlds has also been explored in Ref. [22] and also inflation in the context of a Gauss-Bonnet brane cosmology [23]. More recently, simple inflationary models in the context of braneworld cosmology were analysed against the 2015 Planck data [24, 25].

It is important to compare the dynamics of inflation with scalar fields with the dynamics where higher order fields are considered.

This thesis focus on the study of inflation through dynamical systems considering 3-form fields. I will explore the dynamics of two coupled 3-form fields, in an inflationary context, and see the role of the coupling in comparison with the uncoupled case, already studied in the literature. I will propose a general form for the Lagrangian for $\mathcal{N}$ coupled 3-form fields and show the respective equations of motion. Finally I will also explore the dynamics of a single 3-form in the Randall-Sundrum II braneworld, presenting viable inflationary solutions, comparing with the standard 4 dimensional case, and see how the cosmological observables fit the latest observations made by the Planck satellite [24, 25].

In chapter 2 I will give a brief introduction to some mathematics of general relativity essential in order to understand the basics of cosmology. In chapter 3 I resume the standard model of cosmology, the ΛCDM model. In chapter 4 I review the standard inflationary model, explaining what is inflation, why it is needed and how is it typically studied. In chapter 5 I start by presenting to the reader to the recent 3-form cosmology theory and I follow to describe my original work for this thesis. More precisely in chapter 5.3 I explore the dynamics of two coupled 3-form fields in the standard four dimensional FLRW universe and also consider rotations within the $SO(2)$ group to see if the theory is left unchanged. In chapter 5.4 I propose a general form for the Lagrangian of $\mathcal{N}$ coupled 3-form fields model and show the respective equations of motion. In chapter 6 I familiarize the author with the Randall-Sundrum II, five dimensional, braneworld model where I present the five dimensional Einstein equations and explain the main differences from the standard four dimensional case. In chapter 6.2 I present braneworld inflationary models in the RSII context driven by a single 3-form, confined to the brane, in the light of the Planck 2015 results. Finally in chapter 7 one can find the conclusions.

2 Basics of General Relativity

In general relativity, gravity is an aspect of the geometry of spacetime unlike in Newtonian theory where gravity is a force between particles.

In what follows, some basic aspects and definitions about the spacetime geometry are introduced in order to fully understand Einstein's equations and their meaning.

2.1 Metric

The geometry of the spacetime manifold can be described by the *metric* which is given in terms of a set of coordinates which can be an arbitrary curved coordinate system. The coordinates of the four dimensional spacetime are (x^0, x^1, x^2, x^3), where $x^0 = t$ is a time coordinate. In this thesis I use the notation where the Greek indices denote spacetime coordinates, x^μ, $\mu = 0, 1, 2, 3$, and Latin indices to denote space coordinates, x^i, $i = 1, 2, 3$.

The coordinates are numbers which identify locations but do not specify physical distances. The distance information is in the metric $g_{\mu\nu}$ that gives the square of the line element ds^2 in terms of the coordinate differentials,

$$ds^2 = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu \equiv g_{\mu\nu} dx^\mu dx^\nu, \tag{2.1}$$

where in the last step I introduce the Einstein summation convention of summing over repeated indices and do not write the summation sign. The tensor $g_{\mu\nu}$ is called the *metric tensor.*

The case of Minkowski space, or flat spacetime, in which Einstein's theory of special relativity is most conveniently formulated, the metric tensor in Cartesian coordinates is defined as $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) \equiv \eta_{\mu\nu}$ (I use the signature $(-1, 1, 1, 1)$).

We can define g as the determinant of the metric, $g \equiv \det(g_{\mu\nu})$, and if $g \neq 0$ we can define the inverse of $g_{\mu\nu}$ as,

$$g_{\mu\nu} g^{\nu\alpha} = \delta^\alpha_\mu = g^\alpha_\mu. \tag{2.2}$$

We can raise and lower indices of a tensor using $g_{\mu\nu}$ and $g_{\mu\nu}$.

It will be useful to remember the Jacobi's formula, the rule for differentiating a determinant, that gives,

$$\delta g = \delta \det(g_{\mu\nu}) = g g^{\mu\nu} \delta g_{\mu\nu}. \tag{2.3}$$

2.2 Covariant derivative and the Christoffel symbols

The introduction of the Christoffel symbols becomes necessary when we address the problem that the partial derivative, $\partial_\mu \equiv \partial/\partial x^\mu$, does not tranform as a tensor (is not a tensor) when we consider curved manifolds. We define the *covariant derivative*, or derivative operator on a manifold, of a contravariant vector V^ν as,

$$\nabla_\mu V^\nu = \partial_\mu V^\nu - \Gamma^\nu_{\alpha\mu} V^\alpha \tag{2.4}$$

where we add the connections $\Gamma^\nu_{\alpha\mu}$, known as the *Christoffel symbols* or *metric connections*, that should be nontensorial to cancel out the nontensorial character of the partial derivative in order to preserve the tensorial nature of ∇_μ. That is, an operator which reduces to the partial derivative in flat space but transforms as a tensor on an arbitrary manifold.

An essential postulate of Riemannian geometry is that the length of a vector is unchanged under parallel transport. There are other geometries that do not preserve length for example Weyl geometry but we only need to deal with Riemannian geometry in general relativity. This amounts to assume that the covariant derivative of the metric is zero,

$$\nabla_\mu g_{\nu\rho} = 0. \tag{2.5}$$

If we now use Eq. (2.4) and assume zero torsion, ie $\Gamma^\nu_{\alpha\mu} = \Gamma^\nu_{\mu\alpha}$, we can write the metric connections in terms of the metric and its derivatives as,

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\alpha}(\partial_\rho\, g_{\nu\alpha} + \partial_\nu\, g_{\alpha\rho} - \partial_\alpha\, g_{\nu\rho}). \tag{2.6}$$

It follows from equation (2.6) that the connections are necessarily symmetric, $\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}$. In reality, we are only restricting ourselves to symmetric connections where the torsion vanishes, but, as I said, there are other geometries where torsion does not vanish and the anti-symmetric part of $\Gamma^\mu_{\nu\rho}$ namely,

$$T^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} - \Gamma^\mu_{\rho\nu}, \tag{2.7}$$

is a tensor (unlike the connections $\Gamma^\alpha_{\mu\nu}$) and is called the *torsion tensor* (do not confuse the torsion tensor $T^\mu_{\nu\rho}$ with the energy-momentum tensor $T_{\mu\nu}$). If the torsion tensor vanishes, then the connection is symmetric, $\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}$. We are only interested in torsion-free connections so this is the only time I will refer to the torsion tensor.

2.3 Affine geodesic

Geodesics describe the path (worldline) of a particle acted upon only by gravity. An *affine geodesic* on the spacetime manifold is defined as a curve $x(t)$ that transports their tangent vector parallel to itself,

$$\nabla_{\dot{x}} \dot{x}^\mu = 0, \tag{2.8}$$

where a dot represents the derivative with respect to time $\dot{x} \equiv \partial x/\partial x^0 = \partial x/\partial t$.

Using the definition (2.4) we can write the affine geodesic equation as,

$$\frac{d^2x^\rho}{dt^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = 0. \tag{2.9}$$

Now we want to stop and look at Eq. (2.9) from a Newtonian point of view. We note that the first term in Eq. (2.9) is an acceleration (proportional to a force) so we can informally write,

$$F \propto -\Gamma^\rho_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}, \tag{2.10}$$

that tells us that geodesics can be seen as trajectories of free particles in the spacetime. Equation (2.8) means that the acceleration of the curve has no components in the direction of the surface (its perpendicular to the tangent plane of the surface at each point of the curve). So, *the motion is completely determined by the bending of the surface* (ie by the metric connections $\Gamma^{\rho}_{\mu\nu}$ which is the "correction term" from the flat space). In a fascinating way, we can actually see by Eq. (2.10) the fact that in general relativity the force is seen as a bending of spacetime!

2.4 Riemann, Ricci and Einstein tensors

Covariant differentiation, Eq. (2.2), unlike partial differentiation, does not in general commute. We define the *commutator* of a tensor T^{μ}_{ν} as,

$$\nabla_{\rho}\nabla_{\alpha}T^{\mu}_{\nu} - \nabla_{\alpha}\nabla_{\rho}T^{\mu}_{\nu}. \tag{2.11}$$

Doing the calculation for the commutator (2.11) in the case of a vector X^{μ}, using Eq. (2.4) we find,

$$\nabla_{\alpha}\nabla_{\rho}X^{\mu} = \partial_{\alpha}(\partial_{\rho}X^{\mu} + \Gamma^{\mu}_{\nu\rho}X^{\nu}) + \Gamma^{\mu}_{\sigma\alpha}(\partial_{\rho}X^{\sigma} + \Gamma^{\sigma}_{\nu\rho}X^{\nu}) - \Gamma^{\sigma}_{\rho\alpha}(\partial_{\sigma}X^{\mu} + \Gamma^{\mu}_{\nu\sigma}X^{\nu}), \tag{2.12}$$

and

$$\nabla_{\rho}\nabla_{\alpha}X^{\mu} = \partial_{\rho}(\partial_{\alpha}X^{\mu} + \Gamma^{\mu}_{\nu\alpha}X^{\nu}) + \Gamma^{\mu}_{\sigma\rho}(\partial_{\alpha}X^{\sigma} + \Gamma^{\sigma}_{\nu\alpha}X^{\nu}) - \Gamma^{\sigma}_{\alpha\rho}(\partial_{\sigma}X^{\mu} + \Gamma^{\mu}_{\nu\sigma}X^{\nu}). \tag{2.13}$$

Subtracting the last two equations and assuming $\partial_{\alpha}\partial_{\rho}X^{\mu} = \partial_{\rho}\partial_{\alpha}X^{\mu}$ we obtain the result,

$$\nabla_{\rho}\nabla_{\alpha}X^{\mu} - \nabla_{\alpha}\nabla_{\rho}X^{\mu} = R^{\mu}_{\nu\rho\alpha}X^{\nu}, \tag{2.14}$$

which we can also write,

$$\nabla_{[\rho}\nabla_{\alpha]}X^{\mu} = \frac{1}{2}R^{\mu}_{\nu\rho\alpha}X^{\nu}, \tag{2.15}$$

where $R^{\mu}_{\nu\rho\alpha}$ is defined by,

$$R^{\mu}_{\nu\rho\alpha} = \partial_{\rho}\Gamma^{\mu}_{\nu\alpha} - \partial_{\alpha}\Gamma^{\mu}_{\nu\rho} + \Gamma^{\sigma}_{\nu\alpha}\Gamma^{\mu}_{\sigma\rho} - \Gamma^{\sigma}_{\nu\rho}\Gamma^{\mu}_{\sigma\alpha}. \tag{2.16}$$

Since the left-hand side of Eq. (2.16) is a tensor, follows that $R^{\mu}_{\nu\rho\alpha}$ is a tensor called the *Riemann tensor* or *curvature tensor*. This tensor depends on the metric and its first and second derivative, we can collect the symmetries together and show that,

$$\begin{aligned} R_{\mu\nu\rho\alpha} &= -R_{\mu\nu\alpha\rho} = -R_{\nu\mu\rho\alpha} = R_{\rho\alpha\mu\nu}, \\ R_{\mu\nu\rho\alpha} &+ R_{\mu\alpha\nu\rho} + R_{\mu\rho\alpha\nu} = 0. \end{aligned} \tag{2.17}$$

These symmetries (2.17) reduce the number of independent components from n^4 to $\frac{1}{12}n^2(n^2-1)$, where n is the dimension.

It can be shown that the curvature tensor also satisfies a set of differential identities called the *Bianchi identities*,

$$\nabla_{\mu}R_{\alpha\sigma\nu\rho} + \nabla_{\rho}R_{\alpha\sigma\mu\nu} + \nabla_{\nu}R_{\alpha\sigma\rho\mu} = 0. \tag{2.18}$$

We can also define the *Ricci tensor*, which has a fundamental role in the Einstein equations, by the contraction,

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu} = g^{\rho\alpha}R_{\alpha\mu\rho\nu}. \tag{2.19}$$

Finally, a last contraction defines the *Ricci scalar* or *curvature scalar*,

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{2.20}$$

Now, using Eq. (2.19) and Eq. (2.20), we can define the *Einstein tensor*,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{2.21}$$

which is also symmetric and, by Eq. (2.18), it can be shown that it satisfies the contracted Bianchi identities,

$$\nabla_\nu G^\nu_\mu = 0. \tag{2.22}$$

2.5 Energy-momentum tensor

The energy-momentum tensor describes all properties off matter which affect the spacetime, namely energy density, momentum density, pressure and stress. As we will see, the energy-momentum tensor is the source of the gravitational field in the Einstein equations just as mass density is in Newtonian gravity.

Usually in cosmology, from the symmetry of the spacetime, we consider the EM tensor to be of a perfect fluid (no heat conduction and no viscosity) form,

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}, \tag{2.23}$$

where ρ is the energy density and p is the pressure measured by an observer moving with four-velocity u^μ. The time-time component $T^{00} = \rho$ is the energy density (divided by the square of the speed of light), $T^{0i} = T^{i0}$ gives the momentum density which is equal to the energy flux, T^{ij} gives the flux of momentum i-component in j-direction, in particular T^{ii} represents normal stress, which is pressure, and $T^{ij},\ i \neq j$ represents the shear stress. We can also write,

$$T^\nu_\mu = \begin{bmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}. \tag{2.24}$$

The conservation of energy density and momentum in general relativity is written in terms of the covariant derivative,

$$\nabla_\nu T^{\mu\nu} = 0. \tag{2.25}$$

2.6 Einstein field equations

Having presented the mathematical background, we now want to find out how the curvature of spacetime acts on matter and how matter influences the curvature of spacetime.

Einstein realised that he needed to find the equations that supersede the Poisson equation for the Newtonian potential,

$$\nabla^2 \Phi = 4\pi G \rho, \tag{2.26}$$

where $\nabla^2 = \delta^{ij}\partial_i\partial_j$ is the Laplacian in space, Φ is the gravitational potential, G is the gravitational constant and ρ is the mass density. The generalization should be tensorial and should reduce to Eq. (2.26) in the weak limit ($v \ll c$).

