

UNIVERSIDADE DE LISBOA  
FACULDADE DE CIÊNCIAS  
DEPARTAMENTO DE FÍSICA



# Emergent Behavior In Multiplicative Critical Processes and Applications to Economy

João Pires da Cruz

DOUTORAMENTO EM FÍSICA

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# Emergent Behavior In Multiplicative Critical Processes and Applications to Economy

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Especialmente Elaborada Para a Obtenção do Grau de

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## Emergent Behavior In Multiplicative Critical Processes and Applications to Economy

### **Abstract:**

The main objective of this thesis is to develop a theoretical foundation for the study of economic phenomena based on methods of statistical physics applied to a system composed by set of multiplicative processes. An equivalent of equilibrium is established for such system and proved to behave statistically as in thermal equilibrium. An equivalent to canonical and microcanonical ensembles is realized and the relation with the theory of scale-free complex networks is made. The statistics of more than one century of US economy is studied in the light of these findings and an explanation for inflation and the resilience of wealth inequalities is found. The equivalent of Markov stochastic process on the set of multiplicative processes is established and the corresponding Fokker-Planck equation is derived. Moreover, a relation with self-organized criticality (SOC) is made. The study of market fluctuations is done using SOC models and yielding the same result as the Fokker-Planck approach. Based on these findings, we will argue that the distribution on the fluctuations of prices in organized market cannot follow Levy-stable distributions as stated by Mandelbrot.<sup>89</sup> Finally, some applications to market and credit risk are made.

**Keywords:** Self-Organized Criticality, Statistical Physics, Economics

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## Comportamentos Emergentes Em Processos Multiplicativos Críticos e Aplicações à Economia

### Resumo:

O objectivo principal desta tese é o desenvolvimento de um novo enquadramento teórico para o estudo dos fenómenos económicos baseado em métodos da física estatística aplicada a sistema composto por um agregado de processos multiplicativos. Num tal sistema, um estado equivalente ao estado de equilíbrio emerge e demonstra-se que o seu comportamento estatístico é semelhante a um sistema em equilíbrio térmico. É realizado o estudo dos correspondentes *ensembles* canónico e microcanónico e feita a ligação com a teoria das redes complexas livres de escala. A estatística de mais de um século de economia dos EUA é estudada à luz destes desenvolvimentos e dada uma explicação para os fenómenos da inflação e da resiliência das desigualdades sociais. O equivalente ao processos estocásticos Markovianos no agregado de processos multiplicativos é estabelecido com o desenvolvimento da correspondente equação de Fokker-Planck e é feita a relação com o fenómeno da criticalidade auto-organizada(SOC). O estudo das flutuações nos preços de mercado usando modelos SOC é feito levando ao mesmo resultado esperado pela abordagem equação de Fokker-Planck, o que nos vai permitir no futuro fazer a ligação com conjunto de ferramentas desenvolvidas pela matemática financeira. Baseado nestes resultados, argumentamos que as flutuações dos preços de mercado não podem seguir distribuições Lévy estáveis como propunha Mandelbrot<sup>89</sup> Finalmente, algumas aplicações do enquadramento teórico são apresentadas.

**Palavras Chave:** Criticalidade Auto-organizada, Física Estatística, Economia

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To all the Economists that think that Theoretical Economics is not an issue;

**Thank you!**





*The problem of knowledge is that there are many more books on birds written by ornithologists than books on birds written by birds and books on ornithologists written by birds.*

Nassim Nicholas Taleb



# RESUMO DA TESE

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A surpresa com que os fenómenos económicos dos últimos cinco a seis anos foram recebidos, mostraram ao mundo o pouco que sabemos sobre eles e sobre os mecanismos que lhes dão origem.

No âmbito deste trabalho pretende-se estudar e adaptar o enquadramento teórico associado á explicação de tais fenómenos fazendo uso das mesmas ferramentas usadas pelas ciências naturais, particularmente pela Física, na construção de modelos que reflectam a realidade de forma sólida.

Os economistas definem a Economia como «a ciência social que estuda como a sociedade escolhe afectar os seus recursos escassos para entregar bens e produtos para consumo presente e futuro». <sup>122</sup> Não sendo esta definição completamente unânime entre os economistas, tem uma aceitação alargada o suficiente para que nos possamos restringir a ela. Nesta tese pretende-se despir esta definição da sua metalinguagem e procuramos entender a razão da existência de uma economia para conseguir encontrar um modelo com que possamos encarar tal conjunto de mecanismos como algo explicável numa sequência lógica coerente com os resultados empíricos. Ao contrário do que acontece na actualidade em que os resultados empíricos “explicam” os fenómenos sempre *a posteriori*. A definição com que se trabalha nesta tese é que a “economia é o conjunto de mecanismos pelo qual o ser humano afecta recursos finitos a necessidades infinitas”, que é uma definição em tudo semelhante à anterior mas despida de metalinguagem.

As necessidades infinitas levam a que o ser humano esteja em permanente necessidade de produzir novos recursos em função dos recursos que lhe estão afectos. Isto permite-nos modelar cada agente económico de forma a que os recursos  $x$  que lhe estão afectos segue um processo multiplicativo, i.e.,

$$\frac{dx}{x} = \beta \tag{1}$$

onde  $\beta$  não depende de  $x$ . Uma economia é, então, um gás destes processos multiplicativos que interagem entre si.

Tendo um gás de “partículas” com estas características, podemos recorrer ao enquadramento teórico da Física Estatística para estudar um gás com estas características. Vamos mostrar que um gás de processos multiplicativos se revela estatisticamente em equilíbrio, não no tempo, mas na escala. Mostramos que isto explica também a ocorrência de redes complexas livres de escala na natureza e na sociedade. Este equilíbrio aparente ao longo da escala é também independente do coeficiente multiplicativo  $\beta$ .

Num gás de processos multiplicativos, como definido atrás, dizemos que apresenta uma invariância do logaritmo, uma vez que, da mesma forma que um gás isolado tem energia invariante, um gás de processos multiplicativos tem invariância em  $\log(y)$ , em que  $y = x/x_{max}$ , sendo  $x_{max}$  o maior valor de  $x$  no gás. Com tal princípio, introduzimos *ensembles* canónicos e microcanónicos e estudamos o seu comportamento usando o caso particular das redes

livres de escala como gases de processos multiplicativos, à semelhança do que acontece no enquadramento da Física Estatística com gases "normais". Mostramos que o expoente da rede, que reflecte a correlação entre os vários processos multiplicativos que compõem o gás, corresponde ao inverso da temperatura do gás, usando a analogia com Termodinâmica que a abordagem da Física Estatística permite.

Adicionalmente, mostramos que a inflacção é um fenómeno natural em tal sistema tal como a resiliência das desigualdades sociais que foi observada numa província da Roménia por outros autores.<sup>44</sup>

Com base no descrito acima é estudada a economia dos EUA recorrendo a dados relativos aos últimos 113 anos de emprego, inflacção e um índice bolsista principal, o *Dow Jones Industrial Average*, que foi usado como *proxy* para o estado da economia. Mostramos que descontado o efeito da inflacção e do número de pessoas que contribuem com trabalho, a economia dos EUA comporta-se aproximadamente em invariância logarítmica.

Estando definido um sistema em equilíbrio aparente é lícito procurar o enquadramento teórico para as flutuações em torno de um estado estacionário, à semelhança do que é feito na Física Estatística. Por outras palavras, procurar o enquadramento teórico dos processos Markovianos. Esta tarefa é importante no sentido que é esta a ligação histórica entre a Física Estatística e a Matemática Financeira. Nos anos 70 do sec. XX, Black, Scholes e Merton determinaram o valor de um opção Europeia recorrendo ao princípio, que hoje sabemos errado, de que os preços em mercado organizado seguem um processo estocástico designado por movimento Browniano geométrico. Tal trabalho foi a origem de um ramo da Matemática/Economia chamado Matemática Financeira. Sabendo nós dos pontos anteriores que o estado estacionário se traduz por uma invariância do logaritmo, então podemos determinar a equação de Fokker-Planck associada às flutuações em gases de processos multiplicativos. Mas, sendo um sistema em expansão e cuja solução provamos como única que é estacionária, podemos associar essas flutuações àquilo que se designa por criticalidade auto-organizada (de *Self-Organized Criticality*, SOC).

A SOC é o resultado de um sistema em que se injecta continuamente energia e a liberta por avalanches que afectam o sistema como um todo. Sabendo nós que é matematicamente mais difícil tratar um processo de SOC que uma equação de Fokker-Planck, a verdade é que mostramos que os parâmetros associados à equação de Fokker-Planck não são mensuráveis, ao contrário do expoente da rede complexa que está subjacente à SOC. Como a SOC não depende do nível de energia acumulada a que o sistema entra em avalanche, a SOC torna-se um modelo mais simples de usar que a equação de Fokker-Planck. Mas esta é o ponto de ligação com o conjunto alargado de ferramentas que constituem a matemática financeira actual e será por ela que no futuro o nosso trabalho se vai focar.

Esta relação entre a SOC e os gases de processos multiplicativos é feita estudando as flutuações em vários mercados bolsistas principais, onde demonstramos empiricamente e analiticamente a ligação entre a amplitude das flutuações e a geometria do sistema. O resultado obtido é mesmo quer se esteja a abordar o problema pela equação de Fokker-Planck, quer se esteja a abordar o problema por SOC.

Finalmente, será feita a utilização do enquadramento teórico acima descrito em duas aplicações concretas, uma relacionada com risco de mercado, outra relacionada com risco de crédito.

A questão do risco de mercado é uma questão premente em Matemática Financeira atendendo que se não se assumir que as distribuições das flutuações nos preços de mercado são Gaussianas, então terão que se assumir outras distribuições mais complexas de trabalhar e sem qualquer garantia de que o resultado seja melhor. Com base nos resultados que obtivemos, não conseguimos obter uma distribuição para as flutuações, mas conseguimos encontrar as fronteiras nas quais essas flutuações podem ocorrer. Em termos de quantificação de risco, isto significa que se pode dar a liberdade ao analista de quantificar o risco associado a uma exposição ao risco de mercado da forma que a sua experiência determina porque podemos quantificar o risco da medida de risco estar errada, desde que esse risco esteja dependente de um índice que reflita o comportamento da economia.

No que a risco de crédito diz respeito, estudamos teoricamente a aplicação, pelas entidades reguladoras do sistema bancário, de regras que impõem a subida dos capitais próprios dos bancos para melhorar a resiliência do sistema aos impactos da conjuntura. Recorrendo à modelação acima descrita mostramos que a subida dos níveis de capitais próprios tem exactamente o efeito contrário. Não só não favorece a estabilidade do sistema como o fragiliza perante as alterações da economia envolvente, aumentando a amplitude das crises bancárias.



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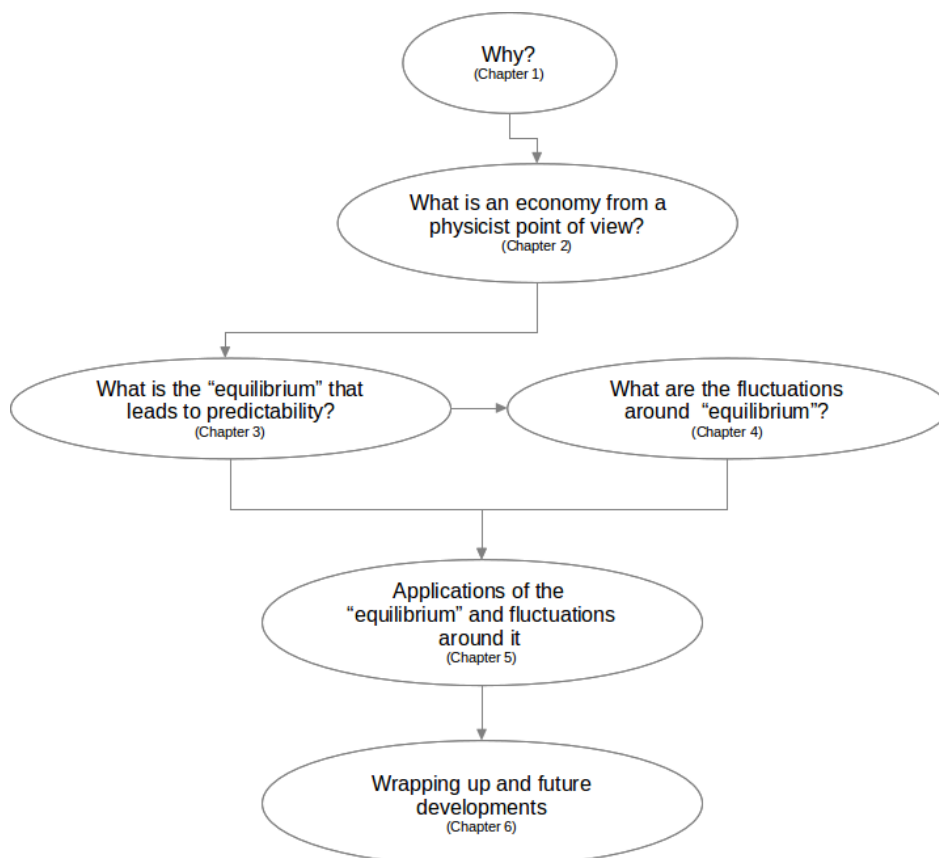
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# TEXT FLOW

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**Figure 1:** Dissertation Scheme



# CONTENTS

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|          |   |            |
|----------|---|------------|
| <b>1</b> | <b>Introduction</b>   | <b>1</b>   |
| 1.1      | An historical tail . . . . .                                | 2          |
| 1.2      | Why is this thesis in Physics? . . . . .                    | 6          |
| <b>2</b> | <b>What is an ‘economy’?</b>                                | <b>7</b>   |
| 2.1      | The meaning of ‘Economy’ . . . . .                          | 8          |
| 2.2      | Why does it exist? . . . . .                                | 11         |
| 2.3      | Economics scientific method . . . . .                       | 13         |
| <b>3</b> | <b>Equilibrium: A concept between Physics and Economics</b> | <b>15</b>  |
| 3.1      | State-Of-the-Art . . . . .                                  | 16         |
| 3.1.1    | Definition of Equilibrium . . . . .                         | 16         |
| 3.1.2    | Statistical mechanics approach . . . . .                    | 18         |
| 3.1.3    | Extensibility . . . . .                                     | 19         |
| 3.1.4    | Equilibrium under probability theory . . . . .              | 22         |
| 3.1.5    | Price ‘Equilibrium’ . . . . .                               | 24         |
| 3.1.6    | Out of Equilibrium Geometries . . . . .                     | 25         |
| 3.2      | Contributions . . . . .                                     | 36         |
| 3.2.1    | Economic Equilibrium and Logarithmic Invariance . . . . .   | 36         |
| 3.2.2    | Wealth Distribution . . . . .                               | 43         |
| 3.2.3    | Inflation . . . . .   | 44         |
| 3.2.4    | Canonical and Microcanonical Economic Ensembles . . . . .   | 46         |
| <b>4</b> | <b>Out of Economic Equilibrium</b>                          | <b>55</b>  |
| 4.1      | State-Of-the-Art . . . . .                                  | 56         |
| 4.1.1    | Stochastic Markov Processes . . . . .                       | 56         |
| 4.1.2    | Ito’s Lemma and Option Pricing . . . . .                    | 63         |
| 4.1.3    | Self-organized criticality . . . . .                        | 65         |
| 4.2      | Contributions . . . . .                                     | 66         |
| 4.2.1    | Brownian Motion on Scale . . . . .                          | 66         |
| 4.2.2    | Dynamics of Stock Markets . . . . .                         | 70         |
| <b>5</b> | <b>Applications</b>   | <b>83</b>  |
| 5.1      | State-Of-the-Art . . . . .                                  | 84         |
| 5.1.1    | Key Concepts in Finance . . . . .                           | 84         |
| 5.1.2    | Market Risk . . . . .                                       | 85         |
| 5.1.3    | Credit Risk . . . . .                                       | 86         |
| 5.2      | Contributions . . . . .                                     | 90         |
| 5.2.1    | The Dynamics of Financial Stability . . . . .               | 90         |
| 5.2.2    | Economic Generated Thresholds . . . . .                     | 103        |
| 5.2.3    | Bounding Market Risk . . . . .                              | 117        |
| <b>6</b> | <b>Conclusion and Discussion</b>                            | <b>125</b> |

**Bibliography**

**129**

# 1

## INTRODUCTION

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*This chapter briefly describes the motivation for this thesis and why is it done in Physics. The best way to describe the motivation is to tell the story on how the previous professional unsuccess lead to the appearance of idea for this work and the economic motives for it.*

### Contents

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|     |  |   |
|-----|--|---|
| 1.1 | An historical tail . . . . .             | 2 |
| 1.2 | Why is this thesis in Physics? . . . . . | 6 |

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## 1.1 An historical tail

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On March 14th, 2008, a 85 years old, 30 000 employees, global investment bank called Bear Stearns was dead. The once pungent investment house that created several of the financial instruments used today, died during a process that last for... one single week. The world was already accustomed to stories of people that got greedy and built huge frauds that dumped centenarian banks, like Bearings, but this was something completely different. This bank did not fall due to greediness. It fell due to ignorance.

To understand the full story we need to go back to 1938 and the FDR New Deal. To improve house building and ownership, the National Housing Act<sup>33</sup> of 1934 and the Housing Act of 1937 created the *Federal National Mortgage Association* also known as 'Fannie Mae', establishing a secondary mortgage market. Regular banks, called 'Savings & Loans', would lend money to families to buy houses keeping the house mortgage as collateral. Then they would 'sell' the loan to Fannie Mae in exchange for cash or a sellable security. This was based on the loan pool formed from the others Savings & Loans and guaranteed by the federal government. The scheme was brilliant: A federal initiative that brought houses to almost everyone that could afford the loan payment. Since then, several changes occurred including a partial privatization of Fannie Mae and the introduction of a competitor called 'Freddie Mac' (Federal Home Loan Mortgage Corporation). Still, the goal was achieved. Houses were built and delivered while funding flowed from wholesale banks to Savings & Loans almost automatically.

In Europe, specially in my home country, Portugal, mortgage lending is a retail banking business without any government support. Banks must take into account the full lending cycle, from raising funds, evaluate the risks and lend to the final debtor. In US there was no opportunity for a bank to make business in this market. Mathematics, however, provided a way. Savings & Loans were selecting customers to whom they lend the money. They used some sophisticated scoring tools to select to whom they could lend and what amount they could lend. This is not a peaceful practice because it violates the sacred principle of American commerce of equal opportunity in service access. As cabs cannot select their customers based on race or social status, the same principle is applied to every public available service, namely banks. So, someone has made the question: what if we do not select the customers and put Savings & Loans aside?

The mathematical answer was simple: 'No problem!'. We can look at the problem as a Bayesian statistics problem. Suppose we consider that there is an uniform probability for the price of houses to raise or to fall. This will be our *a priori* distribution. Then we get a price from the market and obtain the *a posteriori* distribution conditioned to the occurrence of that event. Due to the effect of Fannie Mae and Freddie Mac liquidity services, the housing market in the US was almost monotonically growing and the probability distribution was completely biased. If the probability of an house price to fall is near zero then we can lend money to anyone because, in a short term, the collateral will be more valuable than the loan.

With this overwhelming mathematical proof, the selection process could be disregarded, mortgages could be given at free to customers that did not succeed in selection, the subprime customers. And the banks got into mortgage business. Houses were built and delivered

to customers that could not comply with the corresponding debt. But the mathematical proof was there, no mistakes were made. Subprime pools were resold all over the world as Mortgage Backed Securities (MBS), rating agencies classified them as AAA and regulators from all over the world confirmed the safeness of such investment that started to make part of bank balance sheets in other countries. A full set of derivatives that hedged portfolios of other derivatives that were hedging subprime MBS, and some of them hedging themselves, were accepted as good by regulators due to the incontestability of the mathematical formulation.

Despite some economists warnings that something was terrible wrong, this global blindness continued until the debtors start defaulting on their debts. And when they did, their houses were put in the market. And the more they did, more houses went to the market and prices drop as they never did before. And drop, and drop... AAA mortgage backed securities that were based on the value of the loan/collateral were not getting any liquidity from the house selling and they started to default on interest payment to its investors. The pool of loans was not generating money because debtors did not have it and houses were not selling at the original prices. The dropping of prices started affecting also Freddy Mac and Fannie Mae. A full world of financial sophisticated experts started to appear in the newspapers as complete ignorants or, worse, as crooks. The full set of mathematical tools was wrong!

Between those financial experts were the Bear Stearns officials who plunged the bank into subprime MBS until two of its funds collapsed in the summer of 2007, losing billions of USD. So, it is not only a simple math mistake. The funds collapsed in the summer of 2007 and the bank only fell in March 2008. It is not only wrong on the mortgage side, it must be wrong on the bank side. Why nine months between having the biggest loss in 85 years and collapsed? And why bankrupt in a week after those nine months?

Faraway from all the American turmoil, my main concern in 2004 was the development of portfolio credit risk models due to the upcoming of the Basel II regulation<sup>14</sup> that would impose very strict rules to banks with the aim of upgrading the financial stability. The approach we were following was based on a model called the Merton-Vasicek model<sup>132</sup> which was also the model beneath Basel II consulting documents issued by the Basel Committee for Financial Stability. The model is quite straightforward, we consider two random variables Gaussian distributed, one is the default events in the portfolio and the other some external variable to which we could correlate the first. Since the external variable has only two moments and those are known, the default distribution could be rebuilt using the correlation.

As the American MBS started to make front pages, the feeling of discomfort relating the Basel models started to take over my mind. All the underlying principles are the same (we will describe the model later), the usage of Gaussian distributions, i.e., highly improbable extreme variations, the stationarity of the correlations, the reading of empirical data taken as the absolute truth, disregarding the mechanics of the problem, etc. Yes, it was all there. The external indicator considered was the derivative of the unemployment rate, and then a question should be raised: is the measured correlation equal regardless the amount of the unemployment derivative? Using a very 'economics' line of reasoning, if the derivative of the unemployment rate was 100% in a month, then the correlation with default in the same period would be one. Since the measured correlation was not one for significantly lower derivative, then we should conclude that the correlation was not constant. The conclusion was: 'I was one of the sophisticated incompetents!'



I have reviewed every model I have built since the day I realize that I could be incredibly wrong. Luckily, the banks I have dealt in that particular modeling were not very active on mortgage. More on consumer and enterprise lending, mostly short term lending where the impact of the errors is reduced. But I was not the only one being wrong. The Basel Committee for Banking Stability believed, and still does, that there is nothing wrong with such modeling.

So, everything I have believed to be right, was wrong. I have also discovered other people that were already working on alternative ways of looking at economic and financial problems, like Benoit Mandelbrot,<sup>89</sup> Jean-Phillipe Bouchaud,<sup>22</sup> Sorin Salomon<sup>81</sup> and the most obnoxious man from an economist point of view, the philosopher and mathematician Nassim Nicholas Taleb.<sup>131</sup> Each one with a different approach, they were showing that Gaussian curves cannot be taken as assumptions on models, that empirical approaches are the best way to be bankrupt rapidly and, basically, all what is known today as financial mathematics is a fraud. Taleb even wrote to the His Majesty the King of Sweden in order that the Nobel Medal for Economics was abolished.<sup>131</sup> I am not so radical, but I started to look to other financial problems given as solved, like the Markowitz theorem for asset portfolios.<sup>95</sup> The correlation between the assets is the key feature to establish an optimal diversification to get the compromise between risk and expected profitability. Without any effort<sup>i</sup> it can be shown that correlations are, in fact, the most volatile of the involved quantities, assets included and the theorem has absolutely no value in mitigating risk, which is obtained just by the diversification of assets. The optimization made based on wrong assumptions. The scoring procedures for credit approval are based on empirical data collected in the past and assuming that there are no dependency between debtors. As Bear Stearns officials and some of the biggest banks in Europe can testify, that it is not the case. There is a network of dependencies that is not taken into account in scoring and rating procedures. This fact take us to the reason why Bear Stearns bankrupted abruptly: we are all interconnected. There is not such a thing as uncorrelated events!

Bear Stearns fall in a week, nine months after their funds collapse. The gross amount of affected customers was in the funds. Why did it fall nine months after? The story is more complex than this, but when the funds collapse the affected customers were 'small', when the bank fell the affected customers were called Goldman Sachs and JP Morgan. So two single customers could do what thousands could not. Another thing was missing besides the plain and flat statistics. Some physics was missing in all the economic reasoning. If two customers can do what thousands cannot, this shows that the economy has a geometry which is not compatible with the financial mathematics we use today. Vilfredo Pareto showed in the beginning of the 20th century that the distribution of wealth follows a power-law, i.e., the probability of having a specific wealth is proportional to a negative power the wealth itself. So, as we zoom on wealth the geometry of the economy will be the same and we will find a geometry with fractional dimension.

Thus, Economics seemed to me as an open field where everything that matters is basically wrong. The goal in this PhD thesis is to find out how wrong it is and to deliver some economic value in the process. The work done on finding how deep should one go to fix the errors of the 'economic science'. And, as we will see, we must go very deep.

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<sup>i</sup>Closer confidential material.

Modern economists deny the fundamentals of the science they have tried to build. In one hand, they accept the first principles as unquestionable and, with the other, they accept their field of knowledge to be a ‘free dumping site’ where every non-sense theory and every stupid idea can be accepted. The examples are so many that the problem is choosing one, but there is one that I fancy more called ‘Austerity Multipliers’. The problem generated a huge discussion between several Nobel priced economists about the way government consumption growth influences GDP growth, specially due to sovereign debt crisis in Europe. Too abstract? Well, let us find out what GDP means. According to the real source of knowledge of the 21st century, called *Wikipedia*, the GDP is the market value of all officially recognized final goods and services produced within a country in a given period of time. Reading this, one might think that some one spends the entire year making the inventory of everything produced, and every service made, with it’s corresponding market value. There is a much more simple method which is measuring the money flow. In fact there are several ways of measuring the money flow but, since money flow is physical, then all methods should give the same result! One of the methods is measuring the GDP by summing private consumption, government spending and investment, plus exports minus imports (the money inflow minus the outflow). Unsurprisingly, there is a linear dependence (multiplier) between the variation of GDP and government spending. Linear dependence is precisely what we need to define the measure! Krugman could find a 1.25 multiplier while the IMF could find multipliers between 0.9 and 1.7 and the subject is being discussed for ages. As anyone with some scientific culture can identify immediately, the dependence exists *per definitio* of GDP.

So, what is an economy in the first place? The problem aimed to be solved in this PhD is to find the proper playing field to explain economic phenomena in the same sense we explain any phenomenon in Physics, i.e., the explanation of the phenomena does not depend on the state of nature. In other words, the developed laws should provide ways to predictability. Like is drawn on the scheme in the beginning of this thesis, Fig. 1, we will in Chapter 2 understand what is an economy and why is there an economy, what is right and what is wrong. What are the peculiarities of such a system that make him special when compared to other physical systems of many bodies. For that we will make use of a (very good) book on introductory economic theory, ‘Introductory Economics’ by Arleen and John Hoag, as a model.

Next, in Chapter 3 we will redefine one of the most important concepts in Economics: Equilibrium. We will argue that economic equilibrium is something completely different from what economists and physicists believe to be. Further with such redefinition of economic equilibrium part of the economic strange phenomena can be easily explained because the basic mathematical playing field is established.

Chapter 4 is devoted to near equilibrium economics. The subject is very popular between physicists since Black, Scholes and Merton developed their method for valuating a derivative called European options by making use of Ito’s Lemma.<sup>17,97</sup> However, the issue became controversial since stochastic processes based on Wiener processes fail completely to explain the dynamics of markets.<sup>130,131</sup> Like in any Statistical Physics course, after study the equilibrium we study the fluctuations.

In Chapter 5 some applications are introduced and carefully treated. Of course, such chapter is a never ending one and therefore we concentrate ourselves more on theoretical foundations, specially in Finance to give this work some economic value besides the academic one.

Finally, in Chapter 6 we will draw conclusions and put this work in perspective based on the pros and cons of the present approach and the future work.

## 1.2 Why is this thesis in Physics?

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Why is this work developed in Physics and not in Economics? The answer will be obvious in the text but, in few words, we are looking for predictability; i.e., for models that simplify reality in a way that does not depend on time or in the state of nature. Models that represent reality in a way that when data is collected, it corroborates the theoretical findings, or it denies them, but does not define the model. That is what it is achieved with the Statistical Physics framework which is used abundantly in this thesis. As it should be obvious in the end of this thesis, the study of an economy is all about Physics: Physics of the Multiplicative Processes Systems.

As it will be also obvious, economic objects are multiplicative process, not inanimate molecules of gas. This means that the field to which we will apply the Statistical Physics framework is far from being the traditional inelastic gas, isolated from the neighborhoods or surrounded by an infinite reservoir of energy. Some of the points in this text may seem to academic or even trivial for a reader from Physics, but this has two objectives. The first, to provide readers with backgrounds from Economics or Financial Mathematics the full scope of the concepts involved. Second, we meant to make a very clear statement of the meaning of what we are talking about, specially for the readers with a Physics background. Since we are applying a framework with a valuable set of tools to a system for which it was not destined, each concept must be perfectly clear on the reasons why it can be applied and must be careful and thoroughly introduced.

But is not a physical system fundamentally different from an economic system in the sense that the particles in an economic system can know the ‘laws of movement’ and change the ‘movement’ accordingly? Yes, it is. But we are not going that far, we will not try to predict the position of the market in a given horizon. We are only interested on the physics of the problem. We are as interested in knowing a price of an asset next week as we would be interested in knowing the position of a particular gas molecule in a reservoir. As academically interesting that could be, it is useless. We are just interested on what is reproducible and even if the economic particles can know what is stationary in their system that is irrelevant in their behavior because, as we will see, each particle by itself is unpredictable and on that assumption we can definitely rely on.

# 2

## WHAT IS AN ‘ECONOMY’?

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*In this chapter we will try to provide a description of what is an economic system by translating from theoretical economics literature into something that can be easily and rapidly understood by someone with a Physics background. It is not intend to be a exhaustive description or a state-of-the-art in Economics. First, because we intend to model an economic system as a physical system and we should understand it in its fundamental mechanisms and, second, because any Economics undergraduate book provides such a state-of-the-art. Also we try to argue on why is there an economy, i.e., what cause the existence of such a system in order to sustain our modeling (and Economics modeling) on biological arguments*

### Contents

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|     |                                       |    |
|-----|---------------------------------------|----|
| 2.1 | The meaning of ‘Economy’ . . . . .    | 8  |
| 2.2 | Why does it exist? . . . . .          | 11 |
| 2.3 | Economics scientific method . . . . . | 13 |

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## 2.1 The meaning of 'Economy'

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As human beings, we are surrounded by economic activity. More than we can identify, even when we are at school or watching television we are being a part of the economic world. Almost everything we do in a day is related to our economic role, leaving very few time for the other ones. But what is this world and how can we study it?

Let us start to understand how economists look at the object of their study. Readers from natural sciences should be advised that not every economists agree on the definition of their science or even on its object of study. It is not what we are used to deal with in Physics, but economists do not see any harm on that and, for the time being, we do not want to disagree. We will not tell here the history of how economy was defined or looked at. As it is acceptable that 11th century astronomers did not had the means to understand the universe, 16th century economists did not had the means to understand economy as we understand it today. In a thesis in Economics, it could make sense to make such an exposure in detail, but not here.

We will first concentrate on a widely accepted modern definition of Economics stated by the theoretical economist Lionel Robbins in the first third of the 20th century.<sup>122</sup> According to Robbins, economics is the science that studies human behavior as a relationship between ends and scarce means which have alternative uses. This definition is remarkably elegant and short but, for our purposes we need to reformulate it slightly in other words. A reformulated definition is given by Hoag<sup>63</sup> and we will use it taking in mind the Robbins definition which is equivalent.

According to Hoag, 'Economics is a social science that studies how society chooses to allocate its scarce resources, which have alternative uses, to provide goods and services for present and future consumption'. This definition carries more economics meta-language and therefore allows us to make the bridge between Robbins definition and what it is our intuitive notion of economics.

One thing we should take in mind while reading the definition of economics: What we define as being economy must hold independently from the state of the world, because the economy we have today, the one in which we instinctively act, is the result of the economy we had in the past. The system can be in a different state but the same fundamental mechanisms, the same physics, must be present. The same mechanisms that gave *Homo Sapiens* the new diet or the new tools, gave us money, stock markets or credit default swaps.

It is important to translate the economics meta-language into something more fundamental, so we are going to walkover the concepts in Hoag's definition to understand the mechanisms and link them to Robbins definition. In the end, we ensure that we are all talking about the same thing when we say 'economy'.

Goods and services are Economics' concepts representing something that results from production, i.e., from someones' labor, intended to satisfy a want. If it is tangible like a car it is called a good, if it is intangible like education it is called service. An important point is that they need to be consumed before they can be called a good or service.<sup>63</sup> There is no such

thing as production without a subsequent consumption. And, also, there is no production without incorporation of some kind of labor. As an example, let us take oil. People produce a good to be burned in engines called oil. But, at the same time, oil is useless without another substance which is oxygen. Oxygen is more important for combustion than oil but oil is a good, oxygen is not. Why? Because there is no labor involved in oxygen consumption by engines. Oil incorporates the labor of searching, extracting, refining, transporting, marketing, whatever that turns a liquid rock into a combustible inside a car. Oxygen is - fortunately! - all around us, at least until now.

If goods and services are the result of production, there must be other things that are the input of such process. One is obviously labor which is the human effort put in production. All inputs of the production process are called resources. Economists classify them into four classes: Labor, Entrepreneurship, Land and Capital.

Entrepreneurship is the way we leverage our labor making use of other people's labor to accomplish tasks that we could not make alone, i.e., a particular form of mental labor. The ability to control the allocation of resources is called *wealth*.

Land is the set of natural resources needed for production. Naturally, we need the terrain where we sit to write this dissertation. If we are extracting crude oil we need a place to extract it from. Again, crude is not a (natural) resource in economic terms, the resource is the place we drill, which was found by geologists that made an enormous set of tests and applied complex mathematical modeling to derive a probability of finding crude beneath several hundred meters of rock. The crude itself is just a rock without any economic meaning. For the refineries, the resource is not crude oil, is crude oil out of the earth and transported to the refinery. That is an economic resource because it incorporates labor.

Capital is the set of man-made tools of production, a good which was produced to be consumed with the purpose of production. If we think inside a specific perimeter, this differentiation between goods and services by one hand and capital by another hand, makes sense. Some things are inputs to the perimeter, other are outputs.<sup>134</sup> And we should emphasize that Economics is intended to be applied to all sorts of domains. But if we think in a more holistic way, does it make sense? What is good or service for one agent, is resource for another because there is no production without subsequent consumption. To the farmer an apple is a good, for the blacksmith is a resource to produce the wrench that is needed by the car constructor to produce the car needed to take the apples from the farm to the blacksmith. The only difference between capital and goods/services is the instant in time when it was produced, the point in time when the labor was incorporated. And, as we will see in this dissertation, the ability of modeling everything as forms of labor is considerably useful.

What about land, why do we pay for land? In fact, we do not. Land is also a product that involved some kind of labor, because the resource is not land, is the *control of the allocation of the land*. Its security, its privacy, the exclusivity of access. It was achieved with the labor of some military workers back in the middle ages, to which army invasions and bombardments were added before the lady from the real estate agency presented it to us. But it is the result of production, it has labor incorporated in it.

So, keeping it more mathematical, resources, goods, services are all forms of labor differentiated by the instant in time. If we recall now Robbin's definition, he only refers to means and ends

and the relationship between them, because means and ends are all forms of the same abstract concept: labor. Also, let us recall that the definition of economy should be 'state-of-the-world free'. It should cover the beginning of the economy, from the first labor exchanges until nowadays. Reducing it to the study of exchange of labor is compliant with that requirement and quite useful from a physics perspective.

We now need to address the scarcity issue, focused by Robbins and Hoag, that differentiates cooperation from economy.

The definition economists give to scarcity is not simple. Resources are scarce; meaning that everything that results from labor and it is consumed is scarce. So we know that it is scarce after it is consumed, not before. And the reason why it is consumed is because it is scarce. To unroll this, we should make reference to a study on cooperation on capuchin monkeys.<sup>135</sup> For capuchin monkeys, the food for which they cooperate is scarce, similar to humans. It has the purpose of satisfying a want, hunger. If the food would be available in another way they would not need to cooperate, meaning that in that case food would not be scarce. In this sense the meaning of scarcity is intuitive, the finiteness of resources leads to labor exchange at some point. As soon as the want is satisfied, cooperation ceases because the want is also finite. Hunger disappears as monkeys eat whatever they can find with their cooperation. Is not difficult to conclude that cooperation will not bring anything new to the capuchin monkey society because they will cooperate always to satisfy the same finite amount of wants. They can find another creative forms of cooperation to get food, but they will not produce anything. An equilibrium is reached.

Scarcity, in economy, results from infinite wants. It is the infinite amount of wants that promote labor exchange in a way that the end of a production process is the input of the other. And this is the input for the next production process. Since wants are not finite, the production process never ends. This is why the definition of scarcity is so cyclical, because it must justify an infinite chain of exchanges, unlike the 'natural rareness' that capuchin monkeys are submitted. That is why capuchin monkeys do not have an economy, scarcity in Economics relates finite resources with infinite wants.

It is precisely because we are dealing with infinite wants for finite resources that the word 'allocation' must be present in the definition of economy. There are different kinds of wants and choices must be made. Wants are left behind to allocate resources to the ones we choose, which imply an opportunity cost.

Opportunity cost is a very important concept in economics and means that, for every economic choice we make, we are paying the cost of not choosing an infinite amount of others.

To simplify things, we suggest for the sake of modeling that Economy can be defined as a set of mechanisms of labor exchange performed to satisfy an infinite quantity of needs. In the scope of this dissertation, this is the definition of 'Economy'.

From the late definition of economics by Hoag we are left with the 'social science' concept. For our purposes, that is exactly what we want to get rid off, the idea that there is a special kind of scientific approach just for social matters. In our opinion, there is not. There is science and there are other things. Nevertheless, we must understand it to understand where economist



fail today and where can we, as natural scientists, contribute to give Economics the scientific character that economy allows.

Having defined the economic system, the next important concept to address is money. Economists define money as a resource,<sup>100</sup> something that is used to exchange by other resources. The set of bank notes and coins people carry is called currency. The economists do not have a precise and unique definition of money, but they recognize in money several roles in economy.

The first is that money is a mean of exchange. Since money is a relatively recent invention in the history of the mankind, humans must have lived in a *barter* economy, i.e., an economy without money where resources were exchanged directly without the use of an auxiliary resource. It was, nevertheless, an exchange of labor, but there was the need to match the two forms of resources. For example, exchanging a goat with a dental work. This matching effort no longer exists today due to the use of money.

Second, money provides a unit of account. Since the resources that are exchanged are different, then there is not, in principle, a way of telling if the two amounts of resources are equivalent. Money, provides a way of measuring the *value* of the resources because both are reduced to the same units and therefore are comparable. Money is the *device* used to *measure* the value of labor.

Moreover, money is also a *store* of value. In each note or coin there is a repository of an amount of labor that can be used in the future to exchange by another resource.

Since we defined economy as labor exchange, we must consider money as a form of ‘canned labor’. Each quantity of money encloses in it an amount of labor done in the past. We will come back to this when we focus ourselves in inflation. All in all, the complete message to retain now is that if something is exchanged, is because they share a common base and that common base is labor.

## 2.2 Why does it exist?

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As a physicist, the first question that pop into my mind was: ‘*Why is there an economy?*’. If it exists then there must be a cause or set of causes in the biological substrate that lead to the existence of such a system and, also, a set of necessary conditions that keeps the system existing. These sets of reasons and conditions lie outside the scope of physics. Nevertheless, we need to understand them to support our claim that human beings can be regarded in their economic behavior as predictable economic particles.

It is not easy to find literature on the reason for the existence of an economy. The economy, as we know it, is not a reproducible experiment. There is, though, some literature on cooperation in animal societies<sup>135</sup> in which experiments were made and reported. Some of these authors<sup>62, 135</sup> see cooperation as a form of economy. It is not our goal to get into controversy on the subject, but it is our goal to model economy and we will argue below that cooperation itself does not lead to an economy in the sense of that we are interested in.



To start the arguing, we must anchor ourselves in the biological substrate of the economic particles: the human beings. Using the reasonable well accepted explanation for the evolution of the species by means of natural selection<sup>41</sup> it is known that some species adopt a social behavior, i.e., individual beings group to gain competitive advantages defending themselves from the environmental threats. Between those species that evolve to become social, some developed mechanisms of cooperation.

Cooperation implies that two or more specimens perform individual tasks to achieve a common objective. We can easily find forms of cooperation in which individuals perform the same task, like in wolf packs, or different tasks like with ants or bees. The reason why this specimens organized themselves in societies is that they are more efficient to deal with the environment threats than if they have to face them as individuals and the social ones overcome the 'egoistic' ones in the evolution process.

The way nature engraves a behavior is through what is called an 'instinct', i.e., a non-conscientious impulse to behave in a certain way. Well known examples for us, *Homo Sapiens* are the sexual instinct, that insures genetic diversification in reproduction and, thus, making us more fit to resist pathological threats; and the maternal instinct that provides longer childhoods for a better fitness of the offspring. We know, also, non-conscientious behaviors related with cooperation, like bees fanning their wings to keep a stable temperature in the hive in order to keep infants healthy, or soldier ants protecting the traveling path for labor ants.

The reason for the existence of an economy in the *Homo Sapiens* society must be, in the above sequence of reasoning, the product of natural selection of the fittest and, thus, economic behavior is engraved as an instinct. The argument that there must be an economic instinct, similar to the cooperation one, provides the support to look as human beings as predictable economic particles.

Anthropologists have been studying the moment in the evolution when an economic behavior appeared in the human evolution story. Some<sup>68</sup> argue that the first signs of the existence of distinct economic roles depending on age and gender appeared back in the late Pliocene, i.e., since the beginning of genus *Homo*, while some other researchers reject this idea, arguing that real signs of such differentiation of labor roles only appeared much later.<sup>79</sup> The latter consider as evidences for the existence of an economy the appearance of tools and types of food in the diet reflecting somehow different activities of simple hunting and recollection.

The reason why this current of opinion says that economy popped up much later in the evolution of man is that *Homo Neanderthalensis* does not exhibit any of the features considered as signs of labor differentiation, opposing its contemporary mutant *Homo Sapiens*.<sup>79</sup> The fact is that an highly successful hominid like *Homo Neanderthalensis*, that last for more than 100 000 years, was extinct after *Homo Sapiens* invaded its territory in southern Europe. Regardless the reasons why *Homo Neanderthalensis* disappear, the fact is that the species with an economy was more adapted to the environment than the species without it. But is it plausible that an evolved species like *Homo Neanderthalensis* did not had role differentiation or, at least, was unable to have a cooperative behavior like any primate?

Now we need to separate cooperative behavior from economic behavior. To have a better picture of what makes an economy different from a cooperative behavior we will make use

of a study carried out over capuchin monkeys.<sup>135</sup> On an article by Frans De Wall, entitled ‘How Animals Do Business’ several cases of primates are shown to demonstrate that there is an economy among these animals. We will argue that this is not true because a cooperative behavior is not enough to form an economy. And from this point forward, we will not call economy to cooperative behavior as several authors, like De Wall, do.

De Wall observed that two capuchin monkeys would cooperate to get a cup of food that they would not access if they did not work together. First, one thing is missing from the experiment which is called ‘opportunity cost’. If the monkeys could access the food in an easier way, it is reasonable to assume that they would. It is plausible that, in these conditions, they would not work together to get these particular cups of food. The same thing happens in all animal communities where labor differentiation happens, independently from need to be more mechanical or more creative, like these monkeys. Once the need is satisfied the cooperation ceases, until the need appears again. Soldier ants exchange their security services for food with worker ants and they will always do that. If food was available to fulfill the monkeys hunger there would not be cooperation because the need is always for food.

This makes all the difference. Since the need is always the same and ceases when satisfied, there is an equilibrium around an average need. The need leads to cooperation that leads to satisfaction that leads to the end of cooperation until the need appears again and the cycle repeats. As a mathematical problem it is most likely that any study on the ‘intensity’ of cooperation in cooperative systems will result as a finite variance distribution and any aggregation of such relations will result in a Gaussian distribution, unlike economic systems, as we will show later.

In an economy, like the one *Homo Sapiens* has and, probably, *Homo Neanderthalensis* did not have, the need is always different and, once satisfied, another one appears. The exchanges of labor change their nature in time. The reason why *Homo Sapiens* developed tools and a new diet was, most likely, because his needs causes new tools, new technology which lead to new needs and so on. The exchange of labor in this case does not stop. With or without cooperation. The best explanation for the origin of an economy is a mutation between *Neanderthalensis* and *Sapiens* that made the satisfaction of the need disappear, leaving the human being with a permanent instinct of scarcity. We will argue that scarcity is probably the concept behind non-Gaussian behavior occurring in the economy. And this leads us to the next section.

Despite the fact that the emergence of an economy in nature is not a reproducible experiment, as far as we know, we can say based on the theory of evolution by means of natural selection that some species have cooperation instincts and it is very plausible that *Homo Sapiens* developed economic instincts from those. And this is a step towards predictability.

## 2.3 Economics scientific method

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One of the main issues in addressing economic problems is the considerable volatility of the ‘findings’. Indeed, two persons that study the same subject with the same data can reach two

completely different conclusions and be both right!. If there are no doubts about the purpose and object of the study, then there must be a method issue. In natural sciences, method is a closed subject. As long as the so called scientific method is followed, then one set of data will map univocally with one conclusion. In social sciences this is not the case, specially due to the idea that humans are not inanimate objects that follow pre-existing natural laws. An idea which is statistically wrong and a countersense because if humans were that unpredictable what should be the goal of a social science?

Since it is a question of method, we should inspect the scientific method of Economics, i.e., the scientific method economists believe in, to find the source of such volatility of conclusions. Like natural sciences, physics included, economists use models to simplify reality. Models in Physics are fundamental tools to erase unnecessary complexity from a problem. They are approximations to get the wanted accuracy. To explain the solar system in its macrostructure it is enough to use a set of massive balls. If we need to detail the orbit paths, then the homogeneity of the balls must be dropped.

Models are very important in natural sciences, and we will use them also in this thesis. Still, models are part of the hypothesis and have to be confirmed by data. In some cases, like mathematical physics, they can be put ‘on hold’ if some conclusion is drawn analytically, but data should confirm it sooner or later.

The technique employed by economists is using models to predict future behaviors and any model that explains the evolution of some economic variable is an *equilibrium model*. The notion of equilibrium is also a necessary condition for a Physics model to provide some predictability<sup>1</sup>. However, the two notions of equilibrium are not the same.

In the next chapter we will argue that the notion of equilibrium in Economics and the way it is methodological used is the source of the volatility in conclusions. We will show that from the definition of economics we will redefine economic equilibrium based on mathematical modeling to make it similar to the notion of physical (thermal) equilibrium.

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<sup>1</sup>Even in non-equilibrium Physics, the way to have some predictability is to forge a statistical invariance

# 3

## EQUILIBRIUM: A CONCEPT BETWEEN PHYSICS AND ECONOMICS

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*This chapter is dedicated to the study of the form of equilibrium which provides the invariance of the sample space that physically justify the existence of an economy and the predictability of the phenomena. For that we will translate the definition of economy stated in the previous chapter into a set of multiplicative processes to which we can apply the statistical mechanics framework and show that such an equilibrium appears and we will argue that such an equilibrium provides a justification for several economic phenomena.*

### Contents

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|            |   |           |
|------------|---|-----------|
| <b>3.1</b> | <b>State-Of-the-Art</b>                         | <b>16</b> |
| 3.1.1      | Definition of Equilibrium                       | 16        |
| 3.1.2      | Statistical mechanics approach                  | 18        |
| 3.1.3      | Extensibility                                   | 19        |
| 3.1.4      | Equilibrium under probability theory            | 22        |
| 3.1.5      | Price ‘Equilibrium’                             | 24        |
| 3.1.6      | Out of Equilibrium Geometries                   | 25        |
| <b>3.2</b> | <b>Contributions</b>                            | <b>36</b> |
| 3.2.1      | Economic Equilibrium and Logarithmic Invariance | 36        |
| 3.2.2      | Wealth Distribution                             | 43        |
| 3.2.3      | Inflation                                       | 44        |
| 3.2.4      | Canonical and Microcanonical Economic Ensembles | 46        |

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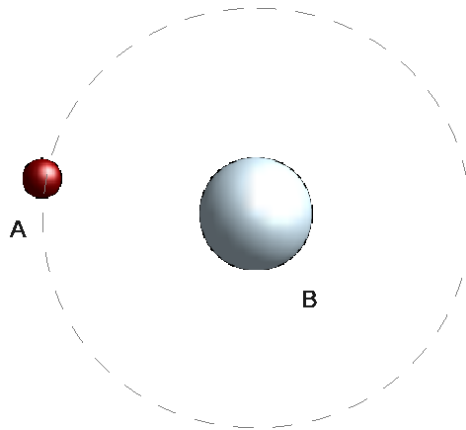
**3.1** State-Of-the-Art

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**3.1.1** Definition of Equilibrium

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What is equilibrium and why is it so important? The word is applied to several situations within several fields of knowledge. And even if we restrict ourselves to physics we will find several meanings in different sub-domains. In mechanics, equilibrium occurs when the sum of all forces applied over the bodies composing the system is zero, leaving all bodies with constant velocity. A system is said to be in thermal equilibrium if the system and the corresponding surroundings are at the same temperature and there is no heat transfers between them.



**Figure 3.1:** *Two dimensional, two body problem with circular fixed orbit.*

As a general definition of equilibrium we help ourselves with Reichl's<sup>118</sup> definition that once a system reaches an equilibrium state all changes cease and the system will remain in that state unless some external influence acts and change it. Despite the intuitive way in which this definition is given, it is not so simple as it looks.

Let us take the example of a two body system in which one moves relatively to another with a circular fixed orbit (Fig. 3.1). Is this system in equilibrium? From a physics point of view, yes. The system will be in that state until the end of times if no external influence perturbs it. But something is changing. The position of the A body is permanently changing, so the definition of equilibrium as 'nothing changes' is not entirely proper for our purposes.

So, in order that the definition holds in one body problem and in many bodies problem the definition of equilibrium must be when all statistical changes cease and the system will remain forever in that state unless some external influence acts and changes it. The introduction of the criteria 'statistical' is important to cover for individual changes that do not change the overall system state.

In practice, we should recall why we need an equilibrium in the first place. The necessary condition for having predictability in a model is to have some kind of stationarity. If we have

a single body system, either we have some deterministic laws that govern the body dynamics and the evolution in time can be completely described providing all predictability power; or we do not have those laws and, without additional information no prediction is possible. These practical issue reduces equilibrium to a pure statistical utility. Even if we are dealing with single body observations and we do not have the full set of deterministic laws, we will add new information from statistics. Either from previous observations of the body in the same unchangeable system state or by observations from the other bodies, again, in the same unchangeable system state.

This is usually a problem in many financial modeling problems such as credit risk. We aim to define the probability of a specific debtor fail to pay back a loan. Since that particular debtor never defaulted on a loan, we must build the probability based on information from other similar debtors at other instants in time. When we do that, we take the strong assumption that the economic system is in a state of equilibrium - since we are looking at the characteristics of the debtors assuming everything else is constant - and all collected information can be 'added' to the one debtor that never defaulted.

Another example, is econometric modeling. We assume that a set of random variables are dependent between each other. With that assumption we then fit our model to empirical data. When we do that we are assuming that the dependence will not change in time. In other words, when is assumed that all information belongs to the same state of the world, we are assuming that the statistics of the system will not change.

But what means to have a statistically unchanged system? It means that the probability distribution of finding a body in one combination of its dynamic parameters remain constant in time. That probability is the invariant feature of an equilibrium system. Taking the example of the two bodies on Fig.3.1 we can see that whatever dynamic parameters we choose according to the problem constraints, the probability for the body A to have some combination of dynamic parameters is invariant. Since the orbit is fixed, the distance between the two bodies is constant, meaning the variable parameter is the angle of the circular coordinate and the probability for that angle to be  $\theta$  radians is  $1/(2\pi)$  and that value of probability does not change in time. Yes, there is an infinite quantity of possible angles and yet the system is at equilibrium, statistically unchanged.

So, there is a straight relation between equilibrium and invariance of probability distributions to take into account when addressing the subject of 'Economic equilibrium'. This relation has been studied in the field of Statistical Physics and Thermodynamics and is fundamental for the understanding of the different types of equilibrium in a many body problem, like in Economics. We should emphasize that the notion of equilibrium as seen by physicists can be dropped bellow due to the several interpretations the word can take. We will keep the word as in 'economic equilibrium', but readers more sensitive in the proper use of the word in physical systems should understand it as 'stationarity'. The correct notion of equilibrium, usually associated with thermal equilibrium is useless for the purpose of explaining economic phenomena because, as we will see, an economy **is not a system in equilibrium** in the sense of thermal equilibrium.

**3.1.2** Statistical mechanics approach

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The subject of what is equilibrium and what is it worth for has been addressed by physicists within the framework of statistical mechanics for more than a century. The framework makes the link between overall thermodynamic system measures, as heat or temperature, and the statistics of the many body problem that underlies those measures, which is fundamental to characterize the mechanics of the equilibrium state.

Some authors call it the ‘statistical mechanics formalism’ which is a very straightforward way to pack several very abstract mathematical concepts that we will address later. First, we specify an overall system state (equilibrium) which we call a ‘macrostate’ characterized by an overall extensive (summable) measure, like energy, and all the possible states of each of the system components. If it is a spin gas system, then each component can have two states. If it is a perfect gas, each component can have an infinity number of states characterized by the momenta of each gas particle. Important is that each of these individual states is compatible with the macrostate of the system.

Second, we build the *ensemble*, the set of all possible states of the system, the ‘microstates’, that comply with the overall system measure. If our problem is a gas of 100 dices that has an overall measure of 600, the ensemble has only one microstate where all dices have turn 6. If our overall measure is 300, then the number of microstates is considerably bigger because there are many combinations of 100 integers between 1 and 6, that lead to a sum of 300. The concept of ensemble is the physics interpretation of the mathematical concept of sample space.

What is particularly interesting in the ensemble approach is that, if the system is in equilibrium then each microstate has a probability of occurrence of  $1/\Omega$  where  $\Omega$  is the total number of microstates. It should be emphasized that each microstate represents mathematically an event which means that the probability of  $1/\Omega$  is a probability measure over the microstates, not on the dynamic parameters that characterize those microstates. For example, if the total dice measure is 300 there is a probability for one dice to have a 6 but that probability is calculated after we have counted how many of the microstates have one dice with 6, how many have two dices with 6 and so on. Undoubtedly, the probability of each microstate is equal to  $1/\Omega$  and all microstates have equal probability. It is intuitive to understand the importance of the equilibrium assumption here, because every single microstate must be compatible with the overall measure and the invariance of the overall measure closes the number of microstates and, with it, the probability measure. From here, we can derive all the characterization of the probability measure.

The statistical mechanics framework brings also solidity to a scientific approach to problems in Economics. Since we start any analysis by enumerating any possible microstate, we are diverging from an empirical approach. We do not care what happened, we care about what is possible to happen. This would make all the difference in the subprime bond risk evaluation, for example.

Since the framework in Statistical Physics was created to address problems in physics (and let us separate Economics from Physics at this stage...), the quantity assumed to be invariant is a physical quantity called energy, according to the laws of thermodynamics. In the next

sections we will detail some more interesting features of the statistical mechanics framework in which we refer to energy, but readers should take in mind that, at this point, for our purposes, energy can be interchanged by any other summable quantity invariant in the system. In the case of Economics it related to labor.

### 3.1.3 Extensibility

Since we are using a framework from physics, some quantities must be assumed as being constant in order to adopt it to our goals. The framework uses two overall parameters that we do not need to understand economic equilibrium - number of particles and volume - that we will assume both to be constants.

Let us define ‘particle’ as the smallest isolated subsystem in the system under study. Let us assume that the energy in the system,  $E$ , is constant. The collection of systems with all the previous parameters constant is called *microcanonical ensemble*. If we divide the system in two subsystems, A and B, just by separating the particles but letting them interact with each other, then the total number of states in the composite system will be the product  $\Omega = \Omega_A(E_A) * \Omega_B(E_B)$ , where  $\Omega_A$  and  $\Omega_B$  are the number of microstates of subsystem A and subsystem B, respectively, and  $E_A$  and  $E_B = E - E_A$  are the corresponding energies. Energy is an *extensive* quantity since the total energy of the system is the sum of the compounding subsystems. But the number of microstates is not, because for each particular microstate of subsystem A we have  $\Omega_B$  possible microstates of subsystem B and vice-versa. If we add more equal subsystems to these ones, by adding particles, the number of possible microstates will grow geometrically with the number of subsystems added while energy will grow linearly. The quantity that is extensive and it is directly related to the number of microstates is  $\log \Omega$ , since  $\log(\Omega) = \log(\Omega_A(E_A) * \Omega_B(E_B)) = \log(\Omega_A(E_A)) + \log(\Omega_B(E_B))$ . This quantity was related by Boltzmann to the thermodynamical quantity of entropy, by establishing the fundamental equation of statistical physics,

$$S = k_B \log \Omega \quad (3.1)$$

where  $k_B$  is called Boltzmann constant and gives physical dimensionality to a pure numerical quantity,  $\log \Omega$ . In other words, entropy has the same physical dimensions as  $k_B$ . From a Statistical Physics point of view this is a very important relation because it translates a macroscopic quantity in terms of the number of microstates.

Since every microstate of the overall system is equally probable, the microcanonical ensemble does not give us any additional information. If we add more equal systems to the existing one that do not exchange energy between each other, i.e., isolated systems, we are adding independent systems with equally distributed energy leading to an overall Gaussian energy distribution, according to Central Limit Theorem, that we will look in more detail on chapter 4. The overall system energy in this case is the sum of the energies of the individual, isolated, systems and the system is in equilibrium because all of its components are, individually, in equilibrium.

Now, we perturb this equilibrium by removing all constraints that inhibit the energy exchange, but keeping the ones that inhibit the exchange of particles and leaving the overall system



isolated. If we want to have a more ‘economical’ view, we are eliminating the borders of a union of countries, forbidding immigration and isolating the union from the rest of the world. This will lead to another equilibrium because new set of microstates is now formed, since the constraints that kept energy constant inside each component no longer hold. The process in which each subsystem will acquire or loose microstates due to the exchange of energy with the surrounding subsystems is called *thermalization*.

If we take a particular subsystem and we assume that it is much smaller than the overall system, this means that the total energy  $E$  is much bigger than the subsystem energy  $E_S$ . Henceforth we will call this subsystem simply ‘system’ and the rest of the overall system as ‘bath’ since we are going to consider it as an infinite reservoir when compared to the system under study. The total number of microstates of the joint set system-bath *if* the energy of the system is  $E_S$  is given by

$$\Omega(E, E_S) = \Omega_S(E_S)\Omega_b(E - E_S) \quad (3.2)$$

where  $\Omega_S(E_S)$  is the number of microstates of the system if the energy is  $E_S$  and  $\Omega_b(E - E_S)$  is the number of microstates of the bath if the energy of the system is  $E_S$ . But, unlike the situation where the system was isolated, now the energy of the system can change due to its interactions with the bath, the only invariant is now the overall energy. Thus, the total microstates of the joint set system-bath is the sum of  $\Omega(E, E_S)$  over the possible values of  $E_S$ .

$$\Omega(E) = \sum_{E_S} \Omega(E, E_S) = \sum_{E_S} \Omega_S(E_S)\Omega_b(E - E_S) \quad (3.3)$$

then we can calculate the probability  $p_S$  for the system to have and energy  $E_S$  by

$$p_S = \frac{\Omega_S(E_S)\Omega_b(E - E_S)}{\int \Omega_S(E_a)\Omega_b(E - E_a)dE_a} \quad (3.4)$$

Since the bath is much bigger than the system, the number of microstates of the bath dominates in both the numerator and denominator of the rhs of Eq. (3.4), thus for a number of discrete states

$$\frac{\Omega_S(E_S)\Omega_b(E - E_S)}{\sum_{E_a} \Omega_S(E_a)\Omega_b(E - E_a)} \sim \frac{\Omega_b(E - E_S)}{\sum_{E_a} \Omega_b(E - E_a)}. \quad (3.5)$$

So, combining with Eq.(3.4),

$$\log(p_S) = \log(\Omega_b(E - E_S)) - \log\left(\sum_{E_a} \Omega_b(E - E_a)\right). \quad (3.6)$$

Since the energy is a constant of the joint set system-bath, then the second parcel of the rhs of Eq.(3.6) is also a constant. Since  $E \gg E_S$  then we can expand  $\log(\Omega_b(E - E_S))$  around  $E_S = 0$ , leading to

$$\log(p_S) = C + \log(\Omega(E)) - E_S \frac{\partial \log(\Omega_b(E_b))}{\partial E_b} \quad (3.7)$$

where  $E_b$  is the energy of the bath. Grouping the constants in a quantity  $1/Z$  independent from  $E_S$ , and denoting the variation of the logarithm of the number of states of the bath with the energy of the bath as  $\beta$ , i.e.,

$$\beta = \frac{\partial \log(\Omega_b(E_b))}{\partial E_b} \quad (3.8)$$

then, combining with Eq.(3.7) the probability for the system to have an energy  $E_S$  is given by

$$p(E_S) = \frac{1}{Z} e^{-\beta E_S} \quad (3.9)$$

which, under the condition that  $p_S$  is a probability density function, i.e.,  $\sum_{E_S} p(E_S) = 1$  yields

$$Z = \sum_{E_S} e^{-\beta E_S}. \quad (3.10)$$

Equation Eq.(3.9) is called the Boltzmann distribution and Eq.(3.10) is called the partition function.

The parameter  $\beta$  characterizes the macrostate of equilibrium and in physics is directly related to the temperature by  $\beta = 1/k_B T$ , where  $T$  is the temperature.<sup>57</sup> The system is said to be in *thermal equilibrium* with the bath. The set of subsystems in thermal equilibrium with a bath is called the ‘canonical ensemble’.

As a parenthesis, we could make the number of particles in the system as variable, exchanging them with the bath. This would give us an additional parcel on the rhs of the Eq.(3.7) representing the variation of the number of microstates of the bath with the number of particles in the bath. An additional equilibrium parameter similar to  $\beta$  called *chemical potential* and the equilibrium would be both thermal and chemical. The set of subsystems in these conditions is called a *grand canonical ensemble* which is out of the scope of this text.

Both configurations, the aggregation of isolated systems and the isolated set of subsystems in a bath, have the same invariant quantity and they are both in equilibrium. The difference between the two is that the first is a population of uncorrelated components and the later a set of correlated components that interact by the imposition of an invariant overall energy. This difference promotes different numbers of microstates and different probability density distributions.

Let us go back to the fundamental relation of statistical physics, Eq.(3.1). Let us assume that each individual isolated component, i.e. a particle, can assume a state that can be characterized by an integer index  $i = 0, 1, 2, \dots$ . The number of microstates is the combination of the  $N$  particles with the several individual states. There are  $n_0$  particles in the state 0,  $n_1$  particles in the state 1,  $n_2$  particles in the state 3, etc. The total number of combinations, i.e., of microstates, is given by

$$\Omega = \frac{N!}{\prod_i n_i!}. \quad (3.11)$$

Using Stirling approximation,  $\log(n!) \approx n \log(n) - n$  we have

$$\log(\Omega) = \log(N!) - \log\left(\prod_i n_i!\right) \quad (3.12)$$

$$\log(\Omega) = N \log(N) - \sum_i n_i \log(n_i) + \sum_i n_i - N. \quad (3.13)$$

The two last parcels of the rhs of Eq.(3.13) disappear because they are equal. Multiplying

and dividing each  $n_i$  by  $N$ , we will have

$$\log(\Omega) = -N \sum_i \frac{n_i}{N} \log\left(\frac{n_i}{N}\right) \quad (3.14)$$

By definition, microstates are all possible states of the isolated system and they represent all combinations of the dynamic parameters that comply with a value of an overall invariant. If we assume the system isolated, we can make an exercise of measuring the relative frequencies  $n_i/N$  and measure the number of possible combinations, independently from the fact that the system is in equilibrium or not.

Let us take the example of a system with 100 particles with minimum energy 1 and energy can take integer figures, meaning 1,2,3.... Also, we impose a constant energy of 200. If we add the condition that one particle must have an energy of 50, then the remaining 150 must be distributed by the other 99 particles. The number of microstates in this case would be higher than if we impose an energy of 50 to another particle, because remaining particles would have only one available energy.

The highest number of microstates is achieved when we do not impose any quantity of energy to any particle. Meaning that with the imposition of an overall invariant extensive quantity we call energy, the number of microstates grows as we remove additional constraints. In other words, with all constraints removed the system goes to the maximum number of microstates. If no limit was imposed to energy, that maximum would be infinite. This result is compliant with the laws of thermodynamics which postulate that an isolated system goes to its maximum entropy.

All the above framework is valid as long as we have an overall extensive invariant and an isolated system of particles. Thus, when all constraints are removed this lead to an extreme in  $\log \Omega$ , the statistics of the system stays constant if no external action changes it (a new constraint is imposed). We have, then, what we called equilibrium.

#### 3.1.4 Equilibrium under probability theory

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Since we are dealing with statistical characterization of a system, the formal approach should be done from probability theory and thus we make a parenthesis to show the parallelism between the concepts we have been using from statistical mechanics and this theory, also known as Kolmogorov probability theory.<sup>4</sup> In this section we will make use of letters we have used in previous sections when the represented concepts are similar.

The first fundamental concept is the *sample space*. The sample space  $\Omega$  is the set of all possible *outcomes* of a random experiment,  $\omega \in \Omega$ , also known as *sample points*. An *event* is a set of outcomes. The parallelism is quite direct, since a microstate is a possible outcome of a random experience consisting on taking a snapshot of a thermodynamical system in one instant. The sample space is the set of all microstates. An event is a particular particle having an energy  $e_k$ , which corresponds to a number of outcomes where the particle has that energy. Meaning that each event is a subset of the sample space  $\Omega$ .

We define over  $\Omega$  a class  $\mathcal{F}$  of events, i.e., a class of subsets of  $\Omega$  and a set function  $P$  that maps  $\mathcal{F}$  into  $[0, 1]$ . Additionally we impose the following conditions over  $\mathcal{F}$  and  $P$ :

1. If  $A \in \mathcal{F} \Rightarrow A^C \in \mathcal{F}$ , where  $A^C$  is the complement of  $A$ . Using the above example of the energy particle  $A^C$  is the set of microstates where the particle has an energy different from  $e_k$ ;
2. If  $A_1, A_2 \in \mathcal{F} \Rightarrow A_1 \cup A_2 \in \mathcal{F}$ . If  $A_1$  is the set of microstates associated to energy  $e_1$  and  $A_2$  is the set of microstates associated to energy  $e_2$ , then the set of microstates of having energy  $e_1$  or having energy  $e_2$  is also part of  $\mathcal{F}$ ;
3.  $P(\emptyset) = 0$ ,  $P(\Omega) = 1$ , if  $A \in \mathcal{F}$  then  $0 \leq P(A) \leq 1$ .
4. If  $A_1, A_2 \in \mathcal{F}$ ,  $A_1 \cap A_2 = \emptyset$ , i.d.  $A_1$  and  $A_2$  are associated to different energies, then  $P(A_1 \cup A_2) = P(A_1) + P(A_2)$ . So  $P$  is countably additive function.
5. If  $A_n \in \mathcal{F}$  and  $A_n \subset A_{n+1}$  for all  $n = 1, 2, \dots$ , then  $\cup_n A_n \in \mathcal{F}$  and  $P(A_n) \leq P(\cup_n A_n)$ .

Under these five conditions  $(\Omega, \mathcal{F})$  is a measure space and  $(\Omega, \mathcal{F}, P)$  a probability space,  $\mathcal{F}$  is a  $\sigma$ -algebra and  $P$  is the probability measure in  $\mathcal{F}$ , meaning that  $P$  maps the events of the random experiment into  $[0, 1] \in \mathbb{R}$  to represent the ‘chance’ of the outcomes. It can be shown that in  $(\Omega, \mathcal{F}, P)$ , the measure  $P$  is unique.

Going back to the set of particles, we should recall that as we remove constraints, the number of microstates grows keeping an overall energy constant. Thus, as constraints are removed, a new sample space is formed, a new  $\sigma$ -algebra  $\mathcal{F}$  and, consequently, a new probability measure  $P$ . So, statistical equilibrium is achieved when the probability measure remains unchanged, which happens when the number of microstates is maximum. Then, in those conditions we can write Eq. (3.14) as

$$\log(\Omega) = -N \sum_i p_i \log(p_i) \quad (3.15)$$

where  $p_i$  is the probability density function of a particle having an energy  $e_i$ . The probability  $p_i$  equals the relative frequency  $n_i/N$  of Eq. (3.14) or, equivalently,

$$\log(\Omega) = -N \langle \log(p) \rangle_{\mathcal{F}} \quad (3.16)$$

where  $p$  the probability density function derived from the probability measure  $P$  and the brackets  $\langle \rangle_{\mathcal{F}}$  represent the average over the events. The probability density function satisfying Eq. (3.16) defines a generic equilibrium state.

The definition of equilibrium we will use from this point forward *is not* the definition of thermal equilibrium from statistical mechanics because we are not interested in thermodynamics. We will use instead a more generic definition based of statistical mechanics framework: A system in equilibrium state has invariant probability measures and the system will remain in that state unless some external influence acts and change it. The state is characterized by a minimum in the average probability density function.

### 3.1.5 Price 'Equilibrium'

In the previous sections we took the statistical mechanics framework and the mathematical formalism to approach the concept of equilibrium. In this section we are going to explore this concept in the scope of Economics and how it is used in the well known and almost unanimously accepted law of price formation by supply and demand equilibrium. For that, we need to introduce the concepts of *price*, *trade* and *market*.<sup>63</sup>

Each economic agent exchanges labor with other agents forming a trade market. Since one agent delivers labor in the form of resources to another agent, he only allocates resources he has and gets the allocation of resources he does not have from another agent. The amount of resources allocated by an agent is called price.

How is it measured? As we saw in the previous chapter, the way how humans implemented a unit of measure was by inventing an abstract resource which function as a measure unit called money. In modern economy a price is denominated in currency units. But in a barter economy a price would also exist, despite it had no significance outside each trade. We usually think that we allocate a money resource to get another resource, but the other resource we get can also be a form of money. For instance, in financial markets we exchange Euros for Dollars and in that exchange a price is defined in currency units in both legs of the exchange. Also, we can exchange Euros in the present by Euros in the future. A loan is a prototypical example of this exchange between present and future. These two different resources have a price.

Scarcity has a key role on how price is formed. Since in trade we exchange two resources, each economic agent decides if it is more scarce for that particular resource or for all the others, including the one he is going to exchange for. If each agent is more scarce of that particular resource then everything else, then the trade is done. The average amount of money all agents allocate to a particular resource in each instant is called the market price.

There is a field of knowledge called Mathematical Economics which is based in the idea of equilibrium, since it is fundamental in the Economics scientific method. The dominant current supports the Walrasian equilibrium (from the name of Léon Walras, a French economist) which states that production and exchange processes are governed by market mechanism to produce a balance similar to a mechanical equilibrium. There are two other dominant currents,<sup>112</sup> the Von Neumann equilibrium (from the name of the well know physicist Jon Von Neumann) and the Neo-Classical. Von Neumann defines economic equilibrium as the point where the production and its technological development grows at a rate equal to the economic growth. Since this notion is defined based on production, the Neo-Classical notion is something in the middle of the previous two, assuming that the production, capital, consumption and investment balances to produce the (growth) equilibrium. The way growth and equilibrium are mixed in the same sentence is, at least, unusual to physicists. We are not going into details, reference<sup>112</sup> provide a good summary of these notions of equilibrium.

Besides the fact that they are not state-of-the-world free, all these definitions, , are definitions of mean-field situations, not statistical equilibrium definitions. They describe force balances resulting from the combined action of the constituents of the system. Since, from the definition

of economy, the system is permanently growing, there is a new balance of forces from one instant to the next, the opposite of the definition of equilibrium. Consequently, the definition of equilibrium used by economists does not insure predictability in the sense we are looking here. Quite the opposite. And any modeling based on such a definition of equilibrium would produce the volatility of results we have already mentioned.

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### 3.1.6 Out of Equilibrium Geometries

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From the definition of Economics, *there is no such thing as a thermodynamical equilibrium in the economy*. The reason is quite simple, if agents are constantly producing to satisfy a never ending scarcity then there is no physical invariant in the system and equilibrium is not possible. In the end of the last century, the materialization of some economic relations like the Internet provided data that could give clues about this out of equilibrium behavior. An important class of 'geometrical' models are the so called *complex networks*.<sup>3</sup>

A complex network is a graph with non-trivial topological characteristics or, in other words, a set of interconnected objects in which the local structure of the connections is not enough to understand the geometry of the entire system. What makes this geometrical figures so interesting is the fact that they can be found in a wide band of systems, specially those when the interconnected objects are humans,<sup>3</sup> websites,<sup>25</sup> airports<sup>82</sup> or imitating human strategies, like algorithmic trading machines.<sup>70</sup>

The interconnected objects are called 'nodes' or, 'vertices', and the connections between the objects are called 'connections', 'links' or 'edges'. If the connection is perfectly symmetrical, i.e. if it is possible to interchange the nodes without changing the nature of the connection the network is an undirected network. If not, it is a directed network. If all connections are equal, the network is an unweighted network. If the connections are different, i.e. some connections are stronger than others, the network is a weighted network. Nodes can be - and generally are - different and can have different natures that do not influence the ability to connect with others. When all types can interact in the same way the network is monopartite. When two types of nodes exist and connections between different types are favored, like in sexual reproduction, the network is bipartite. When several types of nodes are present, like in food webs, the network is multipartite.

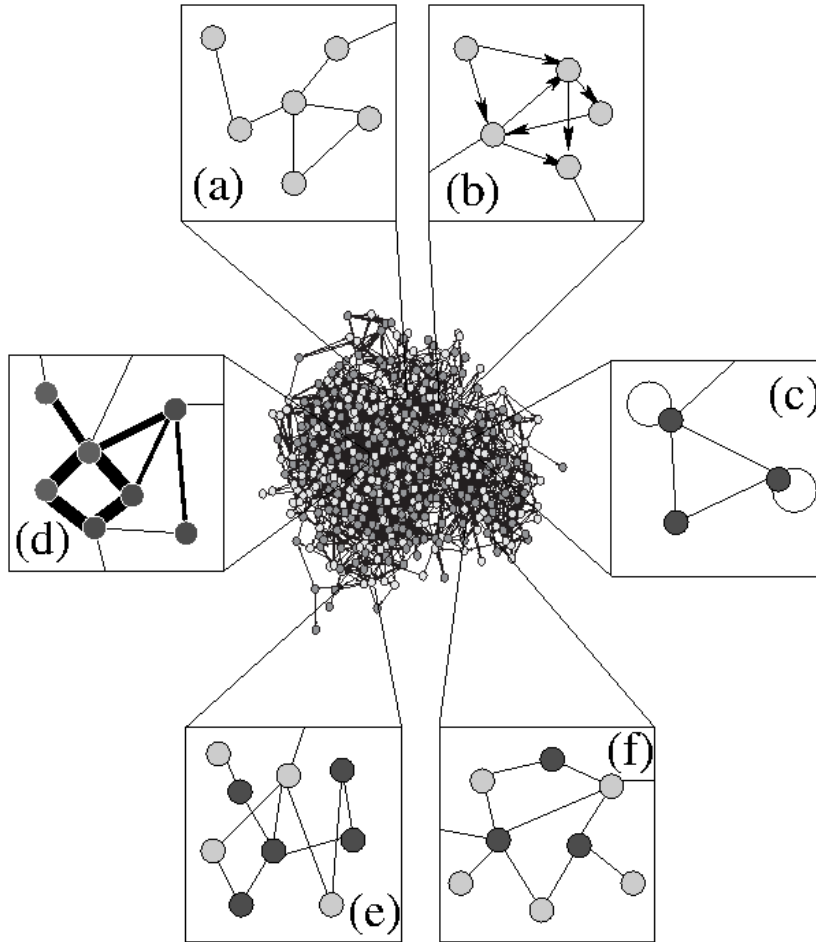
The nature of the network can combine several of the above characteristics, which are based on the type of connections. Thus, based on the nature of the interaction, it is possible to model a physical problem inside the set of the above characteristics. Nonetheless, the geometry of the global network can not be inferred by this characteristics but from more global metrics that we describe next.

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#### 3.1.6.1 Characterization of Networks

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To historically characterize complex networks it is necessary to introduce three main concepts:<sup>3</sup> Small-world, clustering and weight distribution. Small-world characteristic is the evidence



**Figure 3.2:** Illustration of parts of a complex network that are (a) undirected, (b) directed, (c) with self-connections, (d) weighted, (e) monopartite, where the nature of the node does not influence the existence of connection and (f) bipartite, where nodes of one kind only connect to nodes of the other kind. From P.G. Lind<sup>84</sup> with permission.

that despite the existence of a large number of nodes, they are not so separate from each other and in only a small number of steps it is possible to go from one extreme of the network to the other. Regular lattices, do not have such a feature: one needs a large number of steps to go from one extreme to the other. Clustering is the tendency of the nodes to gather around others in clusters, a phenomenon very common in social networks where friends have some common features like geography or profession. Finally, weight distribution is the measure of the distribution of the quantity of first neighbors of the nodes. Unlike regular networks, in complex networks the number of first neighbors is highly variable along the full domain of nodes.

These three features characterize the complex network and can be measured making use of the representation of the network as an adjacency matrix, i.e. the full accountability of which node is connected with which. Considering a network with  $N$  nodes, the network is fully represented by a  $N \times N$  matrix  $a_{ij}$  that we call ‘adjacency matrix’. The elements are binary, positive integer or positive real depending on the several types of connection above described.

The degree of a node is the number of connections it has with its neighbors. In a unweighted network it is the sum of the neighbors, in a weighted network it is the sum of the weights. Using the adjacency matrix, the degree of the node  $i$  is

$$k_i = \sum_j a_{ij} \quad (3.17)$$

From the measure of the degree of each node, it is trivial to build the histogram of degrees which, for a large  $N$  is the probability of a node  $i$  to have a  $k$  degree,  $P(k_i = k)$ . This function is the degree distribution. Using the degree distribution it is possible to measure correlation between the existence of nodes with degree  $k$ . Measuring the average degree of the neighbors of the nodes with degree  $k$  by

$$k_{nn}(k) = \sum_{k'} k' P(k'|k) \quad (3.18)$$

If  $k_{nn}(k)$  is independent of  $k$  then the existence of nodes with the observed degrees is independent.<sup>84</sup>

The existence of clusters in the network is measured by finding if the neighbors of a node are also neighbors of each other, like what happens with friendship networks where our friends are usually friends of each other. The clustering coefficient was created by Watts and Strogatz<sup>136</sup> to measure the triangles<sup>84</sup> of connections (node-neighbor-neighbor or neighbor-node) and given by

$$C(3) = \frac{2m_i}{k_i(k_i - 1)} = \sum_{jn} \frac{a_{ij}a_{jn}a_{ni}}{k_i(k_i - 1)} \quad (3.19)$$

where  $k_i(k_i - 1)/2$  is the maximum possible connections between nodes in a neighborhood radius of 1 (for undirected networks),  $m_i$  is the actual connections existent inside the radius and the last member is the measure of the clustering coefficient using the adjacency matrix. The reference  $C(3)$  is due to the fact that the Eq.(3.19) being used for triangles of nodes. Obviously, a more wide measure of clustering can be used considering more segmented cycles.

The small-world characteristic can be measured using the average shortest path length measure. The average shortest path length is the average path between two nodes measured in connection units and it is obtained numerically over the adjacency matrix.

### 3.1.6.2 Types of Complex Networks

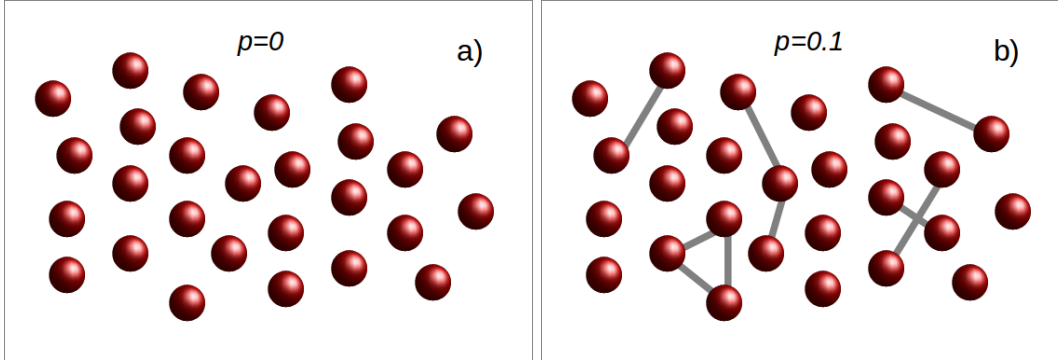
Despite the definition of complex network can cover every non-regular network, the modeling of natural and social phenomena can be done by making use of very few theoretical types of network, namely random networks or Erdős-Rényi networks,<sup>50</sup> small-world networks<sup>136</sup> and scale-free networks.<sup>11</sup> Usually, the relevant characteristics of the problem can be translated into these theoretical models with some additional customization.

The term 'random network' should refer to the global class of networks in which the rules for node interconnecting are probabilistic rules but, with time, it become a reference to the



Erdős-Rényi model for random networks. In this model, to a given node  $i$  all other nodes have equal probability,  $p$ , of interconnect with it. Since there are  $N(N - 1)/2$  possible connections to  $N$  nodes, then the expected number of connections yields

$$E(n) = p \frac{N(N - 1)}{2} \quad (3.20)$$



**Figure 3.3:** Illustration of a Erdős-Rényi network.<sup>3</sup> Nodes connect each other randomly and the emergence of triangles, like the one in b) is directly related to  $p$ . In the limit  $p = 1$  the clustering coefficient,  $C(3)$ , would converge to 1.

The probability of a degree of  $k_i$  of a given node  $i$  to be  $k$  results from the combination of  $k_i$  connections over the remaining  $N - 1$  nodes of the network, the probability of formation of  $k$  connections (which is  $p^k$ ) and the probability of non-formation of the  $k$  connections, i.e. a binomial distribution with parameters  $k_i$  and  $p$ ,

$$P(k_i = k) = \binom{k}{N - 1} p^k p^{N-1-k} \quad (3.21)$$

which with  $N \rightarrow \infty$  will result in a Poisson distribution

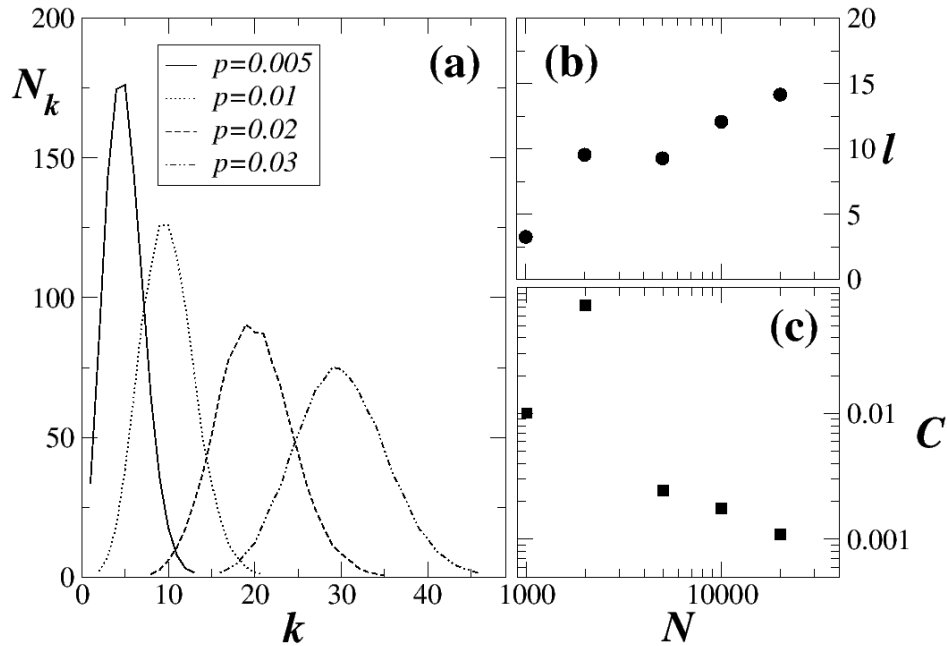
$$P(k_i = k) = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!} \quad (3.22)$$

In terms of clustering coefficient  $C(3)$ , it is quite straightforward that it is equal to  $p$  as the Fig.3.3 can show. The formation of triangles depends on the probability  $p$  since the expression for it, Eq.(3.19) is just the empirical measure of  $p$ , and  $C(3) = p$ .

The average path length, the solution of the Erdős-Rényi is not so straightforward, but it can be found in the reference,<sup>30</sup>

$$l \sim \frac{\log(N)}{\log(pN)} \quad (3.23)$$

meaning that for high probabilities the average path length is close to one and, on the limit  $p = 1$  is actually 1, i.e the minimum average path length possible in any complex network. Small-world networks are networks with the small-world characteristics, as random networks, but with an high clustering coefficient. In nature, complex networks have higher clustering coefficients that are independent of the network size, which seems contradictory with the assumption of being complex, since it reveals some sort of local regularity that characterizes



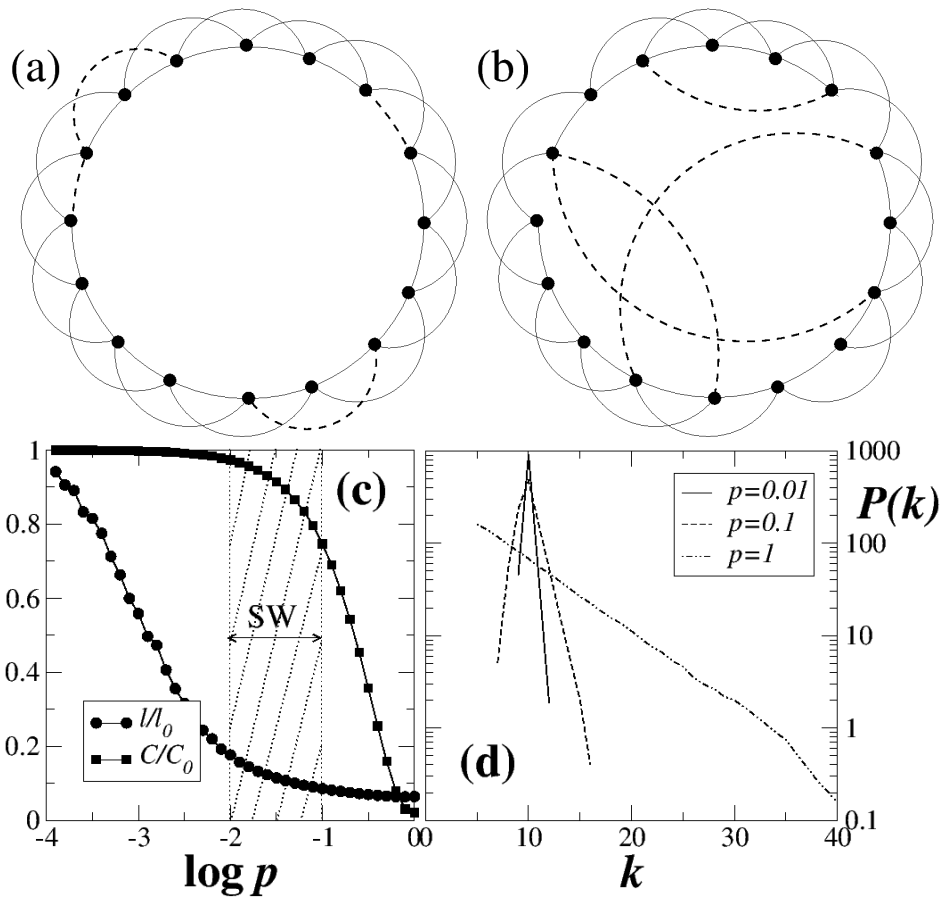
**Figure 3.4:** An Erdős-Rényi Network. (a) The degree distribution for different values of the connection probability  $p$ , with  $N = 2000$  nodes and (b) the average shortest path length  $l$  and (c) the clustering coefficient  $C(3)$ , as a function of the network size  $N$  fixing  $p = 0.01$ . From P.G.Lind<sup>84</sup> with permission.

the full network. What is different from a regular network is that this high clustering combines with a rather small diameter, i.e, with the biggest path between nodes. Watts and Strogatz<sup>136</sup> modeled a network with this characteristics by building an algorithm in which the regular randomness is perturbed. Assuming  $N$  nodes with periodic boundary conditions and connected with  $2k$  nearest neighbors. Then, sequentially, each node remakes  $k$  of its connections with probability  $p$  but, this time, can choose the connected node randomly between the other nodes in the system.<sup>13</sup> With this mechanism part of short range connections are substituted by long ranges connections, reducing the network diameter. The degree of each node can be expressed as the sum of the  $k$  original connections with the connections involved in the remaking process. From this,  $n_1$  were left untouched with  $1 - p$  probability and  $n_2$  are new connections received by the node in the remaking with probability  $p/N$  (composition of the probability for the the connection being remade  $p$  with the probability of the node being chose  $1/N$ ). The distribution for  $n_1$  is a binomial distribution and for  $n_2$  is a Poisson distribution with  $pk$  expectation, which yields<sup>13</sup> for the degree distribution

$$P(c) = \sum_{n=0}^{\min(c-k,k)} \binom{k}{n} (1-p)^n p^{k-n} \frac{(pk)^{c-k-n} e^{-pk}}{(c-k-n)!}, c \geq k \quad (3.24)$$

It is intuitive that average path length of the network will drop as  $p$  grows, as seen in Fig.3.5c), because more short range connections are substituted by long range connections. If  $p = 0$  then the average path length grows linearly with the number of nodes  $N$  in the network and if  $p > 0$ , it can be shown that the average path length grows with  $\ln(N)$ .

In terms of clustering coefficient, for  $p = 0$  each node has  $2k$  neighbors and we can define a



**Figure 3.5:** A Watts and Strogatz (SW) Network. (a) The algorithm that form the network starts with a regular network and (b) a fraction  $p$  of the connections changes one of the nodes. In (c) we have the comparison of the clustering coefficient  $C$  and the average path length  $l$  of the SW network with the clustering coefficient  $C_0$  and the average path length  $l_0$  of a regular network. In (d) we have the degree distribution of a SW network which is highly peaked for small  $p$  (i.e. more close to the regular network) and goes asymptotically to a Poisson distribution for large  $p$ . From P.G. Lind<sup>84</sup> with permission.

clustering normalization constant as the clustering coefficient for  $p = 0$ ,  $C(0)$ , dependent on the number of triangles formed in a radius 1. To determine what is the clustering coefficient for  $p > 0$  then it is equivalent to calculate the probability for the triangle to remain untouched after the remaking of the connections. Since three connections must be left untouched, then  $C(p) \approx C(0)(1 - p)^3$ .

Scale-free networks are the interesting form of complex networks in the scope of this work, due to its omnipresence in nature and society, like airport traffic<sup>82</sup> or internet.<sup>26</sup> They are called scale-free because its degree distribution follows a power-law in the form of

$$P(k) \propto k^{-\gamma}. \quad (3.25)$$

The list of real networks that are approximately scale-free networks is vast and it can be consulted in.<sup>3</sup>

The previous types of complex networks were static, meaning that from a fixed number of nodes, some sort of iterative process lead to the final result. Barabási and Albert introduced one iterative process, in 1999,<sup>11</sup> using the growth of the network. Starting with a small fixed number of nodes, new nodes were added and connected to the existing ones. If the new nodes choose randomly the one to connect, the final distribution would be an exponential distribution on the form  $e^{-\beta k}$ ,<sup>11</sup> so they broke the rule of an uniformly random choice by introducing the choice by ‘preferential attachment’. This type of choice is driven by the previous choices, assuming that the probability for a new node to connect a previous one,  $\Pi(j \rightarrow i)$ , is proportional to the number of connections that the later already received, i.e.

$$\Pi(j \rightarrow i) = \frac{k_i}{\sum_j k_j} \quad (3.26)$$

The idea beneath it is to introduce some sort of imitating behavior intuitively characteristic of humans and it has been interpreted as such, despite the fact that Simon,<sup>126</sup> back in 1955, already formulated this principle to get the distribution of occurrences of words in a text. Nevertheless, the introduction of preferential attachment is fundamental for the understanding of scale-free networks in nature and society, as we can see later in this text. With some simple arithmetics<sup>11</sup> it is possible to deduce that  $\gamma = 3$  in Eq.(3.25). The Barabási-Albert approach consisted on building a differential equation based on the assumption that only one connection is created in each instant and so the number of nodes having  $k_i$  connections is

$$\frac{\partial k_i}{\partial t} = \frac{k_i}{\sum_j k_j} \quad (3.27)$$

Since one connection enters the system with a new node, then the total number of connections summed by all nodes are twice the number of nodes. Also, if we assume that at each tick of the clock we introduce one node then

$$\frac{\partial k_i}{\partial t} = \frac{1}{2} \frac{k_i}{t} \quad (3.28)$$

or

$$k_i = m \left( \frac{t}{t_i} \right)^{\frac{1}{2}} \quad (3.29)$$

where  $t_i$  is the instant the node got into the system and  $m$  is the initial number of links of each node which, since links are only created when the node enters the system, is equal to the rate of creation of links. The following step is even more tricky. The probability of a node to have a degree bigger than  $k$  is equal to the probability of the node got into the system later than  $t(m/k)^{(1/2)}$ , i.e.

$$P(k_i > k) = P(t_i > t(m/k)^{1/2}) \quad (3.30)$$

But nodes enter the system at a constant rate so the probability density function of  $t_i$  is  $1/t$

$$P(k_i > k) = P(t_i > t(m/k)^{1/2}) = 1 - P(t_i \leq t(m/k)^{1/2}) = 1 - \left(\frac{m}{k}\right) \quad (3.31)$$

and finally, the probability density function for a node to have  $k$  neighbors is

$$p(k) = \frac{\partial P(k_i \leq k)}{\partial k} = 2m^2 k^{-3} \quad (3.32)$$

Later works on the growing of networks have found several variations to the preferential attachment mechanism<sup>3</sup> that can provide a wide band of possible values to the exponent  $\gamma$ . From these works, we individualize Dorogovtsev's et al.<sup>46</sup> in which they handled the problem of growing networks with preferential attachment using a master equation approach, i.e. using a time dependent differential equation over the degree distribution. There, they introduced a 'initial attractiveness'  $A$  such that Eq.(3.26) is now written as

$$\Pi(j \rightarrow i) = \frac{A + k_i}{\sum_j (A + k_j)} \quad (3.33)$$

to simulate that any node has an intrinsic attractiveness despite the number of previous connections. With this algorithm, it is possible to get a degree distribution as

$$P(k) \propto k^{-(2+A/m)} \quad (3.34)$$

where  $m$  is the number of connections formed at the entrance of a new node in the network.