Now, for a relativistic generalisation, we need to take into account all forms of relativistic matter, where mass is not the dominant contribution to the energy density and the momentum p can have the same order of magnitude as ρ. So we now know that the generalization of the mass density as a tensor is the energy-momentum tensor $T_{\mu\nu}$ that will tell spacetime how to curve. The spacetime degrees of freedom are given by the metric and we want equations of motion, which are second order, so it can only involve the metric and its first and second derivatives. As a first guess we could think of using the Ricci tensor (2.19) (not the curvature tensor because it has to be of the same type as $T_{\mu\nu}$) and write $R_{\mu\nu} = \kappa^2 T_{\mu\nu}$ (where κ is a constant that will be given by the Poisson equation (2.26) and actually Einstein thought about using this equation at some point. But from the conservation of the energy momentum tensor (2.25) we know that both sides of the equation must have zero divergence, and we can easily see, using Eq. (2.18), that,

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R \neq 0. \tag{2.27}$$

But we can write Eq. (2.27) as,

$$\nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0, \tag{2.28}$$

thus, we can use,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T_{\mu\nu}, \tag{2.29}$$

where we identify the left-hand side being the Einstein tensor (2.21), and therefore we finally get,

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu}, \tag{2.30}$$

that are the *Einstein field equations.*

Einstein equations have to reduce to the Poisson equation in the weak limit ($v \ll c$) and it can easily be shown using the metric,

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)\delta_{ij}dx^i dx^j, \tag{2.31}$$

where $|\Phi| \ll 1$. In this case the Einstein equations (2.10) reduce to the Poisson equation (2.26), where we can set the value $\kappa^2 = 8\pi G$.

3 Standard model of Cosmology

Cosmology is the study of the Universe as a whole at their largest scale. Given obvious difficulties on their study the standard model of Cosmology is based on some basic assumptions. One of the main assumptions is the *Cosmological principle* which states that at large scales (> 100) Mpc the Universe is presented as homogeneous (all places look the same) and isotropic (all directions look the same). This is, at large scales the properties of the Universe look the same for all observers. The Universe is under expansion and the mean distance l of their constituents is given by the Hubble law,

$$\frac{dl(t)}{dt} = v(t) = H_0 l(t), \tag{3.1}$$

where H_0 is the *Hubble* parameter. The dynamics of this expansion is described by the Einstein field equations (2.6). This expansion started from an extremely hot and dense phase where the energy of the Universe, at that time, was dominated by the energy of radiation.

3.1 The FRW model

A common approximation is that there is a slicing of spacetime into spacelike slices, or hypersurfaces, which are homogeneous and isotropic. The proper time t which labels the hypersurfaces is called the *cosmic time*. The Friedmann-Robertson-Walker model, or FRW model, can be described by the spatially homogeneous and isotropic metric,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right], \tag{3.2}$$

where k describes the curvature and $a(t)$ is the scale factor, and this metric (3.2) is called the Friedmann-Robertson-Walker (FRW) metric.

The metric (3.2) can be written in Cartesian coordinates as,

$$ds^2 = -dt^2 + a^2(t)\frac{1}{1+kr^2}\delta_{ij}dx^i dx^j. \tag{3.3}$$

Observations tells us that our Universe is nearly flat. Through this thesis I will usually refer to the FRW metric considering $k = 0$.

In order to find how the scale factor $a(t)$ evolves we need to consider the Einstein equations (2.30). Using the FRW metric (3.2) and considering that the energy-momentum tensor has the perfect fluid form (2.5) the Einstein equations reduce to two ordinary non-linear differential equations,

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \tag{3.4}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p), \tag{3.5}$$

where G is Newton's gravitational constant. Equation (3.4) is usually called the *Friedmann* equation and (3.5) the *Raychaudhuri* equation.

From the conservation of the energy-momentum tensor (2.25) we get the *continuity* equation,

$$\dot{\rho} = 3H(\rho + p) = 0. \tag{3.6}$$

The *critical density* is defined as $\rho_c = \frac{3H^2}{8\pi G}$. When talking about the energy density of the Universe, we often use the dimensionless *density parameter*,

$$\Omega_i = \frac{\rho_i}{\rho_c}, \qquad \Omega = \sum_i \Omega_i, \qquad \Omega_k = -\frac{k}{H^2 a^2}. \tag{3.7}$$

Dividing now Eq. (3.4) by H^2 we can rewrite the Friedmann equation as,

$$\Omega + \Omega_k = 1. \tag{3.8}$$

We can also define the equation of state $w = p/\rho$. There are several types of matter in the Universe (different components) that behave in different forms affecting the evolution of the Universe:

- **Radiation**: matter for which $w = 1/3$, for example the case for a gas of free ultrarelativistic particles in which the energy density is dominated by the kinetic energy. From the continuity equation we arrive at,

$$\frac{\dot{\rho}_\gamma}{\rho_\gamma} = -4\frac{\dot{a}}{a} \quad \Rightarrow \quad \rho_\gamma \propto a^{-4}. \tag{3.9}$$

- **Dust**: matter for which the pressure is zero,or at least negligible $|p| \ll \rho$ i.e. $w = 0$. This is the case for a gas of free non-relativistic particles where the dominant energy density is the mass. From the continuity equation we arrive at,

$$\frac{\dot{\rho}_\gamma}{\rho_\gamma} = -3\frac{\dot{a}}{a} \quad \Rightarrow \quad \rho_\gamma \propto a^{-3}. \tag{3.10}$$

- **Cosmological constant**: Observations of distant supernovae, called SnIa, tell us that the universe is accelerating. For the universe to accelerate we must have $\ddot{a} > 0$ which implies $p < \rho/3$. One type of matter with this behaviour is the vacuum energy, in which $p = -\rho$, or a cosmological constant, depending on which side we consider it in Einstein equations,

$$G_{\mu\nu} \;+\; \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{3.11}$$

$$G_{\mu\nu} \;=\; 8\pi G\left(T_{\mu\nu} - \frac{\Lambda}{8\pi G} g_{\mu\nu}\right) = 8\pi G \tilde{T}_{\mu\nu}, \tag{3.12}$$

  where $\Lambda/8\pi G = \rho_\Lambda = -p_\Lambda$. Substituting in the Friedmann equation we have,

$$a \propto e^{\sqrt{\Lambda}t}. \tag{3.13}$$

Lemaître in 1927 and Hubble in 1929 discovered that the redshifts of galaxies were proportional to their distance. This is, the light from distant galaxies is redder (longer wavelength) when it arrives on Earth. This redshift can be determined with high accuracy from spectral lines in which, their original wavelength λ_0 can be measured in the laboratory. The redshift z is defined as,

$$z \equiv \frac{\lambda - \lambda_0}{\lambda_0} \Rightarrow 1 + z = \frac{\lambda}{\lambda_0} = \frac{a_0}{a}, \tag{3.14}$$

where λ is the observed wavelength. The redshift is observed to be independent of wavelength, and it follows the relation,

$$cz = H_0 d, \tag{3.15}$$

which is the known *Hubble law.*

For a universe with radiation, baryons, dark matter and dark energy, the Friedmann equation can be written as,

$$H^2 = \frac{8\pi G}{3}\left[\rho_m \left(\frac{a_0}{a}\right)^3 + \rho_\gamma \left(\frac{a_0}{a}\right)^4 + \rho_\Lambda\right] - \frac{k}{a^2}. \tag{3.16}$$

Dividing both sides by H^2 and using the relation between redshift and the scale factor we find,

$$H^2 = H_0^2\left[\Omega_m(1+z)^3 + \Omega_\gamma(1+z)^4 + \Omega_\Lambda + \Omega_k(1+z)^2\right], \tag{3.17}$$

where,

$$\Omega_k \equiv -\frac{k}{a_0^2 H_0^2}. \tag{3.18}$$

The Hubble parameter at present is given by,

$$H_0 \approx 67.3 \text{ Mpc}^{-1} \text{ km s}^{-1}. \tag{3.19}$$

4 Inflation

Primordial inflation was proposed by Alan Guth, in 1981, in order to solve some cosmological puzzles such as the horizon, flatness and entropy problems [1, 2]. Inflation was a period of accelerated expansion in the early universe ($\sim 10^{-36}s$ to $\sim 10^{-33}s$ after the Big Bang). In this section I will describe the standard model of inflation considering a single scalar field interacting with its potential, where most literature stands on. For a review see for example the lecture notes *"Inflation and the Theory of Cosmological Perturbations"* by A. Riotto (2002) [37].

4.1 The inflaton

The condition for inflation, i.e., a period of accelerated expansion, $\ddot{a} > 0$ can be satisfied by considering a scalar field ϕ which we will call the inflation. Considering the action,

$$S = \int d^4x\sqrt{-g}\mathcal{L} = -\int d^4x\sqrt{-g}\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi)\right], \tag{4.1}$$

where $\sqrt{-g} = a^3$ for FLRW. The resulting equations of motion are,

$$\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2}\nabla^2\phi + V'(\phi) = 0. \tag{4.2}$$

The energy-momentum tensor reads,

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}. \tag{4.3}$$

So the energy and pressure of the field are defined as,

$$T_{00} = \rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi) + \frac{1}{2a^2}(\nabla\phi)^2, \tag{4.4}$$

$$T_{ii} = p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi) - \frac{1}{6a^2}(\nabla\phi)^2. \tag{4.5}$$

Writing the field as background + perturbation, i.e.,

$$\phi(t,\vec{x}) = \phi_0(t) + \delta\phi(t,\vec{x}), \tag{4.6}$$

where $\delta\phi(t,\vec{x})$ are quantum perturbations around ϕ_0. Setting $\phi_0(t) \to \phi(t)$ for simplicity, now,

$$T_{00} = \rho_\phi(t) = \frac{\dot{\phi}^2}{2} + V(\phi), \tag{4.7}$$

$$T_{ii} = p_\phi(t) = \frac{\dot{\phi}^2}{2} - V(\phi). \tag{4.8}$$

So if $V(\phi) \gg \dot{\phi}^2 \Rightarrow p_\phi \approx -\rho_\phi \approx -V(\phi)$. Inflation is generated by the vacuum energy of the inflaton.

4.2 Slow roll conditions and number of e-folds

To have $\ddot{a} > 0$ we need,

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 > 0 \Rightarrow -\frac{\dot{H}}{H^2} < 1, \tag{4.9}$$

where,

$$-\frac{\dot{H}}{H^2} = \frac{3}{2}\frac{\dot{\phi}^2}{\frac{\dot{\phi}^2}{2} + V} < 1 \quad \Rightarrow \quad V > \dot{\phi}^2 \quad \Rightarrow \quad p_\phi \approx -\rho_\phi. \tag{4.10}$$

In order to ensure inflation $V > \dot{\phi}^2$, and to ensure that we have a sufficient number of e-folds, otherwise the last conditions are violated, it is used the slow roll approximations,

$$H^2 \approx \frac{8\pi G}{3}V \quad \text{and} \quad 3H\dot{\phi} + V' = 0. \tag{4.11}$$

So we will define the first slow roll parameter as,

$$\epsilon \equiv -\frac{\dot{H}}{H^2}. \tag{4.12}$$

Substituting the slow roll approximations in ϵ we have,

$$\epsilon = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2, \tag{4.13}$$

and $\epsilon \ll 1$ for slow roll. Now we should impose that ϵ varies slowly in order to have a sufficient number of e-folds,

$$\frac{d\,\ln\epsilon}{d\,\ln a} = \frac{\epsilon'}{\epsilon} \simeq -2(\eta - 2\epsilon) \approx 0 \quad \Rightarrow \quad \eta \approx \epsilon \ll 1, \tag{4.14}$$

where,

$$\eta \equiv \frac{1}{8\pi G}\frac{V''}{V}, \tag{4.15}$$

is the second slow roll parameter. So we also need $\eta \ll 1$. There is also a third slow roll parameter that comes from the fact that we impose also $\frac{d\,\ln\eta}{d\,\ln a} \approx 0$, defining,

$$\xi^2 \equiv \frac{1}{(8\pi G)^2}\frac{V''V'}{V^2} \quad \Rightarrow \quad \xi^2 \sim \mathcal{O}(\eta^2) \sim \mathcal{O}(\epsilon^2) \ll 1. \tag{4.16}$$

We will define N as the number of e-folds, which represent the number of Hubble times between the horizon exit and the end of inflation, as

$$N \equiv \int_{a_N}^{a_C} d\,\ln a = \int H\,dt = \int \frac{H}{\dot{\phi}}\,d\phi = -\int \frac{3H^2}{V'}\,d\phi = 8\pi G\int_{\phi_N}^{\phi_C}\frac{V}{V'}\,d\phi. \tag{4.17}$$

4.3 Power spectrum

Inflation was first introduced as a possible solution for the horizon, flatness and entropy problems. It happens that inflation also generates the density perturbations spectrum and the gravitational waves spectrum crutial in the formation of structure that we observe today. The space is filled with quantum fluctuations that are like waves with every possible wavelength (Fourier spectrum). During inflation, the wavelength of a fluctuation quickly exceeds the Hubble radius and its amplitude is frozen because fluctuations become causally disconnected. Once inflation ends, the Hubble radius grows quicker than the scale factor and eventually the fluctuation re-enters the horizon.

The power spectrum is a useful quantity to describe the properties of the fluctuations. Considering a generic quantity $g(t,\vec{x})$ with Fourier transform,

$$g(\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} g_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} = \int \frac{d^3x}{(2\pi)^{3/2}} g(\vec{x}) e^{-i\vec{k}\cdot\vec{x}}. \tag{4.18}$$

With $g^*_{\vec{k}} = g_{-\vec{k}}$ we can write,

$$\left\langle g_{\vec{k}} g^*_{\vec{p}} \right\rangle = \delta^{(3)}(\vec{k}-\vec{p}) \frac{2\pi^2}{k^3} \mathcal{P}_g(k). \tag{4.19}$$

where,

$$\mathcal{P}_g = \frac{k^3}{2\pi^2} P(k), \tag{4.20}$$

and P is called the *power spectrum* defined as,

$$P(k) = (2\pi)^3 \left| g_{\vec{k}} \right|^2. \tag{4.21}$$

4.4 Metric scalar fluctuations

Any perturbation in the field ϕ is a perturbation on the energy-momentum tensor,

$$\delta\phi \to \delta T_{\mu\nu} = \frac{1}{8\pi G} \delta G \Rightarrow \delta\phi \to \delta g_{\mu\nu}. \tag{4.22}$$

We will perturb the metric as,

$$g_{\mu\nu} = g^0_{\mu\nu}(t) + \delta g_{\mu\nu}(\vec{x}, t), \tag{4.23}$$

with $\delta g_{\mu\nu} \ll g^0_{\mu\nu}$. We can decompose the perturbations in scalar, vector and tensorial perturbations. The perturbed line element read,

$$ds^2 = a^2[(-1+2A)d\tau^2 + 2\partial_i B d\tau dx^i + ((1-2\psi)\delta_{ij} + D_{ij}E)dx^i dx^j], \tag{4.24}$$

where $D_{ij} = \partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2$.

Using the Einstein equations we define the comoving curvature perturbation,

$$\mathcal{R} = \psi + \mathcal{H}\frac{\delta\phi}{\phi'}, \tag{4.25}$$

where $\mathcal{H} = a'/a$ and the prime means differentiation with respect to conformal time τ.