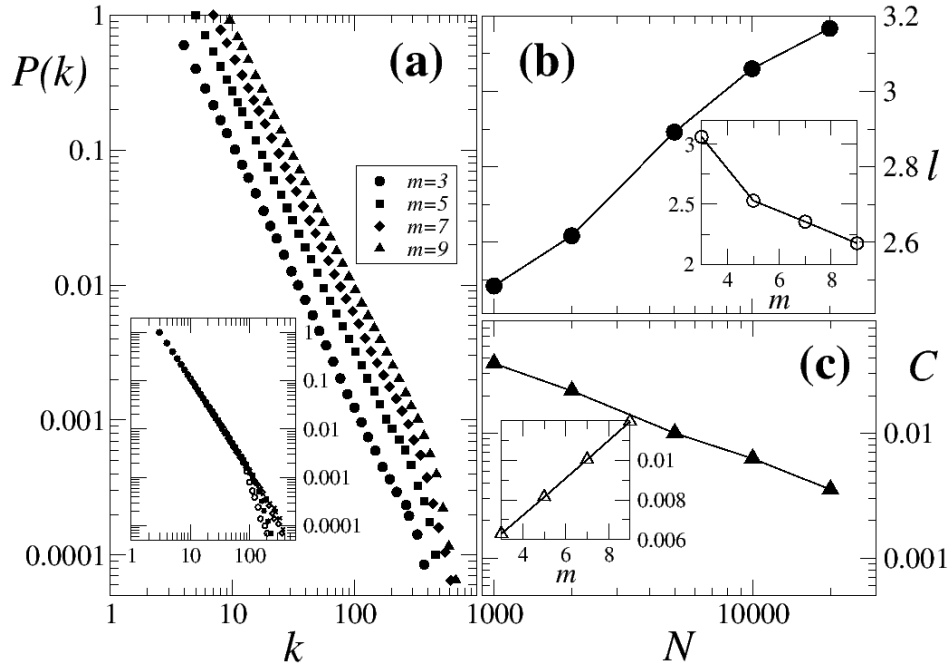
Scale-free networks share the characteristics of the previous types of networks of a short average path length and an high clustering coefficient, as shown in Fig. 3.6.

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### 3.1.6.3 Weighted Networks

The models of complex networks described above are models that are useful to understand the major relevant characteristics of each typical network. Real networks, like the ones we want to address in this work are slightly different. Systems are finite, discrete and connections between the objects are not all equal.

To deal with weighted networks it is necessary to follow three measures, (i) the degree  $k_i$  of the node  $i$  (ii) the weight,  $w_{ij}$ , of a connection between node  $i$  and node  $j$  and (iii) the strength  $s_i$  of the node. The degree of a node has been defined previously, but the last two concepts need additional specification. In an unweighted network the 'weight' of a connection



**Figure 3.6:** Scale-free networks. (a) The degree distribution for different values of the entering number of connections  $m$ , with  $N = 10^4$  nodes. (b) The average shortest path length  $l$  and (c) the clustering coefficient  $C_3$ , as a function of  $N$ . The insets represent the corresponding characteristic with  $m = 3$  constant for several values of  $N$  (<sup>84</sup>) From P.G. Lind<sup>84</sup> with permission.

is binary, one if it exists or zero if it does not. Weighted networks have unequal connections, quantified by the weight  $w_{ij}$ . In that sense, there is a probability distribution of weight built over the weights of the connections in a network.

The strength  $s_i$  of a node  $i$  is the sum of the weights of the connections linking to and from the node. Strength is then defined as

$$s_i \equiv \sum_j w_{ij} \quad (3.35)$$

In the case of an unweighted network the strength and degree have the same value and there is no need to separate the two measures.

Considering a weighted network as a generalization of an undirected network,<sup>45</sup> it is possible to obtain relevant information about the distribution  $R(s)$  of node strength, knowing the degree distribution  $P(k)$  and the weight distribution  $Q(w)$ . Taking the probability generating functions

$$\begin{aligned} \Phi(z) &\equiv \sum_k P(k)z^k, \\ \Psi(z) &\equiv \sum_w Q(w)z^w, \\ \Gamma(z) &\equiv \sum_s R(s)z^s, \end{aligned} \quad (3.36)$$

applying in Eq. (3.36) the definition of strength in Eq. (3.35) yields<sup>45</sup>

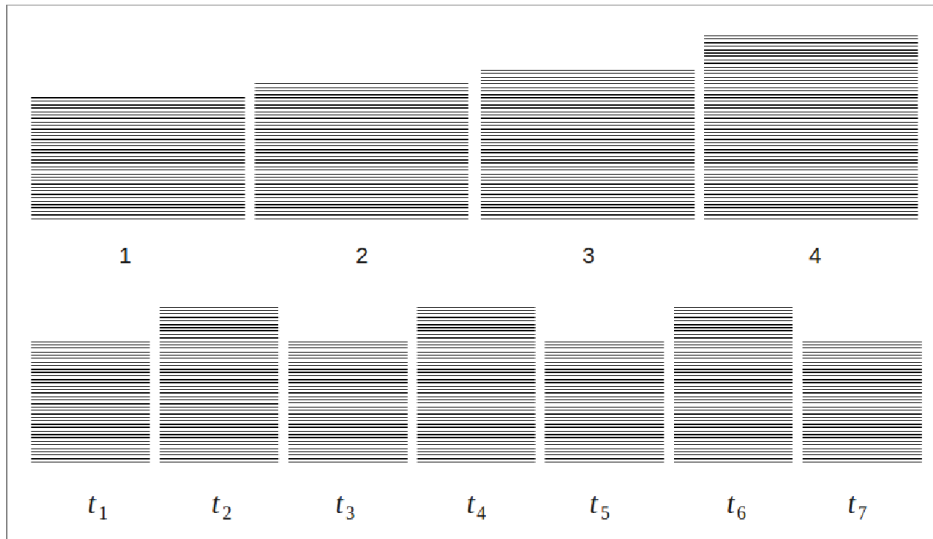
$$\Gamma(z) = \sum_k P(k) (\Psi(z))^k = \Phi(\Psi(z)) \quad (3.37)$$

as  $N \rightarrow \infty$ .

Equation Eq. (3.37) shows that if the degree distribution is a power-law with exponent  $\gamma_k$  and  $Q(w)$  is rapidly decreasing, i.e. has finite moments, then  $R(s) \sim s^{\gamma_s}$  with  $\gamma_s = \gamma_k$ . On the other hand, if both  $P(k)$  and  $Q(w)$  are power-laws with exponents  $\gamma_k$  and  $\gamma_w$  respectively, then the resulting distribution  $R(s)$  is a power-law with exponent given by  $\gamma_s = \min(\gamma_k, \gamma_w)$ . This result, from Dorogovtsev and Mendes,<sup>45</sup> despite the authors assumed it as particular cases, allows us to assume that there are mechanisms of weight growing that keep the degree distribution unchanged. For example, Barrat et al.<sup>12</sup> made a work on world wide airport traffic, based on International Air Transportation Association data to get a similar result approaching from empirical data. Assuming airports as nodes and direct flights as connections, they found that at constant number of nodes, the traffic of airports follows a power-law distribution as long as their degree also follows a power-law distribution.

From what empirical data shows us, there are some forms of economic relations that generate these geometric forms. Also, the Barabási-Albert, together with the results from Dorogovtsev and Mendes, can form a good model for the economic geometry since the preferential attachment mechanism is coherent with the definition of Economy.

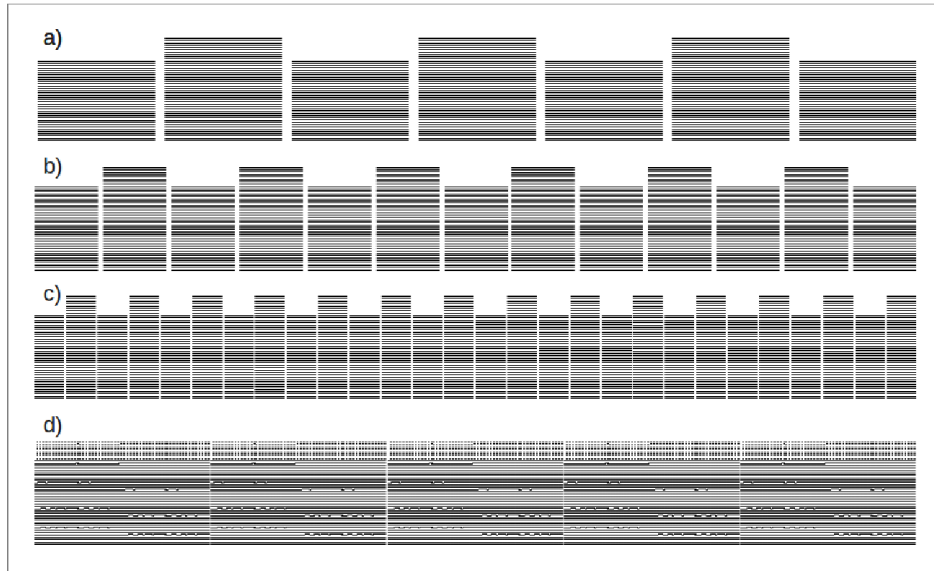
Unfortunately, those approaches are not formally correct because the used framework is only valid in equilibrium or near equilibrium situations, where a  $\partial p/\partial t$  equation can be written knowing that  $p$  is a probability density function. If we represent the microstates as lines in



**Figure 3.7:** Microstates evolution scheme. **Upper** Continuous growing. **Lower** Periodic growing. Description in the text.

Fig.3.7 from  $1 \rightarrow 4$ , if we add energy or particles to an equilibrium system, the number of microstates will grow to establish a new equilibrium with more microstates. Thus, the system in Fig.3.7-4 is not system that it was in Fig.3.7-1. Moreover, the probability measure we take in one case is not the probability measure we take in another. Consequently, Eq. (3.27) is not entirely correct because the  $P$  in that equation is only a probability under a deterministic growth of one node per instant. The Barabási and Albert approach is only correct in those particular circumstances and a generalization to scale-free networks is not correct.

The point here is that when we address the problem of systems outside the equilibrium, like the growing network, a stochastic analysis is far from being trivial. An exception to this is the fluctuating system like in the lower part of Fig.3.7. Let us assume that the system is fluctuating in the sense that the number of microstates changes periodically. When time



**Figure 3.8:** Microstates recurrent evolution scheme. As time goes to infinite the periodicity of the number of microstates becomes less important and the number of microstates can be looked as constant if the characteristic time of the fluctuation is low when compared with the characteristic time of the measurements. In Chapter 4 we will refer to this feature as ‘convergence of autocorrelation to zero’

goes to infinity, the fluctuating number of microstates will be approximately constant and, in that case, differential equation like a master equation<sup>71</sup> can be applied without previous transformations.

So, the reason why network growing models like Barabási-Albert’s need growing in the number of nodes is to keep the probability distribution invariant. The only way to keep the sample space constant is to add a new node and a new link in each instant. Stationarity in this kind of models is not a conclusion, is an assumption that needs to be fed back with the node growing trick to keep the time derivative of a probability density function mathematically supported. In the next section we will show that stationarity is the result of the agregaton of multiplicative processes,<sup>5,61</sup> which is obviously a much more generic case of which the Barabási-Albert model is a very particular case, and we will show that scale-free networks are a peculiar expression of the statistical equilibrium of such agregate that depends on the correlation between each multiplicative process.



## 3.2 Contributions

### 3.2.1 Economic Equilibrium and Logarithmic Invariance

An economy is a permanently growing system where each economic ‘particle’ produces according to its possessed resources. This is called a *multiplicative process*. Many processes in nature and in society are multiplicative processes. The growing of the impact craters in the Moon<sup>139</sup> can be an illustrative example, since the probability of new impacts that promote the growth of the crater rise with area of the crater itself. Thus the process describing the area of the crater follows a multiplicative process. The aggregation of multiplicative processes results in an expanding system in the sense that the linear difference between the maximum and minimum rise with time.

It is a well known fact that the resulting probability distribution of multiplicative process is a power-law distribution,<sup>81,101</sup> here we will show that a probability distribution can only be formulated in a expanding system due to aggregation of multiplicative processes and a power-law distribution with exponent 2 is the solution for the sum of such processes. We also show that several other power-laws can be retrieved from the same system, depending on the overall system measure and how correlation between objects contributes to that overall measure.

Dealing with an expanding system is not a trivial subject since the result is not a physically stationary process because, by definition, it is permanently growing. The main point is that it must be *apparently* stationary because non stationary systems are not physical solutions, in the sense that any monotonic growth, positive or negative, leads to the divergence or collapse of the system. Thus, if the system exists and is expanding, then stationarity is ‘apparent’. Intuitively, we are directed to a scale-free solution where any information retrieved from the system is independent from the size it has at the instant of measure. So, even intuitively, apparent equilibrium implies scale-freeness.

In the scope of this work, a multiplicative process is a process where the growth of a quantity  $x$  depends on the quantity itself, meaning

$$\frac{dx}{x} = \beta \quad (3.38)$$

where  $\beta > 0$  is finite and does not depend on  $x$ . The Barabási-Albert ‘preferential attachment’ is an example of such a process where nodes attract neighbors according to the amount of neighbors they already have (see Eq.(3.28)) and an economic system is another example since production of resources is made on the existing resources.

The finiteness of  $\beta$  brings a problem in a traditional approach to a stochastic process. If  $\beta$  can be infinite, then the full space of events, i.e., of probable states, is completely defined both before and after the multiplication. If not, like in most interesting expanding systems, we can not make a master equation approach because the probability measure in one instant

is not the probability measure in the subsequent instant (see Fig. 3.7). Imposing a finite  $\beta$ , the process is not Markovian and the assumptions for the application of a master equation are not fulfilled.<sup>71</sup> In this section we show why an aggregation of multiplicative processes results in a system that behaves in apparent equilibrium.

We will model the problem of a system of multiplicative processes to argue that it is the only solution of an expanding system that can present an apparent stationarity. To address the problem of expanding systems, let us start by defining a multiplicative object (MO) as an object representing a multiplicative process in the form of Eq.(3.38). Let us also define the event space  $X$  as the set of all possible states  $x > 0$  of an object .

It is not possible, without additional information, to build a probability measure that accounts for the probability for an MO to be in a state  $x \in X$ . A probability measure  $P$  is defined in a probability space  $(X, \mathcal{F}, P)$ <sup>4</sup> together with an event space, in this case  $X$ , and  $\mathcal{F}$  is a  $\sigma$ -algebra (see Section 3.1.4). In the case of an expanding system, the event space  $X$  changes because events, that were previously impossible, are being introduced due to the expansion. The  $\sigma$ -algebra  $\mathcal{F}$  changes structurally and we have a new probability measure  $P$  (see Fig. 3.7 for illustration).

There is one particular case of Eq. (3.38) where the probability measure is structurally invariant under multiplication, if  $\beta = 0$ . This is the *critical* process, where the multiplicative process has an average  $\beta = 0$ <sup>61, 111</sup> which is impossible under the assumption that  $\beta > 0$ .

Now, let us look at the expanding system with a population of  $N > 1$  fixed distinguishable multiplicative processes with initial individual states  $x_i$  uniformly distributed. The space of events of such system  $\Omega$  is a set of admissible states  $x_1, x_2, \dots, x_N$ . Again, we can not build a probability measure that accounts for the probability for the system to be in a state  $x_1, x_2, \dots, x_N$ . But, in this case, there is additional information brought by the population to the problem, can provide a way to transform the  $N$  generic multiplicative process into  $N$  critical processes by means of normalization over the population. Let us take a generic MO  $i$  which is in a state  $x_i$  and let us denote by  $x_{max}$  the highest value of  $x$ . Consider the transformation  $y_i \equiv x_i/x_{max}$  and consequently  $dx_i/x_i = d(y_i x_{max})/(y_i x_{max})$ . Since  $x_{max}$  is also varies because it also represents a multiplicative process, combining with Eq. (3.38) one arrives to

$$\frac{dy_i}{y_i} = \beta - \frac{dx_{max}}{x_{max}} \quad (3.39)$$

We can conclude that if the multiplicative processes in the system only differ by their initial values  $x$ , then we can transform the multiplicative processes in the system in critical multiplicative processes since if  $y_i \equiv x_i/x_{max}$  and  $dx_i/x_i = 0$  then

$$\frac{dy_i}{y_i} = 0 \quad (3.40)$$

Denoting  $Y$  as the space of individual states  $y, ]0, 1]$ , and  $\mathcal{F}'$  as a  $\sigma$ -algebra on  $Y$  then exists a probability measure  $P$  that  $(Y, \mathcal{F}', P)$  is a probability space. We used the fact that the multiplicative processes are together, to get an information from the system  $x_{max}$  and, with that, transform the objects to have a stationary system.

Interesting in the above formulation is that the transformation is independent from the functional form of  $\beta$ , as long as is independent from  $x$  and from  $N$ .

The reciprocal is also true. A set of multiplicative objects is the only expanding system that leads to some sort of stationarity. Let us denote by  $x_{min}$  lower value of  $x$ . Since the system is expanding, the upper boundary is moving apart from the lower boundary, i.e.,  $d(x_{max} - x_{min}) > 0$ . Let us assume that  $dx_i$  is a function of  $x_i > 0$ ,  $dx_i = f(x_i)$ . Moreover, let us assume that  $f(x_i) \sim x_i^\alpha$  with  $\alpha \in \mathbb{R}$ . Then,

$$\frac{dx_i}{f(x_i)} \propto \frac{dx_i}{x_i^\alpha} \quad (3.41)$$

If  $\alpha > 1$  the system is contracting, that contradics the expanding assumption. If  $\alpha < 1$  then the system is expanding. But it can not be transformed into a stationary system, since  $\beta$  in Eq. (3.38) is, in this case, is dependent on  $x_i$ . So,  $f(x_i) \sim x_i$  for expanding and stationarity and aggregation of objects following Eq. (3.38) is the only stationary expanding system.

In practice, what is usually done when we are dealing with a set of multiplicative processes? We take the result of an experiment over a MO system and we normalize the values to its maximum value observed, i.e., we apply Eq.(3.39). We build a set of bins that cover the entire domain of normalized values and measure the frequency of each bin, i.e. frequency of the events. To the normalized result we call ‘probability distribution’.

Each time we do that, we make the transformation  $y_i \equiv x_i/x_{max}$  and Eq. (3.40) holds for the state variable frequency. we say that the system is in *relative conservation*. Despite the fact that a full system of objects is continuously growing through multiplicative processes, it can be represented as invariant as long we impose Eq. (3.40).

Now, we assume that there is a global measure of the system,  $\Lambda$ , related to  $x$  such that  $d \log(x) \propto d \log(\Lambda)$ . Variable  $x$  and measure  $\Lambda$  have logarithmic variations proportional to each other. This enables us to write

$$\frac{dx_i}{x_i} = \frac{1}{\alpha} \frac{d\Lambda}{\Lambda} \quad (3.42)$$

where  $x_i$  is the state of  $x$  in the multiplicative process  $i$ .

A well known example of a problem expressed as Eq.(3.42) is the Barbási-Albert network growth problem where  $x_i$  is the number of first neighbors of each node and  $\Lambda$  is the total number of nodes. Equation (3.42) represents a class of problems where one assumes correlations between the parts of the system that are multiplicative objects. If we assume that no other measure can be taken as observable for the system, at least the sum of all  $x_i$ 's is observable, meaning that there is at least one ‘macroscopic’ observable  $\Lambda = \sum_i x_i$ .

Hence, let us take the probability distribution  $P(x_i < x)$ , the probability of a particle  $i$  to have a measure  $x_i$  lower than  $x$ . Since we want to express every quantity in  $x_{max}$  units, we will divide  $P$  into two parts. The probability distribution depends on  $\Lambda$ , because  $P \propto \Lambda^{-1}$  with a proportionality coefficient, say  $Z$ .

For a better understanding of  $Z$ , if we take a regular histogram,  $\Lambda$  will be the number of total events in the sample and  $Z$  the sum of the frequency of the events lower than  $x$ . Thus,  $Z$  is

also expressed in  $x_{max}$  units.

Hence ,  $P(x_i < x) = \frac{Z}{\Lambda}$  with  $Z > 0$  and consequently

$$\frac{dP(x_i < x)}{d\Lambda} = -\frac{Z}{\Lambda^2} + \frac{1}{\Lambda} \frac{dZ}{d\Lambda} = -\frac{P(x_i < x)}{\Lambda} + \frac{1}{\Lambda} \frac{dZ}{d\Lambda} \quad (3.43)$$

Let us concentrate now on the second parcel of the rhs. Since the system is constantly growing ( $d\Lambda > 0$ ), if  $\frac{dZ}{d\Lambda} < 0$  then the portion of the system below  $x$  is getting lower. Consequently, when  $d\Lambda \rightarrow \infty$ ,  $dZ < 0$  and since  $Z > 0$   $Z \rightarrow 0$ . But  $\Lambda$  is the measure of the system itself and if  $Z \rightarrow 0$  then the system will be reduced to the portion above  $x$ , i.e.,  $\Lambda$  is reduced, which is absurd according to the initial assumptions. The same reasoning is valid for  $\frac{dZ}{d\Lambda} > 1$  and for decreasing systems. If  $\frac{dZ}{d\Lambda} > 1$  then when  $d\Lambda \rightarrow \infty$ ,  $Z \rightarrow \infty$  eliminating all portion of the system above  $x$ , which is also absurd.

From these we can say that there is a number  $0 \leq c \leq 1$  such that  $\frac{dZ}{d\Lambda} = c$ , yielding

$$\begin{aligned} \frac{dP(x_i < x)}{d\Lambda} &= \frac{c - P(x_i < x)}{\Lambda} \\ \Leftrightarrow -\frac{d(c - P(x_i < x))}{c - P(x_i < x)} &= \frac{d\Lambda}{\Lambda}. \end{aligned} \quad (3.44)$$

We can combine Eq.(3.44) with Eq.(3.42) to get

$$\frac{d(c - P(x_i < x))}{c - P(x_i < x)} = -\alpha \frac{dx}{x} \quad (3.45)$$

which leads to

$$P(x_i \geq x) = \left(\frac{x}{x_0}\right)^{-\alpha} - (c - 1) \quad (3.46)$$

$$P(x_i < x) = 1 - \left(\frac{x}{x_0}\right)^{-\alpha} + (c - 1) \quad (3.47)$$

Finally, the probability density function for a system of multiplicative processes is given by

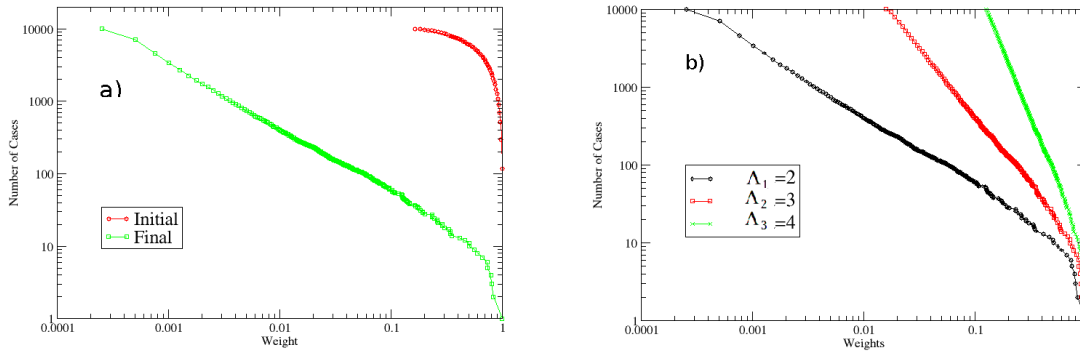
$$p(x_i = x) = \frac{d(P(x_i < x))}{dx} = \alpha \frac{x_0^\alpha}{x^{\alpha+1}} \quad (3.48)$$

which is the expression for a Paretian probability density function with exponent  $\gamma = \alpha + 1$  and initial value  $x_0$ .

The relative logarithmic conservation insures the existence of a probability measure and the continuously growing characteristic of the multiplicative processes give us the expression of that probability measure Eq. (3.48) which, as we showed above, only exists when we consider the aggregation of the multiplicative processes, since all other expanding systems are non stationary and a probability measure cannot exist because the space of events is permanently growing.

When we are dealing with aggregations the issue of independence between the different multiplicative processes is raised, that is the reason why  $\alpha$  appears on Eq. (3.42). If the aggregate observable  $\Lambda$  is merely the sum of all  $x_i$ , then  $\alpha = 1$  and the exponent of the

Eq.(3.48) is  $\gamma = 2$ . This is the fundamental aggregation since it is the ‘natural’ aggregation



**Figure 3.9:** *One single system of multiplicative processes evolving (a) The system starts with uniform distribution and evolves to a power law with exponent  $\gamma = 2$ . In (b) we have three probability measures considering different correlation effects for those measures.*

method and it is independent from any correlation that exists between the multiplicative processes, assuming that  $\beta$  has the same functional form. Nonetheless, there are aggregation measures where  $\alpha$  is not 1. One well known example is the Barabási-Albert network where the measured quantity of aggregation is the total number of links between the multiplicative processes. In this case,<sup>9</sup>  $\alpha = 2$  and the obtained value of  $\gamma$  is 3. Since in Barabási-Albert model the creation of nodes is equivalent to the creation of links, the  $\alpha$  parameter is imposed. In a constant  $N$  system with weighted links formed by preferential attachment we fall into the expanding system in the space of weights.

In summary, if another aggregation observable is used, another correlation factor  $\alpha$  appear, then  $\gamma$  would give a different value. The point is that the aggregate observable will reveal the nature of the correlation between multiplicative processes that we are dealing with and, with that, a different  $\sigma$ -algebra and a new probability measure are produced, with a power-law format but with different exponents, based on the same space of events.

To simulate we assumed a system with  $N$  components each with a initial value  $y_{0i}$  uniformly distributed,  $y_{0i} \sim U(0, 1)$ . At each instant, a MO  $j$  is randomly chosen without any previous criteria and a quantity  $dy_j = \pm\Delta y_j$  is added to the , where  $\Delta$  is arbitrarily chosen in design time and the sign is chosen with probability 1/2. In this way the relative logarithmic conservation is assured. After  $10^9$  iterations three measures are taken,  $\Lambda_1, \Lambda_2$  and  $\Lambda_3$  where

$$\frac{dy_i}{y_i} = \frac{d\Lambda_1}{\Lambda_1} \quad (3.49)$$

$$= \frac{1}{2} \frac{d\Lambda_2}{\Lambda_2} \quad (3.50)$$

$$= \frac{1}{3} \frac{d\Lambda_3}{\Lambda_3} \quad (3.51)$$

In Fig.3.9-a we show that a system with a uniformly distributed quantity evolves, by relative logarithm conservation to a power law distribution. In Fig.3.9-b we show that, depending on

the aggregate observable considered several power-laws distributions can be retrieved from the same evolving system. In Fig.3.10 we show that the system evolves with relative logarithmic conservation to a stationary system.

Since the economy is labor exchange there is a correlation between the growing of the allocated resources to each agent, i.e., the growth of wealth of any agent must be accompanied by the growth in production in other agents. If we assume that there are no losses in the system, then economic growth must be made of changes in two agents. Therefore, that if we denote wealth of an agent  $i$  with  $w_i$  then total economic growth  $\Lambda$  relates with the growth of individual wealth by

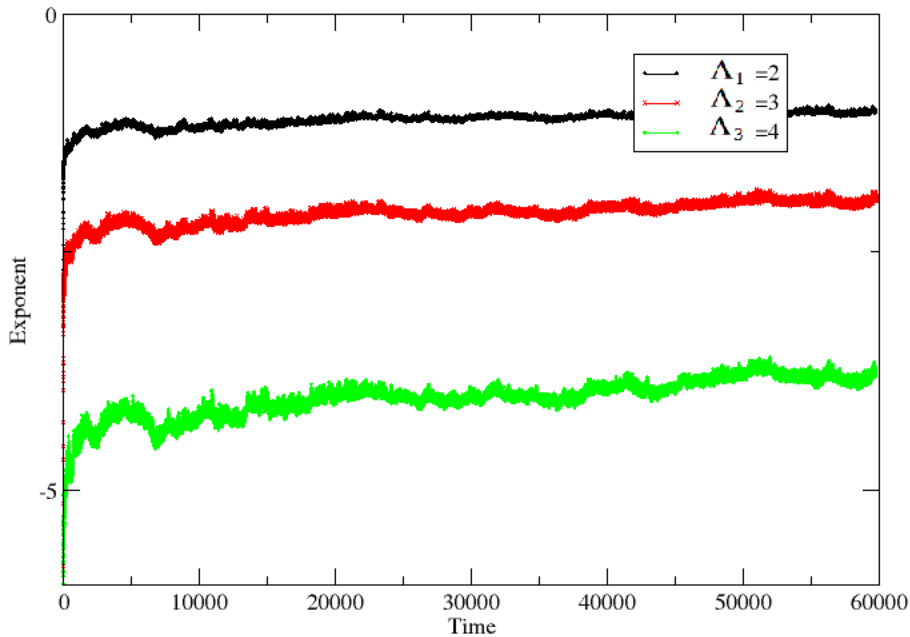
$$\frac{dw_i}{w_i} = \frac{1}{2} \frac{d\Lambda}{\Lambda} \quad (3.52)$$

since the economic growth is affected by the increment in production, but this affects both involved agents. If we took as aggregation observable the sum of individual wealths, then the coefficient in Eq. (3.52) would be  $\frac{1}{1} = 1$ . This would be also the case if the economic growth could be made by the agents independently from each other.

According to Eq.(3.48) Eq. (3.52) leads to a wealth distribution of

$$p(w) \propto w^{-\gamma} \quad (3.53)$$

where  $\gamma = \alpha + 1$  and  $2 < \gamma \leq 3$ , 3 being the value of complete dependence between the agents and 2 the independence case.



**Figure 3.10:** Evolution of the exponents for three correlation values.

So despite the fact that Barabási-Albert approach was not formally correct and there is not need for growing nodes to have a power-law in the connections, the preferential attachment model is good to reproduce economic relations as a microscopic mechanism, since it reproduces the same statistics. In the same way the inelastic impacts of molecules of a gas is a microscopic mechanism for a thermal equilibrium.

Why is it interesting to use preferential attachment as a microscopic mechanism? Let us assume the generic case of a gas where each particle has a different attractiveness, despite the fact that we do not know the nature of that attractiveness. Assuming that no other mechanical influence is present besides the particle-particle attraction, then we can represent the attraction field  $G_i \equiv G(M_i)$  that particle  $i$  exerts over the other ones due to its attractiveness  $M_i$ . The nature of  $M_i$  is irrelevant for our purposes, as later will become evident.

In such a context, if a new particle enters the system, it will suffer the influence of all existing particles, each one with a relative intensity given by

$$I_i = \frac{G(M_i)}{\sum_j^N G(M_j)}. \quad (3.54)$$

where  $N$  is the total number of existing particles, which we assume as  $N \rightarrow \infty$ . At first approximation, the attractive field can be quantified by the attractiveness itself  $G(M_i) \approx M_i$  and

$$I_i = \frac{M_i}{\sum_j^N M_j}. \quad (3.55)$$

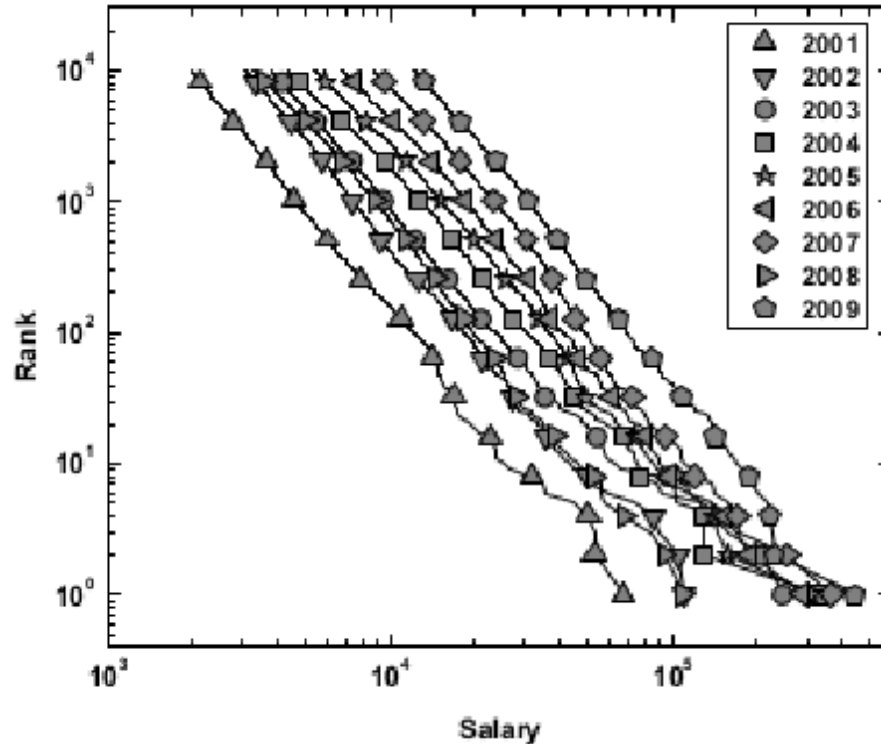
Even not knowing what is the nature of  $M_i$ , the relative intensity  $I_i$  can be safely assumed as measurable since it is dimensionless. Furthermore, Eq. (3.55) is in fact the relative distribution of attractiveness over the system. How to measure this relative attractiveness? By the neighbors the object actually attracts, i.e.,  $k_i$ . Therefore, since no other mechanical influence is present, attractiveness distribution can be translated as the relative number  $k_i$  of neighboring agents trapped by the ‘potential well’ of each agent  $i$ , i.e.,

$$I_i \equiv P_i = \frac{k_i}{\sum_j^N k_j} \quad (3.56)$$

which is perfectly equivalent to a discretization of Eq. (3.55), supported as  $N \rightarrow \infty$ . For a better understanding, since the nature of attractiveness is not known, as it happens in economic systems, Eq. (3.56) says that it is deduced by the number of agents each agent  $i$  attracted, since it is an indirect measure of attractiveness. At the same time, statistically speaking, Eq.(3.56) represents an histogram of the relative frequency of agents attracted by other agents and, by the Law of Large Numbers,<sup>49</sup> it converges to a probability distribution that equals  $P_i$  in Eq.(3.56).

The expression (3.56) is exactly the expression beneath the preferential attachment mechanism we described previously. And, as we empathized,  $\sum k_i$  is the ‘natural’ aggregation observable  $\Lambda$ .

## 3.2.2 Wealth Distribution



**Figure 3.11:** Evolution of salary distribution in the Romanian district of Cluj for the top 10000 wages. From Derzy et al.<sup>43</sup> with permission.

One interesting application is included in what economists call ‘normative modeling’ which is what in current language we call by ‘economical politics’: To produce a set of constraints like taxes or public investments to correct some uncontrolled effects of the free economic growth. We can think of the soviet ‘planned economy’ as an extreme version of normative modeling but in all societies there is some sort of corrections to do, mainly because humans are, in first place, animals with a survival instinct and when survival is in stake economy is left behind. So, corrections must be made in order that humans can be kept in economy.

One of the publicized goals of normative modeling nowadays is the reduction of inequalities or, in other words, to distribute wealth between the people giving each other the ability to allocate resources equally. As we can conclude from the previous sections, there is no such thing as wealth redistribution, in the sense that inequalities are ruled by Eq.(3.53). What the economic equilibrium shows us is that when perturbing the system it can get out from equilibrium temporarily but it will return to it if the perturbation disappears or it will assume a new equilibrium position with the new constraint present. Equilibrium formation in this scope is characterized by Eq.(3.53).

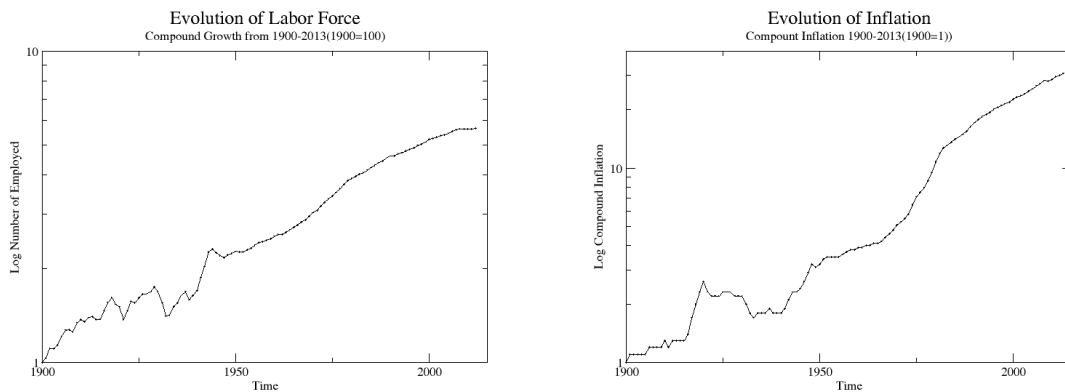
Empirical evidences of this is shown in the Romanian social security database of wages in the district of Cluj by Derzy et al.<sup>43</sup> and summarized in Fig.(3.11). From 2001 to 2009, Romania



got into the European Union and received cohesion funds to develop internal infrastructures, change its currency from Leu to New Leu, suffered the impact of the 2008 crises where wages drop considerably<sup>43</sup> and, still, through all those years the income distribution remained basically constant. The only observed evolution is a translation of the curve but not a considerable difference in wealth distribution that remains invariant, i.e., the exponent of the distribution is the same.

We should emphasize that our approach refers to wealth and the study of Derzsy et al.<sup>43</sup> refers to income. The difference is subtle but relevant. Income is the instantaneous return of some set of resources like education, equity assets, etc. Wealth is the full set of resources, i.e., the ability to have that income, in the present and in the future (in section 4.2.2.2 we will address the issue of time in economic systems in detail). As Derzsy et al.<sup>43</sup> state, the lower income distributions is not power-law distributed which is consistent with this difference since there are minimal wage policies that deviate income from wealth in the lower boundaries.

### 3.2.3 Inflation



**Figure 3.12:** *(Left)* Evolution of the number of employed persons in the US from 1900 to 2013. The number of employed in 1900 was taken as one and the number compound until 2013. Sources: (1900-1945) *Historical Statistics of the United States 1789-1945*;<sup>106</sup> 1946 - Interpolated; 1947-2012 *US Census Bureau*;<sup>107</sup> 2013-Extrapolated. *(Right)* Evolution of the inflation in the US from 1900 to 2013. The value is compound from 1900 until 2013. Sources: (1900-1913) *Historical Statistics of the United States 1789-1945*;<sup>106</sup> 1914-2013 *US Inflation Calculator based on Consumer Price Index published by the Bureau of Labor Statistics*<sup>104</sup>

In the previous sections we showed that in a system of multiplicative objects, despite the fact that it is not possible to have a probability measure in the individual space state, a probability measure rises from the aggregation of those objects and that probability measure is due to what we call relative logarithm conservation that, despite the system is expanding, reveals a stationarity. We also showed that more than one probability measure can result from the system, all power-law distributions being the  $\gamma = 2$  the natural distribution since it results from the summation of the objects. There are other possible probability measures with different exponents depending on how the objects contribute to aggregate observable.

The existence of a stationarity due to relative logarithmic conservation leads naturally to the discussion of inflation. The system itself does not stop expanding, it is absolutely divergent, despite its apparent stationarity. This means that, at a constant number of components, the minimum of the range of values is growing in the positive axis, since the lower component is also growing. The lower value becomes more and more higher in absolute terms. This means that the stationarity of an expanding system implies the existence of an inflation mechanism. In this sense, the empirical data from a real economy, rather than a simulated one, should agree with this.

For that we retrieved data from the US economy. Unlike a simulated economy like the one we use in computers, in a real economy it is not possible to keep a controlled environment. Also, when we talk about human statistics in a century, it is affected by a lot of human related constraints like laws and traditions that contaminates historical data. For example, in US Labor statistics from the beginning of the 20th century children with more than 10 years old were assumed as full part of the labor force, but tradition kept women out of the economy. So, conditions in the beginning of 20th century are completely different from 2013 conditions. Nevertheless we only know one planet with economy in the universe and there is more data from the US economy than from any economy in the world therefore we must use the data that is available.

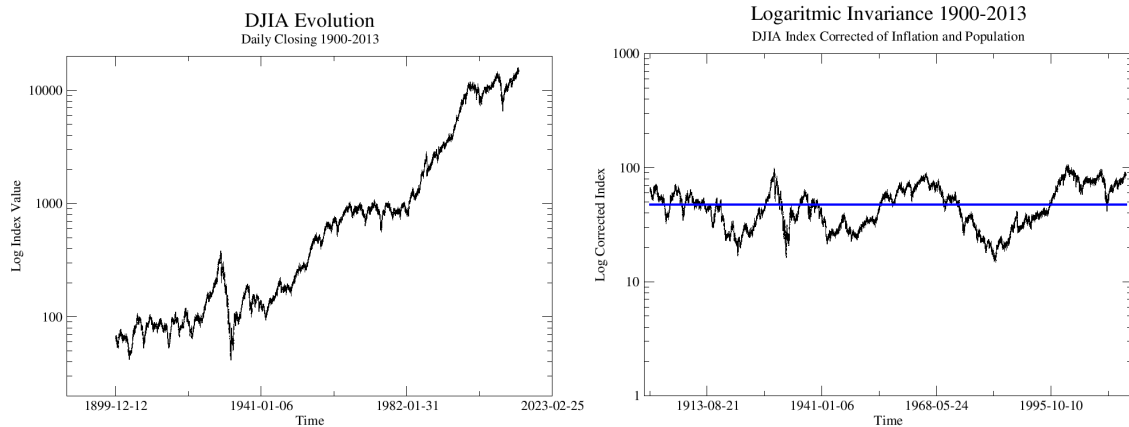
In right plot of Fig. 3.12 we show the evolution of the number of employed persons in the US. Since we cannot have a closed environment with constant number of economic agents  $N$  we should remove the effect of a growing population. For that we used the number of persons exchanging labor as a proxy for the number of agents in the system. Despite the effect of the wars in the beginning, the compound rate of employed populations grows logarithmically in time, as we should expect from any kind of human population.

In left plot of Fig. 3.12 we show the evolution of economic inflation. Inflation is the evolution of prices, i.e., the amount of money allocated for a labor. In other words, the value of money for a constant amount of labor. Since the economy is permanently growing and money represents past labor, we should observe a depreciation of the value of money relatively to the present labor. We can understand this if we think that even the smaller agent in the system is growing and  $dx_{min} = \beta x_{min}$ , therefore  $dx_{min}$ , the minimum amount of introduced labor, also grows. This minimum amount of labor that can be ‘canned’ into money is bigger today than it was yesterday and, consequently, since the facial value of the money bills does not change, the amount of labor it represents must be lower.

In the left plot Fig. 3.13 we show the evolution of the stock market index Dow Jones Industrial Average(DJIA). It is a principal stock market index, meaning that it is the value of a portfolio designed to reflect the total US economy. So the evolution of the index should reflect the growing of the US economy and we will use it as a proxy for it.

In the right plot Fig. 3.13 we show the evolution of the DJIA corrected of inflation and population. We should notice that despite the obvious limitations of the used proxies, the corrected DJIA appears to oscillate around a mean value for 113 years. From this and other similar empirical evidences we assume the existence of a stationary state.

This is the biggest set of data existent and the system is irreproducible. So, the correctness of our assumptions can only be address in the future. Nonetheless, if we take the definition of



**Figure 3.13:** (Left) Evolution of the Dow Jones index from 1900 to 2013. Sources: Federal Reserve Bank of St. Louis<sup>105</sup> (Right) Evolution of the Dow Jones index from 1900 to 2013 corrected from inflation and population.

economy and the associated mechanisms, this is a reasonable and expectable behavior.

In conclusion, the economic equilibrium is translated by a logarithmic invariance of the total labor in the system. Also, inflation is the product of economic equilibrium as a natural emergence of an aggregation of a set of multiplicative processes.

### 3.2.4 Canonical and Microcanonical Economic Ensembles

In this section, we describe the apparent equilibrium, non-growing “ensemble” of multiplicative processes raising a series of important questions concerning the structure of these ensembles, and the possibility for justifying the robustness of their properties, namely the exponent of the degree distribution in scale-free networks. To model this we will refer ourselves to scale-free networks since it is the more studied multiplicative process and it is how human relations have been studied, but all the following is valid for any type of these systems. We prescribe a dynamics based on the constrained maximization of a proper functional, whose most probable macrostate is a scale-free configuration, according to previous sections, and we describe a canonical and microcanonical ensemble of configurations to establish an analogy with thermostatics. We then identify the equivalents of energy, entropy and heat capacity for scale-free networks. Consequently, we establish an equivalent of the Zeroth-law for interacting and merging scale-free networks which may explain why some empirical scale-free networks are so robust in time. We present analytical and numerical results, using Monte Carlo simulations, and we discuss the complementarity between our framework and growing (non-equilibrium) processes. Finally, we show how to use the equilibrium framework for merging scale-free networks, predicting their final exponent value.

The ubiquity of networks in nature underlying many social and natural real-world systems has promoted the increasing interest to understand the emergence and evolution of their structure.<sup>3,18,27,47</sup> While many networks, such as random graphs, are taken as “equilibrium networks”, scale-free networks are typically considered to be the result of some dynamical

process, being therefore a “non-equilibrium” network.<sup>47</sup> Prototypical examples are the World Wide Web,<sup>3</sup> the classical Pareto law<sup>113</sup> and the set of phone calls within societies,<sup>56</sup> that are growing and expanding in time.

However, recently power-laws have appeared in equilibrium distribution of energy in a conservative mechanical system<sup>28</sup> and equilibrium and growing graphs have been addressed as being dual of each other.<sup>19,78,120</sup> Assuming such duality, it is reasonable to address the possibility for describing scale-free networks in nature as resulting from an “equilibrium” ensemble where an equivalent of micro- and macrostates, energy and entropy can be defined.

In this section we present such a description and discuss its consequences. Our multiplicative processes ensemble is derived from the apparent equilibrium which was proposed in the previous sections for explaining the emergence and ubiquity of scale-free networks in nature.

We start from a statistical ensemble of networks, with a time-dependent weight  $w$  for each node, which is used to define an equivalent of energy in one microstate. We then prescribe a dynamics for the re-weighting of the system, such that a functional, which contains the analogue of the entropy of the system, is maximized, while constraining the average total energy and the number of nodes. We present explicitly calculated expressions for the thermostistical variables. From this, we propose the existence of canonical and microcanonical ensembles of such systems, which are scale-free networks with maximum entropy.

For the specific case that the weights correspond to the degrees  $k$  of the nodes, we present a series of Monte-Carlo simulations following the proposed dynamics, both in the canonical and the microcanonical formalism, illustrating that the scale-free networks follow the equivalent of a Zeroth law, i. e. *i*) a “non-equilibrium” system brought in contact with a reservoir relaxes to an “equilibrium” state with the same weight distribution as the reservoir, *ii*) two “equilibrated” systems with different weight distributions brought in contact with each other develop a new “equilibrium” state with a new mixing weight distribution. We then show, *iii*), that the mixing weight distribution is numerically found to be the same as the one calculated from the equivalent of the heat capacity, which contains the second moment of the energy.

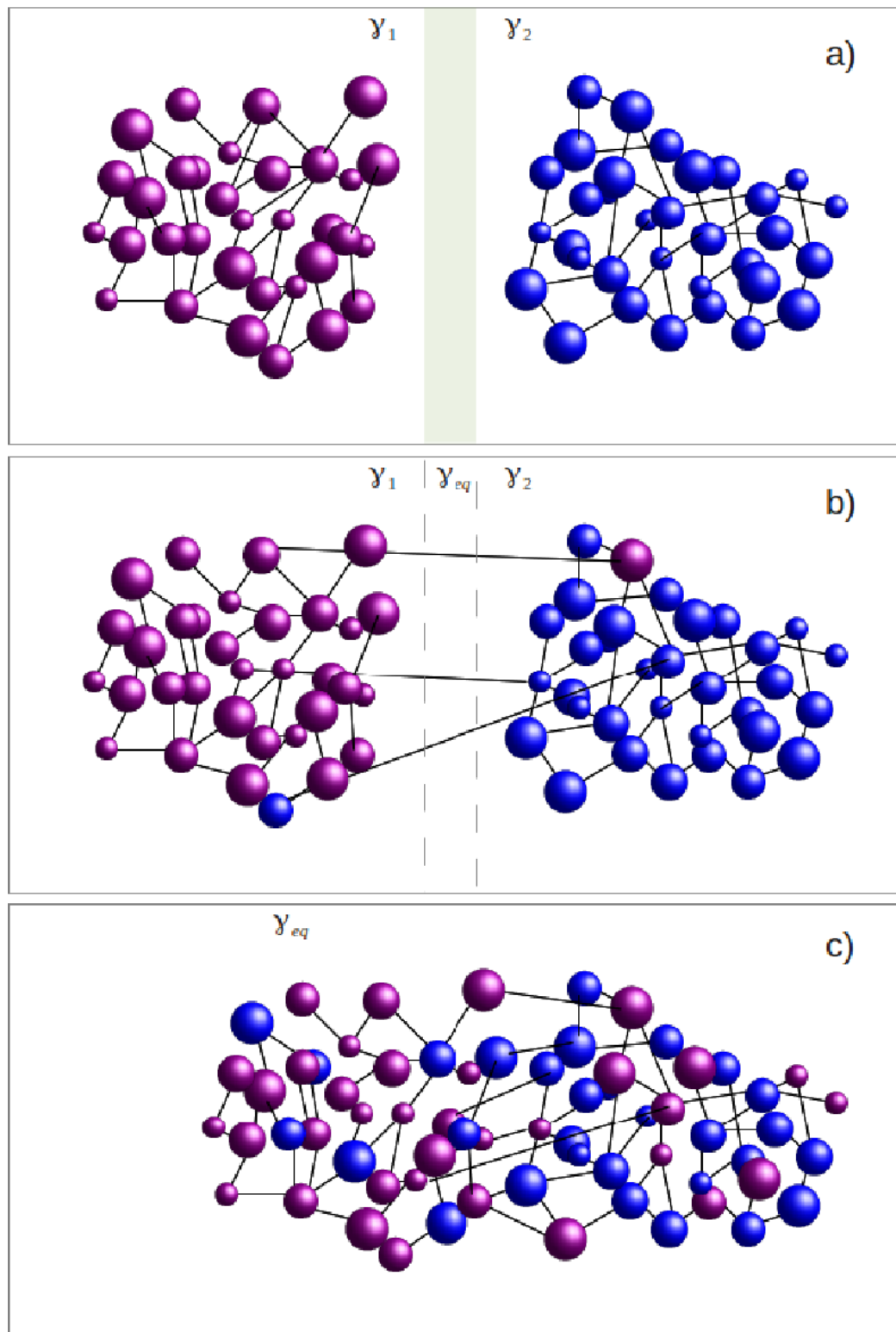
Being able to derive a power-law or scale-free network as an “equilibrium” solution towards which the system tends to evolve when perturbed, opens the possibility to explain in a straightforward way the robustness empirically observed.

We start by considering an ensemble of  $N$  nodes, where each node  $i$  occupies one of  $M$  possible microstates. Each microstate  $j$  is characterized by a scalar  $\hat{w}_j$ , which we call weight. This weight can be the weight of the node for undirected graphs, the in- or out-weight for directed ones, or the sum over the weight of the node links for a weighted graph. To be able to consider weights that change in time, we define a rescaled weight  $w_j$  as,

$$w_j = \frac{\hat{w}_j - \hat{w}_{\min}}{\hat{w}_{\max} - \hat{w}_{\min}}, \quad (3.57)$$

which is equivalent to the transformation done in Eq. (3.39), where  $\hat{w}_{\min}$  and  $\hat{w}_{\max}$  are the minimum and maximum weights, such that  $w_j$  is in the range  $[0, 1]$ . We also define the energy of the microstate  $j$  as

$$\varepsilon_j = \log w_j. \quad (3.58)$$



**Figure 3.14:** When two scale-free networks with weight distributions with exponents  $\gamma_1$  and  $\gamma_2$  are put in contact, they evolve towards an “equilibrium” scale-free state with exponent  $\gamma_{eq}$  (see text).

The macrostate  $W$  is completely defined by the distribution of the  $N$  nodes among the microstates:  $W \equiv W(n_1, \dots, n_M)$ , where  $n_j$  is the occupation number of state  $j$ . For each macrostate there are  $\Omega = \frac{N!}{\prod_j n_j!}$  equivalent configurations. Using Boltzmann's entropy, the entropy  $S$  of a macrostate is  $S = \log \Omega$ .

Having two of such systems, as sketched in Fig. 3.14, one now considers that they evolve in time, i.e. connections are created and destroyed in time according to some criteria, reflecting an interchange of energy among the nodes which, consequently transitate between microstates. We next introduce the canonical ensemble, which describes the distribution among microstates for one single system, taking the other one in Fig. 3.14 as a heat reservoir. After that we will discuss the microcanonical ensemble where both systems in contact are of the same size.

For the canonical ensemble, the most probable macrostate is the one that maximizes the entropy. This maximization has two constraints: conservation of the average total energy respecting logarithmic invariance,  $E = \sum_j n_j \log w_j$ , and of the number of nodes,  $N = \sum_j n_j$ . Thus, the most probable macrostate is an extremum of the functional,

$$\mathcal{F} = \log \Omega - \gamma \sum_{j=1}^M n_j \log w_j + \log \alpha \sum_{j=1}^M n_j \quad , \quad (3.59)$$

where  $\gamma$  and  $\log \alpha$  are the Lagrange multipliers for the constraints in the energy and number of nodes, respectively. For thermodynamic ensembles,  $\gamma$  is the inverse of the reservoir "temperature" and  $\log \alpha$  the ratio between the "chemical potential" and the temperature. Thus,  $\gamma$  is a property of the reservoir, while  $\log \alpha$  is characteristic of the system.

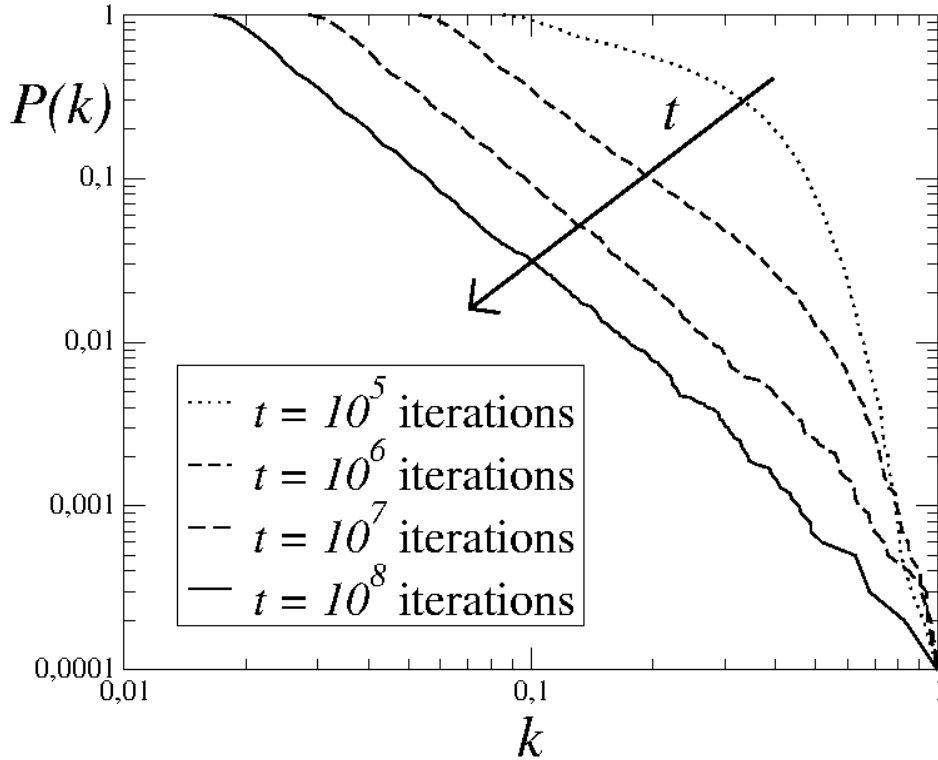
Using Stirling's approximation,  $\log n! \simeq n \log n - n$ , one obtains that the extrema of Eq. (3.59) yield  $n_j = \alpha w_j^{-\gamma}$ . If we define  $\hat{w}_i = k_i$ , where  $k_i$  is the node weight, the weight distribution is a power law of weight exponent  $\gamma$ . In other words, the reservoir sets the weight exponent. The weight distribution is consistent with the expected Boltzmann distribution for canonical ensembles. Note that the node energy is  $\varepsilon_i = \log w_i$ , which is thereby exponentially distributed. In fact, as claimed by other authors,<sup>81,119</sup> if properly interpreted, the Boltzmann distribution yield disguised power-laws.

The canonical formalism developed above, was numerically implemented as follows. One starts with  $N$  nodes, interconnected through  $L$  links, yielding a certain weight distribution,  $\{w_i\}$ . At each iteration, one randomly selects two nodes,  $l$  and  $m$ , and calculates the change  $\Delta E$  in the total energy  $E$  if one link connected to  $l$  is rewired to be connect to  $m$ ,

$$\Delta E = \Delta(\log w_l) + \Delta(\log w_m) \quad . \quad (3.60)$$

This rewiring is executed with probability  $p = \min \{1, \exp(-\gamma \Delta E)\}$ . After some iterations, the network will converge to the desired scale-free weight distribution, conserving the number of nodes and links. Since  $\varepsilon_i = \log w_i$ , the energy is only conserved for movements where  $w_l^{\text{initial}} = w_m^{\text{initial}} + 1$ , where  $w_l^{\text{initial}}$  and  $w_m^{\text{initial}}$  are the weights of nodes  $l$  and  $m$  before the movement. This algorithm is a generalization of the one by Metropolis *et al.*, which respects detailed balance.<sup>99</sup>

Figure 3.15 illustrates the evolution of a canonical scale-free ensemble. Here, we started with a regular random graph with  $N = 10^5$  nodes, each one having a weight  $w = 5$  and put it in the



**Figure 3.15:** The scale-free canonical ensemble: starting with a uniform distribution of weights among  $N = 10^5$  nodes in contact with a reservoir ( $N = 10^7$ ) having a distribution  $P(k) \sim w^{-\gamma}$  with  $\gamma = 3$ , the weight distribution of the system evolves towards the same power-law (see text).

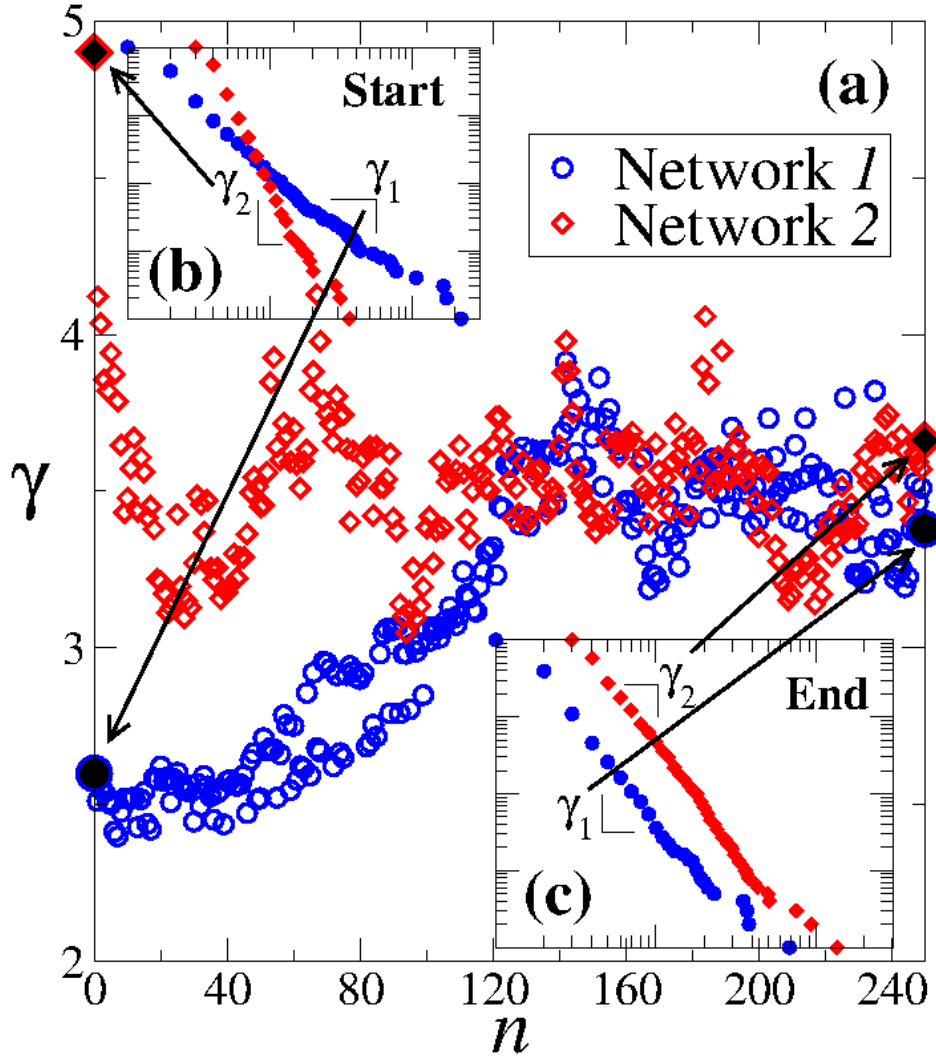
contact with a large scale-free network (reservoir) with  $N = 10^7$  nodes and exponent  $\gamma = 3$ . As one clearly sees throughout the four stages plotted in Fig. 3.15, the weight distribution evolves towards a power-law distribution with the same exponent value.

We consider now the classical problem of thermal equilibration. What is the final weight distribution when two scale-free networks are put in contact? To address this question, as schematically illustrated in Fig. 3.14, we consider two scale-free networks of weight exponent  $\gamma_1$  and  $\gamma_2$ . Since the combined system is isolated, the total energy  $E_{1+2} = E_1 + E_2$  is expected to be conserved, where  $E_1$  and  $E_2$  is the energy of networks 1 and 2, respectively. Figure 3.16a shows the evolution of the weight exponent of the nodes in each network, using the microcanonical algorithm proposed below. The two networks merge into one with a power-law weight distribution of exponent  $\gamma_{\text{eq}}$ .

This situation is described through a microcanonical ensemble, which represents an isolated network at a given energy. To numerically explore the configuration space we propose a generalization of the Creutz algorithm, which respects both detailed balance and ergodicity.<sup>35</sup> Let us consider again the case where the weight corresponds to the weight. Algorithmically, instead of interacting with a heat reservoir like in the canonical case, the network exchanges energy with one energy reservoir, named demon, with energy  $E_{\text{demon}}$ . One starts with one network at the desired energy and  $E_{\text{demon}} = 0$ . As in the canonical simulation, at each iteration two nodes  $l$  and  $m$  are randomly selected and the energy change of the rewiring movement is obtained from Eq. (3.60). If  $\Delta E \leq 0$ , the movement is accepted and the energy difference is transferred to the demon. Otherwise, the movement is only accepted if  $E_{\text{demon}} > \Delta E$ , and,



when it happens, the energy difference is subtracted from  $E_{\text{demon}}$ . To obtain a network with energy  $E$ , one can also start with one network with the desired number of links and nodes and energy  $E_0 < E$  and  $E_{\text{demon}} = E - E_0$ ; After a few iterations the network energy will converge towards  $E$ .



**Figure 3.16:** (Color online) (a) Evolution of the weight exponent for two scale-free networks put in contact. Initially, the two networks have different weight exponents ( $\gamma_1 \neq \gamma_2$ ) (inset (b)) but converge towards a weight distribution with the same weight exponent  $\gamma_{\text{eq}} = \gamma_1 = \gamma_2$  (inset (c)) (see text).

To illustrate the microcanonical ensemble of scale-free networks we implemented the algorithm above for two subsystems, each composed of  $N = 10^5$  nodes and a power-law weight distribution, but having different exponents, namely  $\gamma = 3.5$  and  $\gamma = 5$ . Figure 3.16b shows the initial weight distribution of each subsystem. Iteratively we add or remove a fixed small quantity  $\Delta$  to the energy of a randomly chosen node  $j$ . If the energy decreases (with this add) than the change is accepted, and the demon's energy is updated accordingly. If the energy increases than the change is accepted only if the demon can provide that amount of energy. Both networks, 1 and 2, evolve to states with similar weight distributions, Fig. 3.16c, where we numerically verified that the demon energy converges to a constant value.

With the thermostatistic approach proposed here, one can derive five results for ensembles



of scale-free networks. First, the equivalent to the Zeroth-law of thermodynamics can be expressed as: if networks  $A$  and  $B$  are in equilibrium with network  $C$  then they are also in equilibrium with each other. Here, two networks in equilibrium have the same weight exponent  $\gamma$ .

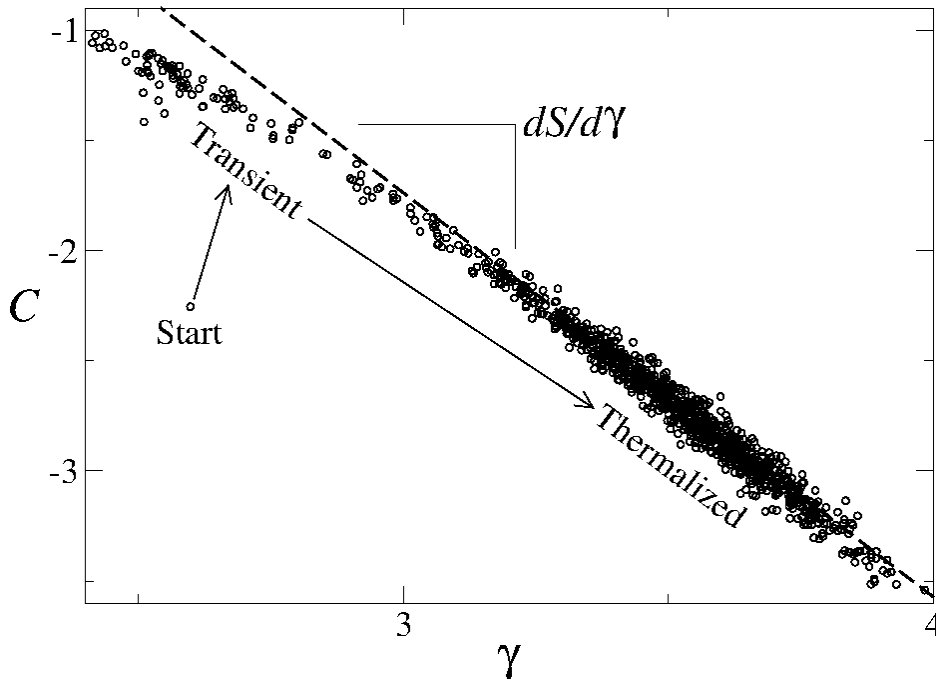
Second, one can derive a closed expression for the entropy of scale-free networks. As discussed before, each macrostate has  $\Omega$  possible equivalent configurations. The entropy is then,

$$S = \log \Omega = S_0 + \gamma \alpha \sum_{i=1}^M w_i^{-\gamma} \log w_i \quad , \quad (3.61)$$

where  $S_0 = N \log \left( \sum_{i=1}^N w_i^{-\gamma} \right)$  is the minimum entropy and the second term is equivalent to the Shannon entropy. This expression for the entropy complements previous works.<sup>114</sup>

Third, in the continuum limit, associated with a large number of microstates  $M$ , the sum in Eq. (3.61) can be approximated by an integral, yielding an expression for  $S$ , solely depending on  $\gamma$  and  $N$ . The differential of the entropy is then,

$$dS = \frac{\partial S}{\partial \gamma} d\gamma + \frac{\partial S}{\partial N} dN \quad . \quad (3.62)$$

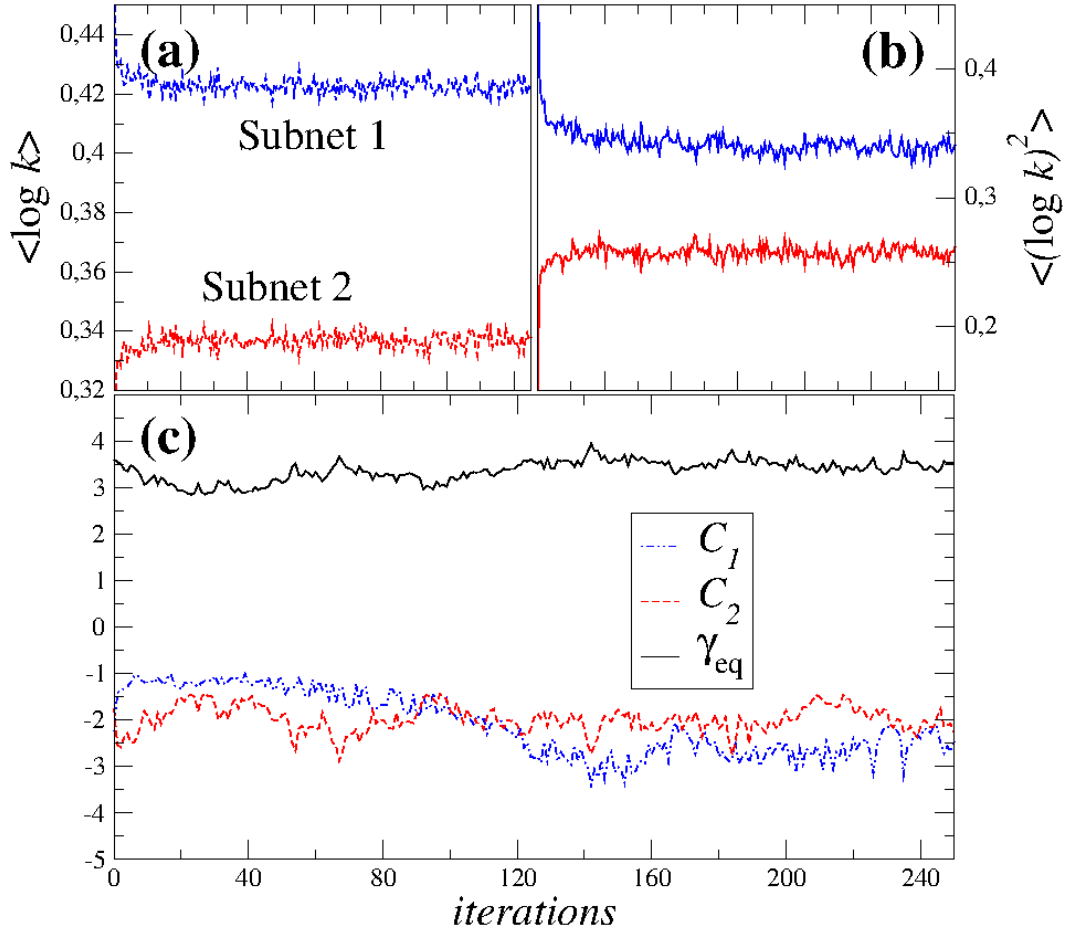


**Figure 3.17:** The heat capacity  $c$  versus exponent  $\gamma$  for the microcanonical ensemble of two scale-free networks. The negative-valued slope of the linear dependence between  $c$  and  $\gamma$  gives the value  $\frac{dS}{d\gamma}$ .

Fourth, we can define a heat capacity for non-growing scale-free networks, which can be obtained from the entropy as,

$$c_N = \gamma \left( \frac{dS}{d\gamma} \right)_N = \gamma \langle \log w \rangle - \gamma^2 \langle (\log w)^2 \rangle \quad . \quad (3.63)$$

In Fig. 3.17 we show the pair  $(c_N, \gamma)$  of the system of interacting networks addressed in Fig. 3.16. One clearly sees a linear dependence with a negative slope ( $(\frac{dS}{d\gamma})_N < 0$ ), which can be understood since  $\gamma$  plays the role of the inverse of a temperature.



**Figure 3.18:** (Color online) Computing equilibrium exponents and heat capacity of scale-free networks: (a) first and second moments of the energy  $\log w$  in two separated scale-free networks; (b) the corresponding heat capacities,  $c_1$  and  $c_2$  (Eq. (3.63)) and equilibrium exponent  $\gamma_{eq}$  (Eq. (3.65)).

Fifth, one can predict the value of the exponent  $\gamma_{eq}$  when two networks of different weight exponents ( $\gamma_1$  and  $\gamma_2$ ) interact. Due to the interaction, internal links of each network are rewired to connect to the other network. The variation of energy  $E_l$ , which accounts for the overall variation of logarithmic variation of the number of connections in network  $l = \{1, 2\}$  obeys the relation,

$$\Delta E_l = c_l (\gamma_{eq} - \gamma_l) \quad , \quad (3.64)$$

where  $c_N^{(l)}$  is the heat capacity of network  $l$ . As the number of links is conserved,  $\Delta E_1 = -\Delta E_2$ . Consequently,

$$\gamma_{eq} = \frac{c_1 \gamma_1 + c_2 \gamma_2}{c_1 + c_2} \quad . \quad (3.65)$$

Figure 3.18a shows the evolution of the quantities  $\langle \log \gamma \rangle$  and  $\langle (\log \gamma)^2 \rangle$  in Eq. 3.63 for computing the heat capacity,  $c_1$  and  $c_2$  in each network, which we plot in Fig. 3.18b together with  $\gamma_{eq}$  calculated from Eq. (3.65). The result for  $\gamma_{eq}$  are consistent with the ones plotted in

Fig. 3.16. Interesting, Eq. (3.65), together with Eq. (3.63), provide a simple way for predicting the equilibrium exponent of two scale-free networks that merge together.

In summary, in this section we revisited the canonical ensemble to fully interpret its constraints when applied to scale-free networks as examples of multiplicative process gases, which are taken as equilibrium distribution due to logarithmic invariance.

We showed that, properly implementing an algorithm for a microcanonical ensemble, the merging of two scale-free networks, with different exponents, evolves towards an equilibrium power-law distribution attained where both scale-free network have the same exponent. The theoretical framework for scale-free networks here introduced provides a procedure for generating power-law networks and for predicting the equilibrium exponent value when two growing scale-free networks merge together and form a unique set. It is interesting to notice that the canonical ensemble assumes the existence of a very large scale-free network, such as the existing WWW. In this scope one may argue that small merging networks, following this thermalization with the reservoir (entire WWW) are pushed to follow the same power-law. Our findings may raise questions even beyond this scope. Indeed, it has been reported that, while the WWW has double several times its size during the last years, the exponent value of its corresponding weight distribution has remained essentially constant.<sup>21,125</sup> Similar phenomena occur in social systems.<sup>38,44</sup>

# 4

## OUT OF ECONOMIC EQUILIBRIUM

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*In this chapter we address the phenomena associated with fluctuations around economic equilibrium, as defined in the previous chapter. We also make the relation between this fluctuations and self-organized criticality by developing the Fokker-Plank equation for fluctuations in a set of multiplicative processes and study how fluctuations in organized markets seem to be in agreement with this approach.*

### Contents

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|  |           |
|--|-----------|
| <b>4.1 State-Of-the-Art</b> . . . . .          | <b>56</b> |
| 4.1.1 Stochastic Markov Processes . . . . .    | 56        |
| 4.1.2 Ito's Lemma and Option Pricing . . . . . | 63        |
| 4.1.3 Self-organized criticality . . . . .     | 65        |
| <b>4.2 Contributions</b> . . . . .             | <b>66</b> |
| 4.2.1 Brownian Motion on Scale . . . . .       | 66        |
| 4.2.2 Dynamics of Stock Markets . . . . .      | 70        |

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## 4.1 State-Of-the-Art

### 4.1.1 Stochastic Markov Processes

#### 4.1.1.1 The Fokker-Planck equation

Let  $\Omega$  be the space of events  $X$ . Using Van Kampen notation,<sup>71</sup> a random variable  $Y_X$  is a mapping between a probability space  $(\Omega, \mathcal{F}, P_X)$  and  $\mathbb{R}$ . A stochastic process is a set of random variables  $Y_X(t)$  sequenced in order by a parameter  $t$ , usually but not necessarily, time. Still, we will always refer to  $t$  as time. The interest in such a sequence of random variables is in way they relate with each other, knowing that it is a sequenced order and  $Y_X(t)$  comes after  $Y_X(t_0)$  if  $t > t_0$ .

If  $p_X(x)$  is the probability density function of  $x \in \Omega$  then the average value of  $Y(t)$ <sup>71</sup> in a specific instant  $t$  is given by

$$\langle Y(t) \rangle = \int Y_X(t) p_X(x) dx. \quad (4.1)$$

Generically, higher moments can be constructed, taking  $t_1 < t_2 < t_3 < \dots < t_n$

$$\langle Y(t_1)Y(t_2)Y(t_3)\dots Y(t_n) \rangle = \int Y_X(t_1)Y_X(t_2)Y_X(t_3)\dots Y_X(t_n)p_X(x)dx \quad (4.2)$$

for the n-th moment.

The simplest and more interesting of the measures on how the sequence of random variables is influenced by the realization in one instant  $t_1$  is the *autocorrelation function*  $k(t_1, t_2)$  that measures the correlation between  $t_1$  and a posterior instant  $t_2$

$$k(t_1, t_2) = \langle \langle Y(t_1)Y(t_2) \rangle \rangle = \langle \langle Y_X(t_1) - \langle Y_X(t_1) \rangle \rangle \langle Y_X(t_2) - \langle Y_X(t_2) \rangle \rangle \rangle \quad (4.3)$$

which, with  $t_1 = t_2$ , yields the time dependent variance  $k(t, t) = \sigma^2(t)$ .

The idea of an ordered sequence impose several probability measures depending on the ‘path’ chosen, from the instant  $t_1$  to  $t_2$ . So we must define a conditional probability  $p(y_2, t_2 | y_1, t_1)$  as the probability density function for the process to have value  $y_2$  at  $t_2$  knowing that at  $t_1$  had as value  $y_1$ . The conditional probability is also called transition probability. This probability is different from the probability of having a value  $y_2$  at  $t_2$  and  $y_1$  at  $t_1$ , which we will represent as  $p(y_2, t_2; y_1, t_1)$ .

In practice, there is only a class of stochastic processes for which it is possible to derive a general theory, the ones that knowing  $t_1 < t_2 < \dots < t_n$  then

$$p(y_n, t_n | y_1, t_1; y_2, t_2; \dots; y_{n-1}, t_{n-1}) = p(y_n, t_n | y_{n-1}, t_{n-1}). \quad (4.4)$$

Meaning that the probability density function of  $y = y_n$  in instant  $t = t_n$  knowing all the past can be reduced to the probability density function of  $y = y_n$  in instant  $t = t_n$  knowing just the previous instant. Such a class of processes is called a Markov process.

A Markov process is fully characterized by two functions, a probability density function at the initial instant,  $p(y_1, t_1)$  and the transition probability that provides the density in the subsequent instants. In a Markov process we can define for  $t_3 > t_2 > t_1$

$$p(y_3, t_3; y_1, t_1) = p(y_1, t_1) \int p(y_3, t_3|y_2, t_2)p(y_2, t_2|y_1, t_1)dy_2 \quad (4.5)$$

Dividing both sides by  $p(y_1, t_1)$ ,

$$\frac{p(y_3, t_3; y_1, t_1)}{p(y_1, t_1)} = \int p(y_3, t_3|y_2, t_2)p(y_2, t_2|y_1, t_1)dy_2 \quad (4.6)$$

The LHS of Eq.(4.6) is equal to  $p(y_3, t_3|y_1, t_1)$  by Bayes Theorem and

$$p(y_3, t_3|y_1, t_1) = \int p(y_3, t_3|y_2, t_2)p(y_2, t_2|y_1, t_1)dy_2. \quad (4.7)$$

Equation (4.7) is the Chapman-Kolmogorov equation and means that the probability for the process to have  $y_3$  at  $t_3$  given the knowledge that it had  $y_1$  at  $t_1$  is the sum of the probability of all paths that go from one value to the other at the given instants. This equation is a necessary condition for the process to be Markov.

A process is said stationary if for any  $\tau \in \mathbb{R}$

$$p(y_1, t_1 + \tau; y_2, t_2 + \tau; \dots; y_n, t_n + \tau) = p(y_1, t_1; y_2, t_2; \dots; y_n, t_n) \quad (4.8)$$

meaning that  $p$  does not depend on the initial instant, but only on  $t_n - t_1$ . In this case we denote by  $T_\tau(y_2|y_1) = p(y_2, t_1 + \tau|y_1, t_1)$ . A non-stationary Markov process having a transition probability which depends on the time difference  $\tau$  is called homogeneous.

We can express the Chapman-Kolmogorov equation by expanding the transition probability  $T_\tau(y_2|y_1)$  for small  $\tau$ ,

$$T_\tau(y_2|y_1) = (1 - a_0\tau')\delta(y_2 - y_1) + W(y_2|y_1)\tau' + \mathcal{O}(\tau'^2). \quad (4.9)$$

Knowing that  $T_\tau(y_2|y_1)$  depends on  $\tau'$  and that  $T_0(y_2|y_1) = \delta(y_2 - y_1)$  then Eq.(4.9) is a first order approximation of  $T_\tau(y_2|y_1)$  with a linear coefficient  $W(y_2|y_1)$  that is the transition probability rate  $dT/dt$  from  $y_1$  to  $y_2$ . The coefficient  $(1 - a_0\tau')$  is the probability that no transition takes place,

$$a_0(y_1) = \int W(y_2|y_1)dy_2. \quad (4.10)$$

Taking the expansion in Eq.(4.9) into Eq.(4.7) then

$$T_{\tau+\tau'}(y_3|y_1) = (1 - a_0(y_3)\tau')T_\tau(y_3|y_1) + \tau' \int W(y_3|y_2)T_\tau(y_2|y_1)dy_2 \quad (4.11)$$

Making  $\tau' \rightarrow 0$ , yields

$$\frac{\partial T_\tau(y_3|y_1)}{\partial \tau} = \int W(y_3|y_2)T_\tau(y_2|y_1) - W(y_2|y_3)T_\tau(y_2|y_1)dy_2 \quad (4.12)$$

or, more ‘useful’<sup>71</sup>

$$\frac{\partial p(y, t)}{\partial t} = \int W(y|y')p(y', t) - W(y'|y)p(y, t)dy' \quad (4.13)$$

which means that the probability density function of being in a state  $y$  grows with all transitions from all other states  $y'$  and decreases with all transitions to all other states  $y'$ . Eq.(4.13) is determined by an initial condition  $p(y, t_1) = \delta(t - t_1)$  and the solution is the transition probability of a Markov process. Is not meant as an equation for a single instant distribution.<sup>71</sup> Eq.(4.13) is the differential form of the Chapman-Kolmogorov equation also called as ‘master equation’.

Denoting by  $r = y - y'$  the size of the jump from  $y'$  to  $y$ , then  $W(y|y') = W(y, r)$  then Eq.(4.13) can be rewritten as

$$\frac{\partial p(y, t)}{\partial t} = \int W(y - r; r)p(y - r, t)dr - p(y, t) \int W(y; -r)dr. \quad (4.14)$$

The fundamental assumption in all the framework is that the jumps are small when compared with  $y$ , meaning that  $W(y'; r)$  varies slowly with  $y'$ . Additionally, a second assumption is taken that  $p(y, t)$  is also a slowly varying function in  $y$  and it is possible to deal with the first integral of the Eq.(4.14) by expanding  $p(y - r, t)$  resulting in<sup>71</sup>

$$\frac{\partial p(y, t)}{\partial t} = -\frac{\partial}{\partial y} [a_1(y)p(y, t)] + \frac{1}{2} \frac{\partial^2}{\partial y^2} [a_2(y)p(y, t)], \quad (4.15)$$

where

$$a_\eta = \int r^\eta W(y; r)dr. \quad (4.16)$$

This form of master equation that complies with the above assumption is called the Fokker-Planck equation. The  $a_\eta$  coefficients are called the ‘jump moments’. Some authors (like<sup>121</sup>) claim that in the general form of Eq.(4.15) the jump moments also depend on time, which is not exact because in that case the Fokker-Planck equation would have a completely different form due to the expansion of  $p(y - r, t)$  in both  $y$  and  $t$ . Under the condition that there is no need for the expansion of  $p(y - r, t)$  in time, then the Taylor expansion for higher approximations, will lead to

$$\frac{\partial p(y, t)}{\partial t} = \sum_\eta \frac{(-1)^\eta}{\eta!} \left( \frac{\partial}{\partial y} \right)^\eta [a_\eta(y)p(y, t)] \quad (4.17)$$

which is called the Kramers-Moyal expansion. It must be emphasized that this expansion is also valid under the same above assumptions.

Both historically and functionally, there is a particular Markov process with a special importance: the Brownian motion. The Brownian motion, named after the British botanist Robert

Brown who observed it for the first time, concerns the movement of heavy particles immersed in a fluid of light molecules.<sup>71</sup>

Brown observed the phenomenon by looking to pollen grains in the microscope but he was unable to explain it. Only seventy years later, the phenomenon was used by Louis Bachelier<sup>6</sup> in his PhD dissertation ‘The theory of Speculation’ to model the evolution of stock market prices. Such application is considered the first application of Brownian motion, despite the fact that the phenomenon was not completely explained.

The full explanation of Brownian motion came with Einstein<sup>48</sup> who made the link between the viscosity of the fluid and the random impact of the molecules on the ‘big’ particle. This is considered a ”milestone” in Statistical Physics plenty documented in the literature, so we will not go into further details.

The Brownian motion represented in the Fokker-Plank has a first jump moment  $a_1$  given by

$$a_1 = \frac{\langle \Delta X \rangle_x}{\Delta t} = 0 \quad (4.18)$$

and a second jump moment  $a_2$  given by

$$a_2 = \frac{\langle (\Delta X)^2 \rangle_x}{\Delta t} = \text{Const.} \quad (4.19)$$

where  $\langle \rangle_x$  means averaging over  $X$  and  $\Delta t$  the time taken for observing the variation  $\Delta X$ . Substituting Eq.(4.18) and Eq.(4.19) in Eq.(4.15) gives

$$\frac{\partial p(y, t)}{\partial t} = \frac{a_2}{2} \frac{\partial^2}{\partial y^2} p(y, t). \quad (4.20)$$

To solve Eq.(4.20), we consider an ensemble of Brownian particles that at  $t = 0$  are all in  $X = 0$ . Eq.(4.20) describes the position of the Brownian particle at  $t > 0$ . In statistical physics a process with  $a_1$  and  $a_2$  given by Eq.(4.18) and Eq.(4.19) respectively is called a Wiener process and the solution considering the initial conditions  $X(t = 0) = 0$  is

$$p(y, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{y^2}{4Dt}} \quad (4.21)$$

with

$$D = \frac{\langle (\Delta X)^2 \rangle_x}{2\Delta t}. \quad (4.22)$$

The solution of Eq.(4.20) is therefore a Gaussian curve propagating in time with a width that grows with  $\sqrt{t}$ .

---

#### 4.1.1.2 The Langevin Equation

The Fokker-Plank equation Eq.(4.15) is a regular partial differential equation defining a probability density function. There is another approach to stochastic processes which is not



based in probability functions, but on dynamic quantities merged into random forces called the Langevin approach.

To better understand this approach we take the example of the Brownian motion. To describe the fluctuations in the velocity of the Brownian particle subjected to friction we write

$$\dot{x} = -\gamma x + L(t). \quad (4.23)$$

where  $\gamma$  is the friction coefficient. The rhs of Eq.(4.23) is the sum of forces over the Brownian particle. The first term is the damping term. The second term incorporates the stochastic contribution, is a mathematical abstraction characterized as being irregular and unpredictable with very simple average properties over the ensemble. Indeed,  $L(t)$  can be treated as a stochastic process itself with  $\langle L(t) \rangle = 0$  and  $\langle L(t)L(t') \rangle \propto \delta(t - t')$ .  $L(t)$  is called the Langevin force and Eq.(4.23) is called the Langevin equation. Despite its simple form it is highly non-trivial due to the introduction of  $L(t)$ . Unlike Fokker-Plank approach that has solutions that are probability density functions, the solutions of the Langevin equation are random functions.

There is a relation between the Langevin equation and the Fokker-Plank equation. They are not equal, because the Fokker-Planck equation describes completely the probability distribution (with the above described approximations) and with the Langevin equation we can not go beyond the first two moments of the probability distribution.<sup>71</sup> It can be shown that Eq.(4.23) with  $L(t)$  Gaussian represent the same Markov process  $x$  as

$$\frac{\partial p}{\partial t} = \gamma \frac{\partial}{\partial x} x p + \frac{\Gamma}{2} \frac{\partial^2 p}{\partial x^2} \quad (4.24)$$

which is the master equation for what is called a Ornstein-Uhlenbeck process.<sup>71</sup>

Generically, the Langevin equation is written as

$$\dot{x} = A(x) + C(x)L(t) \quad (4.25)$$

which by making the transformations  $\bar{x} = \int dx/C(x)$ ,  $A(x)/C(x) = \bar{A}(\bar{x})$  and  $\bar{p} = p(x)C(x)$  can be shown<sup>71</sup> to result in

$$\frac{\partial \bar{p}}{\partial t} = -\frac{\partial}{\partial \bar{x}} \bar{A}(\bar{x})\bar{p} + \frac{\Gamma}{2} \frac{\partial^2 \bar{p}}{\partial \bar{x}^2} \quad (4.26)$$

which is equivalent to a Fokker-Plank equation with constant second jump moment. Equation (4.26) can be expressed back in  $x$  as

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x} A(x)p + \frac{\Gamma}{2} \frac{\partial}{\partial x} C(x) \frac{\partial}{\partial x} C(x)p. \quad (4.27)$$

Equation (4.25) does not have a well defined meaning:<sup>71</sup> since  $L(t)$  is a delta peak that occur at a random instant, when integrating Eq. (4.25),  $C(x)$  can have a value before the peak, after the peak or in the middle of the peak. Different options lead to different integrals and, therefore, different Fokker-Plank equations. If we assume the middle, we have the Stratonovich interpretation which leads to Eq.(4.26).

If we assume that  $C(x)$  takes the value before the peak we arrive to the Itô interpretation yielding

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}A(x)p + \frac{\Gamma}{2} \frac{\partial^2}{\partial x^2} (C(x))^2 p. \quad (4.28)$$

In financial mathematics, usually the Itô interpretation is used Eq.(4.25) written as

$$dy = A(x)dt + C(x)dW(t) \quad (4.29)$$

where  $W(t)$  is a Wiener process. Equation (4.29) is fundamental in Financial Mathematics and it is historically the link between Physics and Finance<sup>133</sup> that partially motivated this work.

#### 4.1.1.3 Schrödinger-like approach to Fokker-Planck equation

One of the ways to use the Fokker-Planck equation is to transform it into a Schrödinger-like equation and to solve it as an eigenvalue problem. Since we will need this later in the text, here we will describe how this is done. The Fokker-Planck equation Eq.(4.15) can be looked on in a more "mechanical" way from the one we described in section 4.1.1.1 in order to make use of mathematical tools already available for other branches of physics. We first rewrite Eq.(4.15) as<sup>121</sup>

$$\frac{\partial p(x, t)}{\partial t} = L_{FP}p(x, t) = -\frac{\partial}{\partial x}S(x, t) \quad (4.30)$$

where

$$L_{FP} = -\frac{\partial}{\partial x}a_1(x) + \frac{\partial^2}{\partial x^2}a_2(x) \quad (4.31)$$

and  $S(x, t)$  is called 'probability current' which can be taken as a flow of probability through space and time. The stationary solutions  $p_{st}(x)$  are the solutions where  $S$  is constant, i.e.,

$$S = a_1(x)p_{st}(x) - \frac{\partial}{\partial x}a_2(x)p_{st}(x) = \text{const.} \quad (4.32)$$

In the case where  $S = 0$ , yields

$$p_{st}^0(x) = \frac{N_0}{a_2(x)} e^{\int \frac{a_1(x')}{a_2(x')} dx'} = N_0 e^{-\Phi(x)} \quad (4.33)$$

where  $N_0$  is a normalization constant and

$$\Phi(x) = \log(a_2(x)) - \int \frac{a_1(x')}{a_2(x')} dx'. \quad (4.34)$$

is called 'potential'.<sup>121</sup>

We can use the potential to rewrite probability current as

$$S(x, t) = -a_2(x)e^{-\Phi(x)} \frac{\partial}{\partial x} \left[ e^{\Phi(x)} p(x, t) \right] \quad (4.35)$$

and, thus, for constant arbitrary  $S$  we have for the stationary solution<sup>121</sup>

$$p_{st}(x) = N_0 e^{-\Phi(x)} - S e^{-\Phi(x)} \int \frac{e^{\Phi(x')}}{a_2(x')} dx' \quad (4.36)$$

One of the constants,  $N_0$  is determined by the fact that the integral of  $p_{st}(x)$  over  $x$  is one. The other must come from the boundary conditions, knowing that stationary solutions can occur only for reflecting boundaries.<sup>121</sup>

Joining Eq.(4.30) and Eq.(4.35) we can represent the operator  $L_{FP}$  as

$$L_{FP} = \frac{\partial}{\partial x} a_2(x) e^{-\Phi(x)} \frac{\partial}{\partial x} e^{\Phi(x)} \quad (4.37)$$

which is not Hermitian (see Ref.<sup>31</sup> for definition) but it can be shown<sup>121</sup> that

$$L = e^{-\Phi(x)/2} L_{FP} e^{\Phi(x)/2} \quad (4.38)$$

is Hermitian and the eigenfunctions of  $L$  are  $e^{-\Phi(x)/2} \phi_n(x)$  and have the same eigenvalues  $\lambda_n$  that the operator  $L_{FP}$  has for its' eigenfunctions  $\phi_n(x)$ . Which means that working with  $L$  gives us access to a full set of mathematical tools usually reserved for quantum mechanics.