The power spectrum for this quantity at superhorizon scales is,

$$\mathcal{P}_{\mathcal{R}} = \frac{4\pi}{m_{\rm Pl}^2\epsilon}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{n_{\mathcal{R}}-1}, \tag{4.26}$$

or,

$$\mathcal{P}_{\mathcal{R}} = A_{\mathcal{R}}\left(\frac{k}{aH}\right)^{n_{\mathcal{R}}-1}, \tag{4.27}$$

where,

$$A_{\mathcal{R}} \equiv \frac{4\pi}{m_{\rm Pl}^2\epsilon}\left(\frac{H}{2\pi}\right)^2. \tag{4.28}$$

At horizon exit $\mathcal{P}_{\mathcal{R}} = A_{\mathcal{R}} \propto H^2/\epsilon$, hence, the spectral index reads,

$$n_{\mathcal{R}} - 1 \equiv \frac{d\,\ln\mathcal{P}_{\mathcal{R}}}{d\,\ln k} = -6\epsilon + 2\eta. \tag{4.29}$$

4.5 Gravitational waves

The line element for tensor perturbations can be written as,

$$g_{\mu\nu} = a^2(\tau)[-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j], \tag{4.30}$$

where $|h_{ij}| \ll 1$.

The power spectrum of gravitational waves, at large scales read,

$$\mathcal{P}_T = A_T\left(\frac{k}{aH}\right)^{n_T}, \tag{4.31}$$

where,

$$A_T \equiv \frac{64\pi}{m_{\rm Pl}^2\epsilon}\left(\frac{H}{2\pi}\right)^2, \tag{4.32}$$

and, the spectral index is,

$$n_T \equiv \frac{d\,\ln\mathcal{P}_T}{d\,\ln k} = -2\epsilon. \tag{4.33}$$

Thus we can write the consistency relation which has to be satisfied,

$$r \equiv \left.\frac{\mathcal{P}_T}{\mathcal{P}_{\mathcal{R}}}\right|_{k=aH} = -8n_T, \tag{4.34}$$

where r is called the tensor to scalar ratio. If this relation is not satisfied by observations we have to consider other scenario for inflation (for example multi-field inflation, non-canonical kinetic terms or higher order fields like 3-forms).

5 3-form Cosmology

In this section I will start by presenting the standard 3-form field model in four dimensions, already studied in Ref. [13], and then study some inflationary scenarios driven by a single 3-form in five dimensions, more precisely in the Randall-Sundrum II braneworld, which I will mainly focus on the differences from the standard four dimensional case. Finnaly I explore the dynamics of two coupled 3-forms, also in an inflationary context, contrasting with the uncoupled N 3-forms case [16] for some suitable choices of the potential.

5.1 Standard single 3-form model

A 3-form, $A_{\mu\nu\rho}$, is a rank 3 totally antisymmetric tensor,

$$A_{\mu\nu\rho} = -A_{\nu\mu\rho}. \tag{5.1}$$

For example, when we define the cross-product as,

$$(\vec{u} \times \vec{v})_i = \epsilon_{ijk} u_j v_k, \tag{5.2}$$

the *Levi-Civita* symbol, ϵ_{ijk}, is a 3-form.

First, lets start by considering a flat *Friedmann-Lemaitre-Robertson-Walker* (FLRW or simply FRW) cosmology, where the line element is given by,

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2, \tag{5.3}$$

where $a(t)$ is the scale factor of the Universe as a function of the cosmic time t.

We shall focus on a theory minimally coupled to Einstein gravity. The action for a single 3-form field $A_{\mu\nu\rho}$ can be written as,

$$S = -\int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R - \frac{1}{48}F^2 - V(A^2)\right), \tag{5.4}$$

where g in the determinant of the metric and $\kappa^2 = 8\pi G$. The first term inside brackets is the standard Einstein-Hilbert lagrangian where R is the *Ricci* scalar. The last two terms are the 3-form lagrangian. I use the notation where squaring means contracting all the indexes, $A^2 \equiv A_{\mu\nu\rho}A^{\mu\nu\rho}$. Finally $F_{\mu\nu\rho\sigma}$ is the generalization of the Faraday form appearing in Maxwell theory,

$$F_{\mu\nu\rho\sigma} = 4\nabla_{[\mu}A_{\nu\rho\sigma]} = \nabla_\mu A_{\nu\rho\sigma} - \nabla_\sigma A_{\mu\nu\rho} + \nabla_\rho A_{\sigma\mu\nu} - \nabla_\nu A_{\rho\sigma\mu}, \tag{5.5}$$

where square brackets denotes antisymmetrization. So F is a 4-form and we have $F^2 \equiv F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma} = \nabla_{[\mu}A_{\nu\rho\sigma]}\nabla^{[\mu}A^{\nu\rho\sigma]}$.

We derive the equations of motion (5.7) by writing the Euler-Lagrange equations for the general 3-form lagrangian (5.4),

$$\frac{\partial\mathcal{L}}{\partial A} - \nabla\cdot\left[\frac{\partial\mathcal{L}}{\partial(\nabla A)}\right] = 0 \quad \Rightarrow \quad -2V'(A^2)A + \frac{1}{6}(\nabla\cdot F) = 0, \tag{5.6}$$

which is equivalent to,

$$\nabla\cdot F = 12V'(A^2)A. \tag{5.7}$$

For the calculation of the energy-momentum tensor it is helpful to remember the Jacobi's formula, the rule for differentiating a determinant, that gives,

$$\delta g = \delta \det(g_{\mu\nu}) = g g^{\mu\nu} \delta g_{\mu\nu}. \tag{5.8}$$

I am also going to use the notation where circling means contracting all but the first index, $(A \circ A)_{\mu\nu} = A_{\mu\alpha\beta} A_{\nu}^{\ \alpha\beta}$.

Using the 3-form lagrangian,

$$\mathcal{L} = -\frac{1}{48} F^2 - V(A^2), \tag{5.9}$$

we can calculate the energy-momentum tensor using the following formula, consequence of Noether's theorem,

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\partial(\mathcal{L}\sqrt{-g})}{\partial g^{\mu\nu}}, \tag{5.10}$$

resulting in,

$$T_{\mu\nu} = g_{\mu\nu}\mathcal{L} + \frac{1}{6}(F \circ F)_{\mu\nu} + 6V'(A^2)(A \circ A)_{\mu\nu}. \tag{5.11}$$

Assuming a homogeneous and isotropic universe (cosmological principle) the 3-form field depend only on time and hence only the space-like components will be dynamical, so we set

$$A_{0ij} = 0, \tag{5.12}$$

and the nonzero components of the unperturbed 3-form are given by,

$$A_{ijk} = a^3(t)\epsilon_{ijk}\chi(t) \quad \Rightarrow \quad A^2 = 6\chi^2, \tag{5.13}$$

where $\chi(t)$ is a comoving field associated with the 3-form and ϵ_{ijk} is the standard Levi-Civita symbol. The dynamics of the Universe is then governed by the behaviour of the scalar quantity $\chi(t)$ which is directly related to the 3-form. When written in terms of this scalar quantity the equations of motion which govern the behaviour of the Universe, and the role of the 3-form potential, are straightforward to interpret.

We can now express the equations of motion (5.7) in terms of the comoving field χ,

$$\ddot{\chi} + 3H\dot{\chi} + 3\dot{H}\chi + V_{,\chi} = 0. \tag{5.14}$$

The third term is a new feature from the 3-form model which is not present in the standard scalar field theory.

From now on, for the rest of this thesis, I will adopt units where $\kappa^2 = 8\pi G = 1$ unless it is convenient otherwise.

Now that we have the energy-momentum tensor, Eq. (5.11), we can calculate the other two evolution equations, Friedmann and Raychaudhuri, through the Einstein equations (2.30), which read,

$$3H^2 = \frac{1}{3}\left(\frac{1}{2}(\dot{\chi} + 3H\chi)^2 + V(\chi)\right), \tag{5.15}$$

$$\dot{H} = -\frac{1}{2}V_{,\chi}\chi. \tag{5.16}$$

We can thus define the energy density and pressure of the field as,

$$\rho_\chi = \frac{1}{2}(\dot{\chi}+3H\chi)^2 + V(\chi), \tag{5.17}$$

$$p_\chi = -\frac{1}{2}(\dot{\chi}+3H\chi)^2 - V(\chi) + V_{,\chi}\chi. \tag{5.18}$$

The equation of state, $w_\chi = \rho_\chi/p_\chi$,can be written as,

$$w_\chi = -1 + \frac{V_{,\chi}\chi}{\rho_\chi}, \tag{5.19}$$

where we see directly that whenever the potential or just its slope vanishes, the field acts like a cosmological constant ($w_\chi = -1$). Furthermore, whenever the slope of V is negative (positive) if χ is positive (negative), the comoving field behaves as a phantom field ($w_\chi < -1$).

In order to study the dynamics of the system, it is useful to express the equations of motion in terms of the dimensionless variables,

$$x \equiv \chi_n, \tag{5.20}$$

$$w \equiv \frac{\chi_n' + 3\chi_n}{\sqrt{6}}, \tag{5.21}$$

where a prime means differentiating in respect to the number of e-folds $N = \ln a(t)$, such that $x' \equiv dx/dN$.The resulting equations of motion are,

$$x' = 3\left(\sqrt{\frac{2}{3}}w - x\right), \tag{5.22}$$

$$w' = \frac{3}{2}\frac{V_{,x}}{V}(1-w^2)\left[xw - \sqrt{\frac{2}{3}}\right], \tag{5.23}$$

subject to the Friedmann constraint,

$$w^2 + y^2 = 1, \tag{5.24}$$

where,

$$y^2 \equiv \frac{V}{3H^2}. \tag{5.25}$$

The critical points of the dynamical system Eqs. (5.22) (5.23) are,

	x	w	$V_{,x}/V$	Description
A	$\pm\sqrt{\frac{2}{3}}$	± 1	any	kinetic domination
B	x_{ext}	$\sqrt{\frac{3}{2}}x_{\text{ext}}$	0	potential extrema

Table 1: Critical points of the dynamical system.

5.2 N uncoupled 3-form fields model

In this subsection I will present the standard model for N uncoupled 3-form fields. This work was described in Ref. [16] which generalizes the background equations related to a single 3-form [11, 12].

We will first consider a flat FLRW metric described by the line element,

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2, \tag{5.26}$$

where $a(t)$ is the scale factor and t being the cosmic time. We will consider the general action for Einstein gravity and the N uncoupled 3-forms, i.e., the generalization of Eq. (5.4), given by,

$$S = -\int d^4x\sqrt{-g}\left[\frac{1}{2}R - \sum_{n=1}^{N}\left(\frac{1}{48}F_n^2 + V_n(A_n^2)\right)\right], \tag{5.27}$$

units where $\kappa^2 = 8\pi G \equiv 1$, where R is the Ricci scalar, $A^{(n)}_{\mu\nu\rho}$ is the n-th 3-form and $F^{(n)}_{\mu\nu\rho\sigma}$ is the strength tensor associated with the respective n-th 3-form. Assuming a homogeneous and isotropic universe, remembering Eq. (5.13), we have,

$$A^{(n)}_{ijk} = a^3(t)\epsilon_{ijk}\chi_n(t) \quad \Rightarrow \quad A_n^2 = 6\chi_n^2, \tag{5.28}$$

where ϵ_{ijk} is the Levi-Civita symbol and χ_n is a comoving field associated with the respective 3-form A_n.

As in the standard single 3-form model (5.1), the strength tensor associated with the n-th 3-form, is given by,

$$F^{(n)}_{\mu\nu\rho\sigma} = 4\nabla_{[\mu}A^{(n)}_{\nu\rho\sigma]}. \tag{5.29}$$

We have the following equations of motion for the N 3-form fields,

$$\ddot{\chi}_n + 3H\dot{\chi}_n + 3\dot{H}\chi_n + V_{n,\chi_n} = 0. \tag{5.30}$$

Because there is dependence on the derivative of the Hubble parameter $\dot{H}$, it is straightforward to see that there is a peculiar coupling of equations (5.30) for each value of n.

For this setting, the evolution equations, Friedmann and Raychaudhuri, read,

$$3H^2 = \frac{1}{3}\left(\frac{1}{2}\sum_{n=1}^{N}(\dot{\chi}_n + 3H\chi_n)^2 + \sum_{n=1}^{N}V_n(\chi_n)\right), \tag{5.31}$$

$$\dot{H} = -\frac{1}{2}\sum_{n=1}^{N}V_{n,\chi_n}\chi_n. \tag{5.32}$$

The total energy density and pressure of the N 3-form fields are,

$$\rho_N = \frac{1}{2}\sum_{n=1}^{N}\left[(\dot{\chi}_n + 3H\chi_n)^2 + 2V_n\right], \tag{5.33}$$

$$p_N = -\frac{1}{2}\sum_{n=1}^{N}\left[(\dot{\chi}_n + 3H\chi_n)^2 + 2V_n - 2V_{n,\chi_n}\chi_n\right]. \tag{5.34}$$

We can also write Eq. (5.30) as,

$$\ddot{\chi}_n + 3H\dot{\chi}_n V^{eff}_{n,\chi_n} = 0, \tag{5.35}$$

where we identify,

$$V^{eff}_{n,\chi_n} \equiv 3\dot{H}\chi_n + V_{n,\chi_n} = V_{n,\chi_n}\left(1 - \frac{3}{2}\chi_n^2\right) - \frac{3}{2}\chi_n \left(\sum_{\substack{m=1\\ m\neq n}}^{N} V_{m,\chi_m}\chi_m\right). \tag{5.36}$$

5.3 Two coupled 3-form fields

In this subsection I will describe the dynamics of two coupled 3-forms and show that it can also bring viable inflationary solutions. I will focus on the role of the coupling on the dynamics compared with the uncoupled case (5.2). I will also explore rotations in the SO(2) group and show that in some particular cases, the coupled model can be expressed in terms of two uncoupled rotated in the field space 3-forms.

5.3.1 Two coupled 3-forms model

Following the same procedure as in (5.2) we will first consider a flat FLRW metric given by Eq. (5.26). The general action for the two coupled 3-form fields minimally coupled to Einstein gravity reads,

$$S = -\int d^4x\sqrt{-g}\left[\frac{1}{2}R - \frac{1}{48}(F^2_{(1)} + F^2_{(2)}) - V_1(A^2_{(1)}) - V_2(A^2_{(2)}) - V_{12}(C)\right], \tag{5.37}$$

where $A^{(1)}_{\mu\nu\rho}$ and $A^{(2)}_{\mu\nu\rho}$ are the 3-forms and $C = A^{(1)}_{\mu\nu\rho}A^{\mu\nu\rho}_{(2)}$. We can see that the coupling is expressed in $V_{12}(C) = V(A^{(1)}_{\mu\nu\rho}A^{\mu\nu\rho}_{(2)})$.

We have now for the Maxwell tensors,

$$F^{(1)}_{\alpha\beta\gamma\delta} = 4\nabla_{[\alpha}A^{(1)}_{\beta\gamma\delta]}, \tag{5.38}$$

$$F^{(2)}_{\alpha\beta\gamma\delta} = 4\nabla_{[\alpha}A^{(2)}_{\beta\gamma\delta]}, \tag{5.39}$$

where antisymmetrization is denoted by square brackets.

Following the same assumption of a homogeneous and isotropic universe, the nonzero components of the 3-forms are given by,

$$A^{(1)}_{ijk} = a^3(t)\epsilon_{ijk}\chi_1 \;\Rightarrow\; A^2_{(1)} = 6\chi_1^2, \tag{5.40}$$

$$A^{(2)}_{ijk} = a^3(t)\epsilon_{ijk}\chi_2 \;\Rightarrow\; A^2_{(2)} = 6\chi_2^2, \tag{5.41}$$

$$C = A^{(1)}_{ijk}A^{ijk}_{(2)} \;\Rightarrow\; C = 6\chi_1\chi_2, \tag{5.42}$$

where χ_n is the comoving field associated with the respective 3-form. We usually refer to $V_n(A^2_{(n)})$ just as V_n.

Considering the lagrangian density (5.37) the Euler-Lagrange equations for the 3-form $A^{(1)}_{\mu\nu\rho}$ read,

$$\frac{\partial \mathcal{L}}{\partial A_{(1)}} - \nabla \cdot \left[\frac{\partial \mathcal{L}}{\partial(\nabla A_{(1)})} \right] = 0 \Rightarrow -2V_1'(A_{(1)}^2)A_{(1)} - V_{12}'(C)A_{(2)} + \frac{1}{6}(\nabla \cdot F_{(1)}) = 0. \quad (5.43)$$

In the same way we obtain the motion equations for the other 3-form $A^{(2)}_{\mu\nu\rho}$ and conclude that,

$$\nabla \cdot F_{(1)} = 12V_1'(A_{(1)}^2)A_{(1)} + 6V_{(12)}'(C)A_{(2)}, \quad (5.44)$$
$$\nabla \cdot F_{(2)} = 12V_2'(A_{(2)}^2)A_{(2)} + 6V_{(12)}'(C)A_{(1)}. \quad (5.45)$$

Comparing with Eq. (5.7) we note that the last two terms in Eq. (5.44) and Eq. (5.45) are only present when there is a coupling. The equations of motion in terms of the comoving fields read,

$$\ddot{\chi}_1 + 3H\dot{\chi}_1 + 3\dot{H}\chi_1 + V_{1,\chi_1} + \chi_2\left(\frac{V_{12,\chi_1}}{\chi_2} + \frac{V_{12,\chi_2}}{\chi_1}\right) = 0, \quad (5.46)$$
$$\ddot{\chi}_2 + 3H\dot{\chi}_2 + 3\dot{H}\chi_2 + V_{2,\chi_2} + \chi_1\left(\frac{V_{12,\chi_1}}{\chi_2} + \frac{V_{12,\chi_2}}{\chi_1}\right) = 0, \quad (5.47)$$

where the last terms in parenthesis express the coupling and are new comparing with the uncoupled case (5.30). These last terms can also be written using the relation,

$$\frac{V_{12,\chi_1}}{\chi_2} + \frac{V_{12,\chi_2}}{\chi_1} = \frac{1}{\chi_2}\frac{\partial V(\chi_1\chi_2)}{\partial(\chi_1\chi_2)}\frac{\partial(\chi_1\chi_2)}{\partial\chi_1} + \frac{1}{\chi_1}\frac{\partial V(\chi_1\chi_2)}{\partial(\chi_1\chi_2)}\frac{\partial(\chi_1\chi_2)}{\partial\chi_2} = 2V_{12,(\chi_1\chi_2)}, \quad (5.48)$$

that will give,

$$\ddot{\chi}_1 + 3H\dot{\chi}_1 + 3\dot{H}\chi_1 + V_{1,\chi_1} + 2\chi_2 V_{12,(\chi_1\chi_2)} = 0, \quad (5.49)$$
$$\ddot{\chi}_2 + 3H\dot{\chi}_2 + 3\dot{H}\chi_2 + V_{2,\chi_2} + 2\chi_1 V_{12,(\chi_1\chi_2)} = 0. \quad (5.50)$$

It is also usefull to keep in mind that if there is no coupling, $V_{12} = 0$, we recover the uncoupled 3-forms case [16].

Varying the action with respect to the metric, we find for the energy-momentum tensor,

$$T_{\mu\nu} = g_{\mu\nu}\mathcal{L} + \frac{1}{6}(F_{(1)} \circ F_{(1)})_{\mu a} + \frac{1}{6}(F_{(2)} \circ F_{(2)})_{\mu a} + 6V_1'(A_{(1)}^2)(A_{(1)} \circ A_{(1)})_{\mu a}$$
$$+6V_2'(A_{(2)}^2)(A_{(2)} \circ A_{(2)})_{\mu a} + 6V_{12}'(C)(A_{(1)} \circ A_{(2)})_{\mu a}, \quad (5.51)$$

which comparing with the uncoupled case we see that a new term, the last one, arises consequence of the coupling.

The evolution equations now read,

$$H^2 = \frac{1}{3}\left\{\frac{1}{2}\left[(\dot{\chi}_1 + 3H\chi_1)^2 + (\dot{\chi}_2 + 3H\chi_2)^2 + 2V_1 + 2V_2 + 2V_{12}\right]\right\}, \quad (5.52)$$
$$\dot{H} = -\frac{1}{2}\left[V_{1,\chi_1}\chi_1 + V_{2,\chi_2}\chi_2 + 4V_{12,(\chi_1\chi_2)}\chi_1\chi_2\right]. \quad (5.53)$$

In both equations a new term arises from the coupling and is expressed through the potential V_{12} and its derivative. We identify the energy density and pressure of the two 3-forms as,

$$\rho = \frac{1}{2}\left[(\dot{\chi}_1 + 3H\chi_1)^2 + (\dot{\chi}_2 + 3H\chi_2)^2 + 2V_1 + 2V_2 + 2V_{12}\right], \tag{5.54}$$

$$p = -\frac{1}{2}\left[(\dot{\chi}_1 + 3H\chi_1)^2 + (\dot{\chi}_2 + 3H\chi_2)^2 + 2V_1 + 2V_2 + 2V_{12} - 2V_{1,\chi_1}\chi_1 - 2V_{2,\chi_2}\chi_2 - 8V_{12,(\chi_1\chi_2)}\chi_1\chi_2\right]. \tag{5.55}$$

Once again we note the arising of a new term from the coupling potential of the fields and its derivative.

5.3.2 Dynamics of the two coupled 3-forms

In order to study the dynamics of the system we will introduce the dimensionless variables,

$$x_n \equiv \kappa\chi_n, \tag{5.56}$$

$$w_n \equiv \kappa\frac{\chi_n' + 3\chi_n}{\sqrt{6}}, \tag{5.57}$$

n is the n-th 3-form and a prime means differentiating in respect to the number of e-folds $N = \ln a(t)$, so that $x_n' \equiv dx_n/dN$.

In this notation, the Friedmann equation read,

$$H^2 = \frac{1}{3}\frac{V_1 + V_2 + V_{12}}{1 - w_1^2 - w_2^2}. \tag{5.58}$$

Let us now define,

$$V^{\text{eff}}_{1,\chi_1} = 3\dot{H}\chi_1 + V_{1,\chi_1} + \chi_2\left(\frac{V_{12,\chi_1}}{\chi_2} + \frac{V_{12,\chi_2}}{\chi_1}\right), \tag{5.59}$$

$$V^{\text{eff}}_{2,\chi_2} = 3\dot{H}\chi_2 + V_{2,\chi_2} + \chi_1\left(\frac{V_{12,\chi_1}}{\chi_2} + \frac{V_{12,\chi_2}}{\chi_1}\right), \tag{5.60}$$

the effective potentials of the 3-forms $A^{(1)}_{\mu\nu\rho}$ and $A^{(2)}_{\mu\nu\rho}$ respectively.

Thus, we can write the equations of motion, (5.49) and (5.50), as

$$H^2 x_n'' + (3H^2 + \dot{H})x_n' + V^{\text{eff}}_{n,x_n} = 0, \tag{5.61}$$

which can be written in the autonomous form,

$$x_1' = 3\left(\sqrt{\frac{2}{3}}w_1 - x_1\right), \tag{5.62}$$

$$x_2' = 3\left(\sqrt{\frac{2}{3}}w_2 - x_2\right), \tag{5.63}$$

$$\begin{aligned} w_1' = \;& \frac{3}{2\sqrt{6}}\frac{1-w_1^2-w_2^2}{V_1+V_2+V_{12}}\Big\{ x_1' \left[V_{1,x_1}x_1 + V_{2,x_2}x_2 + 2(V_{12,x_1}x_1 + V_{12,x_2}x_2)\right] \\ & +3x_1\left[V_{1,x_1}x_1 + V_{2,x_2}x_2 + 2(V_{12,x_1}x_1 + V_{12,x_2}x_2)\right] - 2V_{1,x_1} \\ & -2x_2\left(\frac{V_{12,x_1}}{x_2} + \frac{V_{12,x_2}}{x_1}\right)\Big\}, \end{aligned} \tag{5.64}$$

$$\begin{aligned} w_2' = \;& \frac{3}{2\sqrt{6}}\frac{1-w_1^2-w_2^2}{V_1+V_2+V_{12}}\Big\{ x_2' \left[V_{1,x_1}x_1 + V_{2,x_2}x_2 + 2(V_{12,x_1}x_1 + V_{12,x_2}x_2)\right] \\ & +3x_2\left[V_{1,x_1}x_1 + V_{2,x_2}x_2 + 2(V_{12,x_1}x_1 + V_{12,x_2}x_2)\right] - 2V_{2,x_2} \\ & -2x_1\left(\frac{V_{12,x_1}}{x_2} + \frac{V_{12,x_2}}{x_1}\right)\Big\}. \end{aligned} \tag{5.65}$$

Again, it is good to note that if there is no coupling, $V_{12} = 0$ we recover the uncoupled case, generalized in Ref. [16].

5.3.3 Initial conditions and slow roll regime

Analogous to the scalar field [30] as well as 3-forms [11, 14] the slow roll parameters are given by $\epsilon \equiv -\dot{H}/H^2 = -d\ln H/dN$ and $\eta = \epsilon'/\epsilon - 2\epsilon$. One manner to establish a sufficient condition for inflation is, $\epsilon \ll 1$ and $|\eta| \ll 1$. This conditions are essential in a way that determine the set of possible choices of initial conditions for (5.62),(5.63),(5.64) and (5.65). For this coupled model we have

$$\epsilon = \frac{3}{2}\frac{V_{1,\chi_1}\chi_1 + V_{2,\chi_2}\chi_2 + 2(V_{12,\chi_1}\chi_1 + V_{12,\chi_2}\chi_2)}{V_1+V_2+V_{12}}(1-w_1^2-w_2^2). \tag{5.66}$$

5.3.4 Quadratic potential $V_n \sim \chi_n^2$

Let us consider the following potentials for the fields,

$$V_1 = \chi_1^2, \qquad V_2 = \chi_2^2, \qquad V_{12} = 0.004\,\chi_1\chi_2. \tag{5.67}$$

Observing Figure (1), we can see that a coupling in the order of 10^{-3} has an influence in the duration of inflation, allowing the Universe to inflate by 50 e-folds more, for this particular potential. We can see by the Friedmann equation (5.58) that the fact that we have a coupling makes the system to have more friction and that is expressed in the duration of inflation. The fact that χ_1 and χ_2 have different plateaus comes from a certain asymmetry in the choice of initial conditions for $w_1(0)$ and $w_2(0)$, in Eq. (5.64) and Eq. (5.65), obeying the inflation constraints. Both solutions end in the critical point $(\chi_1, \chi_2) = (0, 0)$ at late times. The fact that interaction is present does not change the solution of the trajectories, compared with the non-interacting case, as we

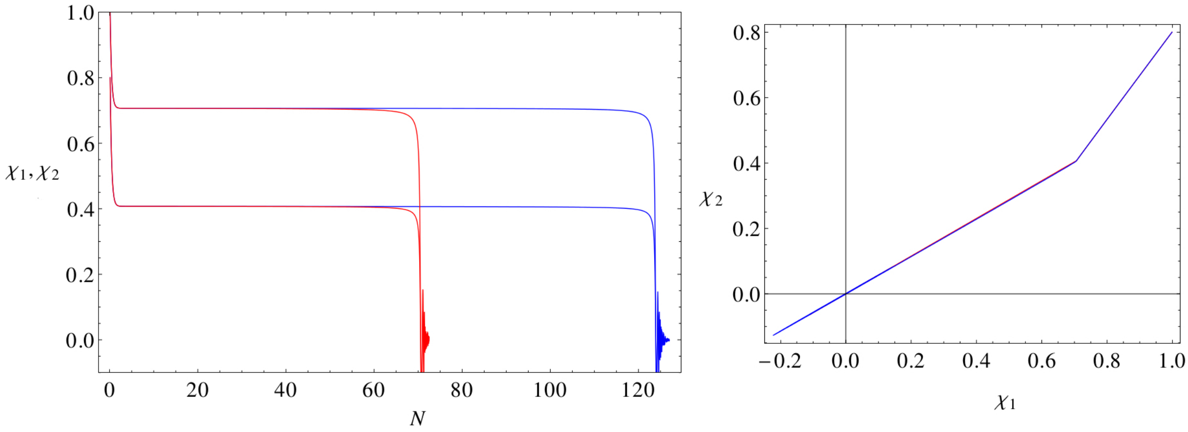

Figure 1: The left panel shows the solutions for the fields with (red) and without (blue) coupling. The red lines stands for the fields χ_1 and χ_2 with no interaction ($V_{12} = 0$), returning the uncoupled case studied in Ref. [16], and the blue line for the fields with coupling, $V_{12} = 0.004\,\chi_1\chi_2$. On the right panel are the set of trajectories evolving in the (χ_1, χ_2) plane for the coupled and uncoupled case. These trajectories are numerical solutions for (5.62), (5.63), (5.64) and (5.65) associated with the solutions in the left panel.

see that the lines coincide in the right panel. We can also see a fixed point in the trajectories that represents the plateaus of the fields where they stay most of the time (inflating).

5.3.5 Exponential potential $V_n \sim e^{\chi_n^2}$

Next we considered the following potentials for the fields,

$$V_1 = e^{\chi_1^2} - 1, \qquad V_2 = e^{\chi_2^2} - 1, \qquad V_{12} = 0.004\,[e^{(\chi_1\chi_2)} - 1]. \tag{5.68}$$

Now, an interaction in the order of 10^{-3} (same as in the quadratic case) has a influence to extend the duration of inflation by 10 e-folds.

The solutions end in the critical point $(\chi_1, \chi_2) = (0, 0)$, shortly after inflation stops, entering in an oscillatory regime, as we can also observe in the right panel.

5.3.6 Rotations within the $SO(2)$ group

It is crucial to study the rotations of the two coupled 3-forms, in which the Lagrangian (5.37) stay invariant, to two uncoupled 3-forms and verify that only the scales in the total potential change.