We will be interested in the case with constant diffusion ( $a_2(x) = D$ ) as

$$L = \frac{D}{2} \frac{\partial}{\partial x^2} - V_S(x) \quad (4.39)$$

where the Schrödinger potential  $V_S(x)$  is given by

$$V_S(x) = \frac{1}{4D} \left( \frac{\partial f(x)}{\partial x} \right)^2 - \frac{1}{2} \frac{\partial^2 f(x)}{\partial x^2}. \quad (4.40)$$

where  $f(x) = \Phi(x)D$

The time dependent solutions for the Fokker-Plank equations can now be written in terms of the eigenfunctions and respective eigenvalues of the operator  $L$ . We can write  $p(x, t) = e^{-\Phi(x)/2} q(x, t)$ , where  $p(x, t)$  is the complete solution for the  $L_{FP}$  operator equation and  $q(x, t)$  the complete solution for the  $L$  operator equation. Denoting  $\Psi_n$  as the orthonormal base of eigenfunctions of the Hermitian operator  $L$  then

$$q(x, t) = \sum_n a_n(t) \Psi_n(x) \quad (4.41)$$

and

$$L \Psi_n(x) = -\lambda_n \Psi_n(x) \quad (4.42)$$

The coefficients  $a_n(t)$  are obtained from trivial variable separation on the Fokker-Plank equation and

$$a_n(t) = a_n(0) e^{-\lambda_n t} \quad (4.43)$$

Eq.(4.41) can be written using Eq.(4.43), yielding

$$q(x, t) = \sum_n a_n(0) e^{-\lambda_n t} \Psi_n(x) \quad (4.44)$$

The coefficients  $a_n(0)$  are calculated using initial condition for Brownian motion<sup>71</sup>  $p(x, 0) = \delta(x - x_0) = e^{-Phi(x)/2} q(x, 0)$ . Thus,

$$a_n(0) = e^{\Phi(x_0)/2} \Psi_n(x_0) \quad (4.45)$$

and the full time-dependent solution for the Fokker-Planck equation  $p(x, t|x_0, t_0)$  is

$$p(x, t|x_0, t_0) = e^{-\frac{1}{2}(\Phi(x) - \Phi(x_0))} \sum_n e^{-\lambda_n |t - t_0|} \Psi_n(x) \Psi_n(x_0) \quad (4.46)$$

As we are going to see later, this approach to the Fokker-Planck equation allows us to determine the stationary solutions and the time dependent solutions for cases where some of the assumptions that underpin the Langevin approach must be dropped, namely the Gaussian nature of stochastic forces.

#### 4.1.2 Ito's Lemma and Option Pricing

There is a field of knowledge called Financial Mathematics and it is undeniable that there is a superposition between what we are developing in our work and what is developed in that field.

In this section, we will briefly describe how Statistical Physics got into Finance and explain why most of both communities have, Physics and Finance, believe that physics played an important role in the development of derivative valuation.

To that end, let us suppose that we have a function  $f(x, y)$ . If  $x$  and  $y$  are regular real variables and  $f$  is well behaved then

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \quad (4.47)$$

Now let us suppose that the function  $f$  depends on a real variable  $t$  but also on a stochastic process  $W$  such that  $dW \sim \sqrt{dt}$  as in Eq.(4.21)). Thus  $f = f(W, t)$  and we can make a two dimensional Taylor expansion of  $f$

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial W} dW + \frac{1}{2} \frac{\partial^2 f}{\partial t^2} (dt)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial W^2} (dW)^2 + \frac{\partial^2 f}{\partial t \partial W} (dt)(dW) + \dots \quad (4.48)$$

Dropping all parcels with order greater than 1 in order to  $dt$ , then

$$df = \left( \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial W^2} \right) dt + \frac{\partial f}{\partial W} dW \quad (4.49)$$

Eq.(4.49) is called the Itô Lemma and plays a very important role in the history of Financial Mathematics and in the way it is related to Statistical Physics.

In the beginning of the 1970's, three economists from the MIT, Fisher Black, Miron Scholes and Robert C. Merton - the first two together and the third by himself - were devoting their research to determine the fair value of a financial instrument called European Option. An option is a contract that gives the owner the right - not the obligation - to buy or sell something. It is what is called a derivative contract because it is associated with that something and the counterpart of that contract has the obligation of following the owner's option. If the owner decides to buy or sell, one says the owner exercises the option. The price which is applied to the buying or the selling is predefined in the contract and it is called the 'strike'. If a predefined date for the exercise exists the option is an European option. If it can be exercised until the predefined date it is a Bermudan option. If it is an option to buy it is called a 'Call'. If it is an option to sell it is called a 'Put'. Finally, the "whatever" thing considered to be bought or sold is called the 'underlying asset'. Black, Scholes and Merton were studying European Call Options.

What is interesting for our purposes is that their approach is based on the assumption that the price of the underlying asset follows a geometrical Brownian motion, meaning that

$$\frac{dS}{S} = \sigma dW \quad (4.50)$$

where  $S$  is the price of the underlying asset. The quantity  $dS/S$  is also called a return and it can be taken as the percentage variation of the price of stocks we hear everyday in the news. The reasoning they followed to evaluate the price of the European Call Option was quite ingenious. They assumed to have a portfolio consisting of a quantity  $h$  of the underlying asset and a short position, i.e., 'selling without having', of the call option. The value of the portfolio is then given by

$$V(t) = hS(t) - c(t) \quad (4.51)$$

where  $c(t)$  is the price for the option.

Since the value of the option depends on a real variable  $t$  and on a Wiener process,  $S(t)$ , these are exactly the same conditions as in Itô Lemma, so using Eq.(4.50),

$$dc = \frac{\partial c}{\partial t} dt + \frac{1}{2} \frac{\partial^2 c}{\partial S^2} \sigma^2 S^2 dt + \frac{\partial c}{\partial S} dS. \quad (4.52)$$

The variations on the portfolio value is given by

$$dV(t) = h dS(t) - dc(t) \quad (4.53)$$

from Eq.(4.51). Substituting  $dc$  with Eq.(4.52) comes

$$dV = h dS - \frac{\partial c}{\partial t} dt - \frac{1}{2} \frac{\partial^2 c}{\partial S^2} \sigma^2 S^2 dt - \frac{\partial c}{\partial S} dS \quad (4.54)$$

Since  $h$  is the quantity of the underlying asset, we are free to build our portfolio in any way

we want and decide that our portfolio has  $h = +\partial c/\partial S$  of underlying asset. So,

$$dV = -\frac{\partial c}{\partial t}dt - \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2 dt \quad (4.55)$$

This is the variation of the value of the portfolio. To establish the price, we must join this information with the value we *want* for the portfolio. The value we want for the portfolio is the value of a portfolio with the same initial value of a deposit on a riskless bank which, in turn, would pay us a fixed rate  $r$ . The variation of value of such portfolio is

$$dV(t) = rVdt \quad (4.56)$$

or, since  $h = +\partial c/\partial S$ , from Eq.(4.53) follow

$$dV(t) = \left(\frac{\partial c}{\partial S}S - c\right) rdt \quad (4.57)$$

Combining with Eq.(4.55) and Eq.(4.57) one arrives to

$$rS\frac{\partial c}{\partial S} + \frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2 = rc \quad (4.58)$$

Eq.(4.58) is called the Black-Scholes equation. The Black-Scholes equation was the first practical link between Statistical Physics and Finance and it is until today and, despite the deserved criticism,<sup>131</sup> the reason why financial markets become heavily populated by physicists<sup>1</sup>. Today, Black-Scholes equation is used in all financial markets to value options and even practitioners that use different methods, keep using it as benchmark of options ‘fair value’.

### 4.1.3 Self-organized criticality

Until now in this section we have been talking about out of equilibrium fluctuations as something very close to equilibrium. It is not difficult to check that most assumptions associated with the approaches we have dealt with imply some sort of fluctuation, namely the Markov property (see Figs. (3.7) and (3.8) in section 3.1.6.3).

A self-organized critical(SOC) system is a non-equilibrium system usually described as a particular case of a critical system in which there is not a control parameter, i.e., one does not need to tune the system for putting it at a critical state. The system converges always to the critical state.<sup>29</sup>

Here we want to look at this kind of systems as a particular case of an out of equilibrium system that, like the ones described in the previous sections, tend to some kind of steady-state.

SOC is typically observed in slowly-driven non-equilibrium systems with extended degrees of freedom and a high level of nonlinearity. Many individual examples have been identified

<sup>1</sup>This is not, by any way, criticism. For the history of how physicists devote to finance what the role they take in finding paths, see Ref.<sup>137</sup>

since the Bak, Tang and Wiesenfeld (BTW) model for SOC in the pioneer paper in Ref.,<sup>8</sup> describing the occurrence and the emergence of power-laws in nature. The phenomenon is better explained by a metaphor than by the definition.<sup>29</sup> The metaphor is the sand pile(see Fig.(4.1)).

Let us imagine a table top in which we drop grains of sand with a constant rate. At the beginning the grains simply settle on the table top and the pile is very approximately flat. The pile starts to grow due to the friction between grains. With more and more grains the pile shapes itself into a pyramid with a slope that starts to increase. But the pile cannot grow indefinitely. From time to time, avalanches occur. The avalanches in the border of the pile make the slope drop and the sand that keeps dropping on the top of the pile make the slope grow.

This is not an equilibrium system, quite the opposite. But it converges somehow to a steady regime with grains that fall from the pile balanced by grains that are added to the top. The sand pile *self-organizes* itself to keep towards the steady state.

The size of each avalanche can be very diversified with many small avalanches, fewer of medium size and less bigger, spanning many orders of magnitude.<sup>29</sup> Also the avalanche is not something localized but it affects the entire system.

The parallelism between the sand pile model and the definition of economy we have used in Chapter 3 should be now clearer. A ‘pile’ of labor to which every economic agent is delivering more labor permanently is a direct mapping that can be done with no effort and there is in fact an history to it which we will describe in section 4.2.2

Surprisingly, such cascade of topplings was reported to be observed in other contexts, assuming the proper interpretation, and even when not completely accepted, SOC established as a strong candidate for modeling the phenomenology of such out of equilibrium systems. For example, evolution of species seems to lay around a self-organized critical state of periods with almost no mutations and then periods with arbitrarily large sequences of mutations exhibiting a power-law distribution.<sup>7</sup> Another important example is earthquake statistics, which reflects in the known Gutenberg-Richter law<sup>59</sup> previously derived empirically, a power-law which agrees with SOC phenomenology.

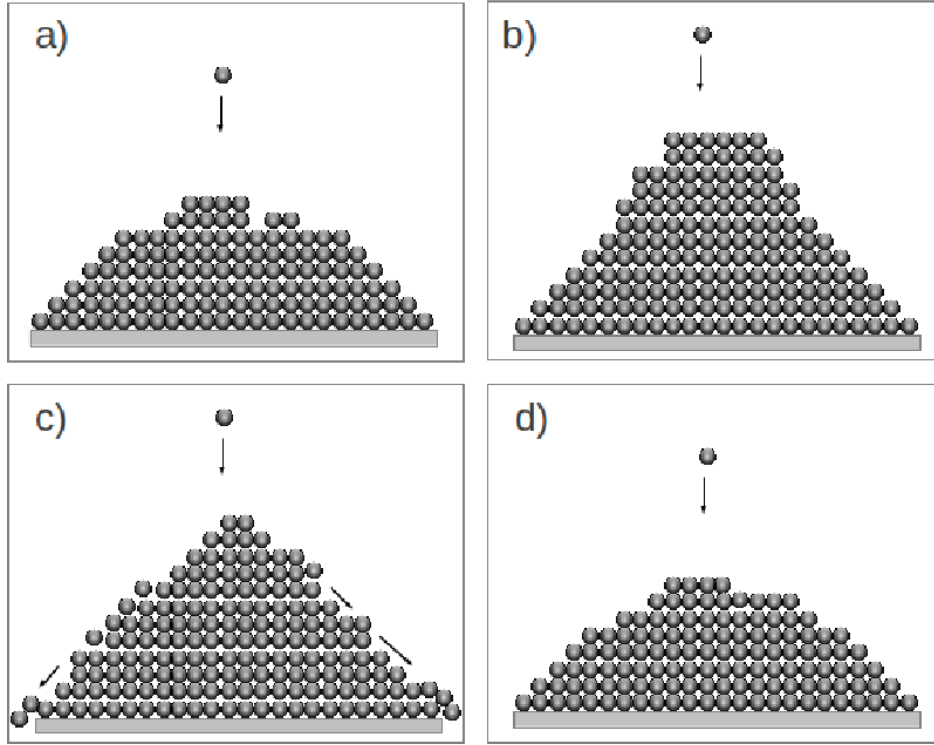
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## 4.2 Contributions

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### 4.2.1 Brownian Motion on Scale

From the Chapter 3 we have seen that a gas of  $N$  multiplicative processes - consistent with the definition of an economy - would form an apparent equilibrium by normalization of the growing quantity  $x_i$ , with  $i = 1, 2, \dots, N$ . Denoting  $x_{max}$  as the highest  $x_i$  then  $0 < x_i/x_{max} \leq 1$  or, equivalently,  $-\infty < y_i = \log(x_i/x_{max}) \leq 0$ . Also, we have seen that the overall characterization of the system, namely the exponent  $\gamma$  and the invariant quantity  $\sum y_i$



**Figure 4.1:** **a)** A sand grain is dropped at a fixed rate over a sand pile over a table. **b)** As grains keeps falling over the pile, the pile grows raising the energy of the system. **c)** At some point, the angle between the pile and the table becomes higher than the tension between the grains can support and an avalanche occurs, making grains fall over the pile and leave the system. **d)** The pile, without the grains involved in the avalanche, starts growing again with the falling grains.

do not depend on the multiplier coefficient  $\beta$  in Eq. (3.38). Thus, there is no mechanism to produce fluctuations in such a system. Nevertheless, fluctuations are present in the economy. The mechanism that can provide such fluctuation is SOC, as we will see in this section.

We can easily see that  $y_i$  can be expressed not in terms of  $x_{max}$  but in terms of the minimum of  $x_i$ ,  $x_0$ , without losing its meaning. Let us assume that a random multiplicative process  $i$  is activated at each instant to multiply according to Eq. (3.38), i.e.,  $dx_i = \beta x_i$ . When  $x_0$  grows, all other values  $y_i$  drop. Since the system is expanding, i.e.,  $d(x_{max} - x_0) > 0$ , for any generic process  $i$  there is an attractive field that pushes  $x_i/x_0 \rightarrow 0$ . This is equivalent to say that, in first approximation, we can assume that exists a field in the direction of  $-x_i/x_0$ . For the development of the reasoning, let us also assume that there is a mechanism that promotes fluctuation.

Since we aim to build the Fokker-Planck equation that reflects the fluctuations on such a system, we can make use of the Brownian motion model under the effect of an attractive field.<sup>71</sup> Thus, we represent the first jump moment as

$$a_1(y) = \frac{\langle \Delta y \rangle_y}{\Delta t} = \nu \quad (4.59)$$



and a constant second jump moment,

$$a_2(y) = D = \text{const.} \quad (4.60)$$

The Fokker-Plank equation is then

$$\frac{\partial}{\partial t} p(y, t) = \nu \frac{\partial}{\partial y} [p(y, t)] + \frac{D}{2} \frac{\partial^2}{\partial y^2} [p(y, t)]. \quad (4.61)$$

Making the transformation for a Schrödinger-like equation, using Eq.(4.40) yields

$$V_S(y) = \frac{\nu^2}{2D} \quad (4.62)$$

and, from Eq.(4.34)

$$\Phi(y) = \frac{\nu}{D} y + \log\left(\frac{D}{2}\right) \quad (4.63)$$

The second term of Eq.(4.63) is a constant and it can be dropped since  $\Phi(y)$  is only defined up to one additive constant, leading to a Fokker-Planck equation written as

$$L\Psi_n = -\lambda_n \Psi_n \quad (4.64)$$

From Eq.(4.39)

$$L = \frac{D}{2} \frac{\partial^2}{\partial y^2} - \left[ \frac{2\lambda_n}{D} - \frac{\nu^2}{D^2} \right] = \frac{D}{2} \frac{\partial^2}{\partial y^2} - k_n^2 \quad (4.65)$$

where  $\lambda_n$  are the eigenvalues of  $L$ ,  $\Psi_n$  are the respective eigenfunctions. From Eq.(4.65) we conclude that  $\Psi_n$  functions have the form

$$\Psi_n(y) = A \cos(k_n y) + B \sin(k_n y). \quad (4.66)$$

Until now we have assumed that fluctuations exist despite the fact that we can not describe the mechanism responsible for it. Now we make use of the boundary conditions to show that to have fluctuations we need an additional boundaries.

One of the boundaries can be assumed directly from the definition of  $y$ . Since  $y$  is the logarithm of a normalized quantity there exists a reflecting boundary at  $y = 0$ . Another boundary comes from the fact there must be a negative finite lower boundary that prevents the system to grow indefinitely or, since we are talking about normalized quantities, prevents that the difference between the maximum and the minimum is infinite. This lower boundary (threshold) gives us the second reflecting boundary to obtain both a stationary solution and time dependent solutions for the Fokker-Plank equation. It can be shown<sup>121</sup> that the operator  $L$  in Eq.(4.64) has only eigenvalues bigger or equal than 0, being the lower one equal to zero if a stationary solution exists. The stationary solution is, according to Eq.(4.33)

$$p_s(y) = N_0 e^{-\Phi(y)} \quad (4.67)$$

which, expressed as  $x$  is given by

$$p_s(x) = N_0 e^{-\frac{\nu}{D} \log(\frac{x}{x_0}) - \log(\frac{D}{2})} \left( \frac{dy}{dx} \right) = N_0' \frac{x_0^{\frac{\nu}{D}}}{x^{\frac{\nu}{D}+1}}. \quad (4.68)$$

Equation (4.68) is equal to Eq. (3.48) with  $\alpha = \nu/D$ . The stationary solution for the Fokker-Planck equation for fluctuations near logarithmic invariance is the power-distribution that is the result of the set of correlated multiplicative processes. And the correlation of the processes is expressed in terms of the drift coefficient  $\nu$  and the diffusion coefficient  $D$ .

To study the time-dependent solutions we use Equation (4.46) for a transition from a value  $y_0$  at instant  $t_0$  to  $y$  at instant  $t$  which is given by<sup>24,121</sup>

$$p(y, t|y_0, t_0) = e^{-\frac{1}{2}(\Phi(y)-\Phi(y_0))} \sum_n e^{-\lambda_n |t-t_0|} \Psi_n(y) \Psi_n(y_0) \quad (4.69)$$

Since  $k_n = \sqrt{2\lambda_n/D - \nu^2/D^2}$ , substituting Eq.(4.66) into Eq.(4.69) and assuming that the  $\lambda_n$  values are close enough to assume a continuity

$$p(y, t|y_0, t_0) \propto e^{-\frac{1}{2}(\Phi(y)-\Phi(y_0))} \frac{1}{\pi} \int e^{-\left(\frac{Dk^2}{2} + \frac{\nu^2}{2D}\right)|t-t_0|} (\cos(ky) \cos(ky_0) + \sin(ky) \sin(ky_0)) dk. \quad (4.70)$$

Cross products of sin and cos were not written since they are zero after integration.

Thus

$$\begin{aligned} p(y, t|y_0, t_0) &\propto e^{-\frac{1}{2}(\Phi(y)-\Phi(y_0))} \\ &e^{-\frac{\nu^2}{2D}|t-t_0|} \cdot \\ &\frac{1}{\pi} \int e^{-\left(\frac{Dk^2}{2}\right)|t-t_0|} (\cos(ky) \cos(ky_0) + \sin(ky) \sin(ky_0)) dk \\ &\propto e^{-\frac{3}{2}(\Phi(y)-\Phi(y_0))} e^{-\frac{\nu^2}{2D|t-t_0|}|t-t_0|^2} \frac{e^{-\frac{(y-y_0)^2}{2D|t-t_0|}}}{\sqrt{2D|t-t_0|}} \end{aligned} \quad (4.71)$$

yielding

$$p(y, t|y_0, t_0) \propto e^{-\frac{3}{2}\frac{\nu}{D}(y-y_0)} P_{B_t(\nu, D/2)} \quad (4.72)$$

where  $P_{B_t(\nu, D)}$  is the probability density function associated with transitions of a Brownian motion with drift  $\nu$  and diffusion  $D$ .

Since we define  $y = \log(x)$  we have finally the expression for continuous  $x$

$$p(x, t|x_0, t_0) \propto \left(\frac{x_0}{x}\right)^{-\left(\frac{3\nu}{2D}\right)} \frac{1}{x} P_{B_t(\nu, D/2)} \quad (4.73)$$

and for discrete  $x$

$$p(x, t|x_0, t_0) \propto \left(\frac{x_0}{x}\right)^{-\left(\frac{3\nu}{2D}\right)} P_{B_t(\nu, D/2)} \quad (4.74)$$

We should emphasize that  $p(x, t|x_0, t_0)$  is the probability density function for a change from  $x_0$  to  $x$ , a transition probability.

If we pick the discrete case, then we can write Eq.(4.72)

$$p(r, t) \propto p_s(x)^{\frac{3}{2}} P_{B_t(\nu, D/2)} \quad (4.75)$$

which establishes that the time dependent solution for the Fokker-Planck equation in multiplicative objects system. The exponent of the power law associated with the fluctuation probability is 1.5 times bigger than the stationary one and the amplitude of the fluctuation drops in time with the same rate as in Brownian motion fluctuation.

The expression is consistent with the findings in the sand pile model<sup>29</sup> by fitting the field  $\nu$  and the diffusion  $D$  coefficients to the particular conditions of the problem. Nonetheless, Eq.(4.75) provides an interesting opening for future work since it combines the power-law associated with complex systems and the Brownian motion model that underpins Financial Mathematics. In the next section we will approach the stock market price fluctuations problem using SOC by explicitly by imposing a threshold influencing the economic link growing and the results will be consistent to the ones we present in this section.

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#### 4.2.2 Dynamics of Stock Markets

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##### 4.2.2.1 Observing SOC in financial data

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Organized markets are the main target of researchers due to the short term variations that allows, in principle, to look at a time series independently from the underlying mechanisms and, thus, to apply directly the tools they already know. At least this is the belief of a good share of the physicists and applied mathematicians that study the subject.

The application of statistical physics to finance and economy was boosted in the last decades, particularly with the analysis of financial data in 1973 by Black, Scholes and Merton.<sup>17,97</sup> More recently,<sup>54,55</sup> such application found important developments with the introduction of procedures for quantitative description of financial data. These procedures derive a Fokker-Planck equation for the empirical probability distributions, catching the typical non-Gaussian heavy tails of financial time-series, across time-scales. However, as Mandelbrot pointed out,<sup>91</sup> other typical features are observed in the variation of prices, namely scale invariance behavior,<sup>93</sup> which cannot be explained by means of a cascade model. The modeling of financial index dynamics and other complex phenomena taking into account non-Gaussianity and scale invariant behavior has been indicated as an appealing problem to address in the scope of finance analysis and statistical physics,<sup>74,93</sup> particularly in what concerns the emergence of Self-organized Criticality (SOC)<sup>8</sup> in financial markets.

The fluctuations in economic systems, described for instances by financial markets indices or prices were also reported by Mandelbrot,<sup>91</sup> Mantegna and Stanley<sup>93</sup> and other to show scale invariance behavior. But to the best of our knowledge there is still no clear connection between the power-laws seen in financial markets and SOC. Sornette<sup>130</sup> suggests an association between the observable heavy-tails in the return distributions and a SOC behavior but in

fact that is the only point of contact. There are several possible causes for the appearance of heavy-tails in distributions, SOC is one of them. In this section we will make the connection between SOC and the heavy-tails by the value of the exponents.

Using an agent model to characterize two groups of traders, Lux and Marchesi presented some evidence that scaling in finance emerges as a result of the interactions between individual market agents.<sup>87</sup> Agent-based models of financial markets have been intensively studied and agent-based models are a well known and well established subject in the IT industry. It has been claimed that agent models are able to complement the approach borrowed from social sciences, where from one theory a specific model is derived and applied to empirical data.<sup>102</sup> An agent model is typically constructed by starting to identify the statistical features in the data of interest and then to implement the necessary ingredients in the model for generating data with the same features.

In this section we aim to show that two fundamental assumptions in Economics are sufficient for the emergence of a self-organized critical state in the social environments where humans trade, an environment that we call henceforth the financial system and that we map into a set of interconnected agents. Further, we will show that under such assumptions, the model is also able to reproduce the scaling features observed in empirical data.

The first assumption is that humans are attracted to each other to exchange labor due to biological specialization that made the species more efficient when each individual could perform the tasks to which is more capable.<sup>85</sup> The second assumption is that the amount of labor exchanged by each of the two involved agents in an economic relation is ruled by the law of supply and demand.<sup>85</sup>

Based on these two principles and translating them to the physical context, we will show that differently from the Brownian particle approach the assumption for the system to be in equilibrium cannot be sustained. As stated in Chapter 3, such configuration corresponds to a multiplicative process gas that presents a an apparent equilibrium.

Furthermore, since labor must be produced in order that agents can establish labor exchanges, on each agent there is an “energy” dissipation that must be finite for physical reasons. This additional principle introduces a threshold and a system where agents are impelled to create more economic exchanges introducing labor corresponds to a self-organized critical system.<sup>8</sup> Though the term energy used in the economical context is not the same as physical energy, since it does not necessarily satisfy the thermodynamic constraints, we will use the term in the economical context only. For example, human labor is assumed as “energy” delivered by the agent to a neighbor, which rewards the agent with an energy that the agent accumulates. The balance between the labor produced for the neighbors and the reward received from them may be positive (agent profits) or negative (agent accumulates debt). For simplicity, we omit henceforth all quotation marks.

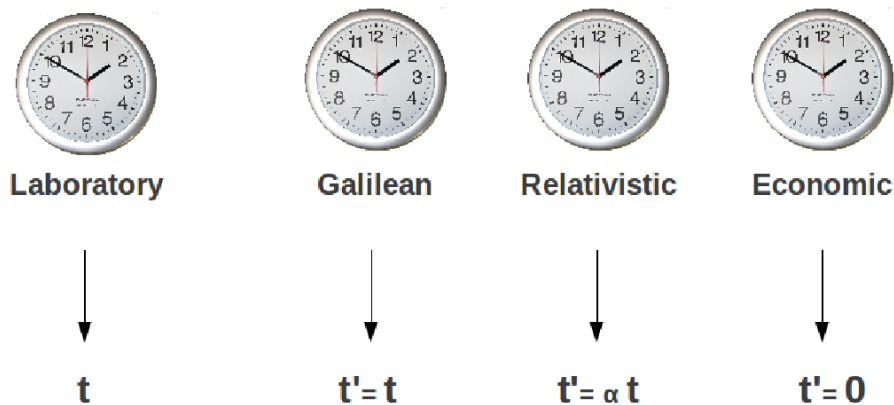
Since the main concern in risk management deals with the distribution of the so-called drops in financial market indices, we restrict ourselves to that side of the distribution. In a financial market index, one drop is defined as the decrease of the index from one time-step to the next one. Each drop on financial markets results from the economical crash of one or more agents, being represented by the collapse of those agents. Such local crashes appear at no particular

moment and may cascade into an avalanche of arbitrary size, i.e. a succession of an arbitrary large number of local economical crashes, taken as a global economical crisis.

#### 4.2.2.2 Time in Economic Systems

The idea of having two clocks instead of one in an experiment is not unusual in physics. Since the discovery of relativistic time dilatation between two observers traveling in a relative uniform motion we need to have a clock attached to a referential at rest (time  $t$ ) and a clock attached to a referential of the system under study (time  $t'$ ). Depending on the conditions of the experiment, we can make an approximation to Galilean relativity and take the clocks as being the same ( $t = t'$ ), but in the general case  $t \neq t'$ . For instance, in a relativistic system  $t = \alpha t'$  (see Fig. 4.2 for illustration). Furthermore, all these considerations are also applied to space, meaning that in general, in all physics problems, we have two rulers, the one of the referential at rest and the one of the system under study.

To get a better understanding of the differences in the context of economic systems let us call the ‘external clock’ to the clock attached to the referential at rest and ‘internal clock’ to the clock attached to the system under study.



**Figure 4.2:** Illustration of time dilatation/ compression.  $t$  represents how measures from the system are recorded in ‘laboratory’ referential.  $t'$  represents how time is seen in the system referential compared with the laboratory time interval.

Unlike any physical particle, economic particles are human beings that have memory and are able to forecast. They are human beings able to produce and consume, so the idea of having particles with memory and prediction is quite intuitive. But when we talk about particles in physics they do not have memory and prediction. If we isolate any physical particle we can be sure that it follows a Markovian process. It can be characterized in each instant by its position and momentum and these variables in the next instant will not be dependent on past or future values, but only on present ones. In other words, any physical system can be taken as Markovian because we can always take all motion equations from all particles to build a deterministic process.

To model economic particles as physical particles we make use of the two clocks. From the perspective of the external clock time goes as in physics. It is relatively to this clock that we build time series of economic metrics. But unlike relativistic particles where the internal clock dilates time relatively to the external clock, in economic particles the internal clock shrinks time intervals to an infinitesimal quantity, at each instant of the external clock. Why? Because at each instant of physical time, economic particles take all stock of resources taken from the past and produce for present and future consumption (see the definition of Economics in Chapter 2). In each instant, all physical time axis is smashed into an infinitesimal time quantity in the internal clock.

Why is this important in modeling? Because economic particles exchange labor asynchronously (e.g. a loan is given assuming future interest payments and stock market shares are bought assuming future dividends). This exchange of labor is based on the amount of information each economic particle holds. But regardless of the time shrinking in the internal clock, the external clock keeps ruling the measures of the system, as we are used to see in every price time series. These separation of clocks is fundamental to understand market price fluctuations as an economic phenomenon and the reason why physicists always try to explain stock market prices, keeping themselves away from other financial subjects. Stock markets prices are exactly the result from all past information and future perception, they are at each instant the result of time shrinking. Agents sell stocks because they take to the present the prediction of losses in the futures. To model any other economic event we need time compression because without it there is no exchange and without exchange there is no economy.

These past paragraphs can be somewhat abstract for readers with Economics background and counterintuitive for readers with a Physics background but it is a key point of modeling economic systems.

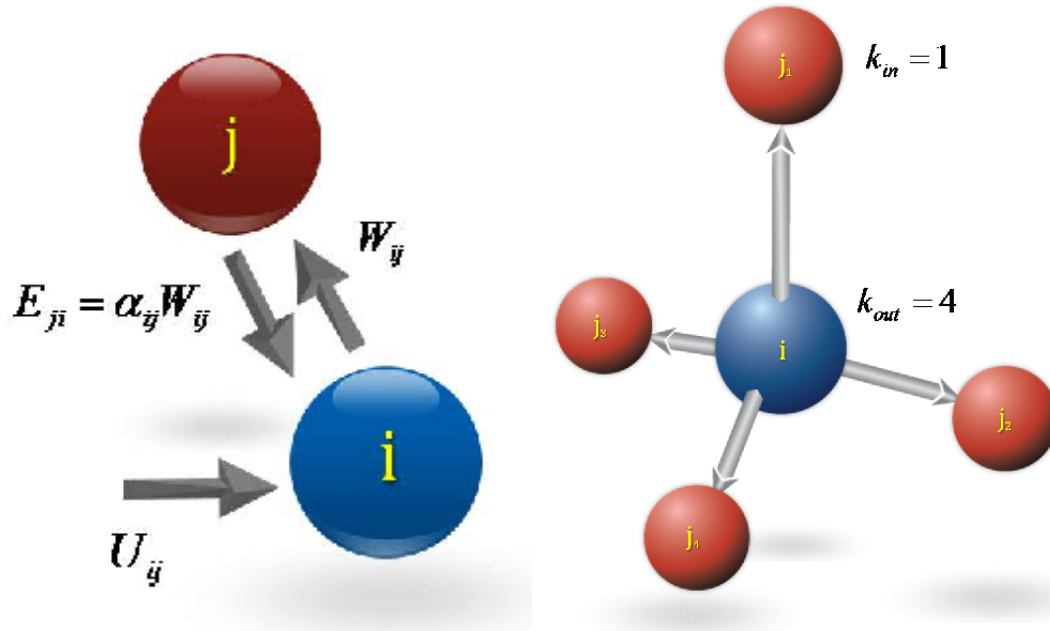
#### 4.2.2.3 Minimal Model for Economic Relations

In this section we introduce an agent-model that considers a collection of  $N$  agents operating as energy transducers into the economic space in the form of labor. Agents, labeled  $i$  and  $j$  in Fig. 4.4 (left), establish among them bi-directed connections characterized as follows. When agent  $i$  delivers labor  $W_{ij}$  to agent  $j$  it receives in return a proportional amount of agent  $j$  labor,  $E_{ij} = \alpha_{ij}W_{ij}$ . Both quantities  $E_{ij}$  and  $W_{ij}$  can be regarded as forms of energy, in the economic space. Henceforth we consider all interchange of energy in units of  $W_{ij} = 1$ .

The factor  $\alpha_{ij}$  is a (dimensionless) measure of the labor price, defined as

$$\alpha_{ij} = \frac{2}{1 + e^{-(k_{out,i} - k_{in,j})}} \quad (4.76)$$

where  $k_{out,i}$  and  $k_{in,j}$  are the number of outgoing connections of agent  $i$  and the number incoming connections of agent  $j$ , respectively (see Fig. 4.4). A large (small)  $k_{in,j}$  indicates a large (small) supply for agent  $i$  and a large (small)  $k_{out,i}$  indicates a large (small) demand of agent  $i$ .



**Figure 4.3: Left:** Illustration of an economical connection between two agents  $i$  and  $j$ . Agent  $i$  transfer labor  $W_{ij}$  to agent  $j$  receiving an energy  $E_{ij} = \alpha_{ij} W_{ij}$  where  $\alpha_{ij}$  measures how well the labor is rewarded: agent  $i$  has an outgoing connection (production) and agent  $j$  an incoming connection (consumption). This interaction attributes to agent  $i$  an amount of “internal energy”  $U_{ij} = W_{ij} - E_{ij}$  that can be summed up over all agents connections up to a threshold  $U_{th}$  beyond which it is distributed among the neighbors (see text). **Right:** The transfer of labor is done according to a preferential attachment scheme: the agent prefers to work for agents which have already a significative number of labor connections to them.

When  $\alpha_{ij} = 1$ , agent  $j$  returns to agent  $i$  the same amount of labor it receives from agent  $i$ . This happens when  $k_{in,j} = k_{out,i}$ , yielding  $W_{ij} = E_{ij}$ , i.e. when there is local balance between supply and demand. This value  $\alpha_{ij} = 1$  is the middle value between the asymptotic limits  $k_{in,j} \gg k_{out,i}$  ( $\alpha_{ij} \sim 0$ ) and  $k_{in,j} \ll k_{out,i}$  ( $\alpha_{ij} \sim 2$ ) which satisfy basic economic principles.<sup>85</sup> Namely, in the limit  $k_{in,j} \gg k_{out,i}$ , the labor of agent  $j$  is in much greater demand than the labor of  $i$ , and thus agent  $j$  is in a position to pay  $i$  very little in return. In other words, for  $\alpha_{ij} < 1$  the labor of agent  $i$  is paid by  $j$  below the amount of energy  $W_{ij}$  it delivers, i.e. agent  $i$  loses from the connection (trade) and loses a certain amount of energy,  $U_{ij} = W_{ij} - E_{ij} > 0$ . For  $\alpha_{ij} > 1$  the opposite occurs, i.e. when  $k_{in,j} < k_{out,i}$ , agent  $i$  has more supply of labor than agent  $j$  has demands for its labor. In the limit  $k_{in,j} \ll k_{out,i}$ , the value of  $\alpha_{ij}$  could in principle be any finite value larger than one. To guarantee an equal range length for both the situation when agent  $i$  profits from agent  $j$  and the situation when agent  $i$  loses from agent  $j$ , we consider the range  $\alpha_{ij} \in [0, 2]$ , which implies the limit  $\alpha_{ij} = 2$  for  $k_{in,j} \ll k_{out,i}$  as can be seen from Eq. (4.76).

The energy balance for each trade,  $W_{ij} - E_{ij}$ , translates into an energy balance for each node that takes into account all outgoing and incoming connections the agent has:  $U_i = \sum_{j \in \mathcal{V}_{out,i}} (W_{ij} - E_{ij}) + \sum_{j \in \mathcal{V}_{in,i}} (E_{ji} - W_{ji})$  where  $\mathcal{V}_{out,i}$  and  $\mathcal{V}_{in,i}$  are the outgoing and incoming vicinity of agent  $i$ , with  $k_{out,i}$  and  $k_{in,i}$  neighbors respectively.

We also assume that each agent  $i$  chooses its neighbors according to a preferential attachment

scheme, following the Barabási-Albert-Jeong method.<sup>10</sup>

This scheme is as follows: one starts with a small amount of agents totally interconnected, and adds iteratively one agent with one connection to one of the previous agents, chosen from a probability function proportional to their number of connections. Thus, agents having a large amount of connections are more likely to be chosen for a new connection than other agents.

Here, this topology underlies the empirical observation in economic-like systems that agents are more likely to deliver their work to agents receiving already significantly amount of work. See Ref.<sup>10</sup> for several examples such as the Internet and the airport network among other. See Fig. 4.3.

Since we drop the assumption of an equilibrium system, these particles we call agents can experiment the boundaries of the system and leave it if they go too far from the expected general equilibrium conditions.<sup>103</sup> Thus, as agents build new economic connections the factors  $\alpha_{ij}$  and the energy of agents change. The number of incoming connections (consumption) can overcome the outgoing ones (production) up to a certain threshold  $U_{th,i}$  that is related to how much the system allows the agent to accumulate debt.

Following standard economical reasoning,<sup>98</sup> the amount of debt an agent may accumulate is directly related with the volume of its overall business, i.e., with the broadness of its influence in the system.

Representing this influence by the turnover  $T_i = k_{out,i} + k_{in,i}$ , we fix a threshold  $d_{th} = U_{th,i}/T_i$  against which we compare the measure  $d_i = U_i/T_i$ .

Under these assumptions we consider that when  $d_i < d_{th}$  the agent collapses and an avalanche takes place. This collapse induces the removal of all the  $k_{in,i}$  consumption connections of agent  $i$  from the system, implying

$$U_i \rightarrow U_i + \sum_{j \in \mathcal{V}_{in,i}} (1 - \alpha_{ji}) \quad (4.77a)$$

$$T_i \rightarrow T_i - k_{in,i} \quad (4.77b)$$

$$U_j \rightarrow U_j - (1 - \alpha_{ji}) \quad (4.77c)$$

$$T_j \rightarrow T_j - 1, \quad (4.77d)$$

where  $j$  labels each neighbor of agent  $i$ . The collapse of agent  $i$  generates a new energy balance on agent  $j$ . If  $j$  does not collapse, the avalanche stops and is saved as an avalanche of size  $s = 1$ . If  $j$  collapses the avalanche continues to spread to the next neighbors.

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#### 4.2.2.4 Emergence of SOC in Financial Networks and Empirical Financial Indices

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Having presented the model, we show next that under the above assumptions the system remains at a critical state. To this end, we make use of a mean-field approach for such a



system. The factors  $\alpha_{ij}$  are substituted by the average value  $\alpha = \langle \alpha_{ij} \rangle$  yielding for each agent

$$U_i = (1 - \alpha)(k_{out,i} - k_{in,i}). \quad (4.78)$$

Due to the preferential attachment scheme,<sup>10</sup> in the initial state of the system, the outgoing connections follow a  $\delta$ -distribution  $P_{out}(k) = \delta(k - k_{out})$  and the incoming connections follow a scale-free distribution  $P_{in}(k) = k_{in}^{-\gamma_{in}}$ , where  $\gamma_{in}$  is the exponent of the degree distribution. As the system evolves, the number of agents remains constant but at each event-time  $n$  one new connection joining two agents is introduced, with both agents independently chosen according to the preferential attachment scheme mentioned above. Thus, through evolution both consumption and production networks are pushed to a degree distribution of the form  $P(k) \sim K_0 k^{-\gamma}$ , where  $K_0$  is the initial number of outgoing connections a node has.

As equated in Eqs.(4.77), a collapsing agent  $i$  has all its consumption connections removed from its neighbors. Thus, the collapse of a neighbor occurs if  $k_{out,j} - k_{in,j} > d_{th}(k_{out,j} + k_{in,j})$  and  $k_{out,j} - 1 - k_{in,j} \leq d_{th}(k_{out,j} - 1 + k_{in,j})$ , yielding

$$\omega k_{in,j} < k_{out,j} \leq \omega k_{in,j} + 1 \quad (4.79)$$

with  $\omega = \frac{1+d_{th}}{1-d_{th}}$ . Taking a collapsing node, the probability  $P_{br}$  for a neighbor to also collapse is the probability for the above condition to be fulfilled. Since all connections are formed by preferential attachment,

$$P_{br}(k_{in,j}) = P(\omega k_{in,j} < k_{out,j} \leq \omega k_{in,j} + 1) \approx K_0(\omega k_{in,j})^{-\gamma}. \quad (4.80)$$

To know if one collapsing agent triggers an avalanche one needs to estimate the expected number of neighbors that the agent brings to collapse, due to its own collapse. If this expected number would be smaller than one, the systems would need to consume an infinite amount of energy from the environment. If the expected number would be larger than one, the entire system would typically be extinct by one large avalanche. The avalanche of collapsing agents form a branching process and assuming that the system cannot consume an infinite amount of energy from the environment and that the system survives the avalanches, the expected value of collapsing agents from a starting one must be equal to one<sup>61</sup> and therefore

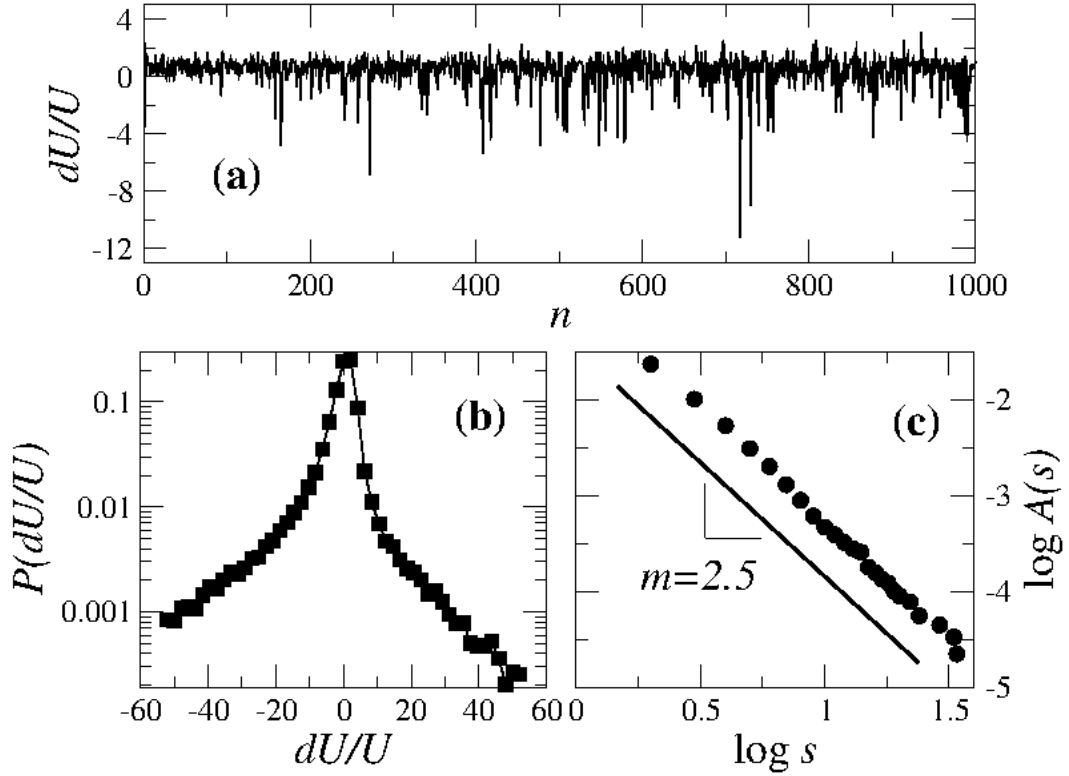
$$\sum_{k_{in,i}=1}^{\infty} k_{in,i} P(k_{in,i}) P_{br} = 1 \quad (4.81)$$

which yields  $\sum_{k_{in,i}=1}^{\infty} k_{in,i}^{-\gamma} = \left(\frac{\omega}{K_0}\right)^{\gamma}$ , i.e. the system remains in the critical state<sup>61</sup> as

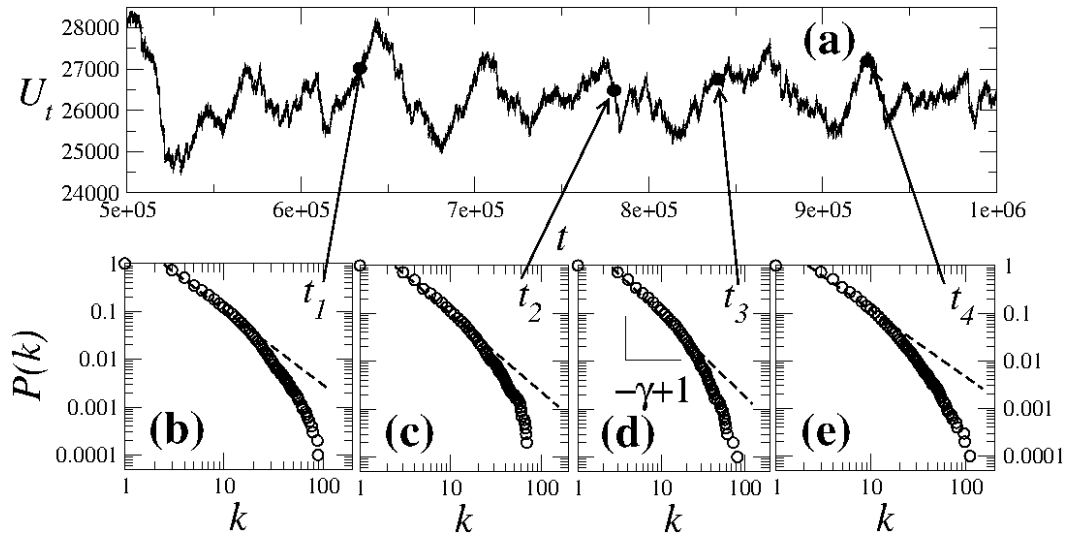
$$\omega^{\gamma} = K_0^{\gamma} \zeta(2\gamma - 1) \quad (4.82)$$

where  $\zeta$  is the Riemann zeta-function. Condition (4.82) closes our model, relating economic growth ( $K_0$ ), topology ( $\gamma$ ) and the allowed level of debt( $\omega$ ).