In this case, for a quadratic potential, we can show that the two coupled 3-form model can be expressed as two uncoupled rotated 3-forms.

Let us consider the following orthogonal rotation matrix with determinant 1, within the SO(2) group,

$$\begin{bmatrix}\sigma_1\\ \sigma_2\end{bmatrix} \equiv \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix}, \tag{5.69}$$

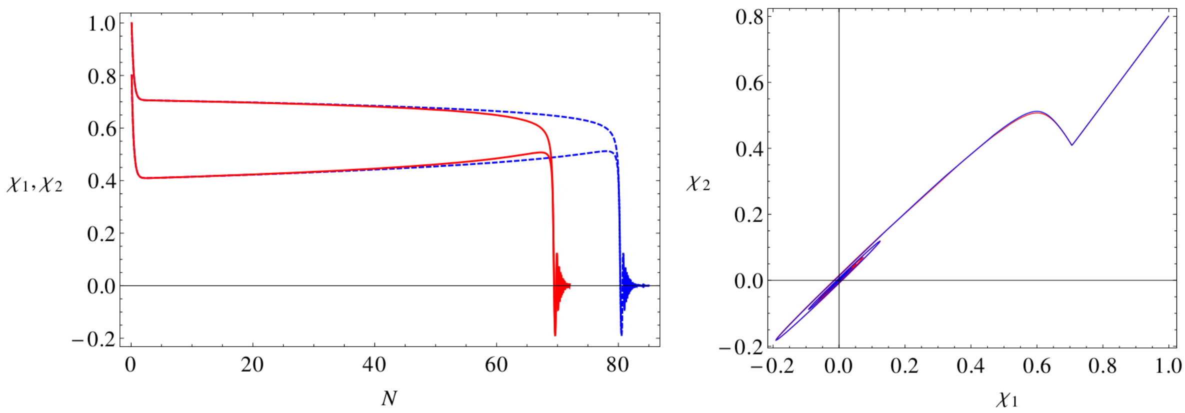

Figure 2: Once again, on the left panel, the red lines stands for the fields χ_1 and χ_2 with no interaction ($V_{12} = 0$), returning the uncoupled case studied in Ref. [16], and the blue, dashed, line for the fields with coupling, choosing $V_{12} = 1.2 \times 0.004\, e^{(\chi_1 \chi_2)}$. On the right panel we present the set of trajectories evolving in the (χ_1, χ_2) plane for the coupled and uncoupled case.

where σ_1 and σ_2 are the new rotated fields associated with the χ_1 and χ_2 fields of our model.

The kinetic term in Eq. (5.37) is invariant under the rotation (5.69). The total potential can be expressed as 2 uncoupled 3-forms for some particular choices of the scales. For the choice that was used in (5.3.4) the total potential can be express as,

$$
\begin{aligned}
V_t &= \begin{bmatrix}\chi_1 & \chi_2\end{bmatrix}\begin{bmatrix}1 & 0.002\\ 0.002 & 1\end{bmatrix}\begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix} = \begin{bmatrix}\sigma_1 & \sigma_2\end{bmatrix}\begin{bmatrix}\frac{249}{250} & 0\\ 0 & \frac{251}{250}\end{bmatrix}\begin{bmatrix}\sigma_1\\ \sigma_2\end{bmatrix}, \\
&\theta = n\pi - \frac{3\pi}{4}.
\end{aligned} \tag{5.70}
$$

We can see that the rotated 3-forms are uncoupled as the second square matrix in (5.70), for the coefficients of the potential, is diagonal.

In general, for nonlinear potentials we cannot express the total potential in terms of uncoupled 3-forms because of their nonlinearity.

5.4 $\mathcal{N}$ coupled 3-form fields model

In this subsection I will propose a general form for $\mathcal{N}$ coupled 3-form fields and show the respective equations of motion.

I consider the following action for $\mathcal{N}$ coupled 3-forms, $A_1, A_2, ..., A_{\mathcal{N}}$ minimally coupled to gravity,

$$S = -\int d^4 x\sqrt{-g}\left[\frac{1}{2}R - \sum_{i=1}^{\mathcal{N}}\left(\frac{1}{48}F_i^2 + V_i(A_i^2)\right) - \mathbb{U}^{\mathcal{N}}\right], \tag{5.71}$$

units where $\kappa^2 = 1$, R is the Ricci scalar, F_k is, once again, the strength tensor associated with the respective kth 3-form,

$$F^k_{\mu\nu\rho\sigma} \equiv 4\nabla_{[\mu}A^k_{\nu\rho\sigma]}, \tag{5.72}$$

and $\mathbb{U}^{\mathcal{N}}$ is the total coupling potential that, for this model, is given by,

$$\mathbb{U}^{\mathcal{N}} \equiv \prod_{i=1}^{\frac{\mathcal{N}-(\mathcal{N} \bmod 2)}{2}} \prod_{\substack{\mu_1=1 \\ \mu_{2i}\geqslant...\geqslant\mu_1 \\ \exists\, j\in\mathbb{N},\,\forall\, l\neq j:\, \mu_j\neq\mu_l}}^{\mathcal{N}} ... \prod_{\mu_{2i}=1}^{\mathcal{N}} \left[1 - \alpha_{\mu_1...\mu_{2i}}V_{\mu_1...\mu_{2i}}(A_1...A_{2i})\right] - 1, \tag{5.73}$$

where $\alpha_{\mu_1\mu_2...\mu_{2i}}$ are constants such that, depending on our theory for the couplings we do not want to consider we can set the respective constants to zero. The first product goes up to $\frac{\mathcal{N}-(\mathcal{N} \bmod 2)}{2}$ to ensure that if $\mathcal{N}$ is odd, than the product goes up to $\frac{\mathcal{N}-1}{2} \in \mathbb{N}$ to consider all the $2i$ even combinations of the couplings and up to $\frac{\mathcal{N}}{2}$ if $\mathcal{N}$ is even. For example, for three 3-forms we only want to consider combinations up to two 3-forms (V_{12},V_{13},V_{23}) because we cannot contract all the indexes of an odd number of 3-forms. The condition $\mu_{2i} \geqslant \mu_{2i-1} \geqslant ... \geqslant \mu_2 \geqslant \mu_1$ ensures that we do not have repetitions like V_{12} and V_{21} which express the same coupling. Finally the last condition $\exists\, j \in \mathbb{N}, \forall\, l \neq j : \mu_j \neq \mu_l$ ensures that there is at least one coupling, i.e. excludes terms like V_{11} or V_{4444} already considered in the action (5.71), but includes terms like V_{1444}. Note that for the case of a single 3-form model we have an empty product $\prod_{i=1}^{0}$ in (5.73), in which by convention its value is one leading to $\mathbb{U}^1 = 1 - 1 = 0$ recovering the single 3 form field model (5.1). The last -1 term is to avoid having a term acting as a cosmological constant in the Lagrangian (5.71)

The equations of motion resulting from the action 5.71 reads,

$$\nabla \cdot F_k = 12A_kV_k(A_k^2) + 6[\mathbb{U}^{\mathcal{N}}{}'(A_k)], \tag{5.74}$$

where a prime denotes derivative with respecto to the parameter,

$$\mathbb{U}^{\mathcal{N}}{}'(A_k) = \frac{\partial\, \mathbb{U}^{\mathcal{N}}}{\partial A_k}. \tag{5.75}$$

5.4.1 Example: $\mathbb{U}^{\mathcal{N}}$ for the 3 and 4 coupled 3-forms model

$$
\begin{aligned}
\mathbb{U}^3 &= [1-\alpha_{12}V_{12}][1-\alpha_{13}V_{13}][1-\alpha_{23}V_{23}]-1 \\
&= \alpha_{12}V_{12}+\alpha_{13}V_{13}+\alpha_{23}V_{23}+\alpha_{12}V_{12}\alpha_{13}V_{12}+\alpha_{12}V_{12}\alpha_{23}V_{23}+\alpha_{13}V_{13}\alpha_{23}V_{23}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{U}^4 = \; & [1-\alpha_{12}V_{12}][1-\alpha_{13}V_{13}][1-\alpha_{14}V_{14}][1-\alpha_{23}V_{23}][1-\alpha_{24}V_{24}][1-\alpha_{34}V_{34}][1-\alpha_{1112}V_{1112}] \\
& [1-\alpha_{1113}V_{1113}][1-\alpha_{1114}V_{1114}][1-\alpha_{1122}V_{1122}][1-\alpha_{1123}V_{1123}][1-\alpha_{1124}V_{1124}] \\
& [1-\alpha_{1133}V_{1133}][1-\alpha_{1134}V_{1134}][1-\alpha_{1144}V_{1144}][1-\alpha_{1222}V_{1222}][1-\alpha_{1223}V_{1223}] \\
& [1-\alpha_{1224}V_{1224}][1-\alpha_{1233}V_{1233}][1-\alpha_{1234}V_{1234}][1-\alpha_{1244}V_{1244}][1-\alpha_{1333}V_{1333}] \\
& [1-\alpha_{1334}V_{1334}][1-\alpha_{1344}V_{1344}][1-\alpha_{1444}V_{1444}][1-\alpha_{2223}V_{2223}][1-\alpha_{2224}V_{2224}] \\
& [1-\alpha_{2233}V_{2233}][1-\alpha_{2234}V_{2234}][1-\alpha_{2244}V_{2244}][1-\alpha_{2333}V_{2333}][1-\alpha_{2334}V_{2334}] \\
& [1-\alpha_{2344}V_{2344}][1-\alpha_{2444}V_{2444}][1-\alpha_{3334}V_{3334}][1-\alpha_{3344}V_{3344}][1-\alpha_{3444}V_{3444}]-1
\end{aligned}
$$

6 Randall-Sundrum braneworld

Extra dimensions were proposed in the early twentieth century by Nordstrom and a few years later by Kaluza and Klein which combined principles of quantum mechanics and relativity.

Randall and Sundrum originally suggested a two-brane scenario in five dimensions with a highly bulk geometry. This is the so called Randall-Sundrum I (RSI) model. Writing the solution of Einstein equations on the positive tension brane and sending the negative tension brane to infinity, an observer confined to the positive tension brane recovers Newton's law if the curvature scale of the AdS is smaller than a millimeter. In contrast with the Kaluza-Klein mechanism where all extra-dimensional degrees of freedom are compact, in this model the higher-dimensional space is non-compact.

The braneworld cosmology in which I will focus is the so called Randall-Sundrum II (RSII) model first proposed by Lisa Randall and Raman Sundrum [17] where the standard model particles are confined to a $(3+1)$ dimensional hyper-surface, positive tension, called 3-brane, embedded in a 5 dimensional anti-de Sitter (negative cosmological constant) space-time, called bulk, in which only gravity and other exotic matter can propagate (see Fig.3). The 4 dimensional Einstein-Hilbert action is reproduced at low energies, recovering the standard 4 dimensional Friedmann equation (3.4). Our Universe my be such brane-like object. For a review see Ref. [18] in which this subsection is based on.

They also proposed a two brane model (RSI) in which the hierarchy problem, i.e. the large discrepancy between the weak force and gravity, is addressed.

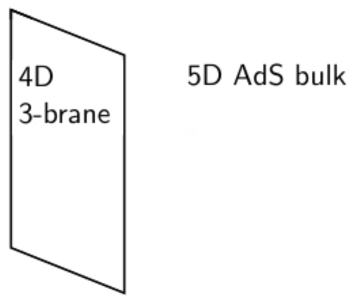

Figure 3: Randall-Sundrum II braneworld.

6.1 Model

We will concentrate in the so called Randall-Sundrum II (RSII) model. It was shown that there is a continuum of Kaluza-Klein modes for the gravitational field, contrasting with the discrete spectrum if the extra dimension is periodic. In consequence, the force between two masses confined to the brane will have a new correction term and the potential energy between them is given by,

$$V(r) = \frac{Gm_1m_2}{r}\left(1 + \frac{l^2}{r^2} + \mathcal{O}(r^{-3})\right), \tag{6.1}$$

where l is related to the five-dimensional bulk cosmological constant Λ_5 therefore is a measure of the curvature scale of the bulk spacetime. As said before, there is no deviation from Newton's law on length scales larger than a millimeter. Thus, l has to be smaller than that length scale.

For this theory we will consider the brane action and the Einstein-Hilbert action,

$$S = S_{\rm EH} + S_{\rm brane} = -\int dx^5 \sqrt{-g^{(5)}} \left(\frac{R}{2\kappa_5^2} + \Lambda_5 \right) - \int dx^4 \sqrt{-g^{(4)}} \lambda, \tag{6.2}$$

where λ is the brane tension, which is constant and positive, R is the Ricci scalar, κ_5 is the five-dimensional gravitational coupling constant, $g^{(4)}$ and $g^{(5)}$ are the four and five dimensional determinants of the metric respectively. The brane is located at $y = 0$ and is assumed a Z_2 symmetry (identifying y with $-y$). Since we are looking for solutions to the five-dimensional Einstein equations, the four-dimensional universe derived for this theory should look like our own universe, so should appear flat and static. The ansatz for the metric reads,

$$ds^2 = e^{-2K(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2, \tag{6.3}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the four-dimensional Minkowski metric, $e^{-2K(y)}$ is the warp factor which depends only on the five dimension y, written, for convenience, as an exponential.

The Einstein equations, derived from the action (6.2), result in two independent equations,

$$6K'^2 = -\kappa_5^2 \Lambda_5, \tag{6.4}$$
$$3K'' = \kappa_5^2 \lambda \delta(y). \tag{6.5}$$

The first equation results in,

$$K(y) = \sqrt{-\frac{\kappa_5^2}{6} \Lambda_5} y, \tag{6.6}$$

which gives us the information that $\Lambda_5 < 0$, which explains the reason why the five dimensional bulk is Anti-de Sitter (AdS). If we integrate Eq. (6.5) from $-\epsilon$ to ϵ, take the limit $\epsilon \to 0$ and make use of the Z_2 symmetry, we get,

$$6K'|_0 = \kappa_5^2 \lambda, \tag{6.7}$$

now using Eq. (6.6) gives,

$$\Lambda_5 = -\frac{\kappa_5^2}{6} \lambda^2, \tag{6.8}$$

which is the fine-tuning between the brane tension and the bulk cosmological constant for static solutions to exist.

We will derive the cosmological equations making use of the bulk equations only. The bulk metric reads,

$$ds^2 = a^2 b^2 (dt^2 - dy^2) - a^2 \delta_{ij} dx^i dx^j, \tag{6.9}$$

which is consistent with isotropy and homogeneity on the brane (at $y = 0$). The functions a and b depend on t and y only. This metric results in the Einstein equations for the bulk,

$$\begin{aligned}
a^2b^2G^0_0 &= 3\left[2\frac{\dot{a}^2}{a^2}+\frac{\dot{a}\dot{b}}{ab}-\frac{a''}{a}+\frac{a'b'}{ab}+\kappa b^2\right] = a^2b^2\left[\rho_B+\rho\bar{\delta}(y-y_b)\right], && (6.10)\\
a^2b^2G^5_5 &= 3\left[\frac{\ddot{a}}{a}-\frac{\dot{a}\dot{b}}{ab}-2\frac{a'^2}{a^2}-\frac{a'b'}{ab}+\kappa b^2\right] = -a^2b^2T^5_5, && (6.11)\\
a^2b^2G^0_5 &= 3\left[-\frac{\dot{a}'}{a}+2\frac{\dot{a}a'}{a^2}+\frac{\dot{a}b'}{ab}+\frac{a'\dot{b}}{ab}\right] = -a^2b^2T^0_5, && (6.12)\\
a^2b^2G^i_j &= \left[3\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}-\frac{\dot{b}^2}{b^2}-3\frac{a''}{a}-\frac{b''}{b}+\frac{b'^2}{b^2}+\kappa b^2\right]\delta^i_j && (6.13)\\
&= -a^2b^2\left[p_B+p\bar{\delta}(y-y_b)\right]\delta^i_j, && (6.14)
\end{aligned}$$

where a dot represents derivative with respect to time t, and a prime derivative with respect to the fifth dimension y. The bulk energy-momentum tensor T^a_b has been kept general. For the Randall-Sundrum model we will consider $\rho_B = -p_B = \Lambda_5$ and $T^0_5 = 0$.

Integrating the 00-component, over y from $-\epsilon$ to ϵ, taking the limit $\epsilon \to 0$, and use the Z_2 symmetry gives,

$$\frac{a'}{a}|_{y=0} = \frac{1}{6}ab\rho. \tag{6.15}$$

Integrating now the ij-component making use of Eq. (6.15) tells,

$$\frac{b'}{b}|_{y=0} = -\frac{1}{2}ab(\rho+p). \tag{6.16}$$

Equations (6.15) and (6.16) are called the junction conditions. The other components should be compatible with these conditions. From equation (6.12), i.e., the 05-component, to $y = 0$, leads to the continuity equation,

$$\dot{\rho}+3\frac{\dot{a}}{a}(\rho+p) = 0, \tag{6.17}$$

which represents matter conservation on the brane.

The 55-component gives,

$$\frac{\ddot{a}}{a}-\frac{\dot{a}\dot{b}}{ab}+\kappa b^2 = -\frac{a^2b^2}{3}\left[\frac{1}{12}\rho(\rho+3p)+q_B\right]. \tag{6.18}$$

in which, changing to cosmic time $d\tau = ab\,dt$, assuming $a = e^{\alpha(t)}$, and using the energy conservation results in Ref. [31],

$$\frac{d(H^2e^{4\alpha})}{d\alpha} = \frac{2}{3}\Lambda_5e^{4\alpha}+\frac{d}{d\alpha}\left(e^{4\alpha}\frac{\rho^2}{36}\right), \tag{6.19}$$

in which $aH = da/d\tau$. Integrating Eq. (6.19) shows,

$$H^2 = \frac{\rho^2}{36}+\frac{\Lambda_5}{6}+\frac{\mu}{a^4} \tag{6.20}$$

where the last term μ appears in the derivation above as an integration constant, and this last term, containing μ, is called the *dark radiation* term (see Ref. [32]). A version of Birkhoff's theorem tells us that if the bulk spacetime is AdS then $\mu = 0$ [33]. If the bulk is AdS-Schwarzschild then $\mu \neq 0$ but a measure of the bulk black hole's mass. Splitting the total energy-density and pressure into parts coming from matter and brane tension, $\rho = \rho_M + \sigma$ and $p = p_M - \lambda$, we find for the Friedmann equation,

$$H^2 = \frac{8\pi G}{3}\rho_M\left[1 + \frac{\rho_M}{2\lambda}\right] + \frac{\Lambda_4}{3} + \frac{\mu}{a^4}, \tag{6.21}$$

from which, using the energy conservation, the Raychaudhuri's equation reads,

$$\frac{dH}{d\tau} = -4\pi G(\rho_M + p_M)\left[1 + \frac{\rho_M}{\lambda}\right], \tag{6.22}$$

where,

$$\frac{\Lambda_4}{3} = \frac{\lambda^2}{36} + \frac{\Lambda_5}{6}, \tag{6.23}$$

and comparing the last equation with the fine-tuning relation, i.e. equation (6.8), we see that $\Lambda_4 = 0$ in this case.

For the rest of this work I will consider $\mu = \Lambda_4 = 0$ and, from the next section to the rest of this thesis, I will refer to ρ_M simply as ρ for convenience.

The most interesting feature of the Friedmann equation (6.21) is that the universe expands faster at high energies (early times), i.e. for $\rho_M \gg 2\lambda$, with expansion rate $H \propto \rho_M$. At low energies we always recover the standard case where $H \propto \sqrt{\rho_M}$, i.e. for $\rho_M \ll 2\lambda$ (late times). This is one important feature of braneworlds.

It is important to note that at the time of nucleosythesis this braneworld effects on the Friedmann equation must be negligible, otherwise, if the expansion rate is modified, the results for the abundances of the light elements are unacceptably changed. This leads to the bound $\lambda \geq (1\text{MeV})^4$. But the stronger constraint comes from tests of deviation from the Newton's law [34] (assuming the RS fine-tuning relation Eq. (6.8)) and gives $\lambda \geq (100\text{Gev})^4$.

6.2 3-form in Randall-Sundrum II

In this section I will consider a single 3-form confined to the brane, in the Randall-Sundrum II scenario, and I will present some inflationary scenarios in this five dimensional model.

As the 3-form $A_{\mu\nu\rho}$ is confined to the brane, I will consider an action where I include the 3-form lagrangian density (5.9) in the brane part of the action for the RSII braneworld Eq. (6.2), and will consider the same minimally coupling to Einstein gravity theory,

$$S = -\int d^5x\sqrt{-g^{(5)}}\left(\frac{R}{2\kappa_5^2} + \Lambda_5\right) - \int d^4x\sqrt{-g^{(4)}}\left(\lambda - \frac{1}{48}F^2 - V(A^2)\right). \qquad (6.24)$$

Since the 3-form is confined to the four dimensional brane we can follow the same initial approach as the standard four dimensional case (5.1) where wa associate to the 3-form a comoving field χ whose equations of motion are given by Eq. (5.14) and the energy density and pressure are given by Eq. (5.17) and Eq. (5.18) respectively. give The main difference now to be considered is the modified Friedmann equation,

$$H^2 = \frac{1}{3}\rho\left(1 + \frac{\rho}{2\lambda}\right), \qquad (6.25)$$

(units where $\kappa = 1$) where ρ is the energy density and λ the positive brane tension, as mentioned in (6.1). The fact that the expansion rate is larger at high energies ($\rho \gg 2\lambda$) means that the friction term in Eq. (5.14) is larger in that regime. This means that the field $\chi(t)$ rolls slower, for the same initial conditions, and inflation can last longer in this five-dimensions set up than in the four-dimensional case. The Friedmann equation in the standard cosmology is reproduced in the limit of low energies, $\rho \ll 2\lambda$.

6.2.1 Dynamics of the 3-form on the brane

We want to study the dynamics of the 3-form in the nature of dynamical systems. In order to do that, we rewrite the equations of motion (5.14) as a system of first order differential equations, introducing the dimensionless variables,

$$x \equiv \kappa\chi, \qquad (6.26)$$

$$y^2 \equiv \frac{V}{\rho}, \qquad (6.27)$$

$$w \equiv \frac{\dot{\chi} + 3H\chi}{\sqrt{2\rho}}, \qquad (6.28)$$

$$\Theta \equiv \left(1 + \frac{\rho}{2\lambda}\right)^{-1/2}, \qquad (6.29)$$

where x represents the comoving field χ, y and w are, respectively, the normalized potential and kinetic energies and Θ represents the correction term in Eq. (6.25). These variables are subject to the constraint, that follows from Eq. (5.17),

$$w^2 + y^2 = 1. \qquad (6.30)$$

Using equations (5.17), (6.26), (6.28) and (6.29), the modified Friedmann and Raychaudhuri equations can be written as,

$$H^2 = \frac{1}{3}\frac{V}{(1-w^2)}\Theta^{-2}, \tag{6.31}$$

$$\dot{H} = -V_{,x}x\left(\Theta^{-2}-\frac{1}{2}\right). \tag{6.32}$$

Substituting for ρ in Eq. (6.30) using Eq. (6.27) and Eq. (6.29), we obtain the useful relation for Θ in terms of the x and w variables,

$$\Theta^2 = \frac{1-w^2}{1-w^2+\frac{V}{2\lambda}}. \tag{6.33}$$

The dynamical system for the equations of motion (5.14) reads,

$$x' = 3\left(\sqrt{\frac{2}{3}}\Theta w - x\right), \tag{6.34}$$

$$w' = \frac{3}{2}\frac{V_{,x}}{V}(1-w^2)\left[xw-\Theta\sqrt{\frac{2}{3}}\right], \tag{6.35}$$

where a prime means differentiating in respect to the number of e-folds $N = \ln a(t)$, so that $x' = dx/dN$.

At low energies, $\rho \ll 2\lambda$ and therefore $\Theta \approx 1$, we end up recovering the four-dimensional equations studied in Ref. [16] even though the variables were normalized to H^2 instead of ρ as we do here. We would like to see now, how the presence of this correction term, Θ, affects the dynamics of the system in comparison with the evolution in the four-dimensional case.

6.2.2 Critical points

Let us assume for now that Θ evolves sufficiently slow such that we can take it to be a constant within a few e-folds. We will see later that this assumption is actually supported by the numerical solutions. We can then identify the *instantaneous* critical points of the dynamical system established by Eq. (6.34) and Eq. (6.35). These are shown in Table 2.

	x	w	$V_{,x}/V$	Description
A	$\pm\sqrt{\frac{2}{3}}\Theta$	± 1	any	kinetic domination
B	$x_{\rm ext}$	$\sqrt{\frac{3}{2}}\frac{1}{\Theta}x_{\rm ext}$	0	potential extrema

Table 2: Instantaneous critical points of the dynamical system.

The critical points A do not exist for the standard scalar field models [28] and result from the extra $3\dot{H}\chi$ term in the equation of motion (5.14). One of the eigenvalues vanishes, hence, we cannot infer anything regarding its stability from the linear analysis without specifying the form of the potential. The critical point B corresponds to the value of the field at the extrema of the potential, therefore, its stability is strongly dependent on whether it corresponds to a minimum or a maximum of the potential.

From the analysis of the critical points we can see that, in the five dimensional set up, the critical points have a dependence on the correction term Θ. This means that as the energy decreases, the instantaneous critical points move along the phase space and approach the four dimensional case at low energies, $\Theta = 1$.

In Fig 4 is shown the phase space portrait for a potential of the form $V = e^{\chi^2} - 1$. Comparing these figures we, again, note that the critical points A (upper and lower blue dots) are shifted along the x axis as the system evolves and will eventually end at $x = \pm\sqrt{2/3}$ (4 dim case). As we will see in (6.2.4), at the critical points A (top and bottom blue dots), the universe inflates and critical point B (central yellow dots) corresponds to the attractor and potential minimum for this potential where reheating happens as usual [29].

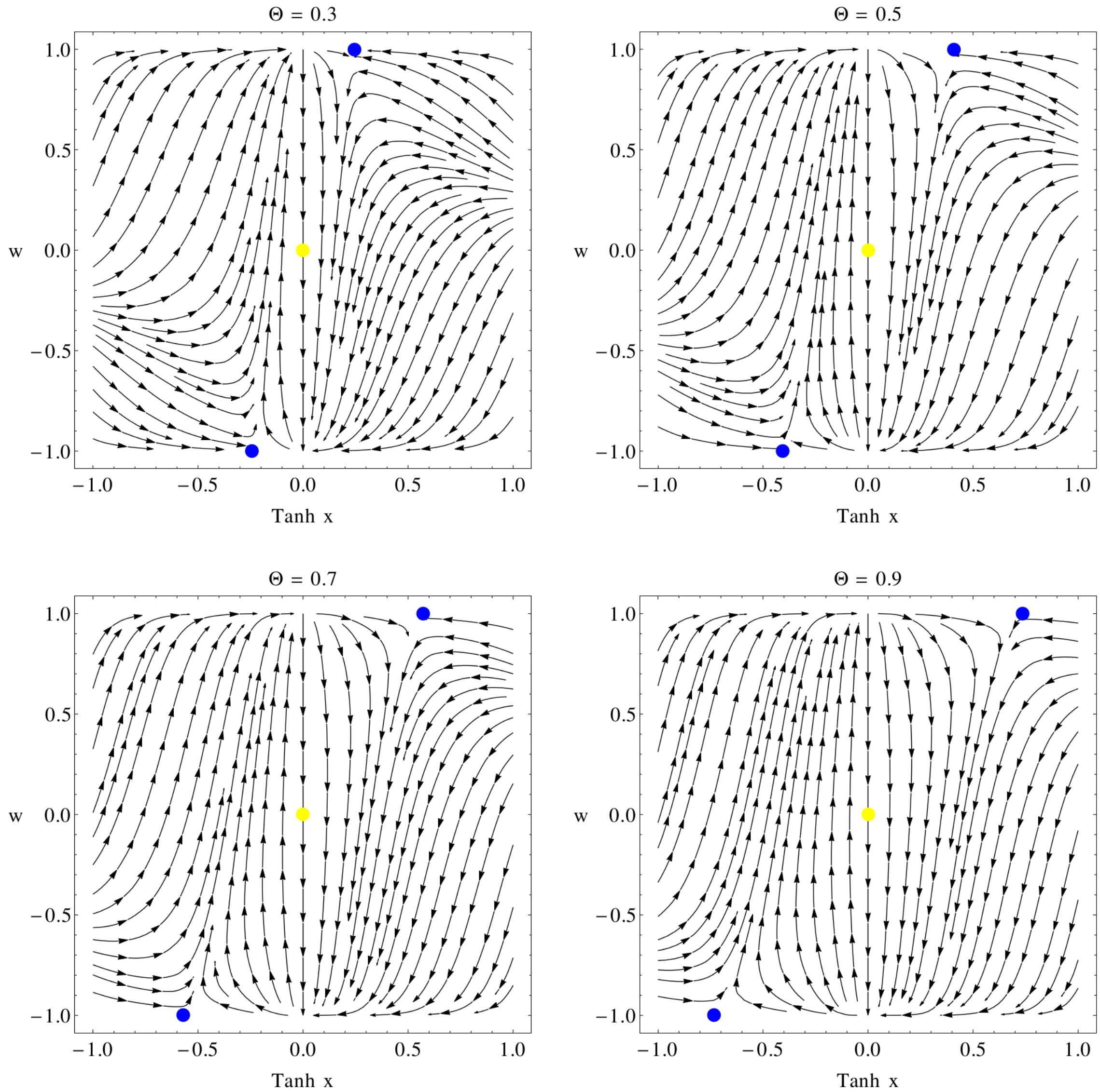

Figure 4: Phase space $(\tanh(x), w)$ for $V = e^{\chi^2} - 1$ at different stages, i.e. different values of Θ.