From Otter's theorem<sup>111</sup> for branching processes the distribution for the avalanche size expressed as number of agents  $r$  is given by  $P(r) \propto r^{-\frac{3}{2}}$ . Since the energy of our system is expressed as connection number, the number of collapsed agents in an avalanche is given by

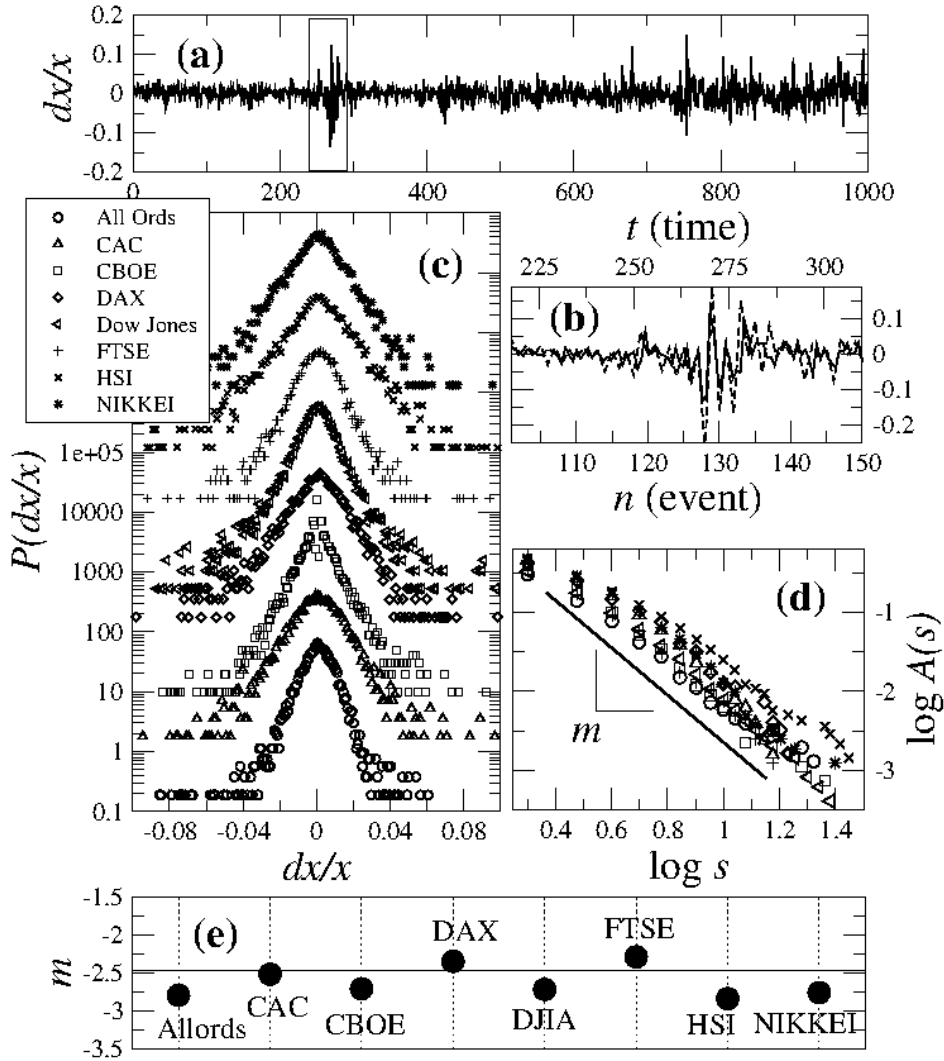


**Figure 4.4:** (a) Evolution of the variation of the total internal energy, which shows (b) probability density function and (c) avalanche size distribution similar to the ones observed for empirical data (compare with Fig. 4.3). Here  $N = 1000$ ,  $K_0 = 1$  and  $W_{ij} = 1$  for all  $i$  and  $j$ .



**Figure 4.5:** (a) Time series of the total internal energy  $U$  of the agents. For the four instants  $t_1$ ,  $t_2$ ,  $t_3$  and  $t_4$  we show the observed cumulative degree distribution  $P(k^* \geq k) \propto k^{-\gamma+1}$  yielding exponents (b)  $\gamma = 2.6 \pm 0.5$ , (c)  $\gamma = 2.7 \pm 0.5$ , (d)  $\gamma = 2.6 \pm 0.5$ , (e)  $\gamma = 2.6 \pm 0.5$ , all of them according to the theoretical prediction (check Eq. (4.83)). The deviations from the power-law for large  $k$  are due to the avalanches (crisis) in the system (see text).

$r = NK_0(\omega k_{in})^{-\gamma}$ , where  $N$  is the total number of agents in the system. If all agents in the



**Figure 4.6:** (a) Time evolution of the logarithmic returns of the DJIA index (partial). The box is zoomed in (b) to emphasize the contrast between real data (solid line) throughout time  $t$  and the succession of events (dashed line) where the variation changes sign. (c) Probability density functions for some important financial indices, including interest rate options (CBOE). (d) Distribution of avalanche sizes detected throughout the evolution of each financial index showing critical behavior with (e) an exponent  $m$  approximately invariant and similar to the one obtained for branching processes and to our model.

avalanche has the same degree, then, since degree and the number of agents are both discrete variables,  $P(k_{in}) = P(r(k_{in}))$  where  $r(k_{in})$  is the number of agents expressed as a function of  $k_{in}$ . Meaning that  $P(k_{in}) \propto k_{in}^{-\frac{3}{2}\gamma}$ . Since agents have different degrees, we can divide the avalanche into disjoint partitions  $r_j$  where each node have exactly a degree  $j$ . Since the partitions are disjoint,  $P(k_{in}) = \sum P(r_j(k_{in}))$  yielding  $P(k_{in}) = P(\sum_j r_j(k_{in})) = P(r(k_{in}))$

Therefore, the degree distribution is given by  $P(k) \propto k^{-\frac{3}{2}\gamma}$ . and the avalanche size distribution reads

$$P(k \geq s) \propto \int_s^{+\infty} k^{-\frac{3}{2}\gamma} dk \propto s^{-\frac{3}{2}\gamma+1} \equiv s^{-m}. \quad (4.83)$$

Equation (4.83) relates the exponent characterizing the network topology with the exponent

taken from the avalanches, using first principles in economy theory, translating the topology of the economic network into the heavy-tails of the return distributions.

Next we show that for the typical values of  $\gamma$  found in empirical networks, our model reproduces the values of  $m$  predicted in Eq. (4.83). For that, we define a macroscopic quantity for the internal energy, which accounts for all outgoing connections in the system at each time step:

$$U = \sum_{i=1}^N \sum_{j \in \mathcal{V}_{out,i}} (W_{ij} - E_{ij}). \quad (4.84)$$

The quantity  $U$  varies through time, and its evolution reflects the development or fail of the underlying economy, similar to a finance index. Alternatively,  $U$  can be calculated from the incoming connections. This macroscopic quantity will be used to characterize the state of our economic-like system.

Figure 4.4a shows a sketch of the evolution of a typical time-series for the logarithmic returns  $dU/U$ . As can be seen from Fig. 4.4b the distribution of the logarithmic returns is non-Gaussian with the heavy tails observed in empirical data.<sup>74</sup> In Fig. 4.4c the cumulative distribution  $A(s)$  of the avalanche size  $s$  is plotted showing a power-law whose fit yields  $A(s) \sim s^{-m}$  with an exponent  $m = 2.51$  ( $R^2 = 0.99$ ). Looking again to Eq. (4.83) and borrowing from the literature<sup>10</sup> the values of  $\gamma$  of empirical networks which lay typically in the interval  $[2.1, 2.7]$ , one concludes that the exponent should take typical values  $m \in [2.15, 3.05]$  which agrees with the results from our model.

Figure 4.5 shows how the overall index dynamic emerges from the underlying network mechanics. As agents connect each other by preferential attachment, the topology of the system is pushed to a power law degree distribution. On the other hand, avalanches push the system away from it. Thus, the system undergoes a structural fluctuation that generates a fat tail distribution of the index, expressed by Eq. (4.83) and shown in Fig. 4.6, rather than a Gaussian one. Figure 4.5a shows a typical set of successive  $U_t$  values taken from our model. The cumulative degree distribution  $P(k)$  at the marked instants  $t_1$ - $t_4$  are shown in Fig. 4.5b-4.5e. The dashed lines guide the eye for the scaling behavior observed at the lower part of the degree spectrum. In all cases  $\gamma \sim 2.6$ . Varying the threshold one observes other values for exponent  $\gamma$  (not shown). For large degree  $k$  the distribution deviates from the power-law, due to the drops of connections for agents experiencing an economical crash. Nonetheless, the slope  $-\gamma + 1$  of the dashed line yields values in the predicted range, within numerical errors.

Next we address the observation that the results obtained from our model in Fig. 4.4 do agree with the analysis done on eight main financial indices, as shown in Fig. 4.6.

The time-series of the logarithmic returns (Fig. 4.6a) must first be mapped in a series of events.

One event is defined as a (typically small) set of successive instants in the original time-series having the same derivative sign, either positive (monotonically increasing values) or negative (monotonically decreasing values). Each time the derivative changes sign a new event starts. Figure 4.6b shows an zoom of the original time series in Fig. 4.6a (solid line) with

the corresponding series of events. In the continuous limit, events would correspond to the instants in the time-series with vanishing first-derivative.

Further, to be comparable to empirical series, we consider in our analysis a sampling of data which takes one measure of the original series from the system each five iterations.

The non-Gaussian distributions of the logarithmic returns (Fig. 4.6c) were extracted from the logarithm returns of the original series of each index, as in Ref.<sup>74</sup> The characteristic heavy tails observed in<sup>74</sup> are observed for short time lag (hours or smaller), where in Fig. 4.6c the daily closure values are considered. The power-law behavior of the avalanche size (Fig. 4.6d) is indeed similar to the simulated results. Moreover the exponents  $m$  have all approximate values, plotted in Fig. 4.6e, around the simulated value  $m = 2.51$  (solid line), and predicted by Eq. (4.83).

All empirical indices are sampled daily but in different time periods. For FTSE 6498 days in London stock market were considered, starting on April 2nd 1984 and ending on December 18th 2009. For DJIA 20395 days in New York stock market were considered, starting on October 1st and ending on December 18th 2009. For DAX 4815 days in Frankfurt stock market were considered, starting on November 26th 1990 and ending on December 18th 2009. For CAC 5003 days in Paris stock market were considered, starting on March 5th 1990 and ending on December 18th 2009. For ALLORDS 6555 days in Australian stock market were considered, starting on August 3rd 1984 and ending on June 30th 2010. For HSI 5701 days were considered in Hong Kong stock market, starting on December 31st 1986 and ending on December 18th 2009. For NIKKEI 6386 days were analyzed in Tokyo stock market, starting on January 4th 1984 and ending on December 18th 2009. For CBOE IR10Y 12116 days in Chicago derivative market, starting on January 2nd 1962 and ending on June 30th 2010.

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#### 4.2.2.5 Conclusions

In this section, we have showed that, based only on first principles of economic theory and assuming that agents form an open system of economic connections organized by preferential attachment mechanisms, one is able to reach the distribution of drops observed in financial markets indices, including stocks and interest rate options. Assuming that the preferential attachment mechanism is part of the growing of economic connections, the resulting self-similar topology allows us to assume that the total economy system may present a similar topological structure as the sub-economy around financial markets and, thus, market indices can be taken as good proxies for the total economy.

We presented evidence that the distribution of drops in financial indices reflects the degree distribution taken from the trading network of the economical agents. In other words, the topology and structure of economic-like networks strongly influences the frequency and amplitude of economical crisis. Quantitatively, we showed that the two exponents characterizing the degree distribution and the distribution of drops, respectively, obey a scaling relation. Further, we showed how the scaling relation can be derived from a mean-field approach, assuming that the avalanche is a branching process of the economical agents and measuring its amplitude from the expected number of trading connections that are lost.

It should be noticed that the mean-field approach does not provide insight on local variations of both exponents. Differences between the different indices are also related to the different social-economic realities beneath them, including e.g. growing periods or crisis. For example, despite the fact that, in general, all economies have the same behavior, during a growing period the structure of the economic network shows a broader degree distribution (see e.g. Fig. 4.5e) corresponding to a higher value of the proxy (financial index).

The results averaged for sufficiently long time show that the numerical model here introduced reproduces the exponent  $m \sim 5/2$  for the distribution of the drops observed in empirical financial indices. The value of the exponent  $m$  gives indications of how large are crisis in the corresponding economy. Subsequently in chapter 5 we will show that the exponent  $m$  is bounded by  $[2, 3.5]$ .

Two remarks are due here. First, as stated in the introduction, we only dealt with drops, occurring for a lower threshold of the difference between consumption and production at one single agent. Though, the interpretation of the total “energy” in the system as a financial index for the market can only be closed if the so-called “booms” are also considered in the evolution of the financial proxy. The booms were not incorporated in our model, since we were concerned with risk management. To incorporate them an additional threshold in the consumption would be needed.

Second, our results show that the full topology underlying economic-like systems plays an important role in the evolution of the proxies characterizing the economical state.

The results we achieved here are analytically equal to the Fokker-Planck equation approach of section 4.2.1. The Fokker-Planck approach has the additional benefit of explaining the evolution of the agents degree in Fig. 4.5 but we need to define two parameters to complete the approach. Using the one in this section we have one parameter  $\gamma$  which in fact can be measured from the society (see Chapter 3). The point here is that, despite the fact that formally the Fokker-Planck equation is the correct way to handle the problem and the link to the full set of Financial Mathematics tools, the SOC approach leads in a much more easier way to the same results.



# 5

## APPLICATIONS

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*In the previous chapters we have been dealing with theory, both in Economics and Physics. One of the main goals of this work was to deliver economic value to the institutions that payed for it, both Closer and the Portuguese tax payer. Thus, this chapter is devoted to some applications.*

### Contents

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|            |   |           |
|------------|---|-----------|
| <b>5.1</b> | <b>State-Of-the-Art</b> . . . . .             | <b>84</b> |
| 5.1.1      | Key Concepts in Finance . . . . .             | 84        |
| 5.1.2      | Market Risk . . . . .                         | 85        |
| 5.1.3      | Credit Risk . . . . .                         | 86        |
| <b>5.2</b> | <b>Contributions</b> . . . . .                | <b>90</b> |
| 5.2.1      | The Dynamics of Financial Stability . . . . . | 90        |
| 5.2.2      | Economic Generated Thresholds . . . . .       | 103       |
| 5.2.3      | Bounding Market Risk . . . . .                | 117       |

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## 5.1 State-Of-the-Art

### 5.1.1 Key Concepts in Finance

The field of Economics that deals with the movement of scarce resources over time and states of nature is called finance. Finance is the economy of uncertainty since the exchanges are done asynchronously. For example, a loan is an exchange of present money, that the lender delivers to the debtor, for future money that the debtor delivers to the lender. Obviously, the amount of money is not equal. When both sides of the exchange are synchronous then we talk about what is called a financial derivative, typically because both legs of the exchange are based on future events. For example, a swap is a contract where both counterparts are lenders and debtors.

Since we are talking about moving resources in time, the key concept of finance is the time value of money.<sup>23</sup> As we saw in Chapter 3 there is a natural inflation associated with economic equilibrium caused by the production of new resources from the existing ones. This means that if we have a 10 € bill today, the economic value of the bill tomorrow will be lower. To keep its' value, we need to allocate the 10 € bill and the labor it represents, to the production of something and, with such allocation, retrieve the same resource plus part of the product. Since we are talking about multiplicative processes, this part of the production that we retrieve from the allocation of our money is also proportional to the amount of allocated money.

The way society allocates money from where is available to where it is needed for production is through agents. Agents deliver labor to channel the money, providing a way for money allocation and a way from allocating money to production, receiving part of it in exchange for their labor. The act of allocating money through this agents is called 'deposit', the act of getting money from this agents to be allocated to production is called 'loan' and, finally, this intermediate agents are called financial institutions or, more commonly, 'banks'. We will be back to the importance of banks further ahead so we will not detail here more. Important for us now is that banks compensate deposits by giving more money to depositors and get compensated by getting more money from the debtors. The multiplicative coefficient that defines the amount of the compensation in both cases is called 'interest rate' and the amount of the compensation is called 'interest'. The reason why we need to introduce banks before we define interest rate is that banks provide a way of getting an average interest rate due to the quantity of exchanges they provide, including the interest rate applied when they lend money between them. In fact, this average interest rate serve as a market reference for pricing other types of loan. Associated with Euro currency, this average rate is called EURIBOR.

So, the economy has the knowledge of the price of money allocation. Let  $r$  be the average interest rate for the allocation of money during a time horizon of one year and the amount we allocate as  $x$ . We know that after one year the amount of money we have must be  $(1 + r)x$ . So the economic connection is based in the exchange of  $x$  in the present for  $(1 + r)x$  in the

future. Meaning that  $x$  today and  $(1+r)x$  one year from now have the same economic value. To  $(1+r)x$  we call ‘future value’ of  $x$  and, vice-versa,  $x$  is the ‘present value’ of  $(1+r)x$ . In other words, if we will have a 10 €bill one year from now, the present value of that bill is  $10/(1+r)$  €. Thus, every financial operation can be reduced to the present and to a synchronous exchange, as we will see below.

Now we take a two year time horizon. Then the future value of the 10 €bill will be  $10(1+r)(1+r) = 10(1+r)^2$ . Generically, the future value of money for a time horizon of  $n$  years is obtained multiplying for the factor  $(1+r)^n$ . Analogously, the present value of money received/paid  $n$  years from now is obtained by multiplying for the factor  $(1+r)^{-n}$ , so called the discount factor. If we consider periods shorter than one year, then is just a matter of dividing the rate  $r$  from the number of such periods in one year. Plus, if we want to consider that time is continuous, we make  $\lim_{n \rightarrow \infty} (1+r/n)^n = e^{rt}$ .

Let us consider now a long term contract, like a mortgage loan, we will have several monthly payments through a long period of time, tens of years. The present value of such a contract is just the sum of all the future payments multiplied by the discount factor applied to it. If we want to know how much the contract values we subtract to the present value of the contract the amount of money already received and that gives us the ‘net present value’ or NPV.

Now let us consider a more complex financial instrument like a stock market share. A stock market share represents a share of a company property, meaning that the holder is entitled to receive dividends, i.e., the part of the company profits corresponding to its share of the company. Then the present value of the share is, like the mortgage loan, the present value of all future payments. So why does the price of shares in the stock market fluctuate? The reason is that we are always talking about future and future has different perceptions. The stock market price just reflects the average perception of the economic agents that trade the share of what the future dividends will be, based on present results and news about the company and the market. And this is what a finance problem is all about, the uncertainty on the present value of money.

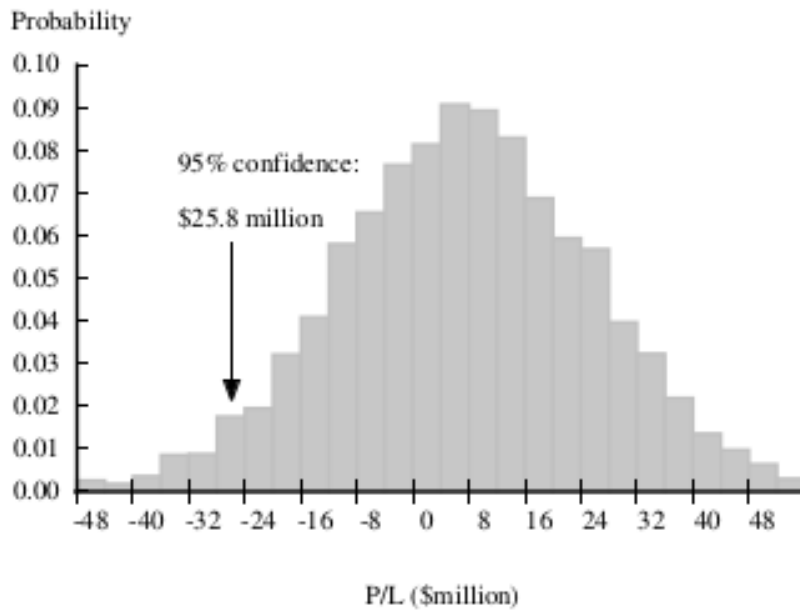
If there is uncertainty relating the present value (PV) of money, that means that instead of dealing with a scalar value we are actually dealing with a distribution. From that distribution we can retrieve the expected value of PV, but also all the possible values that it can assume due to future events. Some of those values makes the NPV be positive, i.e., we get a larger value than the one we allocate in the first place, but we can also get less value than the one we allocate. That possibility of losing value in an economic exchange due to future events is called ‘financial risk’. The aim in a financial investment is to get the maximum expected return with the minimum risk.<sup>23</sup>

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### 5.1.2 Market Risk

Market risk is the possibility of incurring losses due to price changes in the market. For example, the price fluctuations of oil or of a stock market share. Currently, there are several techniques to model market risk, the chief among those being Value-at-Risk (VaR).<sup>96</sup>

$\text{VaR}(\alpha, T)$  is a real function of the time horizon  $T$  and level of confidence  $\alpha$  that delivers a quantification of the risk associated with a financial instrument or portfolio of financial instruments. VaR was created as a part of a methodology that was born inside J.P. Morgan, in 1994, called RiskMetrics,<sup>86</sup> a set of tools with the aim of managing market risk. Quoting Riskmetrics "VaR is a measure of the maximum potential change in the value of a portfolio of financial instruments with given probability over a pre-set horizon". It answers the question "how much can I lose with probability  $\alpha$  in time horizon  $T$ ?". So, for computing VaR we have to find the distribution of the value in a time horizon  $T$ . An example of such computation is shown in Fig.(5.1).



**Figure 5.1:** *How to calculate Value-At-Risk.<sup>86</sup> We construct the distribution of the returns for the time horizon based on historical prices and see what is the biggest loss with  $\alpha\%$  certainty. Source: Longerstaey and Zangari.<sup>86</sup>*

Since VaR started being used until today, Gaussian distributions have been used to model the value distributions of each asset and, consequently, the value distribution of the portfolios. Let us recall that the Brownian motion model supports most of Financial Mathematics and, in this sense, the assumption of having such a distribution is perfectly logic. But today due to that assumption the criticism around VaR is considerable.<sup>131</sup> Simultaneously, there is no practical and credible alternative and therefore VaR is still looked as the best of all bad ways for quantifying risk.

### 5.1.3 Credit Risk

Unlike market risk, where every information is related to public traded assets and thus available and abundant, credit risk must be quantified with almost no information. Credit

risk is the possibility of incurring losses due to the inability of the counterpart of the asset allocation to fulfill previous arrangements or, in other words, the debtor fails to pay back the loan (it 'defaults'). Thus, since each counterpart is an individual (private or company) it is not possible to get a distribution of such events since it deals with defaults of someone that never defaulted or, at most, happened once or twice. So the problem here is to get information we can have, to substitute the one we do not have.

<HELP> for explanation. DL18 Corp CSDR  
Enter # <GO> for historical ratings. Page 1/2

**Foreign Currency LT Debt**  
Region - Eastern Europe

|                | MOODY'S  | S&P     | DCR       | FI      |
|----------------|----------|---------|-----------|---------|
| Bulgaria       | 1)B2     | 15)B    | 29)NR     | 43)B+   |
| Croatia        | 2)Baa3   | 16)BBB- | 30)NR     | 44)BB+  |
| Cyprus         | 3)A2     | 17)A+   | 31)NR     | 45)NR   |
| Czech Republic | 4)Baa1   | 18)A-   | 32)A-     | 46)BBB+ |
| Estonia        | 5)Baa1   | 19)BBB+ | 33)NR     | 47)BBB  |
| Hungary        | 6)Baa1   | 20)BBB  | 34)BBB    | 48)BBB  |
| Latvia         | 7)Baa2   | 21)BBB  | 35)NR     | 49)BBB  |
| Lithuania      | 8)Ba1    | 22)BBB- | 36)NR     | 50)BB+  |
| Moldova        | 9)B2 *-- | 23)NR   | 37)NR     | 51)B    |
| Poland         | 10)Baa1  | 24)BBB  | 38)BBB-   | 52)BBB+ |
| Romania        | 11)B3    | 25)B-   | 39)NR     | 53)B-   |
| Russia         | 12)B3    | 26)SD   | 40)B- *-- | 54)CCC  |
| Slovakia       | 13)Ba1   | 27)BB+  | 41)NR     | 55)BB+  |
| Slovenia       | 14)A3    | 28)A    | 42)NR     | 56)A-   |

COLOR DENOTES A RATING CHANGE WITHIN THE LAST 30 DAYS (Pos/Neg/Neutral)

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Princeton: 609-279-3000 Singapore: 226-3300 Sydney: 2-9777-8686 Tokyo: 3-5201-8900 Sao Paulo: 11-5048-4300  
[741-93-0 08-Sep-99 19:16:12]

**Bloomberg**

Figure 5.2: Rating publication for sovereign debt of eastern countries' states. Source Wilmott<sup>140</sup>

If the counterpart we are dealing with is a bank or a state it is not easy to get information because events involving banks or states are very rare in history and Statistics is useless. Nonetheless, there are some metrics that experts can relate to a higher propensity to default, like large public deficit in a state, a patent legal action or a large amount of existing debt in a company. Those experts, called rating agencies, use this kind of metrics to say that a particular debtor is comparatively worse than another and establish a qualitative classification called rating (see Fig. 5.2). Knowing that the top rating is the best with zero probability of default in one year and the last the worse rating with a probability one for default, then we can build a distribution based on this values and on a transition matrix. The table in Fig. 5.3 is a transition matrix and assumes that the credit risk of a counterpart follows a discrete Markov process, so just by matrix multiplication we can get any time horizon bigger than the one that is represented in the matrix. This approach has many strong mathematical assumptions beneath it, the probabilities of the extreme ratings, the Markov assumption, the stationarity of the matrix elements, etc. Nevertheless, ratings are the accepted risk notation in banking regulation.

If the counterpart we are dealing with is small, like a private or a small company, then we can substitute the information we do not have by the information relating events with similar debtors. There are several approaches to do that.

|         | AAA     | AA      | A       | BBB     | BB      | B       | CCC     | Default |
|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| AAA     | 0.90829 | 0.08272 | 0.00736 | 0.00065 | 0.00066 | 0.00014 | 0.00006 | 0.00012 |
| AA      | 0.00665 | 0.90890 | 0.07692 | 0.00583 | 0.00064 | 0.00066 | 0.00029 | 0.00011 |
| A       | 0.00092 | 0.02420 | 0.91305 | 0.05228 | 0.00678 | 0.00227 | 0.00009 | 0.00041 |
| BBB     | 0.00042 | 0.00320 | 0.05878 | 0.87459 | 0.04964 | 0.01078 | 0.00110 | 0.00149 |
| BB      | 0.00039 | 0.00126 | 0.00644 | 0.07710 | 0.81159 | 0.08397 | 0.00970 | 0.00955 |
| B       | 0.00044 | 0.00211 | 0.00361 | 0.00718 | 0.07961 | 0.80767 | 0.04992 | 0.04946 |
| CCC     | 0.00127 | 0.00122 | 0.00423 | 0.01195 | 0.02690 | 0.11711 | 0.64479 | 0.19253 |
| Default | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 1       |

**Figure 5.3:** A transition matrix is the register of the probability of migration from one rating to another based on historical events. Source Wilmmot<sup>140</sup>

One is by finding correlations in historical records between the events and characteristics of the debtor, making a similar approach to what we described for bigger debtors. This approach is called ‘scoring’ and it is quite popular in retail banking. The problem here is the volatility of the correlations due to the critical nature of the economy, but most of the credit associated with this practice is short term lending (like consumer credit) with very small amounts.

A second approach is to consider the small debtors as a pool of equal objects and measure the risk associated with the entire pool. This approach underpins the banking regulation in small credits, as the rating-based approach is used for large debtors, so all banks in the world have their risk controlled by regulators with this model.<sup>108–110</sup> The pool approach is based on the option theory and is known as the Merton-Vasicek model<sup>132</sup> which we explain next.

Let us assume that to a generic debtor we can attribute a ‘financial health’ and that exists a level  $\xi$  henceforth called Merton level, below which the debtor defaults. In option theory language the lender sells a put option to the debtor and when the debtor defaults he or she delivers its assets to the lender. Merton thought this model to be applied to public listed companies whose shares have a market price. But Vasicek applied it to pools of debtors by assuming that their financial health follows a Brownian motion.<sup>132</sup>

So let us represent the financial health of a debtor as a random variable  $Y$ . The financial health is influenced by the environment and by intrinsic characteristics of the debtor. The environment we represent by a random variable  $X$  called the ‘systemic factor’. The intrinsic characteristics we represent by a random variable  $\varepsilon$  called the ‘idiosyncratic factor’. Environment and intrinsic characteristics are considered to be uncorrelated.

The financial health is then given by

$$Y = \rho X + \sqrt{1 - \rho^2} \varepsilon \quad (5.1)$$

where  $\rho$  is the correlation between the debtor’s financial health and the systemic factor. Equation (5.1) is typically used to define a Gaussian random variable that is known to depend on the sum of two other Gaussian random variables.<sup>116</sup>

Consequently we can write the idiosyncratic factor as

$$\varepsilon = \frac{Y - \rho X}{\sqrt{1 - \rho^2}}. \quad (5.2)$$

Since there is a threshold below which the debtor defaults, then the condition for default is

$$\varepsilon \leq \frac{Y_{th} - \rho X}{\sqrt{1 - \rho^2}} \quad (5.3)$$

where  $Y_{th}$  is the threshold level.

Here we make a transformation to a systemic factor that generates defaults, rather than financial health since we cannot measure financial health. Since the systemic factor is yet undefined we can change the signal in the numerator of the RHS from  $-\rho X$  to  $\rho X$ .

If there is a threshold below which debtors default then if we take a pool of  $N \rightarrow \infty$  debtors we can substitute  $Y_{th}$  by the measure of actual defaults.

Assuming that the pool is big enough and  $Y$  is Gaussian distributed, then  $Y_{th} = \Phi^{-1}(PD)$  where  $\Phi^{-1}$  is the inverse cumulated normal function and  $PD$  is the measured number of defaults within a time period  $T$  which, under this conditions, can be taken as the expected value of defaults.  $PD$  stands in regulatory language as probability of default and it is assumed as an expected value of the default distribution.

We are left with a distribution,  $X$ , since  $\Phi^{-1}(PD)$  is a scalar.  $X$  gives  $\varepsilon$  the distribution characteristics, meaning that the statement *with  $\alpha\%$  confidence* translates into the inverse cumulated normal function of  $\alpha$ . Thus Eq.(5.3) gives the domain of idiosyncratic factor in which the debtor defaults as

$$\varepsilon \leq \frac{\Phi^{-1}(PD) + \rho\Phi^{-1}(\alpha)}{\sqrt{1 - \rho^2}}. \quad (5.4)$$

Since  $X$  is assumed to be normally distributed, the resulting probability distribution for default is

$$P(\alpha, T) = \Phi \left[ \frac{\Phi^{-1}(PD) + \rho\Phi^{-1}(\alpha)}{\sqrt{1 - \rho^2}} \right] \quad (5.5)$$

which is an adimensional VaR quantity and  $T$  in the RHS of Eq.(5.5) results from measurement of  $PD$ .

Banking regulation calls the probability measure in Eq.(5.5) as the conditional probability and to  $PD$  unconditional probability. Equation (5.5) is the base for the Basel Accords on banking stability. The amount of flaws in the model is considerable since Gaussian distributions are assumed everywhere. As we will see further below, this level actually exists and it is important to exchange the criticality threshold assumption for an empirical evidence of its existence.

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## 5.2 Contributions

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### 5.2.1 The Dynamics of Financial Stability

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#### 5.2.1.1 Introduction

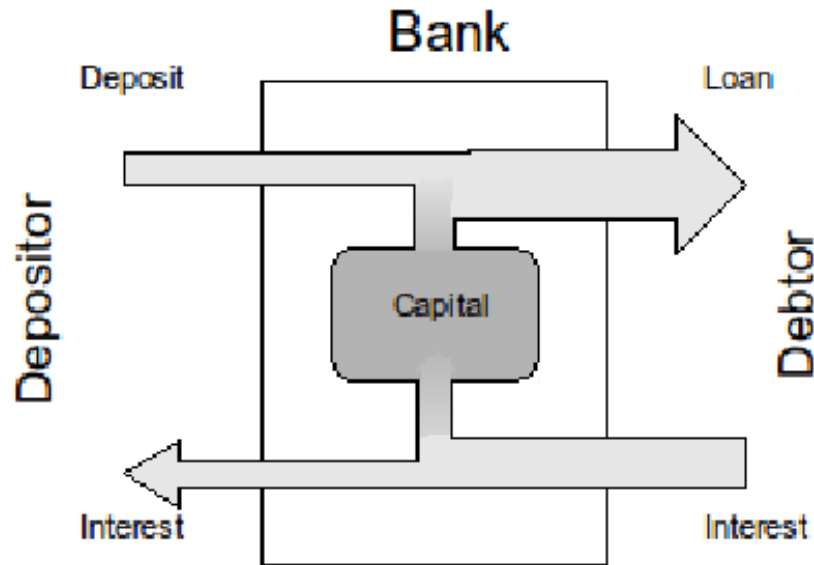
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The well-being of humankind depends crucially on the financial stability of the underlying economy. The concept of financial stability is associated with the set of conditions under which the process of financial intermediation (using savings from some economic agents to lend to other economic agents) is smooth, thereby promoting the flow of money from where it is available to where it is needed. This flow of money is made through economic agents, commonly called ‘banks’, that provide the service of intermediation and an upstream flow of interest to pay for the savings allocation. Because the flow of money that ensures financial stability occurs on top of a complex interconnected set of economic agents (network), it must depend not only in individual features or conditions imposed to the economic agents but also on the overall structure of the entire economic environment. The role of banking regulators is to protect the flow of money through the system by implementing rules that insulate it against individual or localized breaches that happen when a bank fails to pay back to depositors. However, these rules do not always take into account the importance of the topological structure of the network for the global financial stability. In this section, we will present quantitative evidence that neglecting the topological network structure when implementing financial regulation may have a strong negative impact on financial stability.

The event of not paying back the money owed is called ‘default’. In order that downstream defaults do not generate the default of a particular bank, each bank holds an amount of money as a reserve for paying back its depositors. In other words, a part of the money one bank sends downstream is its own money. This share of own money is called ‘capital’ (see Fig 5.4). Looking to one single bank, if it has a large amount of capital, one reasonably expects that the bank will also cover a proportionally large debtor default, guaranteeing the deposits made by its depositors. On the contrary, if the capital level of the bank is small, a small debtor default is sufficient to put the bank with no conditions for guaranteeing the money of all its depositors. Loosing such conditions, the bank enters a situation called bankruptcy or insolvency. Usually, bank regulators base their rules in such arguments.

In 1988, a group of central bank governors called the Basel Committee on Banking Regulation unified the capital level rules that were applied in each of the member countries and defined a global rule to protect the banking system that was becoming global at the time.<sup>108</sup> Roughly speaking, these rules imposed a minimum capital level of 8% without any empirical reason. A few years later the accord came under criticism from market agents who felt that it did not differentiate enough between the various debtors, i.e. between the entities whom the bank lends money, and a second version of the accord<sup>109</sup> was finished in 2004 to become effective in





**Figure 5.4:** Illustration of a bank ‘apparatus’ for money flow. A bank lends money to debtors using money from depositors and also its own money, the capital. In return, debtors pay interest to the bank, which keeps a part to itself and pays the depositors back.

2008. In this second version, banks were allowed to use the Merton-Vasicek model<sup>132</sup> based on the Value-at-Risk paradigm<sup>53</sup> to weight the amount of lent money in the calculation of the necessary capital according to the measured risk of the debtor. Thus the 8% percentage was now calculated over the weighted amount and not over the total amount. This version of the accord became effective at the beginning of the 2008 financial turmoil; with regulators under severe criticism from governments and the media, in 2010 the Committee issued a new version<sup>110</sup> tightening capital rules.

At the same time, since the beginning of the 2000s the academic community has been very critical of the capital rules, particularly because the VaR paradigm, on which such rules are based, assumes that returns are normally distributed and “does not measure the distribution or extent of risk in the tail, but only provides an estimate of a particular point in the distribution”.<sup>40</sup> In fact, there is a huge amount of evidence<sup>20,22,91,130</sup> that the returns of economic processes are not normally distributed, having typically heavy tails. According to the Central Limit Theorem,<sup>49</sup> if returns are heavy-tail distributed, then the underlying random variables have infinite variance or a variance of the order of the system size.<sup>92</sup> In economic systems, random variables are related to measurements taken from economic agents. Thus, the infinite variance results from long-range correlations between the economic agents. We will argue that this single fact compromises the stability of the flow and brings into question the effectiveness of capital level rules.

Physics, and in particular statistical physics, has long inspired the construction of models for explaining the evolution of economies and societies and for tackling major economic decisions in different contexts.<sup>55,123</sup> The study of critical phenomena and multi-scale systems in physics led to the development of tools that proved to be useful in non-physical contexts, particularly in financial systems. One reason for this is that fast macroeconomic indicators, such as



principal indices in financial markets, exhibit dynamical scaling, which is typical of critical physical systems.<sup>93</sup>

In this section we will address the problem of the financial stability using statistical physics models that explain the occurrence of large crises, in order to show that the resilience of the banking system is not necessarily improved by raising capital levels. Our findings have a concrete social importance, since capital is the most expensive money a bank can provide to its debtors. Capital belongs to the shareholders, who bear the risk of the business and keep the job positions. So it must be remunerated above the money from depositors who do not bear these risks. Consequently, more capital means more costs on the flow of the money and, in the end, more constraints to economic development.

We start in section 5.2.1.2 by describing an agent-based model<sup>123</sup> which enables us to generate the critical behavior observed in economic systems. In section 5.2.1.3 we describe the observables that account for the economic properties of the system, namely the so-called overall product and business level.<sup>138</sup> Furthermore, the agent-based model as well as the macroscopic observables, are discussed for the specific situation of a network of banks and their deposits and loans. One important property in financial banking systems is introduced, namely the minimum capital level, defined here through the basic properties of agents and their connections. In section 5.2.1.4 we focus on the financial stability of the banking system, showing that raising capital levels promotes concentration of economic agents if the economic production remains constant and it destroys economic production if that concentration does not occur. Finally, we present specific situations where each agent seeks the stability of its economic production after a raise in capital levels, leading to a state of worse financial stability, i.e. a state in which large crises are more likely to occur. In section 5.2.1.5 we draw the conclusions.

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#### **5.2.1.2** Minimal model for avalanches of financial defaults

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The model introduced in this section is based on the previous chapters, on the fundamental feature, described in Chapter 2, that human beings have developed through natural selection, in order to be able to fight environmental threats collectively: economic behavior.

In this scope, let us assume that the economic environment is composed of elementary particles called ‘agents’ and all phenomena occurring in it result from the interaction of those particles. Let us also assume that agents are attracted to interact, exchanging an observed quantity that takes the form of money, labor or other effective means used in the exchange. This type of model where the decision concerning an exchange is made by the exchanging agents alone is called a “free-market economy”.

We represent these interactions or trades between agents through economic connections, and call the exchanged quantity ‘economic energy’. Though the “energy” used here is not the same as physical energy, we will use the term in the economical context only. Notice, however, that human labor is assumed to be “energy” delivered by one individual to those with whom he/she interacts, which reward the individual with an energy that he/she accumulates. The balance between the labor (“energy”) produced for the neighbors and the reward received from

them may be positive (agent profits) or negative (agent accumulates debt). For details see section 4.2.2.3. This analogy underlies the model introduced in the following, where we omit the quotation marks and consider entities more general than individual, which we call agents. Agent-based models of financial markets have been intensively studied, see e.g. Refs.<sup>117,123</sup> and references therein.

Economic connections between two agents are in general not symmetric and there is one simple economic reason for that: if a connection were completely symmetric there would be no reason for each of the two agents to establish an exchange. In several branches of Economics we have different examples of these asymmetric economic connections like production/consumption, credit/deposit, a labor relation, repo's, swaps, etc. In the next section, we will focus on a specific connection, namely in credit/deposit connections.

Since each connection is asymmetric we distinguish the two agents involved by assigning two different types of economic energy. Hence, let us consider two connected agents,  $i$  and  $j$ , where  $i$  delivers to  $j$  an amount of energy  $W_{ij}$  and receives an amount  $E_{ij} \neq W_{ij}$  in return. We call these connections the outgoing connections of agent  $i$ . The connections where agent  $i$  receives from  $j$  an amount of energy  $W_{ji}$  and delivers in return  $E_{ji}$  we call incoming connections.

The energy balance for agent  $i$  in one single trade connection is, from a labor production point of view,  $U_{ij} = W_{ij} - E_{ij}$ . Having two different types of energy, we choose  $W_{ij}$  as the reference to which the other type  $E_{ij} = \alpha_{ij}W_{ij}$  is related through the coefficient  $\alpha_{ij}$ . Without loss of generality, we consider that one connection corresponds to the delivery of one unit of energy,  $W_{ij} = 1$ , yielding:

$$U_{ij} = 1 - \alpha_{ij}. \quad (5.6)$$

The definition for the coefficient  $\alpha_{ij}$  is given in Eq.(4.76)

In the used model we disregard the economic details of agents and connections, keeping the model as general as possible. Still, this generalization is not different in its essence from the one accountants must use to provide a common report for all sorts of business, with the difference that they use monetary units and we use dimensionless energy units.

Because each agent typically has more than one neighbor, the total energy balance for agent  $i$  is given by

$$U_i = \sum_{\text{all neighbours}} U_{ij} = \sum_{j \in \nu_{out,i}} (1 - \alpha_{ij}) + \sum_{j \in \nu_{in,i}} (\alpha_{ji} - 1) \quad (5.7)$$

where  $j$  runs over all neighbors of agent  $i$ , and  $\nu_{out,i}$  and  $\nu_{in,i}$  are, respectively the outgoing and incoming vicinities of the agent.

This total energy balance  $U_i$  is related to the well-known financial principle of net present value (NPV): When an agent holds a deposit he or she supposedly pays for it (by definition) and most (but not all) accounting standards<sup>65</sup> assume it as a negative entry on the accounting balance. Here, we model deposits as a set of incoming connections from the same agent in which all associated cash-flows were already discounted. In this way, if we could think of a balance sheet totally built with NPV's we would be near  $U_i$ .

As we noted previously, economic energy is related to physical energy in the sense that the agents must absorb finite amounts of physical energy from the environment to deliver economic

energy. Consequently, the economic energy balance  $U_i$  of agent  $i$  must be finite. The finiteness of  $U_i$  for each agent is controlled by a threshold value, below which the agent is no longer able to consume energy from its neighbors, i.e. below which it loses all its incoming connections. Furthermore, since this threshold reflects the incoming connections, it should depend on how many incoming connections our agent has. With such assumptions, we introduce the quantity

$$c_i \equiv \frac{U_i}{\sum_{j \in \nu_{in,i}} (\alpha_{ji} - 1)} \quad (5.8)$$

for ascertaining if the agent is below a given threshold  $c_{th}$  or not. We call this quantity  $c_i$  the ‘leverage’ of agent  $i$ . Unlike we did previously in section 4.2.2.3, here  $U_i$  is divided by the total product of the incoming connections solely and not by the ‘turnover’. This choice is made to be in line with the way banking regulators define leverage. Still, this alternate definition does not change the critical behavior observed in this particular version of the model. For the case that the mean-field approximation  $\alpha_{ij} \sim \langle \alpha \rangle$  holds, the leverage  $c_i$  depends exclusively on the network topology, yielding  $c_i = \frac{k_{out,i}}{k_{in,i}} - 1$ .

Leverage has a specific meaning in Economics, which is related to the quantity  $c_i$ : it measures the ratio between own money and total assets.<sup>23</sup> Thus, each agent has a leverage  $c_i$  which varies in time and there is a threshold  $c_{th}$  below which the agent ‘defaults’ or goes bankrupt, losing its incoming connections with its neighbors. Since the bankrupted agent is connected to other agents, the energy balances must be updated for every affected agent  $j$ . Bankruptcy leads to the removal of all incoming connections of agent  $i$ , reducing the consumption of the bankrupted agent to a minimum, i.e. keeping one single consumption connection,  $k_{in,i} = 1$ . This situation implies that agent  $i$  and its neighbors  $j$  should be updated as follows:

$$c_i \rightarrow k_{out,i} - 1 \quad (5.9a)$$

$$k_{out,j} \rightarrow k_{out,j} - 1 \quad (5.9b)$$

$$c_j \rightarrow c_j - \frac{1}{k_{in,j}}. \quad (5.9c)$$

We keep the agent with one consumption connection in the system also to avoid the divergence of  $c_i$  as defined in the context of financial regulation.<sup>108–110</sup> Such a minimum consumption value has no other effect on the problem we will be dealing with in the next section.

The bankruptcy of  $i$  leads to an update of the energy balance for neighbor  $j$ , which may then also go bankrupt, and so on, thereby triggering a chain of bankruptcies henceforth called an ‘avalanche’. See illustration in Fig. 5.5.

The concepts of leverage and leverage threshold are used by R.C. Merton<sup>98</sup> and O. Vasicek<sup>132</sup> in their credit risk models, which are the theoretical foundation for the Basel Accords.<sup>108–110</sup> Namely, Merton assumed this threshold for pricing corporate risky bonds using a limit on debt-equity ratio and Vasicek generalized it to a “debtor wealth threshold” below which the debtor would default on a loan.