Other way than can be used to study the stability of the critical points is by defining the effective potential,

$$V_{\text{eff},\chi} = 3\dot{H}\chi + V_{,\chi}, \tag{6.36}$$

and analyse the potential and effective potential.

The potential and the corresponding effective potential for $V = e^{\chi^2} - 1$ is illustrated in Fig.5. We can observe the shift in the value of the instant critical points as the energies decrease, i.e. as Θ approaches unity, where the critical points are $x = \pm\sqrt{\frac{2}{3}}$ as we can also verify in Table 2.

One interesting feature regarding the dynamics of a 3-form in RSII is that the Θ dependence of the dynamics can change the stability of the critical points as the energy decreases. For example, in Fig.6, we traced the Landau-Ginzburg potential

$$V(\chi) = (\chi^2 - c^2)^2, \tag{6.37}$$

with $c = 0.5$ (solid), and its effective potential (dashed) at different values of Θ. We observe that at early times the potential minima at $x = \pm 0.5$ are initially unstable and, as the energy decreases, they become stable.

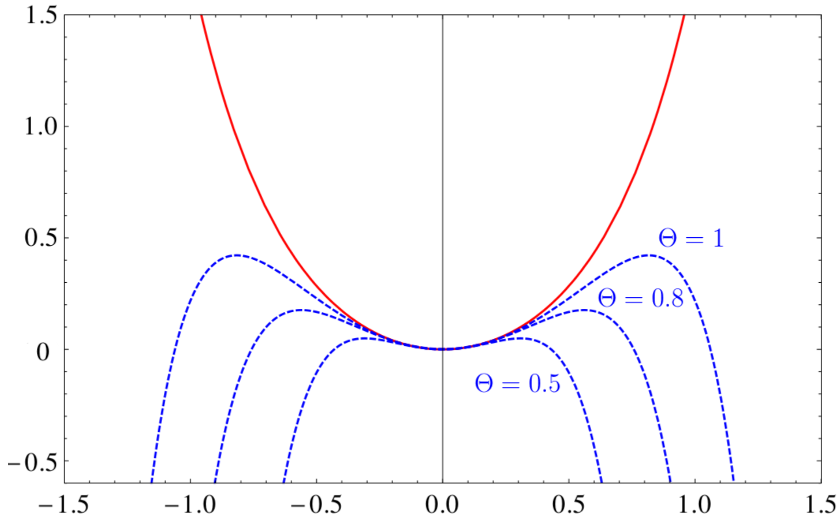

Figure 5: Potential $V(\chi)$ (solid red line) and effective potential V_{eff} (dashed blue lines) for the potential $V = e^{\chi^2} - 1$ for different values of Θ.

6.2.3 Initial conditions and slow roll regime

In order to study inflation we need to understand how the slow-roll parameters are modified in this set up. Analogously to the scalar field as well as 3-forms [11, 14] the parameters are defined by $\epsilon \equiv -\dot{H}/H^2 = -d\ln H/dN$ and $\eta = \epsilon'/\epsilon - 2\epsilon$. One manner to establish a sufficient condition for inflation is, $\epsilon \ll 1$ and $|\eta| \ll 1$, which must last for at least ≈ 50 e-folds. For our RSII model we have,

$$\epsilon = \frac{3}{2}x\frac{V_{,x}}{V}(1-w^2)(2-\Theta^2), \tag{6.38}$$

$$\eta = \frac{x'(V_{,x} + V_{,xx}x)}{V_{,x}x} + 6x\frac{V_{,x}}{V}(1-w^2)\frac{\Theta^2 - 1}{2-\Theta^2}, \tag{6.39}$$

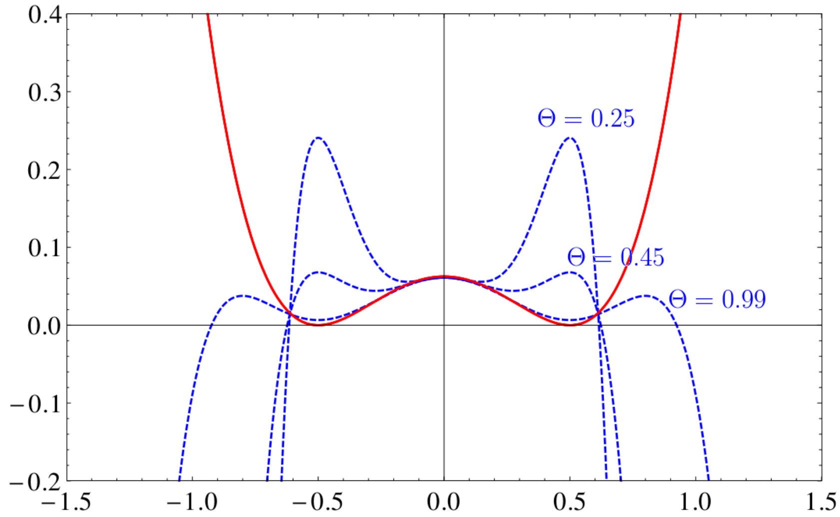

Figure 6: Potential $V(\chi)$ (solid line) and effective potential V_{eff} (dashed lines) for the potential $V = (\chi^2 - 0.5^2)^2$ for different values of Θ.

where the terms in Θ signal the new contributions to the slow-roll parameters.

6.2.4 3-form inflation on the brane

I will now present inflationary solutions for the system (6.34)–(6.35). I will also compare the evolutions between the four and five dimensional cases. Observing Fig.7 we note that inflation happens when the field is on the plateau of the evolution that for the four dimensional case is flat and corresponds to the critical point $\chi = \pm\sqrt{2/3}$ [16]. We can make sure that the field is inflating by observing the value of the slow roll parameter ϵ in Fig.8 which $\epsilon \ll 1$ during inflation, ending it at $\epsilon = 1$. For the RSII case, however, the plateau has a gentle slope due to the dependence of the instantaneous critical points on Θ (we saw that $\chi = \pm\sqrt{2/3}\Theta$) up to the point in which $\chi = \pm\sqrt{2/3}$. We can also note that, for the same initial conditions, inflation lasts about 30 e-folds longer in the five dimensional set up due to the fact that there is additional friction to the field's evolution. When inflation ends, the field goes to the attractor $\chi = 0$ which is the potential minimum (critical point B in Table 2).

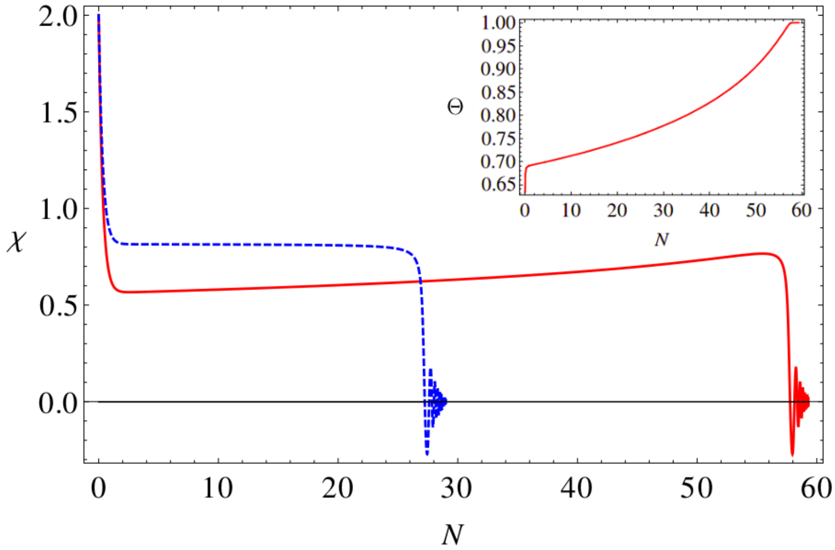

Figure 7: Solutions for the system (6.34)–(6.35) for the four dimensional case (dashed blue line) i.e. for $\Theta = 1$ already studied in Ref. [16] and for the RSII model (solid red line) when Θ is given by Eq. (6.29) for $V = V_0[e^{\chi^2} - 1]$, $V_0 = 10^{-14}$, $\lambda = 10^{-12}$ and for the initial conditions $(x_0, w_0) = (2, 0.9055)$. The smaller panel shows the change in Θ, for the RSII model, as the system evolves.

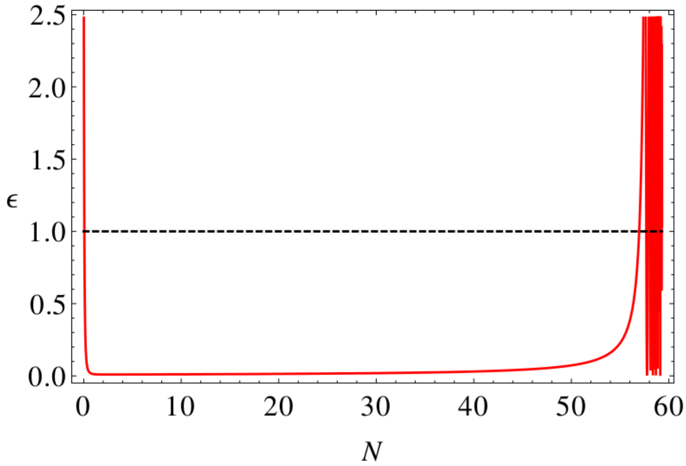

Figure 8: Variation of the slow roll parameter ϵ for the solutions of the system (6.34)–(6.35) for the RSII model for $V = V_0[e^{\chi^2} - 1]$, $V_0 = 10^{-14}$, $\lambda = 10^{-12}$ and for the initial conditions $(x_0, w_0) = (2, 0.9055)$. The black dashed line marks $\epsilon = 1$ for reference.

6.2.5 Cosmological Perturbations

We have seen, from the Friedmann equation (6.25), that early universe cosmology in RSII is modified at high energies, where the ρ^2 term dominates. As a result we expect the early evolution to be different from the standard four dimensional case.

It was shown in previous work [18] that for scalar perturbations, fluctuations in the bulk spacetime are not important during inflation. So I will only consider the standard four dimensional scalar perturbations on the brane, where the 3-form is confined.

For the 4D brane we can consider general perturbations about the FLRW metric where the line element can be written as [12],

$$ds^2 = -(1+2\psi)dt^2 + 2b_i dx^i dt + a^2(t)(1-2\phi)dx^i dx_i + a^2(t)h_{ij}x^i x^j, \tag{6.40}$$

where the two scalar perturbations ψ and ϕ are the Bardeen potentials in the longitudinal (sometimes called Newtonian) gauge, b_i is a transverse vector and h_{ij} is a transverse and traceless tensor describing the tensor perturbations.

Einstein equations for the scalar perturbations read,

$$\begin{aligned}
\frac{1}{2}\delta G^0_0 &= -\frac{\nabla^2}{a^2}\phi + 3H(\dot{\phi} + H\psi) = -4G\delta\rho, && (6.41)\\
\frac{1}{2}\partial_i\delta G^i_0 &= -\frac{\nabla^2}{a^2}(\dot{\phi} + H\psi) = 4\pi G(\rho+p)\frac{\theta}{a}, && (6.42)\\
\frac{1}{2}\delta G^i_i &= \ddot{\phi} + H(3\dot{\phi} + \dot{\psi}) + (2\dot{H} + 3H^2)\psi - \frac{\nabla^2}{3a^2}(\phi - \psi) = 4\pi G\delta p, && (6.43)\\
\frac{1}{2}\delta G^j_i &= -\nabla^2(\phi - \psi) = 12\pi G(\rho+p)\varpi. && (6.44)
\end{aligned}$$

The first equation, G^0_0 component, represents the energy constraint, the second, G^0_i component, shows the momentum constraint involving the velocity perturbation θ, the third, G^i_i component, is the trace of the spatial components, and the last one, G^j_i component, gives the shear propagation for the shear ϖ.

We must also account for perturbations in the matter sourcing the universe's evolution, which in this case is the 3-form field. To parametrize the fluctuations of the 3-form it is done a similar decomposition as for the metric. The four degrees of freedom in a 3-form turn out to be two scalar and two vector degrees of freedom. The most general form of the perturbed 3-form is given by,

$$\begin{aligned}
A_{0ij} &= a(t)\epsilon_{ijk}(\alpha_{,k} + \alpha_k) && (6.45)\\
A_{ijk} &= a^3(t)\epsilon_{ijk}(\chi(t) + \alpha_0), && (6.46)
\end{aligned}$$

where α_k is a transverse vector and thus has two independent degrees of freedom. We easily see that, as usually, the vector and scalar perturbations decouple at linear order so it can be neglected. So now we have the relation,

$$A^2 = 6\left[\chi^2 + 2\chi(\alpha_0 + 3\chi\phi)\right]. \tag{6.47}$$

The equations of motion (5.7) now yields the equation of motion for α_0,

$$\ddot{\alpha}_0 + 3H\dot{\alpha}_0 + (3\dot{H} + V_{,\chi\chi})\alpha_0 - \frac{\nabla^2}{a^2}(\dot{\alpha} - 2H\alpha) + (\dot{\chi} + 3H\chi)(3\dot{\phi} - \dot{\psi}) + 3(V_{,\chi\chi}\chi - V_{,\chi})\phi + 2\psi V_{,\chi} = 0, \tag{6.48}$$

the constraint (5.5),

$$\dot{\alpha}_0 + 3H\alpha_0 + (3\phi - \psi)(\dot{\chi} + 3H\chi) + \left(\frac{V_{,\chi}}{\chi} - \frac{\nabla^2}{a^2}\right)\alpha = 0, \tag{6.49}$$

and, for last, we have the identity $\nabla \cdot (\nabla \cdot F) = 0$ that gives the additional constraint,

$$\frac{\partial}{\partial t}\left(\frac{V_{,\chi}\alpha}{\chi}\right) - V_{,\chi\chi}(\alpha_0 + 3\chi\phi) - V_{,\chi}\psi = 0. \tag{6.50}$$

Now we can calculate the perturbed components of the energy-momentum tensor,

$$\begin{aligned}
-\delta T^0_0 &= \delta\rho = (\dot\chi + 3H\chi)\left[\dot\alpha_0 + 3H\alpha_0 + (\dot\chi + 3H\chi)(3\phi - \psi) - \frac{\nabla^2}{a^2}\alpha\right] \\
&+ V_{,\chi}(\alpha_0 + 3\chi\phi), \qquad (6.51)\\
\delta T^i_i &= \delta p = -(\dot\chi + 3H\chi)\left[\dot\alpha_0 + 3H\alpha_0 + (\dot\chi + 3H\chi)(3\phi - \psi) - \frac{\nabla^2}{a^2}\alpha\right] \\
&+ V_{,\chi\chi}\chi(\alpha_0 + 3\chi\phi), \qquad (6.52)\\
\delta T^0_i &= -V_{,\chi}\alpha_{,i} \qquad (6.53)\\
\delta T^i_j &= 0. \qquad (6.54)
\end{aligned}$$

From the last equation, T^i_j component, we note that 3-form does not generate anisotropic stress $\phi = \psi$, i.e. $\varpi = 0$ in Eq. (6.44).