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**5.2.1.3** Macroproperties: overall product and business level
 

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Let us consider a system of  $L$  interconnected agents which form the environment where each agent establishes its trades. We call henceforth this environment the operating neighborhood. We can measure the total economic energy of the system by summing up all outgoing connections to get the overall product  $U_T$ , namely

$$U_T = \sum_{i=1}^L \sum_{j \in \mathcal{V}_{out,i}} (1 - \alpha_{ij}), \quad (5.10)$$

where  $\mathcal{V}_{out,i}$  is the outgoing vicinity of agent  $i$ , with  $k_{out,i}$  neighbors. The quantity  $U_T$  varies in time and its evolution reflects the development or failure of the underlying economy. Instead of  $U_T$ , we consider the relative variation  $\frac{dU_T}{U_T}$ , also known as ‘return’ in a financial context. We can also measure the average business level per agent, defined as the moving average in time of the overall product:

$$\Omega = \frac{1}{L} \frac{1}{T_S} \int_t^{t+T_S} U_T(x) dx \quad (5.11)$$

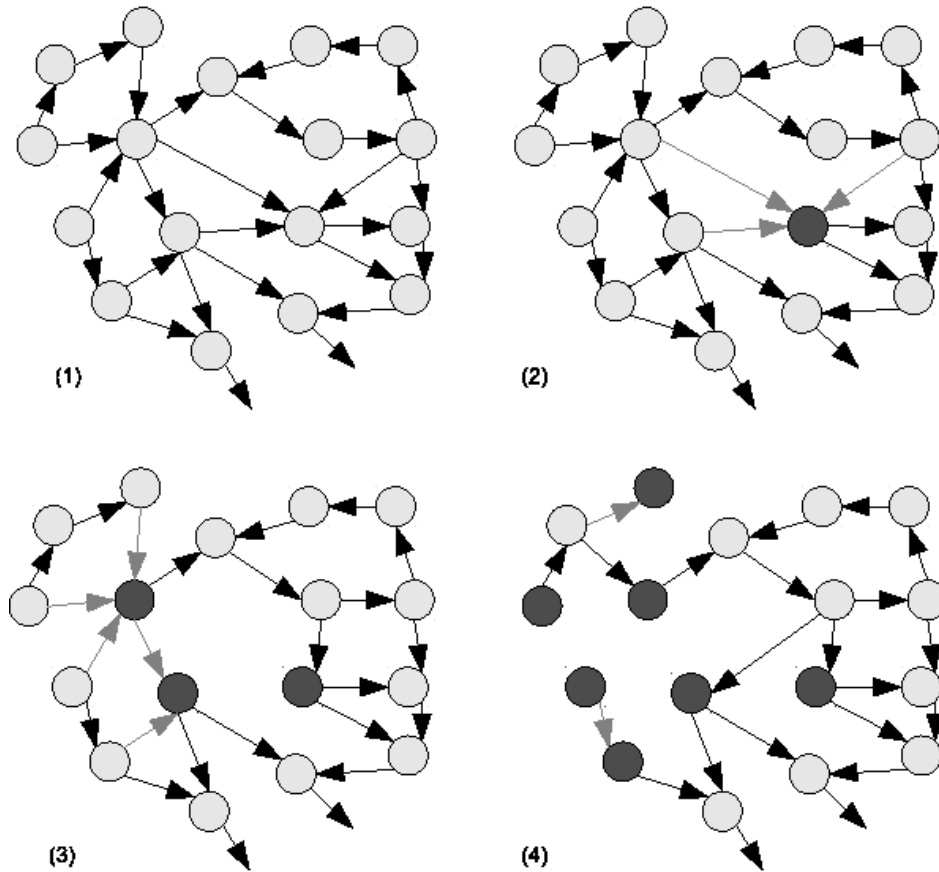
where  $T_S$  is a sufficiently large period for taking time averages. Similar quantities are used in Economics as indicators of individual average standard of living.<sup>138</sup> In the continuum limit, the time derivative of the business level  $\Omega$  gives the overall product uniformly distributed over all agents.

At each time step a new connection is formed, according to the standard preferential attachment algorithm of Barabási-Albert:<sup>11</sup> For each connection created one agent is selected using a probability function based on its previous outgoing connections, expressed as

$$P(i) = \frac{k_{out,i}}{\sum_{l=1}^L k_{out,l}} \quad (5.12)$$

and one other agent is selected by an analogous probability function built with incoming connections. Such a preferential attachment scheme is associated with power-law features observed in the Economy long ago<sup>64,113</sup> and is here motivated by first principles in economics that agents are impelled to follow: an agent having a large number of outgoing connections is more likely to be selected again to have a new outgoing connection, and likewise for incoming connections.

As connections are being created, a complex network of economic agents emerges and individual leverages (see Eq. (5.8)) are changing until eventually one of the agents goes bankrupt ( $c_i < c_{th}$ ) breaking its incoming connections and changing the leverage of its neighbors, who might also go bankrupt and break their incoming connections and so on. See Fig. 5.5. This avalanche affects the total overall product, Eq. (5.10), because the dissipated energy released during the avalanche is subtracted. This total dissipated energy is given by the total number of broken connections, and measures the ‘avalanche size’, denoted below by  $s$ . Since the avalanche can involve an arbitrary number of agents, and is bounded only by the size of the system, the distribution of the returns  $\frac{dU_T}{U_T}$  will be heavy-tailed, as expected for an economic system. See Fig. 5.8 below.



**Figure 5.5:** Illustration of a bankruptcy avalanche. Each arrow points to the agent for which it is an incoming connection. Any agent in the economic environment is part of a complex network (1), and is susceptible to go bankrupt, which will destroy its incoming connections (2). Consequently, new energy balances must be updated for the affected neighbors, whose leverage can go over the threshold (3). Since these neighbors also have neighbors of their own, connection will continue to be destroyed until all agents again have a leverage above the threshold (4).

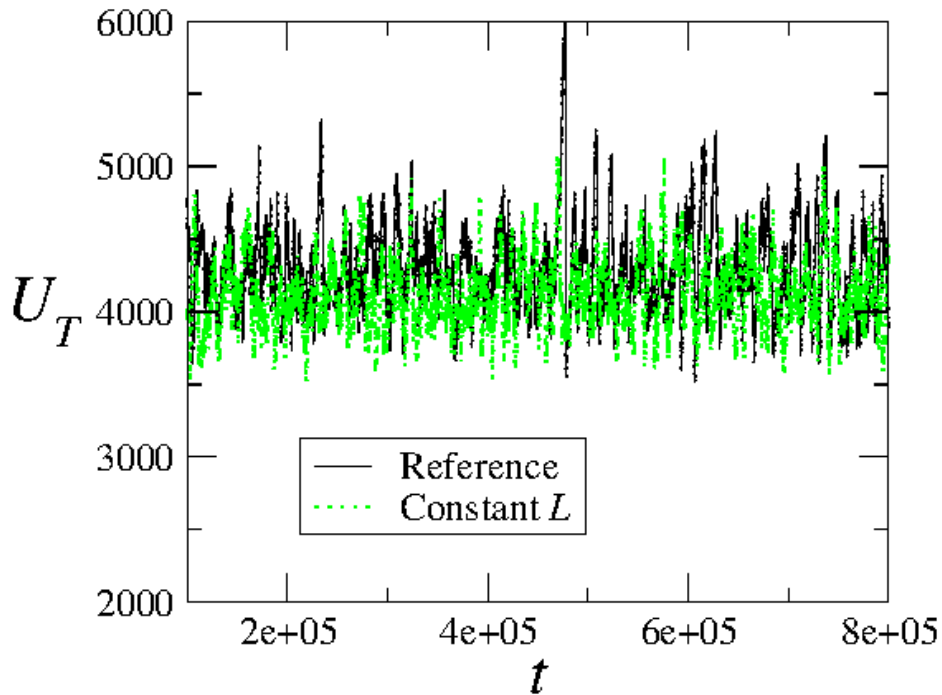
Until now we have been dealing with generic economic agents that make generic economic connections between each other. No particular assumption has been made besides that they are attracted to each other to form connections by the mechanism of preferential attachment and that the economic network cannot have infinite energy. From this point onward, we will differentiate some of these agents, labeling them as ‘banks’. To this end we fix the nature of their incoming and outgoing connections: The incoming connections are called ‘deposits’, the outgoing connections are called ‘loans’. We should emphasize that we are not singling out this kind of agent from the others. Banks are modeled as economic agents like any other. We have only named its incoming and outgoing connections, which we could also do for all the remaining agents, as consuming/producing, salary/labor, pension/contribution, etc, to model every single business we could think of. We are choosing this particular kind of agent because banks are the object of banking regulation and the aim of financial stability laws.

The threshold leverage  $c_{th}$  for one bank represents its ‘minimum capital level’. The capital of one bank is really an amount of incoming connections, which are equivalent to deposits, because shareholders are also economic agents. This means that the ‘minimum capital level’ in the model will be much higher than in real bank markets because we are disregarding

shareholders and adding the remaining energy deficit to fulfill  $c_{th}$ . Therefore, we cannot map directly the levels obtained in the model onto the levels defined in banking regulation. We can, however, uncover the behavior of economic agents in scenarios difficult to reproduce without such a model.

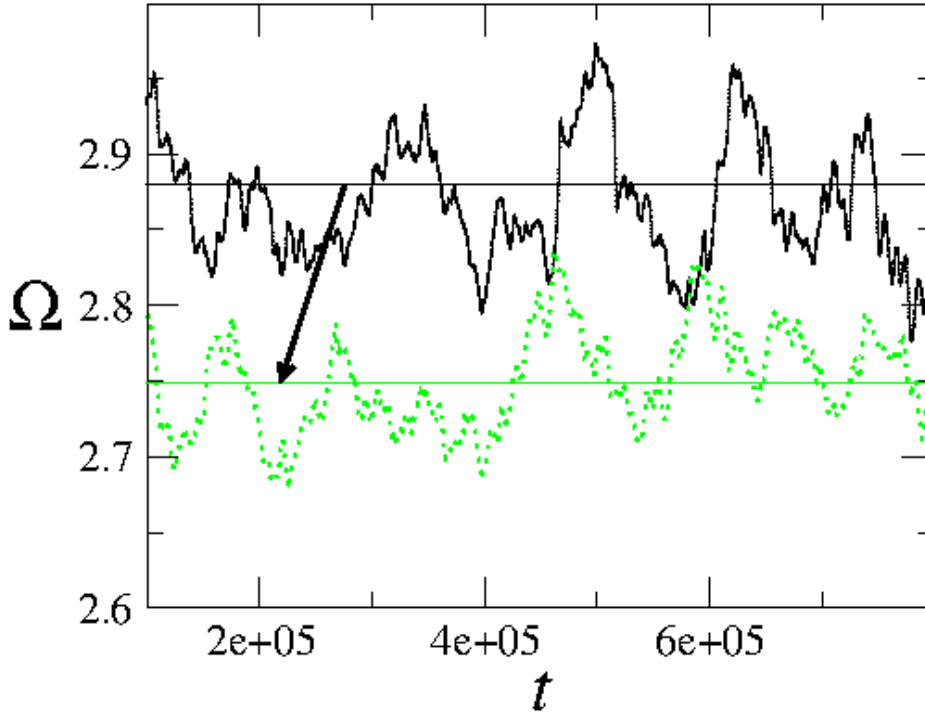
#### 5.2.1.4 Raising the minimum capital level

In this section we use the model described above in different scenarios, i.e. for different sizes of the operating neighborhood and different minimum capital levels. From Eq. (5.8) one sees that the leverage of one agent is always larger than  $-1$ . Since we deal with bankruptcy we are interested in negative values of  $c_{th}$ , which reduces the range of leverage values to  $[-1, 0]$ . Our simulations showed that a representative range of values for both the threshold and the size  $L$  of the operating neighborhood is  $[-0.72, -0.67]$  and  $[500, 2000]$  respectively. For each pair of values  $(L, c_{th})$  the system evolves until a total of  $1.5 \times 10^6$  connections are generated. We discard the first  $10^5$  time-steps which are taken as transient.



**Figure 5.6:** Illustration of the effect of raising the minimum capital level on the overall product  $U_T$ , at constant  $L = 1500$ . Raising  $c_{th}$  from  $-0.71$  (solid line) to  $-0.69$  (dotted line) does not significantly change the overall product.

Figures 5.6 and 5.7 illustrate the evolution of the overall product  $U_T$  and business level  $\Omega$  for a situation in which the minimum capital level is raised, while keeping the size of the operating neighborhood constant. The solid line shows the initial situation with lower minimum capital level and the dashed line the final situation with higher minimum capital level. From Fig. 5.6 we can see that if the size of the operating neighborhood is kept constant, the quasi-stationary level of the overall product does not significantly change.



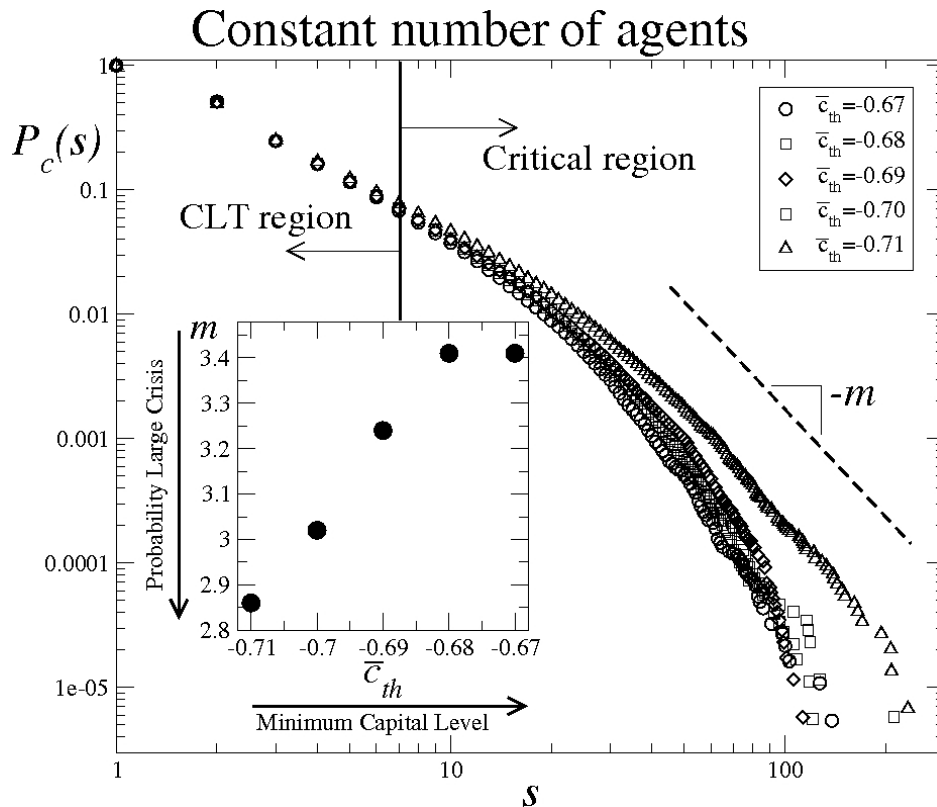
**Figure 5.7:** Illustration of the effect of raising the minimum capital on the business level at constant  $L = 1500$ . Raising  $c_{th}$  from  $-0.71$  (solid line) to  $-0.69$  (dotted line) decreases the business level from  $\Omega = 2.88$  to  $\Omega = 2.75$ .

Following this observation we next investigate the evolution of the return distribution for  $U_T$ , considering an increase of the minimum capital level at constant size  $L$  of the operating neighborhood. To this end we compute the cumulative size distribution of avalanches, i.e. the fraction  $P_c(s)$  of avalanches of size larger than  $s$ . Numerically, the size  $s$  of an avalanche is found by summing all connections destroyed during that avalanche. The value of  $P_c(s)$  is then obtained by identifying the avalanches whose size is greater than  $s$ .

Figure 5.8 shows the cumulative size distribution of avalanches for different minimum capital levels, keeping  $L = 2000$ . For small avalanche sizes, the Central Limit Theorem holds<sup>92</sup> and thus all size distributions match independently of the minimum capital level. For large enough avalanches ('critical region'), the size distributions deviate from each other, exhibiting a power-law tail  $P_c(s) \sim s^{-m}$  with an exponent  $m$  that depends on the minimum capital level  $c_{th}$  (inset). As expected,<sup>39</sup> the exponent found for the avalanche size distribution takes values in the interval  $2 < m < 7/2$ .

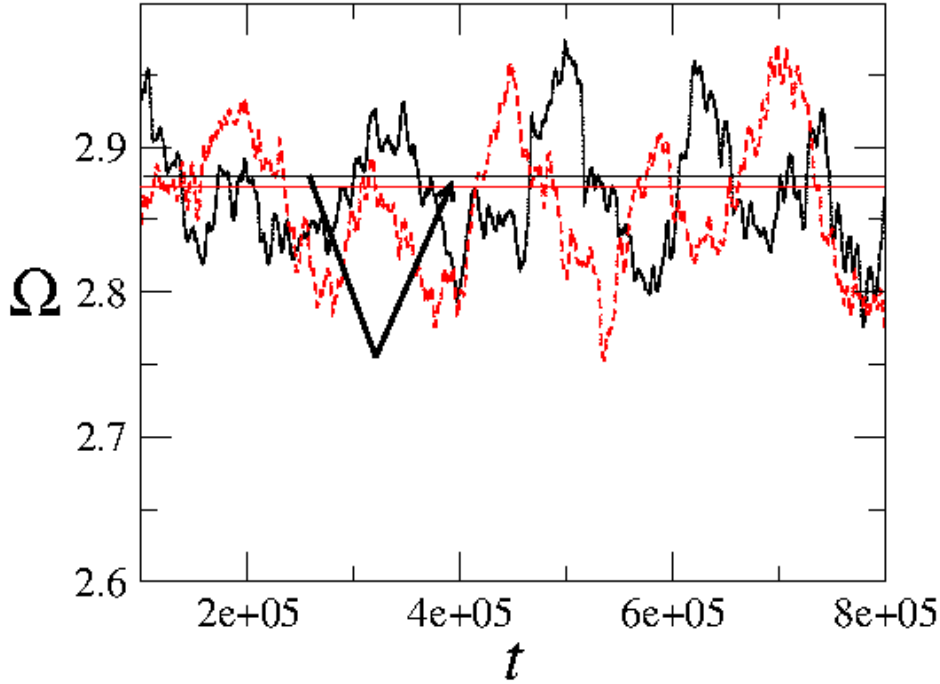
As can be seen in the inset of Fig. 5.8 the exponent increases in absolute value for larger minimum capital levels, indicating a smaller probability for large avalanches to occur. However, this scenario occurs only when the size of the operating neighborhood is kept constant and, as shown in Fig. 5.7, the increase of the minimum capital level is also accompanied by a decrease of the business level. This means that each agent has less economic energy or, in current language, is poorer.

Assuming that agents do not want to be poorer despite regulatory constraints, and therefore try to keep their business levels constant (Fig. 5.9), a natural reaction against raising the minimum capital level is to decrease the number of neighbors with whom the agent establishes



**Figure 5.8:** Avalanche (crises) size distributions for different scenarios of minimum capital level, keeping the operating neighborhood unchanged for each agent. The different distributions match at small sizes, in the region where the Central Limit Theorem (CLT) holds, and deviate from each other for larger crises (critical region). In the critical region one observes (inset) that increasing the minimum capital level decreases the probability for a large avalanche to occur, which supports the intentions of the Basel III accords. However, in this scenario one assumes that each bank will have a simultaneous decrease of their business level (see text and Fig. 5.7). A more natural scenario would be one where each bank reacts to the rise in the minimum capital level in such a way as to keep its business level constant, which leads to a completely different crises situation (see Fig. 5.11).





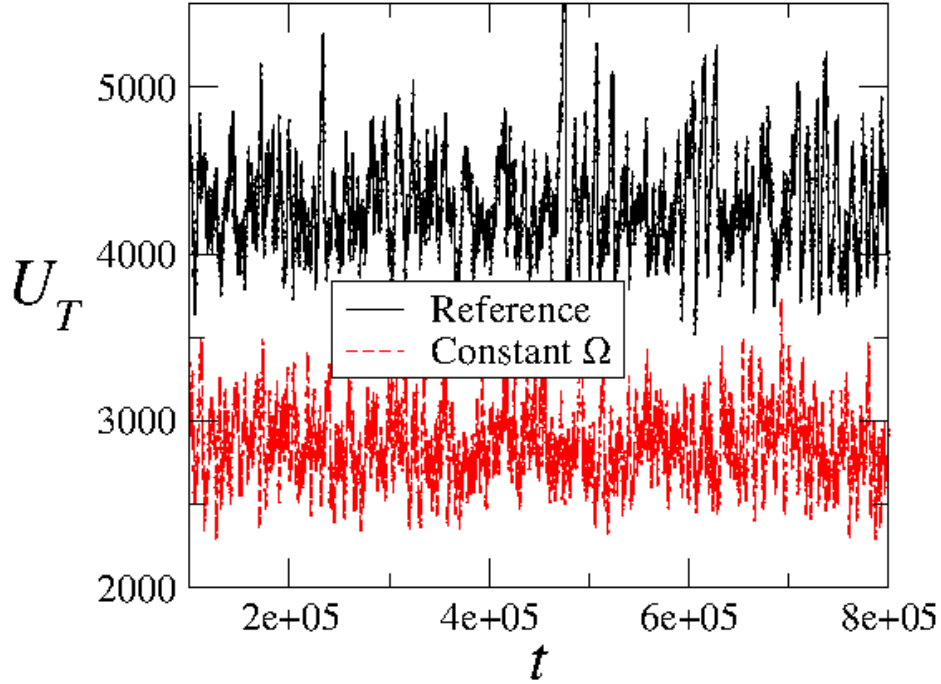
**Figure 5.9:** Unlike in Figs. 5.6 and 5.7 it is possible to raise the minimum capital level  $c_{th}$  from  $-0.71$  (solid line) to  $-0.69$  (dashed line), while keeping the business level  $\Omega$  constant. In the case plotted,  $\Omega \sim 2.88$

trade connections, i.e. to decrease the size of the operating neighborhood (Fig. 5.10). In Economics this is called a concentration process,<sup>42</sup> which typically occurs when the regulation rules are tightened up. In such a scenario where the size of the operating neighborhood is adapted so as to maintain the business level constant, the distributions plotted in Fig. 5.8 are no longer observed. In particular, the exponent  $m$  does not increase monotonically with the minimum capital level as we show next.

Figure 5.11a shows the critical exponent  $m$  and the business level per agent  $\Omega$  as functions of the minimum capital level  $c_{th}$  and the operating neighborhood size  $L$ . For easy comparison, both quantities are normalized in the unit interval of accessible values.

The critical exponent shows a tendency to increase with both the minimum capital level and the operating neighborhood size. The business level, on the other hand, decreases when the minimum capital level or the neighborhood size increase. Considering a reference state  $F_0$  with  $c_{th,0}$ ,  $L_0$  and  $\Omega_0$  there is one isline of constant minimum capital level,  $\Gamma_{c_{th}}^0$ , and another of constant operating neighborhood size,  $\Gamma_L^0$ , crossing at  $F_0$ . Assuming a transition of our system to a larger minimum capital level at isline  $\Gamma_{c_{th}}^f$  while keeping  $L$  constant, i.e. along the isline  $\Gamma_L^0$ , one arrives at a new state  $F_L$  with a larger critical exponent, which means a lower probability for large avalanches to occur, as explained above. However in such a situation the new business level  $\Omega_f$  is lower than the previous one  $\Omega_0$ .

On the contrary, if we assume that the transition from  $F_0$  to the higher minimum capital level occurs at constant business level, i.e. along the isline  $\Gamma_\Omega^0$ , one arrives to a state  $F_\Omega$  on the isline  $\Gamma_{c_{th}}^f$  for which the critical exponent is not necessarily smaller than for the initial state.



**Figure 5.10:** Keeping the business level constant at  $\Omega \sim 2.88$  and raising the minimum capital level from  $-0.71$  (solid line) to  $-0.69$  (dashed line) leads to a decrease of the operating neighborhood, which is reflected in a lower overall product.

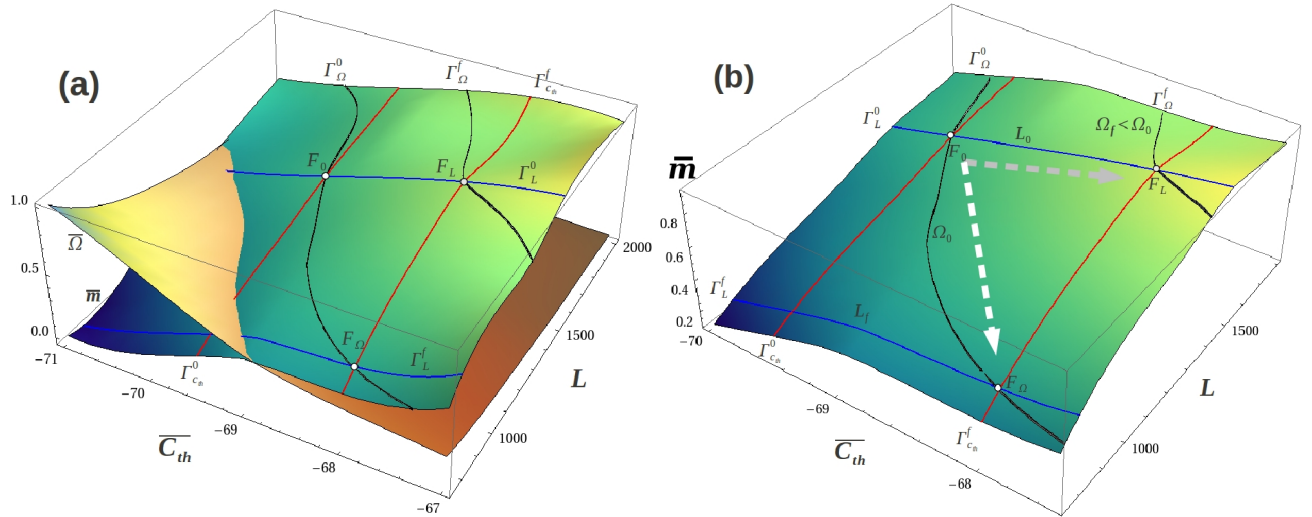
From economical and financial reasoning, one typically assumes that, independently of external directives, under unfavorable circumstances economical and financial agents try, at least, to maintain their business level. This behavior on the part of agents leads to a situation which contradicts the expectations of the Basel accords and raises the question of whether such regulation will indeed prevent larger avalanches from occurring again in the future. To illustrate this, Fig. 5.11b shows a close-up of the  $m$ -surface plotted in Fig. 5.11a.

For the reference state  $F_0$  one finds an exponent  $m = 2.97 \pm 0.18$ . An increase of the minimum capital level at constant operating neighborhood size (state  $F_L$ ) yields  $m = 3.34 \pm 0.09$ , while increasing the minimum capital level at constant business level (state  $F_\Omega$ ), yields  $m = 2.79 \pm 0.09$ , which corresponds to a significantly higher probability that large avalanches will occur.

#### 5.2.1.5 Discussion and conclusion

In summary, raising the minimum capital levels may not necessarily improve banking system resilience. Resilience may remain the same if banks go after the same business levels, as one should expect, according to economic reasoning. Indeed, since business levels are part of the achievement of any economic agent that enters a network of trades, each agent will try, at least, to maintain this level, independently of regulatory constraints.

Furthermore, our findings can solve the apparent contradiction between the credit risk models that serve as the theoretical foundation for bank stability accords and the definition of capital



**Figure 5.11:** (a) Normalized critical exponent  $\bar{m}$  and business level  $\Omega$  as functions of the minimum capital level  $c_{th}$  and system size  $L$ . For an initial financial state  $F_0$ , an increase of the minimum capital level takes the system along one of the infinitely many paths between the initial and final isolines at constant minimum capital level,  $\Gamma_{c_{th}}^0$  and  $\Gamma_{c_{th}}^f$  respectively. (b) If such a path follows the isoline at constant system size,  $\Gamma_L^0$ , the critical exponent increases and thus the probability for large avalanches decreases. Simultaneously however, its business level decreases ( $\Omega_f < \Omega_0$ ), which runs against the natural intentions of financial agents. On the contrary, if the path is along the isoline at constant business level,  $\Gamma_\Omega^0$ , as one naturally expects the financial agents would do, the critical exponent does not change significantly, meaning that large financial crises may still occur with the same probability as before (see text).

levels. In fact, bank stability accords impose on banks an adapted version of Merton-Vasicek model<sup>132</sup> in which it is assumed that each agent has a leverage threshold above which it defaults on credit. The assumption of this threshold combined with a first principle of Economics – that the Economy emerges from the exchanges between agents – naturally leads to an interplay between agents that can propagate the effect of one default throughout the entire economic system.

Economic systems have long-range correlations and heavy-tailed distributions that are not compatible with a linear assumption that raising individual capital levels will lead to stronger stability. Because of the interdependency, this assumption is probably valid only in two situations: when it is impossible for an individual to default; and when individuals behave independently from each other (random trade connections). Both situations do not occur in real economic systems.

These findings can inform the recent governmental measures for dealing with the effects of the 2008 financial crises. In particular, governments have shown<sup>110</sup> a tendency for imposing a higher capital investment from banks. If the threshold is increased, while the total amount of trade remains constant, there will be fewer trade connections between the banks and their clients, which leads to smaller avalanches in the evolution of the financial network. On the other hand, if the total amount of trade is assumed to grow, following the rise in minimum capital, the probability of greater avalanches will also increase to the level where it was before or even to a higher level.

The scale-free topology of the economic network plays a major role in the determination of the

size distribution of the avalanches. At the same time, the scale-free topology emerges naturally from the rules introduced, which are motivated by economic reasoning, namely the principles of demand and supply. Still, one could argue that for bank regulation purposes, a different (imposed) topology for the connections between financial agents would help to prevent large crises. For example, if the economic network is structured as a random Erdős-Rényi network,<sup>50</sup> in which every economic agent has the same probability of being chosen to form an economic connection, the system would not have avalanches. In such a model, since connections are equally distributed throughout the system, all agents would have statistically the same balance. In other words, for each bankruptcy the expected number of child bankruptcies in the avalanche would have either zero size or the size of the system. Thus, with Erdős-Rényi topology, one expects still the danger of triggering such a large chain of insolvencies able to collapse the entire system.

Directives more oriented to the connection topology emerging in the financial network could be a good alternative. Interestingly, although controversial, our claims point in the direction of IMF reports in November 2010,<sup>115</sup> where it is argued that rapid growth in emerging economic periods can be followed by financial crises, and also to recent theoretical studies on the risk of interbank markets.<sup>60,67</sup> Indeed the recent IMF Memorandum on Portuguese economic policy<sup>32</sup> already includes directives that reveal IMF's concern not only with tuning capital buffers and other local properties but also with monitoring the banking system as a whole, and in particular keeping track of the financial situation of the largest banks in the network. We believe that such global networking measures are much more trustworthy than local ones.

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### 5.2.2 Economic Generated Thresholds

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#### 5.2.2.1 Introduction

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The amount of capital of a company is considered an important measure of the health of a company. Not only of banks<sup>38</sup> but also of all other companies. It is one of the major variables in the application to bank credit and, consequently, an important part of rating determination methodologies.<sup>72</sup> In our previous works<sup>37-39</sup> we have been dealing with criticality assuming the existence of a threshold in the economic system that, when experimented by economic agents, they bankrupt and cease to be a part of the economic dynamics, i.e., stop forming economic connections of production and consumption.<sup>37</sup> That threshold can be expressed in forms of a *capital level*, the amount of resources owned by the agent<sup>63</sup> itself. The existence of such minimum was postulated in the financial context by Merton<sup>98</sup> and used by Vasicek<sup>132</sup> for the development of credit risk models that are used today in banking regulation<sup>108,109</sup> as we saw before.

The existence of such thresholds, in conjunction with the permanent growth of the economic connections (see Chapter 3) is enough to explain the negative half of the return distributions.<sup>37</sup> But we are short of one threshold for the positive side of the return distribution that should be related also with capital levels or, at least, highly dependent on capital levels.

To study the occurrence of such capital levels we deal here with accounting data. Accounting is a report of the economic health of a particular agent in which we can obtain the measures of the resources allocated to or by it. For the definition of *resource* readers should see Hoag's book<sup>63</sup> in reference or Chapter 2. The particular economic agent corresponding to the accounting report is referred in this section as *the company*.

A resource allocated by the company to other agents is called an *asset* and it is expected that the company receives, in the future, more resources than the initial ones for that allocation. It is an economic connection.<sup>37</sup> The total amount of assets we call *total assets*,  $A$ , of the company. A resource allocated by another agent to the company is called a *liability* and it is expected that the company delivers, in the future, more resources than the initial ones for that allocation. It is also an economic connection.<sup>37</sup> To the total amount of liabilities we will call *total liabilities*,  $L$ , of the company. In principle, a company must put its own resources in this resource allocation game. For example, to start a company, shareholders give their money to the company to buy machinery. This amount of company own resources can be also looked as resources allocated to the company by the shareholders that expect to receive more resources than the initial ones and it is called *the capital*,  $C$ , of the company. It is expected from an accounting report that the total allocated resources *by* the company equals the resources allocated *to* the company, i.e.,

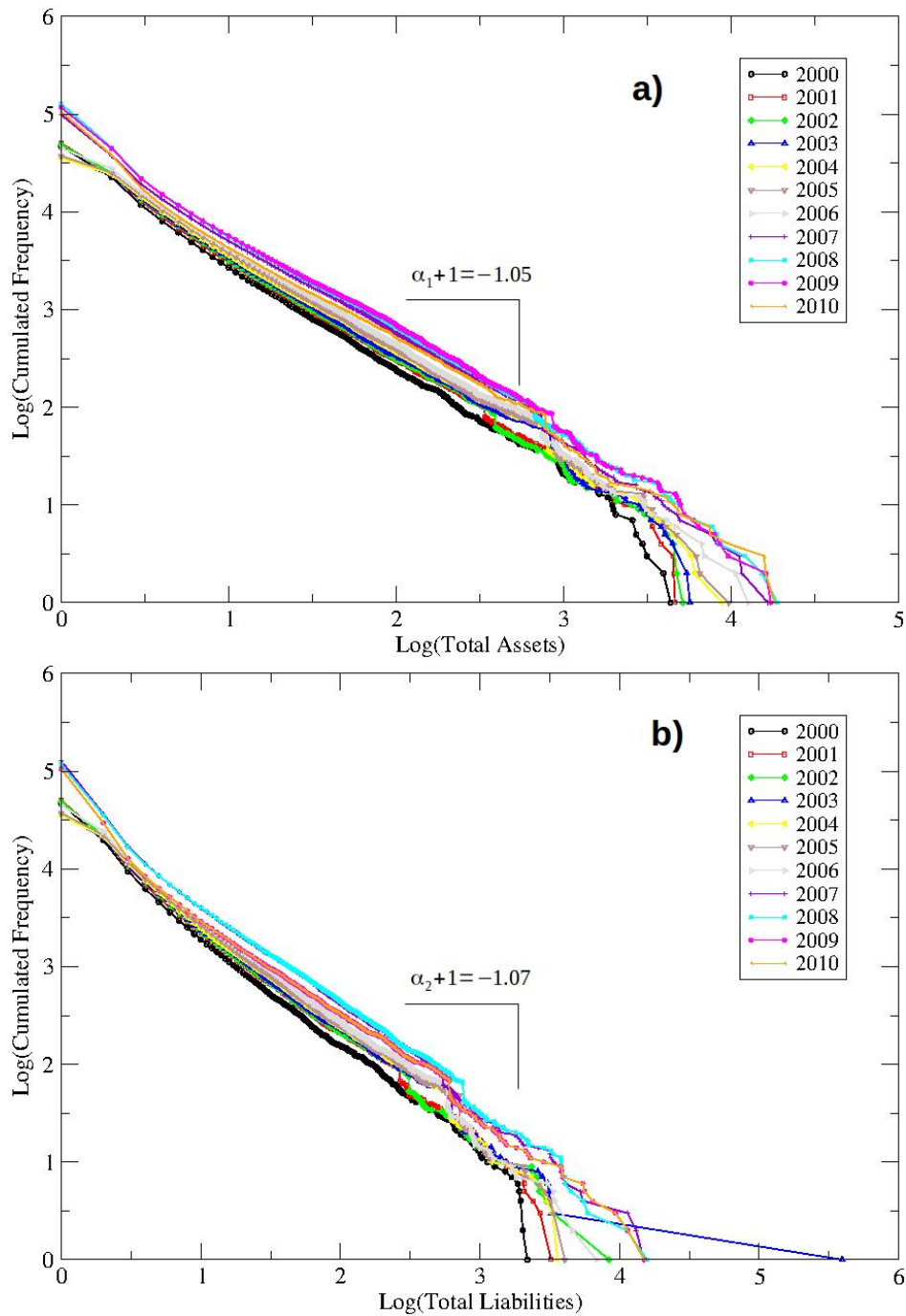
$$A = C + L. \quad (5.13)$$

It is common to remove the dimensionality from the capital measures by using dimensionless capital ratios,  $A/L$  or  $L/A$ , which are better for company comparison and risk measurement. We will refer to these ratios as *asset-to-debt* and *debt-to-asset*, respectively.

In this section, we complete the approach on the study of criticality in economy, by explicitly showing that in fact the natural capitalistic market yields thresholds for local individual capitals, independently from regulations. We address this problem by studying the total assets and total liabilities of more than half a million balance sheets of Portuguese companies and their corresponding asset-to-debt and debt-to-asset ratios. Our results show that a long range correlations between assets and liabilities exist and further that such long range correlations evidences the existence of a thresholds for insolvency and over-capitalization which causes the observed critical behavior.

#### 5.2.2.2 Data Analyzed

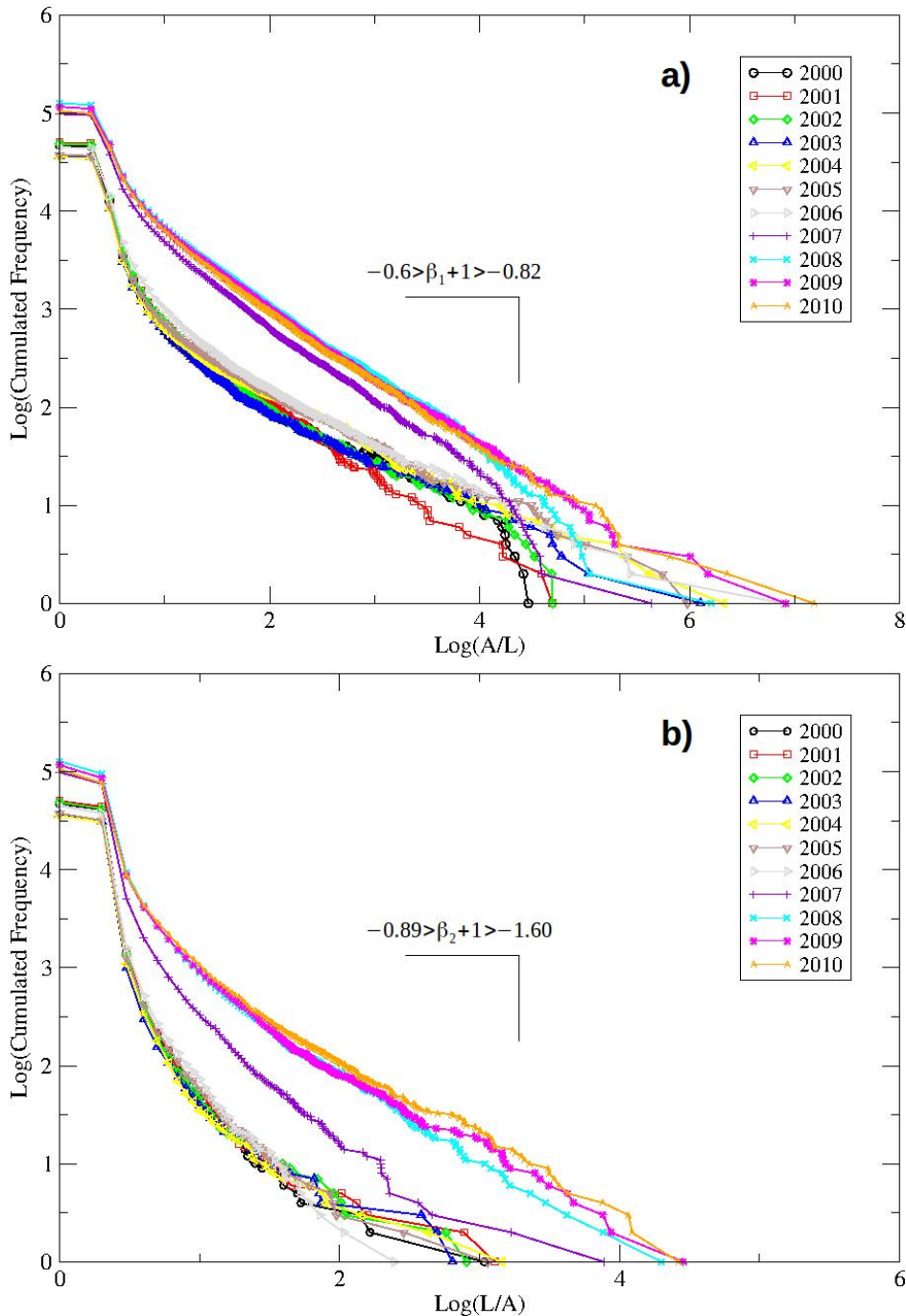
We analyze a sample consisting of end-of-year balance sheet records of more than  $4 \times 10^4$  Portuguese companies in 2000 to more than  $1.3 \times 10^5$  in 2010 in a total of over  $7 \times 10^5$  registers. Each register gives the total assets and the total liabilities of one company at the end of one specific fiscal year, which in Portugal is usual for companies to take the end of calendar year for that. According to the Portuguese National Statistics Institute (INE),<sup>66</sup> in its 2009 report on business activities, the Portuguese economy registered on that year 349511 companies, summing a turnover of approximately 318 billion Euros and employing 2.9 million individuals. The data we analyze comprises approximately one third of the total possible universe for



**Figure 5.12:** (a) Empirical volume distribution of total assets  $A$ , (b) and of total liabilities  $L$ , For the data set of Portuguese companies, both variables distribute according to Pareto's law with a very closed range of values. Both assets and liabilities have approximately the same exponent value,  $\alpha_1 + 1 \simeq -1.05$  and  $\alpha_2 + 1 = -1.07$  respectively. The data sets contain the full register of assets and liabilities from  $\sim 7 \times 10^4$  Portuguese companies, between 2000 and 2010. [Data made available by Millennium BCP.]



that year, and was collected by Millennium BCP bank (BCP.LS). This bank is the biggest private bank in Portugal and one of the earlier adopters of risk evaluation policies in the world, using balance sheet analysis in credit risk evaluation since 1999 and keeps a database with the companies that asked for credit to the bank. Balance sheet data is public accessible information under the Portuguese law, so no banking secret was violated for this study.



**Figure 5.13:** (a) Empirical distribution of asset-to-debt  $A/L$ , (b) and of debt-to-asset  $L/A$ . Unlike total assets and total liabilities, the ratios do not present a stable distribution in time. The exponents reveal a relatively wide range of values (one order of magnitude above  $A$  and  $L$  individually). [Data made available by Millennium BCP.]

Registers with either zero total assets or zero total liabilities were removed, reducing the sample

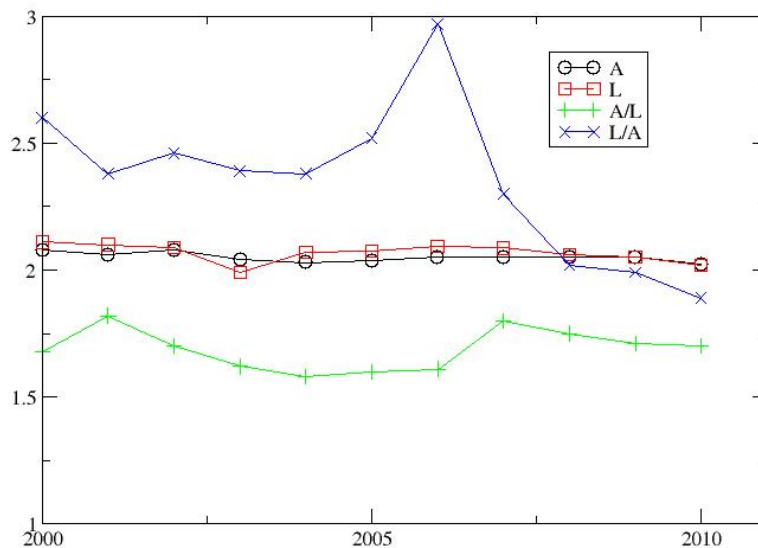
by approximately 4000 registers. We extract their cumulative distributions and compare them with the one of the ratio between total assets and total liabilities.

We start with the cumulative distributions. As shown in Fig. 5.12, both the total assets and liabilities volumes follow a Paretian distribution with approximately the same exponent, namely  $\alpha_1 + 1 = -1.05$  for the distribution of assets and  $\alpha_2 + 1 = -1.07$  for the distribution of liabilities.

In Fig. 5.13 we plot the distribution of both ratios, finding again a significant range of values which follow Paretos law. However, the exponent derived for both ratio are different than the ones found for the asset and liabilities distribution:  $-0.6 > \beta_1 + 1 > -0.82$  for the asset-to-debt ratio and  $-0.89 > \beta_2 + 1 > -1.6$  for the debt-to-asset ratio. In Fig. 5.14 we plot the evolution in time of the distribution exponents for total assets, total liabilities, asset-to-debt and debt-to-asset for the Portuguese companies.

The evolution for total assets and total liabilities corroborates the results from Chapter 3. In fact, the stability of the exponents for the company domain reproduce the ones for the Romanian social security<sup>43</sup> for individuals. In the company domain there is no minimum wage or social support, which explains the non-existence of a different behavior for the lower range and for the tail of the distribution,(see Fig. 5.12).

The evolution for the exponents of the distributions of asset-to-debt and debt-to-asset show that these are considerable more volatile than the marginals,  $A$  and  $L$ . That, by itself, suggests that the distribution of the ratios can be indicative of the state of the economy. But what it is interesting for our purposes is the fact that the distribution exponents are different between themselves and different from the ones for the distributions of total assets and total liabilities.



**Figure 5.14:** Evolution of the distribution exponents in time.



Here we raise the first question: is there a relationship between the exponents observed for the risk indicators and the corresponding “marginal” distributions of total assets and total liabilities? We will in the next section answer positively to this question by showing that the exponent of the ratio distributions is related to the exponent of the marginals through the non-zero correlation between total assets and total liabilities.

The second question is: what conditions for total asset and total liabilities should be fulfilled for the emergence of such a difference between the ratios? In this section we will argue that the emergence of threshold levels is evident from the study of the ratios  $A/L$  and  $L/A$ , from which we will argue that the correlation between  $A$  and  $L$  is not constant through their domains.

### 5.2.2.3 Model

In this Section we put the accounting variables describing total assets and total liabilities in a more general context. We start by considering two random variables,  $X$  and  $Y$ , following the Pareto law with the same exponent, namely their cumulative density functions are defined by

$$f_X(x) = \frac{(\alpha - 1)x_0^{\alpha-1}}{x^\alpha} \quad (5.14a)$$

$$f_Y(y) = \frac{(\alpha - 1)y_0^{\alpha-1}}{y^\alpha} \quad (5.14b)$$

with  $x \geq x_0 > 0$  and  $y \geq y_0 > 0$ , where  $x_0$  and  $y_0$  are the minimum values for total assets and total liabilities, respectively.

Next, consider the random variable  $R = X/Y$  and derive explicitly its distribution. Since both random variables  $X$  and  $Y$  are strictly positive, the probability density function (PDF) of ratio  $R$  is given by<sup>36</sup>

$$p_R(z) = \int_0^{+\infty} y f_{X,Y}(zy, y) dy \quad (5.15)$$

where  $f_{X,Y}$  is the joint probability density function of  $(X, Y)$ ,  $x$  and  $y$  are the realizations of  $X$  and  $Y$  and  $z = x/y$  is the realization of  $R$ .