The rotational perturbations, in the absence of other vector sources, evolve like,

$$\begin{aligned}
\dot b_i + Hb_i &= 0 \qquad (6.55)\\
\frac{\nabla^2}{a^2}b_i &= V_{,\chi}(\chi b_i - \alpha_i), \qquad (6.56)
\end{aligned}$$

thus the vector perturbations decay and can be ignored. Since the 3-form does not generate tensor perturbations they evolve separately and we will talk about them in a few moments.

Because there is no anisotropic stress, $\phi = \psi$, we can now derive an evolution equation for the Bardeen potential ϕ in a closed form. Equation (6.45) can be used to eliminate $\dot\alpha_0$ from the system, equations (6.41) and (6.51) may then be used to eliminate α_0. We can eliminate α using Eq. (6.42) and Eq. (6.53). Finally plugging the solutions into Eq. (6.43) with the right hand side given by Eq. (6.52) we get,

$$\ddot\phi + \left(H - \frac{\ddot H}{\dot H}\right)\dot\phi + \left(2\dot H - \frac{\ddot H H}{\dot H}\right)\phi = \left(1 - \frac{\ddot H}{\dot H}\frac{\chi}{\dot\chi}\right)\frac{1}{a^2}\nabla^2\phi, \qquad (6.57)$$

where the right hand side of Eq. (6.57) is simply $\delta p_\chi/2$ in the comoving gauge.

One feature of consider 3-form fields is that the dynamical speed of sound, c_s^2, can vary and can be expressed to the derivatives of the potential. On the rest frame we read,

$$c_s^2 = \frac{\delta p_\chi}{\delta\rho_\chi} = \frac{V_{,\chi\chi}\chi}{V_{,\chi}}, \qquad (6.58)$$

where it is used the background relations in the 3-form dominated universe.

By inspecting Eq. (6.57) we see that we can describe the scalar fluctuations of the field with only one degree of freedom. This is due to the symmetries of the FLRW metric. It happens that the kinetic term has a gauge symmetry which reduces the number of physical degrees of freedom in the absence of the potential. Even when the potential is present, the symmetry is partially efficient. This is because the potential depends only on A^2 and the spatial components of A are forced to vanish in the FLRW background (5.12) so their fluctuations α cannot contribute at the linear order to the quadratic invariant A^2 (6.47).

To consider quantum fluctuations during inflation we must find the gauge invariant variable to describe the degree of freedom we have. It is conventional to refer to the curvature perturbation ζ [13] given by,

$$\zeta = -H\frac{\dot{\phi} + H\phi}{\dot{H}} + \phi. \tag{6.59}$$

The 2-point correlation function for the curvature perturbation is defined as,

$$\langle \zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\rangle = (2\pi)^5\delta^3(\mathbf{k}_1 - \mathbf{k}_2)\frac{P_\zeta}{2k_1^3}, \tag{6.60}$$

where P_ζ is the power spectrum of the curvature perturbation given by,

$$P_\zeta \equiv \frac{1}{2\pi^2}k^3 |\zeta_k|^2 = \left.\frac{1}{2(2\pi)^2\epsilon}\frac{H^2}{c_s M_{\rm Pl}^2}\right|_*, \tag{6.61}$$

where ϵ is the slow roll parameter (6.2.3), and $*$ indicates that the expression is evaluated at horizon crossing $c_s k = aH$. The power spectrum now has a dependence on the sound speed. The spectral index n_s is then given by,

$$1 - n_s = 2\epsilon + \frac{\dot{\epsilon}}{\epsilon H} + \frac{\dot{c}_s}{c_s H}. \tag{6.62}$$

In the Randall-Sundrum model, however, the amplitude of the tensor modes are modified and read [20],

$$A_T^2 = \frac{4}{25\pi M_{\rm Pl}^4}H^2F^2(H/\mu)|_*, \tag{6.63}$$

where F is a correction function,

$$F(x) = \left[\sqrt{1+x^2} - x^2\ln\left(\frac{1}{x} + \sqrt{1+\frac{1}{x^2}}\right)\right]^{-1/2}, \tag{6.64}$$

and

$$x_0 \equiv \frac{H}{\mu} = \left(\frac{3}{4\pi\lambda}\right)^{1/2} HM_{\rm Pl}, \tag{6.65}$$

or in terms of our dynamical variable Θ,

$$x_0 = \sqrt{4\frac{\Theta^{-2} - 1}{\Theta^2}}. \tag{6.66}$$

For $x_0 \ll 1$, $F(x_0) \simeq 1$ and Eq. (6.63) reduces to the standard cosmology formula, and for $x_0 \gg 1$, $F(x_0) \simeq \sqrt{3x_0/2}$. Finally, the tensor to scalar ratio is then,

$$r \equiv \frac{A_T^2}{A_\zeta^2} = 16c_s|\epsilon|F^2(x_0). \tag{6.67}$$

We are now ready to compare the cosmological parameters, scalar to tensor ratio and spectral index, for our inflationary setting with the 2015 Planck data [26]. First we consider a form of the scalar potential which has been proven in Ref. [16] to lead to

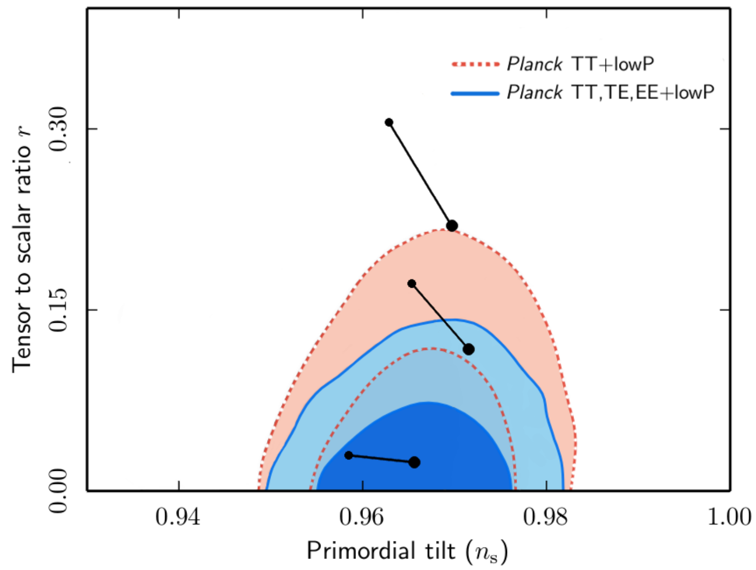

Figure 9: Comparison of the spectral index and the tensor to scalar ratio against the recent Planck 2015 data [26] for 50 (small dot) and 60 (large dot) e-folds for different values of the brane tension λ. We considered the potential in (6.68) with $b = -0.245$. The bars represent, from bottom to top, the solutions with $\lambda = 5\times10^{-4}$, $\lambda = 1.5\times10^{-7}$ and $\lambda = 8 \times 10^{-8}$ units $\kappa^2 = 1$).

a viable cosmology in the four dimensional set up (although for a two 3-form system) and to produce a good fit to the Planck 2013 results,

$$V = V_0(\chi^2 + b\chi^4), \tag{6.68}$$

where V_0 and b are free parameters.

In Fig.9 the bottom bar represents the prediction for the five dimensional case with $\lambda = 5 \times 10^{-4}$. With this value of the brane tension, the evolution quickly reaches $\Theta \approx 1$ which means that this case is practically indistinguishable from the four dimensional solution. When we lower the brane tension and consequently increase the five dimensional effects, we observe that the predictions worsen due to the presence of the correction $F^2(H/\mu)$ in Eq. (6.63), which enhances the tensor to scalar ratio. For $\lambda = 8 \times 10^{-8}$, corresponding to $\lambda \simeq (2 \times 10^{17}\ \mathrm{GeV})^4$, (corresponding to the upper bar) the predictions are beyond the Planck TT+lowP contour limits. We find a lower bound, for 60 e-folds, of $\lambda \geq 1.26 \times 10^{-7}$, corresponding to $\lambda \geq (2.3 \times 10^{17}\ \mathrm{GeV})^4$, for the inflationary predictions to be within the Planck TT,TE,EE+lowP contour limits.

In Fig. 10 we present the relation between the spectral index and the logarithm of the brane tension λ. As expected, n_s is not very sensitive to the value of λ, as it was shown in Ref. [13], using the dual of this theory to a scalar field, we can fix the number of e-folds as 60 and in this case, $n_s \approx 0.97$. Also in Fig. 10 we analyse how the brane tension and the tensor to scalar ratio are related as λ is lowered for 60 e-folds. For $\lambda < 10^{-6}$, r quickly increases due to the presence of F^2 in Eq. (6.67), making the predictions in conflict with Planck data, as we also saw in Fig. 9.

When we lower the brane tension, in order to keep the power spectrum of scalar perturbations fixed as $\mathcal{P}_\zeta(k_0) = 2.196 \times 10^{-9}$, for the pivot scale chosen at $k_0 = 0.002$

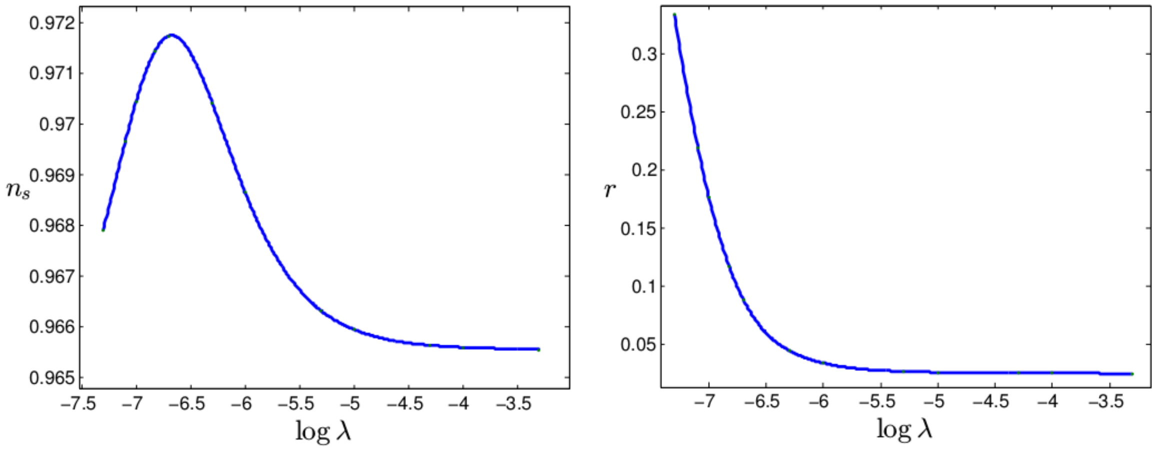

Figure 10: Left panel: $\log\lambda$ vs n_s, for the potential (6.68), with $b = -0.245$, for 60 e-folds, for different values of the brane tension λ. Right panel: $\log\lambda$ vs r, for the potential (6.68), with $b = -0.245$, for 60 e-folds, for different values of the brane tension λ.

Mpc^{-1}, we also have to change the V_0 in order to keep this fine tuning. In Fig. 11 we show the relation between λ and V_0 in order to keep fixed $\mathcal{P}_\zeta(k_0) = 2.196 \times 10^{-9}$.

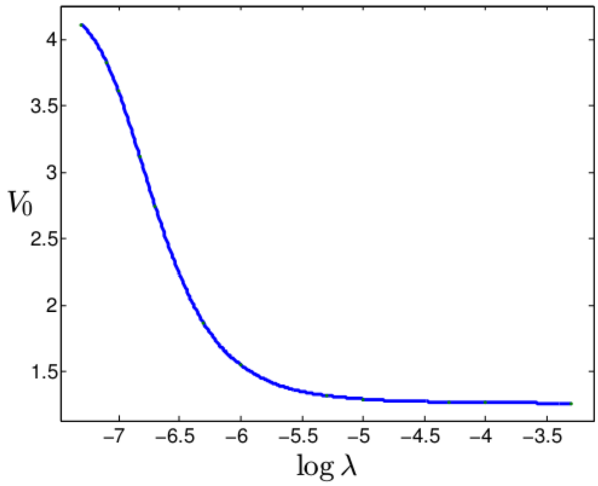

Figure 11: $\log\lambda$ vs V_0, for the potential (6.68), with $b = -0.245$, for 60 e-folds, for different values of the brane tension λ.

7 Conclusions

In this thesis I explored the dynamics of inflation driven by two coupled 3-forms compared with the uncoupled case [16]. It was shown that the presence of the coupling gives rise to a new term in the equations of motion, (5.49) and (5.50), of the 3-forms. I proceeded with the calculation of the stress-tensor (5.51) that also gives rise to a new term resulting from the coupling. In particular I explored two solutions, for a quadratic and an exponential potential, of the equations of motion (5.62), (5.63), (5.64) and (5.65), for quadratic and exponential potentials, with a particular choice of initial conditions in order to have inflation. It is shown that a small coupling, in the order of 10^{-3}, has the influence to extend the duration of inflation of about 20 to 50 e-folds.

I have also proposed a general form for the Lagrangian of $\mathcal{N}$ coupled 3-form fields. By introducing a new term (5.73) that express the couplings of multiple 3-forms, imposing some restrictions in order to avoid repetitions in the coupled potentials, I also provided the equations of motion (5.74) for this theory.

I explored the main differences between the dynamics of a single 3-form in the Randall-Sundrum II braneworld and the standard four dimensional case [11]. Rewriting the equations of motion for the 3-form model in terms of a system of first order differential equations (6.34) and (6.35). By defining a set of useful variables (x, y, w, Θ) I have identified what we called the instantaneous critical points which now have a dependence on the correction term, Θ, arising from the modified Friedmann equation. I described the phenomena that take place at high energies by showing the phase space of the system at different stages of the universe, or in other words, for different values of Θ, and by interpreting them as a modification to the effective potential. I also observed that in five dimensions the behaviour, or more precisely the stability, of some instantaneous critical points can change as the energies decrease. I presented an inflationary solution for the potential in (6.68) and computed the respective cosmological parameters tensor to scalar ratio (6.67) and spectral index (6.62). Finally I was able to fit the cosmological predictions with the recent Planck 2015 data [26] for a choice of parameters and saw that, as expected, the effects of the braneworld bring the observables away from the central region of the data contours. I found a lower bound for the brane tension for the potential (6.68) such that the observable values remain inside the contours of the Planck TT,TE,EE+lowP.

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"Learn from yesterday, live for today, hope for tomorrow. The important thing is not to stop questioning."

-Albert Einstein