In case  $X$  and  $Y$  are independent it follows from Eqs. (5.14) that

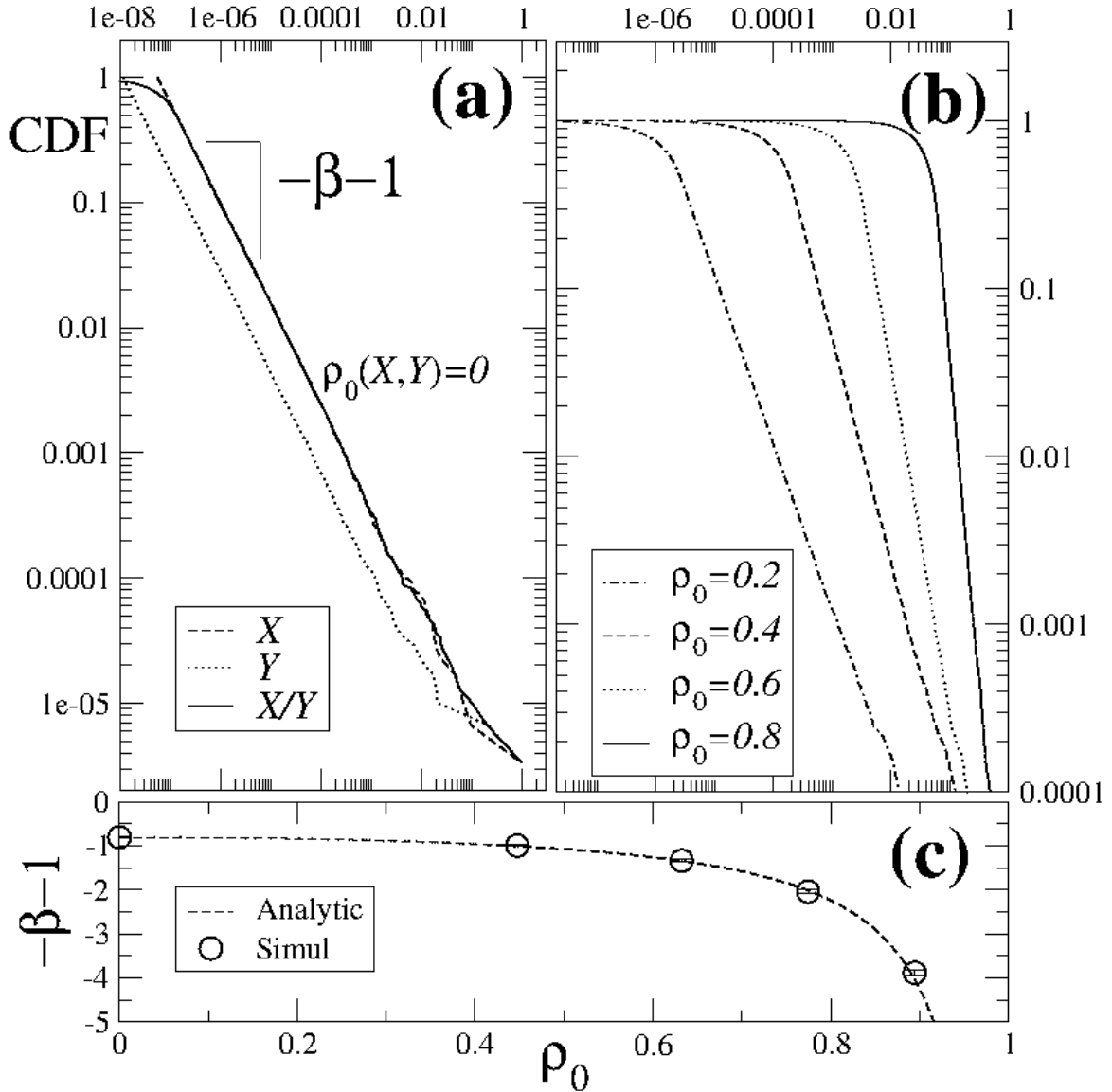
$$f_{X,Y}(zy, y) = f_X(zy)f_Y(y) = (\alpha - 1)^2 \frac{(x_0 y_0)^{\alpha-1}}{(zy^2)^\alpha} \quad (5.16)$$

and substituting Eq. (5.16) in Eq. (5.15) yields

$$P_R(z) \propto z^{-\alpha} \int_{y_0}^{+\infty} y^{-2\alpha} dy \propto z^{-\alpha} \quad (5.17)$$

where condition  $y \geq y_0 > 0$  is used. Thus, since this the integral on the right-hand side of Eq. (5.17) exists ( $\alpha > 0$ ) we can conclude that if the total assets and the total liabilities are

uncorrelated from each other ( $\rho = 0$ ) and follow Pareto's distribution with the same exponent, their ratio will also follow a Pareto distribution with the same exponent value.



**Figure 5.15:** Simulated ratio distributions of correlated Pareto distributions. In (a) we show the distribution of two random variables,  $X$  and  $Y$ , following Pareto's distributions with the same exponent and being independent from each other, i.e. their correlation  $\rho(X, Y) = 0$ . The result for the ratio distribution (solid line) is an exponent  $\beta$  that equals the exponent for  $X$  and  $Y$ . See text for the analytical proof of this. Again all variables are normalized to maximum value observed in our simulations. In (b) we plot the distribution for two Pareto distribution with the same exponent as in (a) but having a non zero correlation value, namely  $\rho = 0.2, 0.4, 0.6$  and  $0.8$ . Clearly the exponent  $\beta$  varies with correlation  $\rho$ . In (c) one sees that the exponent computed for the four examples in (b) through a least square fit (circles) is well predicted by Eq. (5.20) represented by the dashed line. For details on the methods for such correlated distribution see our section devoted to our methods.

This is however not the case of Fig. 5.14. The ratio show significant deviations from the exponent observed for total assets and total liabilities separately. The hypothesis that distributions are uncorrelated must therefore be dropped.

From an economic reasoning it makes sense that the independence is dropped. We will make

use of Fig. 5.16 to explain this. From Eq. (5.13) seems as mathematically trivial that  $A$  and  $L$  are not independent. But that is not so straightforward since  $C$  can be a cushion for every change in  $A$  or  $L$ . And, if there are no constraints for  $C$ , then  $A$  and  $L$  are independent. The constraints result from the economic nature of the measures. In fact,  $A$  and  $L$  are not limited in growth since they represent economic connections and they should grow from the set of multiplicative processes constituents of an economy as described in chapter 3. But as we mentioned in the definition of capital, the shareholders allocated their resources to the company to get more resources in return. Meaning that the growth of  $A/L$  is constrained by the shareholder pressure to get the results from the company. Therefore, the distribution for the upper range of  $A/L$  values (for the lower range of  $L/A$  values) is "pressured" not to grow in the same manner as  $A$  or  $L$ . Also, the creditors allocated their resources to the company to get more resources in return, not to lose them. In that sense, they need the company to own resources to pay back the credits. The growth of  $L/A$  is constrained by this "pressure" of the creditors. This "pressures" the distribution for higher values of  $L/A$  (lower values of  $A/L$ ).

The nature of these "pressures" is functionally the same, considering the company and its neighborhood in a complex network modeling such as the ones described in Chapters 3 and 4. The differences rely on the type of economic relation the neighboring agents have with the company, shareholder or creditor, but both relations are resource allocation and, thus, economic connections.<sup>37</sup> Thus, some correlation is formed between  $A$  and  $L$  that depend on their ratios resulting from the two mechanisms.

Next, we consider the more general case of ratio distributions, defined in Eq. (5.15), together with the assumption of Paretian marginal distributions, but where the random variables  $X$  and  $Y$  are characterized by a constant correlation coefficient  $\rho$  for their logarithmic returns,  $\log X$  and  $\log Y$  respectively.

The typical case for a joint probability density function with Paretian marginals is a so-called bivariate Pareto distribution of type II.<sup>77,94</sup>

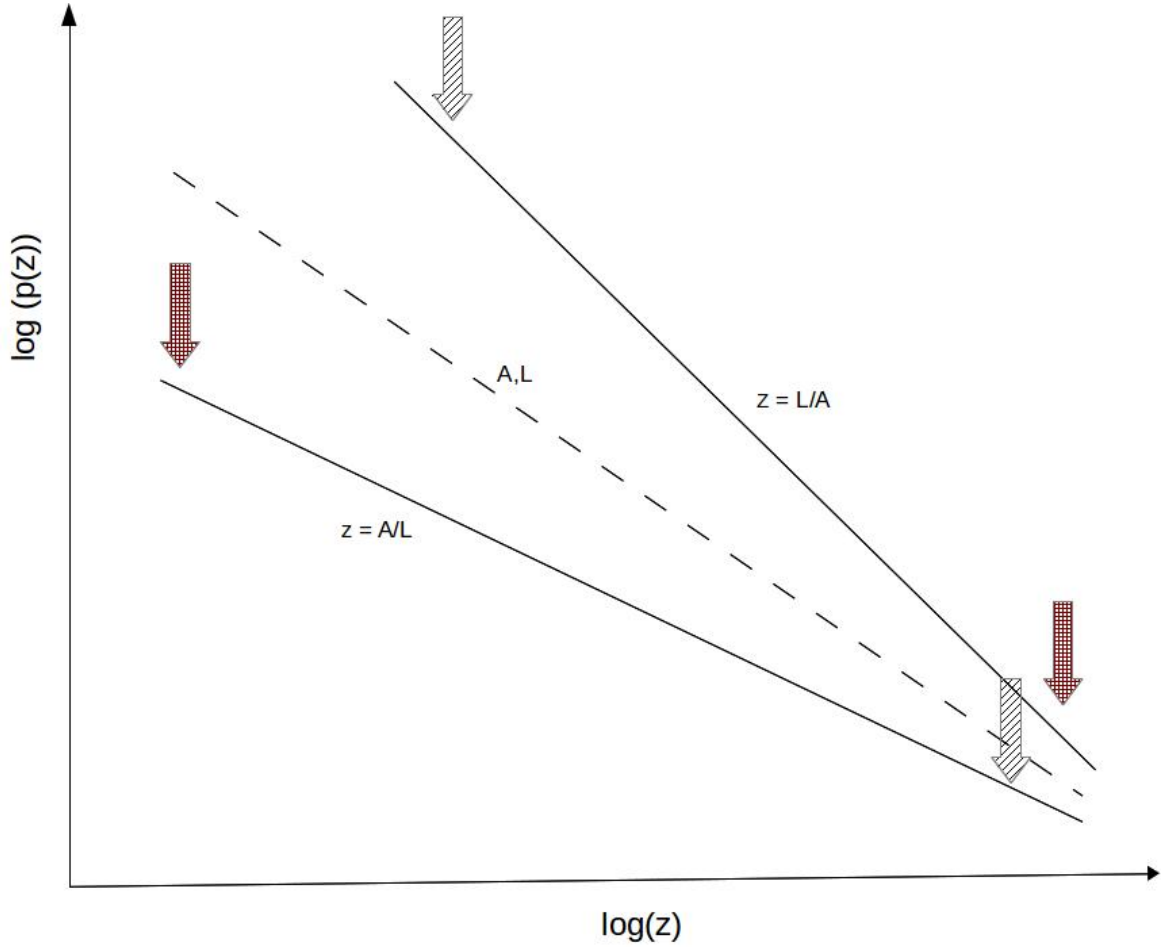
$$f_{X,Y}(x,y) = \frac{(\alpha - 1)^2}{(1 - \rho)xy} \left[ \left( \frac{x_0}{x} \right)^{\alpha-1} \left( \frac{y_0}{y} \right)^{\alpha-1} \right]^{1/(1-\rho)} I_0[W] \quad (5.18)$$

where  $I_0(W)$  is the modified Bessel function of the first kind of order zero<sup>58</sup> and

$$W = \frac{2(\alpha - 1)\sqrt{\rho \log\left(\frac{x}{x_0}\right) \log\left(\frac{y}{y_0}\right)}}{1 - \rho}. \quad (5.19)$$

In Eq. 5.19 we assume the same exponent  $\alpha$  for both marginal distributions  $X$  and  $Y$  and consider the rules for joint gamma distributions.<sup>73</sup>

From Eq.(5.15) and Eq.(5.18), the differences in exponents on inverse ratios like  $A/L$  and  $L/A$  suggest that the correlation between  $\log(A)$  and  $\log(L)$  is different when we look at the results of  $A/L$  and  $L/A$ , which seems to be contradictory. To overcome this contradiction we address the hypothesis that Pearson correlation,  $\rho$ , is not the proper measure of the existing dependence between  $A$  and  $L$  and that is shown by the asymmetry of the exponents of the distributions of  $A/L$  and  $L/A$ . Therefore, we add a new measure, based on Eq.(5.18), which we call "piecewise correlation",  $\rho_z$ , in which we intend to capture an asymmetric relation



**Figure 5.16:** Schematic representation for the distributions for total assets, total liabilities, asset-to-debt and debt-to-asset. The arrows with 45 degree hatching represent the shareholders pressure to get their return on investment. The arrows with the square hatching the lower capital level(see text).

between  $A$  and  $L$  of the type  $A \geq \zeta L$ , with  $\zeta$  being a constant. In this particular case, for  $A, L > 0$  the Person correlation would be non-zero, since there is an actual relation between  $A$  and  $L$ . We emphasize that  $\rho_z$  is not an actual Pearson correlation in the sense that we can measure it around  $z$ . It is just a form of decomposition of  $\rho$  as expressed in Eq.(5.18) to give us information about the asymmetry, which is the interesting point for us in this scope.

Substituting Eqs. (5.18) and (5.19) in Eq. (5.15) and defining  $\beta$  as

$$\beta = \frac{\alpha - 1}{1 - \rho} + 1 \quad (5.20)$$

leads to

$$p_R(z, \rho) = h(z, \rho)G(z, \rho) \quad (5.21)$$

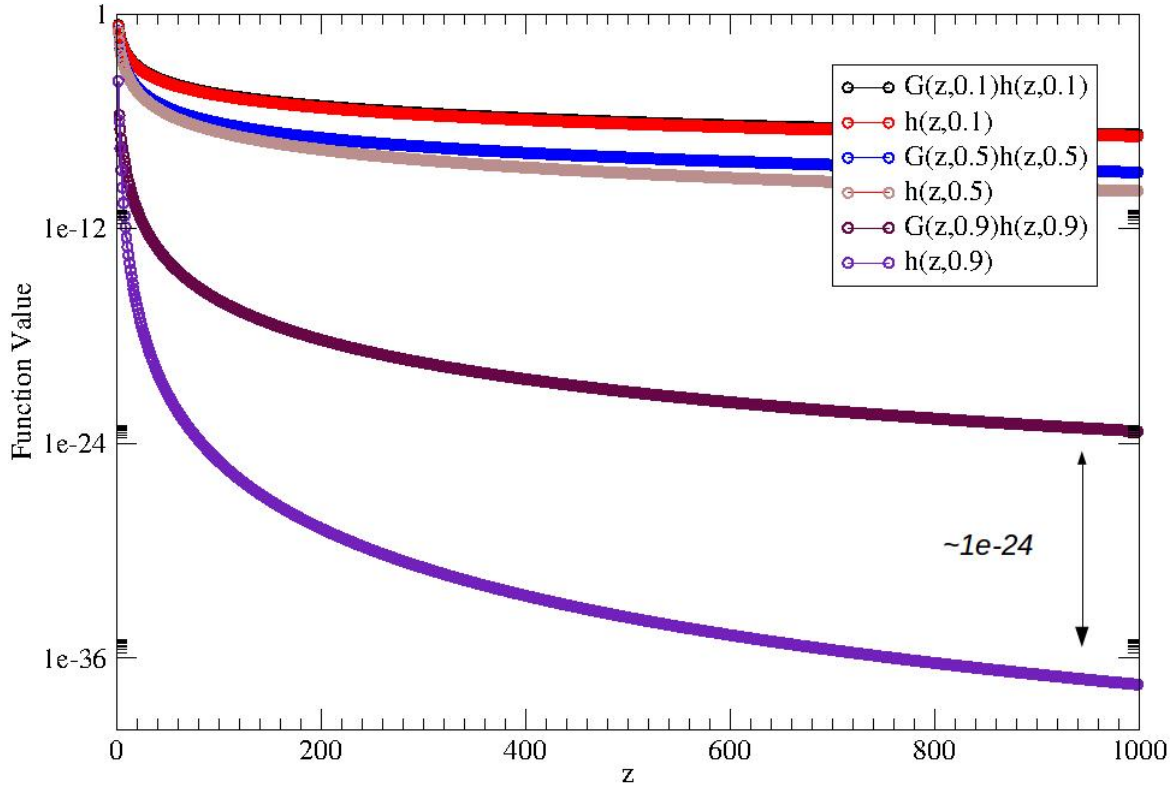
where  $h(z, \rho)$  is the power-law

$$h(z, \rho) = (\alpha - 1)(\beta - 1)z^{-\beta} \quad (5.22)$$

and function  $G(z, \rho)$  is defined by

$$G(z, \rho) = \int_0^\infty y^{-(2\beta-1)} I_0[W] dy. \quad (5.23)$$

Though complicated it can be numerically calculated since the integral is finite. From Fig. 5.17 we can see the difference between  $h(z, \rho)$  and  $G(z, \rho)$  for  $\rho = 0.1, 0.5, 0.9$  To simplify Eq.(5.21)



**Figure 5.17:** Log-Lin plot of  $G(z, \rho)h(z, \rho)$  and  $h(z, \rho)$  for  $z \rightarrow \infty$ . Since  $h(z, \rho)$  drops to zero much faster than the Bessel modified function of the first kind of order zero the difference between multiplying by  $G(z, \rho)$  is negligible. Since  $G(z, \rho)$  has a minimum of 1 as  $z \rightarrow 0$  then we can change  $G(z, \rho)$  by a constant with a value 1.

we will make the approximation  $G(z, \rho)h(z, \rho) \sim \chi h(z, \rho)$  with  $\chi \sim 1$  being a constant, since the differences between  $G(z, \rho)h(z, \rho)$  and  $h(z, \rho)$  are very small according to the numerical results presented on Fig. 5.17, yielding

$$p_R(z) = (\alpha - 1)(\beta - 1)\chi z^{-\beta} = -(\alpha - 1)\chi \frac{\partial}{\partial z} z^{-\beta+1}. \quad (5.24)$$

Integrating both sides of Eq.(5.24) with respect to  $z$

$$\int_0^z p_R(z') dz' = -(\alpha - 1)\chi \int_0^z \frac{\partial}{\partial z'} z'^{-\beta+1} dz', \quad (5.25)$$

where  $z'$  the integration variable. Thus,

$$\int_0^z p_R(z') dz' = (\alpha - 1)\chi z^{-\beta+1}, \quad (5.26)$$

$$\log \left[ \int_0^z p_R(z') dz' \right] = \log [(\alpha - 1)\chi] + -(\beta - 1) \log(z), \quad (5.27)$$

Substituting  $\beta$  and manipulating, yields

$$\rho_z = 1 + (\alpha - 1) \frac{\log(z)}{\log \left[ \int_0^z p_R(z') dz' \right] - \log [(\alpha - 1)\chi]}. \quad (5.28)$$

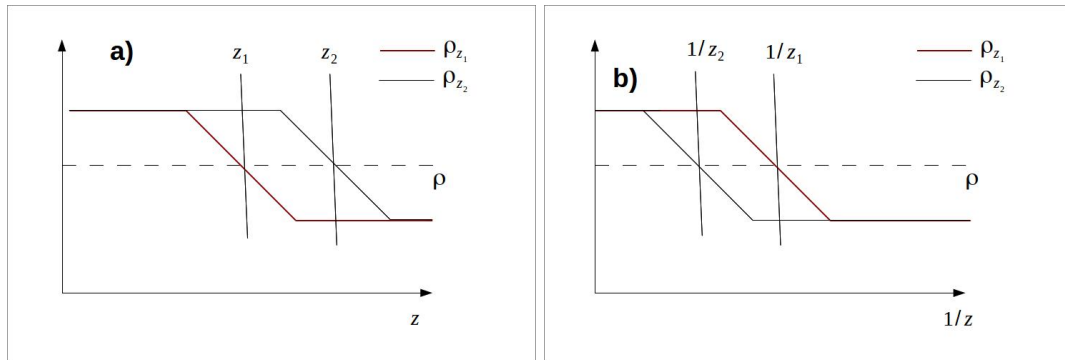
The quantity  $\rho_z$  it is assumed to be constant in the interval  $(z, z + dz)$ . If we assume  $\chi = 1$  then we can rewrite Eq.(5.28) as

$$\rho_z = 1 + (\alpha - 1) \frac{\log(z)}{\log \left[ \frac{P_R(z)}{(\alpha - 1)} \right]}. \quad (5.29)$$

where  $P_R(z)$  is the cumulative distribution function  $P(R \leq z)$  and the inverse ratio  $1/z$

$$\rho_{\frac{1}{z}} = 1 + (\alpha - 1) \frac{\log(1/z)}{\log \left[ \frac{P_{1/R}(1/z)}{(\alpha - 1)} \right]}. \quad (5.30)$$

where  $P_{1/R}(1/z)$  is the cumulative distribution function  $P(\frac{1}{R} \leq \frac{1}{z})$ .



**Figure 5.18:** Schematic explanation of how exponents of ratio distributions are influenced by cumulative distribution on the ratio and it is translated by piecewise correlation. In **a)**  $\rho_{z_1}$  represent the evolution of piecewise correlation where the range above  $\rho$  is similar to the range below  $\rho$  and  $\rho_{z_2}$  an evolution of piecewise correlation where the ranges are considerably different. In **b)**, the same representation in  $1/z$  domain.

Equations (5.29) and (5.30) reduce both to Eq.(5.20) if we consider then both distributions are Pareto distributions with the same exponent. We are interested in using Eq.(5.29) and Eq.(5.30) with the empirical cumulative distribution functions,  $P_R$  and  $P_{1/R}$ , since we know that the exponents of the distributions are not equal. Thus, we can take the Pearson correlation as reference measure and rewrite Eq.(5.29) and Eq.(5.30) as

$$\rho_z - \rho = (\alpha - 1) \log(z) \left[ \frac{1}{\log \left[ \frac{Q_R(z)}{(\alpha - 1)} \right]} - \frac{1}{\log \left[ \frac{P_R(z)}{(\alpha - 1)} \right]} \right] \quad (5.31)$$

and

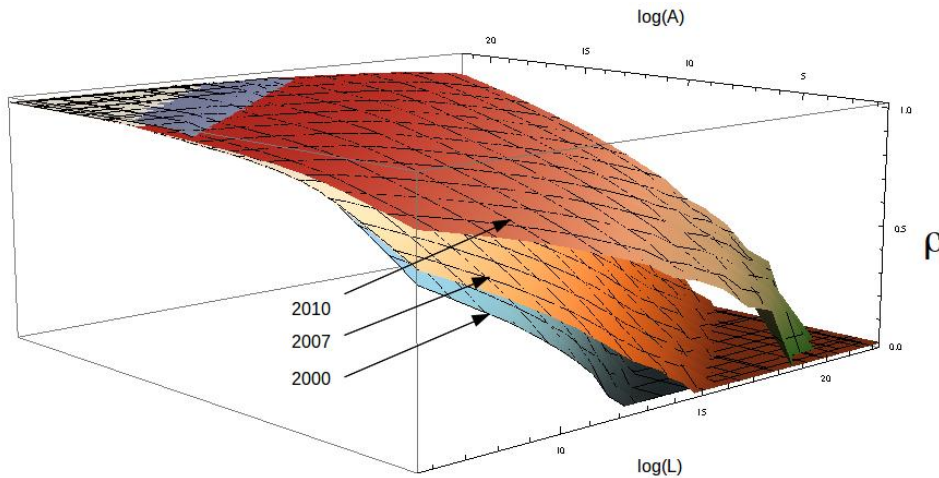
$$\rho_{1/z} - \rho = (\alpha - 1) \log(1/z) \left[ \frac{1}{\log \left[ \frac{Q_{1/R}(1/z)}{(\alpha - 1)} \right]} - \frac{1}{\log \left[ \frac{P_{1/R}(1/z)}{(\alpha - 1)} \right]} \right], \quad (5.32)$$

where  $Q_R(z)$  and  $Q_{1/R}(1/z)$  are the cumulative distribution functions on  $z$  associated with an homogeneous  $\rho_z = \rho_{1/z} = \rho$  over the full  $z$  domain, i.e.,  $\beta_1 = \beta_2 = \beta$  as we seen above.

In Fig. 5.18 we present a schematic explanation on the asymmetry of exponents based on piecewise correlation calculation. In the case of the  $z$  domain, piecewise correlation  $\rho_z$  is higher than the Pearson correlation  $\rho$  when the cumulative density function  $P_R(z)$  is lower for a particular value of  $z$  than the expected cumulative density function  $Q_R(z) = 1 - z^{-\beta}$  for a Pareto distribution with exponent  $\beta$  (see Eq.(5.31)). If there are ranges in the  $z$  domain with values of  $P_R(z)$  lower than  $Q_R(z)$  then there are ranges with  $P_R(z)$  higher than  $Q_R(z)$  because with  $z \rightarrow \infty$  then  $P_R(z) = Q_R(z) = 1$ . Making the difference between Eq.(5.29) and Eq.(5.30) yields

$$\rho_z - \rho_{1/z} = (\alpha - 1) \log(z) \left[ \frac{1}{\log \left[ \frac{P_R(z)}{(\alpha-1)} \right]} + \frac{1}{\log \left[ \frac{P_{1/R}(1/z)}{(\alpha-1)} \right]} \right] \quad (5.33)$$

From Eq.(5.33) and from the schema in Fig. 5.18 we can see that the maximum asymmetry, the absolute value of the difference between  $\rho_z$  and  $\rho$ , occurs when the difference to the  $\rho$  line in Fig. 5.18 is maximum *for both* ratios, i.e., the  $\rho_1$  line. The  $\rho_2$  line will produce a smaller difference between  $\rho_z$  and  $\rho_{1/z}$  and, in the limit where the difference is zero there is only one distribution exponent for the ratios.

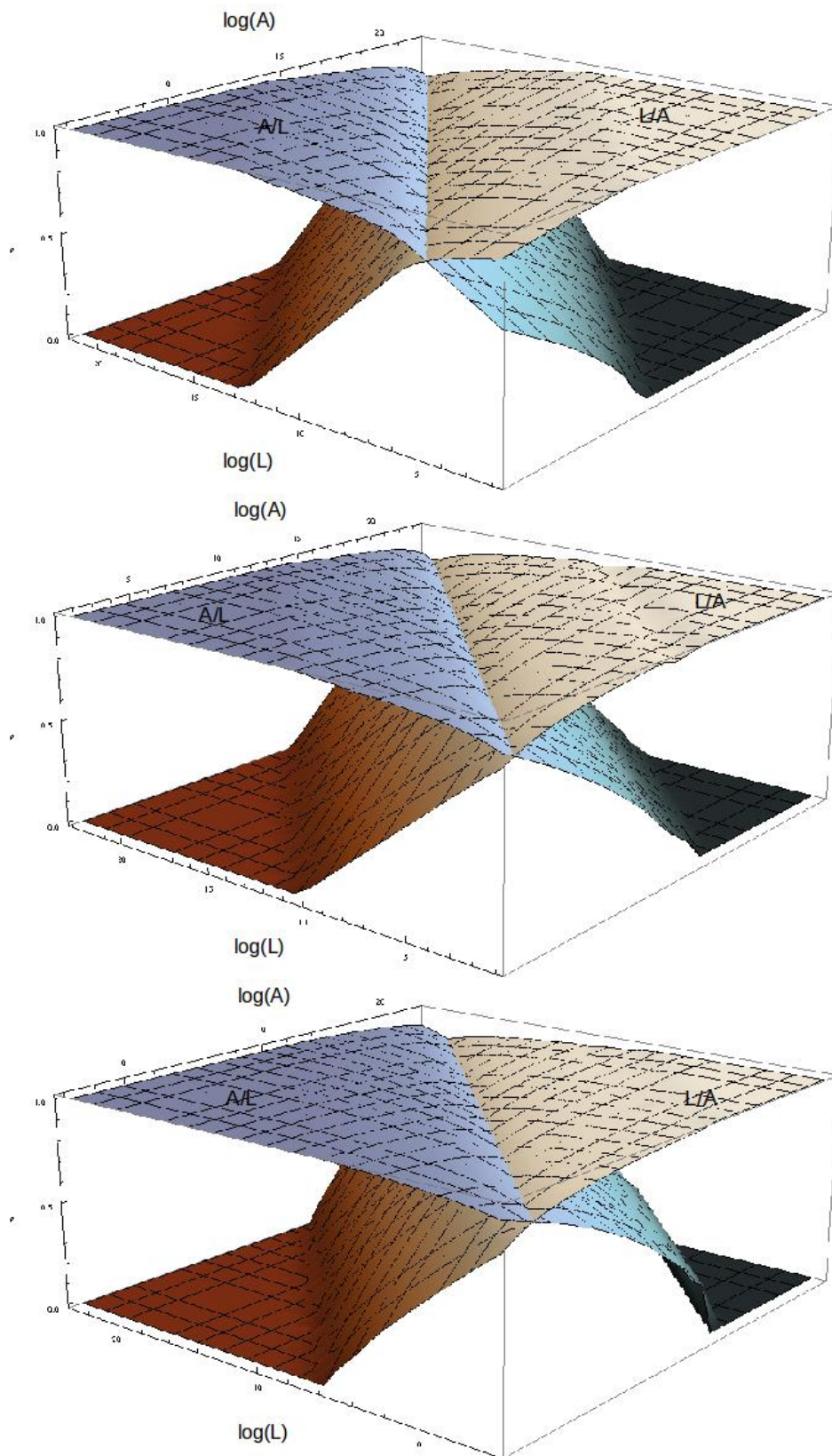


**Figure 5.19:** Evolution of piecewise correlation in time for  $A/L$ . As the crisis deepens the piecewise correlation spans a wider range of values in the  $\log(A) \times \log(L)$  domain

Thus, our technique is to measure directly from data the cumulative distribution functions to show that the measured piecewise correlations evidence the existence of a threshold that changes in time. Schematic explanation of how exponents of ratio distributions are influenced by cumulative distribution on the ratio and it is translated by piecewise correlation. In Fig. 5.18a  $\rho_{z_1}$  represent the evolution of piecewise correlation where the range above  $\rho$  is similar to the range below  $\rho$  and  $\rho_{z_2}$  an evolution of piecewise correlation where the ranges are considerably different. In Fig. 5.18b, we have the same representation in  $1/z$  domain.

In Fig. 5.19 we plot the evolution of the piecewise correlation  $\rho_z$  over the time. The evolution of the high correlation area in the  $\log(A) \times \log(L)$  domain is evident. From 2000 to 2010





**Figure 5.20:** (a) Empirical piecewise correlation  $\rho_z$  for both ratios in 2000, (b) in 2007, (c) and in 2010. The domain of  $A, L$  where  $A/L$  correlation is maximum is much bigger in 2010 than in 2000. Calculations made based on the Millennium BCP data.



we can observe that the area spanned by high correlation grows almost covering the entire domain. This means by

In Fig. 5.20 we represent  $\rho_z$  for  $A/L$  and  $L/A$  in three moments in time chosen by their relevance in the evolution of the exponents shown in Fig. 5.14. In 2000 the exponent for the distribution of  $L/A$  was higher than the exponent for  $A$  and  $L$ , while the exponent for the distribution of  $A/L$  was lower. If we make use of our schematic representation of the two “pressures”, Fig. 5.16, this would mean that the “pressure” in higher values of  $A/L$  (the one produced by shareholders) is more effective than the “pressure” in the lower values of  $A/L$  (the one produced by the system of creditors). In 2010, in the deepening of the Portuguese sovereign debt crisis, the distribution exponent for  $L/A$  is lower than the one for  $A$  and  $L$  which, in the same schematic reasoning would suggest that the “pressure” in the higher values of  $A/L$  become less effective than the “pressure” in the lower values of  $A/L$ .

These schematic reasoning can also be retrieved from Fig. 5.20. As we can see, from 2000 to 2010 the area in the  $A, L$  domain spanned by the higher correlation values of  $A/L$  grows, as the area for higher correlation values of  $L/A$  shrinks, showing that one of the thresholds is advancing and the other is retracting from the  $A, L$  domain.

Why is the term ‘threshold’ used in the last sentence? We should emphasize that what is represented in Fig. 5.20 is a dependence representation. Meaning that a growing value of that piecewise correlation implies that the quantities represented on the axis tend to a linear dependence. If we assume that the maximum correlation is 1 then Fig. 5.20 shows that at the minimum of  $A/L$  we have  $A = \zeta L$  where  $\zeta$  is a positive constant. On all other points where correlation is less than 1, that means that  $A > \zeta L$  or  $A < \zeta L$ . But since the ratio we are studying is  $A/L$ , we know that as  $A/L \rightarrow \infty$  then  $A > L$ , since  $A > 0, L > 0$ . From this reasoning we conclude that  $A \geq \zeta L$ . The same reasoning can be used for  $L/A$  and the threshold for the threshold of stockholders “pressure”.

If we want to be mathematically rigorous we should not use the expression  $A \geq \zeta L$  since in our approach  $A$  and  $L$  are random variables. The correct way to express it is  $A \geq_{st} L$  and  $L \leq_{st} A$ .<sup>75</sup> The symbols  $\geq_{st}$  represents

$$A \geq_{st} L \Leftrightarrow P(A > x) \geq P(L > x), \forall x \quad (5.34)$$

where  $P$  is the cumulative distribution function. An analogous expression is defined for the  $\leq_{st}$  symbol. In practice this means that it is possible that some individuals in the system overcome the threshold, but the probability is very low (see Fig. 5.20).

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#### 5.2.2.4 Summary and Conclusion

In summary, we study a database of balance sheet records from a representative number of Portuguese companies. We obtained the empirical evidence that both asset and liability distributions are Paretian and stable in time, corroborating for companies the result obtained in<sup>43</sup> for individuals and the results of chapter 3, both analytical and empirical.

The ratio of two inverse ratios was studied and are also approximately Paretian, with a completely different exponent and volatile in time. We showed that it indicates the existence of a correlation between the two random variables representing the accounting measures, as expected from the nature of the quantities. Moreover, we show that there exist a difference between the exponents of the ratio total assets/total liabilities and total liabilities/ total assets, that indicates that the correlation is not constant through the full domain of the quantities in study.

We suggest that there are two economic mechanisms that can promote that variable correlation. We modeled correlation between total assets and total liabilities as a function of each of the ratios under study and showed that correlation changes over their domain. The form of theses correlation functions changes in time and we have shown that these economic generated thresholds move in the ratio domain due to the economic environment. The existence of theses two thresholds contribute for the explanation of the form of stock market return distributions,<sup>37</sup> the under-capitalization for the negative side, the over-capitalization for the positive side.

The data we collected does not have a economic sector segregation, meaning that what is imposed over the banking system (see section 5.2.1) is in fact an emergent behavior in the economy dependent on the dynamics of the system. The results from this study can be used in future work to predict the movement of the thresholds and, with that, add some help to model credit risk in portfolios. Also, a future discussion on the consequences of defining administratively minimum capital levels should be made in light of these findings and the ones from previous work.<sup>38</sup>

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### 5.2.3 Bounding Market Risk

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#### 5.2.3.1 Introduction: a note on agent-models for social systems

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Similarly to other fields in social sciences, most of the research made in finance and economics has been dominated by an epistemological approach, in which the behavior of the economic system is explained by a few key characteristics of the behavior itself, like the amplitude of price fluctuations or the analytical form of the heavy-tailed return distributions.<sup>20,93</sup> These key characteristics motivated researchers to assume such distributions as  $\alpha$ -stable Lévy distributions or truncated  $\alpha$ -stable Lévy distributions.<sup>92</sup> The reason for this assumption is given by the more general version of the central limit theorem – sometimes not so well known – which states that the aggregation of a growing number of random variables converges to a  $\alpha$ -stable Lévy distribution.<sup>76</sup> If these random variables have finite variances then the resulting aggregation is a 2-stable Lévy distribution, i.e. a Gaussian distribution. If the variances are infinite – or of the order of the system size – then  $\alpha < 2$  and the so-called heavy-tailed shape emerges as a result of the aggregation. Further, non-Gaussian (heavy-tailed) distributions are associated with correlated variables and therefore it is reasonable to assume that measurements on aggregates of human activities will result in a  $\alpha$ -stable Lévy distribution, since humans

are strongly correlated with each other. Henceforth, we refer to  $\alpha$ -stable Lévy distributions with  $\alpha < 2$  as Lévy distributions and with  $\alpha = 2$  as Gaussian distributions.

Without leaving an epistemological approach, we could address the study of the resulting distributions by ignoring the previous arguments and construct a function that fits any set of empirical data just by building up fitting parameters until the plotted function fit the empirical data. Such approach would be the best one, if economic processes were stationary. Unfortunately they are not,<sup>22,37,130</sup> unless some transformation is made over the measured quantities (see Chapter 3). Thus, we cannot disregard the underlying mechanisms generating the data we are analyzing.

Since heavy-tails are observed in the returns of economic variables, one would expect that practitioners use Lévy distributions. The particular case of Gaussian distribution was the first to be considered for modeling price of European options, through the well known Black-Scholes model<sup>16</sup> proposed in 1973. This model ended a story started already in 1900 with Bachelier and his *Theory of Speculation*<sup>6</sup> where Brownian motion was used to model stock price evolution. The Black-Scholes model for option-pricing is however inconsistent with options data, since stock-price behavior is essentially not Gaussian. To overcome the imperfections of the Black-Scholes model, more sophisticated models were proposed since 1980s and 1990s, which basically assume processes more general than Brownian processes. These processes are called Lévy processes<sup>124</sup> and the probability distributions of their increments are infinitely divisible, i.e. one random variable following that probability distribution can be decomposed into one sum of an arbitrary integer number of independent identically distributed random variables.

Still, despite considerable progresses on modeling financial data with Lévy processes, practitioners continue to show a strong preference for the particular class of finite moment's distributions and there are good reasons for that. Assuming that Lévy distributions are good representations of economic variables fluctuations, a model based on them is closed when one fits the distribution to empirical data choosing properly the parameter values, which represent the valuable information for financial insight and decision making. However, as said above, fitting is no good when the series are not stationary: There is no guarantee that today's fitting will be the same as tomorrows. Since working with a Gaussian curve is more straightforward than working with a Lévy distribution and needs less parameters for curve fitting, there is no practical gain in abandoning Gaussian distribution to model the distribution of fluctuations according to a prescribed mathematical model, even though it is not entirely correct. In other words, if a Lévy distribution is fitted to empirical data of a non-stationary process one will carry basically the same *model risk*, as if a Gaussian distribution is used.

On a more ontological approach, when modeling financial and economic networks, random variables are translated into agents. Agent-based models for describing and addressing the evolution of markets has become an issue of increasing interest<sup>52</sup> and appeals for further developments.<sup>60,69,123,141</sup> They enable one to access three important questions.<sup>22</sup> First, the system is able in this way to be decomposed into sellers and buyers, a common feature of all finance systems. Second, one enables non-stationary regimes to occur, as in real stock markets. Third, by properly incorporating the ingredients of financial agents and the trades among them one can directly investigate the impact of trades in the price, according to some prescribe scheme.

In this section we use an agent model for the individual behavior of single financial agents, at a microscopic scale, in a way that the collective behavior generates an output in accordance with the observed curves of macroscopic variables, namely the financial indices. Several of such bottom-up approaches were thoroughly investigated.<sup>87,141</sup> The Solomon-Levy model<sup>80</sup> defines each agent as a wealth function  $\omega_i(t)$  that cannot go below a floor level, given by  $\omega_i(t) \geq \omega_0 \bar{\omega}(t)$  where  $\bar{\omega}(t)$  is the agent average  $\omega$  at instant  $t$  and  $\omega_0$  is a proper constant. The imposition of the floor based on the mean field  $\bar{\omega}(t)$  means that on average  $\langle |\omega_i(t) - \bar{\omega}(t)| \rangle \sim N$  and, by basic statistics,  $\text{var}(\omega(t)) \sim N^2$ . Consequently, the result of the Solomon-Levy model, despite the interesting idea of the introduction of a floor similar to what was done by Merton<sup>98</sup> in the agent dynamics, will surely be a  $\alpha$ -stable distribution with a power law heavy-tail, i.e.  $\alpha < 2$ . Percolation based models like Cont-Bouchaud<sup>34</sup> or Solomon-Weisbuch,<sup>127</sup> by the nature of the phenomena, also brings up variations of the order of the system size, leading also to Lévy-type distributions.

In our approach, we follow the above considerations, to address the following question: what are the fundamental assumptions, common e.g. to all economic systems, that naturally lead to the emergence of macroscopic distributions that are characterized by heavy-tails? Taking an economic system as a prototypical example for the emergence of heavy-tailed distributions, we argue that there are three fundamental assumptions.

First, agents tend to trade, i.e. to interact. Human beings are more efficient in doing specialized labor than being self-sufficient and for that they need to exchange labor. The usage of the expression ‘labor’ can be regarded as excessive by economists, but we look at it as the fundamental quantity that is common to labor, money or wage. Something must be common to all these quantities; if not, we wouldn’t exchange them. The physicists can regard such fundamental quantity as an ‘economic energy’(see Chapter 3 for details).

Second, we only consume and produce a finite amount of the overall product that exists within our environment. This assumption justifies the emergence for each agent of a maximum production and minimum consumption. If an agent transposes that finite amount he should not be able to consume anymore. This assumption is a natural consequence of the developments of Chapter 4.

Third, human agents are different and attract differently other agents to trade. For choosing the way “how” agents attract each other for trading, we notice that this heterogeneity should reflect some imitation, where agents tend to prefer to consume (resp. produce) from (resp. to) the agents with the largest number of consumers (resp. producers). The number of producer and consumer neighbors reflects, respectively, supply and demand of its labor. With such observation it is reasonable to assume that combining both kinds of neighbors should suffice to quantify the price of the labor exchanged.

Heavy-tailed distributions have been subject to intensive research activity till very recently, e.g. when addressing the formation and construction of efficient reservoir networks,<sup>88</sup> which shows self-organized criticality with critical exponents that can be explained by a self-organized-criticality-type model. In this section, we deal with heavy tails found in economic systems and show that heavy-tailed return distributions are due to the economic organization emerging in a complex economic network of trades among agents governed under the above three assumptions.

Further, the model reproducing empirical data is also of the self-organized-criticality-type model, but its main ingredients result from economical reasoning and assumptions.

Our central result deals in particular with the return distribution found in both data and model described on section 4.2.2.4: we show that the power-law tails are characterized by an exponent that can be measured and is constrained by upper and lower bounds, which can be analytically deduced. In section 3.2.1 we argued that correlation between the multiplicative processes influences the exponent of the resulting power-law distribution, here we will show that also for geometrical reasons the exponent  $\gamma$  is bounded.

The knowledge of such boundaries is of great importance for risk estimates: by deriving upper and lower bounds, one avoids either underestimates, which enable the occurrence of crisis unexpectedly, as well as overestimates, which prevent profit maximization of the trading agents.

### 5.2.3.2 Bounding values for the avalanche size distribution

All indices in Fig. 4.4c take values around the model prediction  $m = 5/2$ , see section 4.2.2.4, and lay within the range  $m_{\min} \equiv 2 < m < \frac{7}{2} \equiv m_{\max}$ . From Eq. (4.83), one concludes that the above range of  $m$ -values corresponds to the range  $2 < \gamma < 3$ , which is a typical range of exponent values observed in empirical scale-free networks specially in the economic ones like airports,<sup>83</sup> Internet<sup>3</sup> and international trade of products and goods.<sup>15</sup> With such observations we ask: What are then the topological causes underlying the emergence of those bounding values? In this section, we derive  $m_{\min}$  and  $m_{\max}$ , applying renormalization methods<sup>128</sup> to the case of undirected networks.

The bounding values of  $m$  result directly from bounded values of  $\gamma$  (see Eq. (4.83)), and this latter values can be derived under the assumption that the degree distribution is scale invariant.

As mentioned in section 4.2.2.4, links are either outgoing or incoming and the probability for an agent to have  $k$  outgoing links – or correspondingly  $k$  incoming links – depends on the scale one is considering: at each scale  $p$  there is a fraction  $P(p, k)$  of agents with  $k$  connections. Figure 5.21 illustrates three successive scales  $p = 1, 2$  and  $3$  for directed networks. When the connections are directed, from one scale to the next there are  $N$  admissible connections.<sup>128, 129</sup> Undirected networks can be regarded as compositions of two directed networks, since the degree law  $P(p, k)$  is the probability for a agent to have  $k$  start links or  $k$  end links *indistinctly*. Consequently, when going from one scale to the next, the renormalization generates  $N^2$  admissible states leading to  $\frac{dN^2 P(p, k)}{dp} = 0$ .

Therefore, the self-similar transformation of the agent degree, i.e. the number of links in a agent will be ruled by

$$N^2 P(k) dk = N_p^2 P(k_p) dk_p \quad (5.35)$$

where  $k$  and  $k_p$  symbolize, respectively, the total and renormalized number of links and  $N$  and  $N_p$  are the correspondent number of agents.



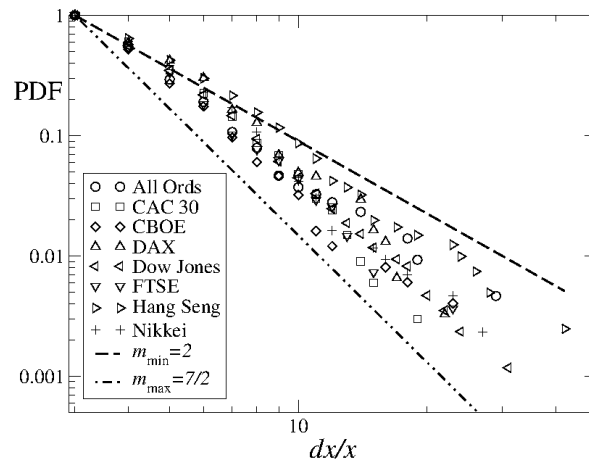
**Figure 5.21:** Illustration of renormalization in complex networks. Starting at connection  $\bar{A}\bar{B}$  between two clusters of agents, one scales down finding each cluster composed by two sets of agents,  $A_1$  and  $A_2$  on the left and  $B_1$  and  $B_2$  on the right connected again by  $\bar{A}_1\bar{A}_2$  and  $\bar{B}_1\bar{B}_2$  respectively. Each set of agents of this new scale can also be decomposed in two connected sets and so on downscale. For undirected networks, the number of probable states grows with  $N_p^2$ , with  $N_p$  being the renormalized number of agents.

The power-law in Eq. (4.83) is invariant under renormalization, i.e.  $P(k) \sim k^{-\gamma}$  and  $P(k_p) \sim k_p^{-\gamma}$ . Defining  $l_p$  as the distance between agents at a given scale  $p$ , as the average number of links separating a randomly chosen pair of agents at a given scale  $p$ , the fractal dimension  $d_B$  of the network can then be calculated using the box-counting technique:<sup>51,128</sup>

$$N_p = N l_p^{-d_B}. \quad (5.36)$$

Similarly, the number of links scale as

$$k_p = k l_p^{-d_k}. \quad (5.37)$$



**Figure 5.22:** The bounding heavy-tails with  $m_{min} = 2$  and  $m_{max} = 7/2$  delimiting the return distributions of the stock indices in Fig. 4.83c.

And finally, substituting Eqs. (4.83), (5.36) and (5.37) in Eq. (5.35) yields

$$\gamma = 1 + 2 \frac{d_B}{d_k}. \quad (5.38)$$

This result retrieves a topological constraint for the value of  $\gamma$  and, consequently, for the “weight” of the heavy-tail in the degree distribution. It is known that whenever the above results holds the corresponding degree distribution is invariant under renormalization (see Supplementary Material of Ref.<sup>129</sup>). At each scale the number of connections of each agent varies between two limit cases, one where each agent connects to only one neighbor, and another where everybody is connected with everybody else, within the same scale. In the first limit case, each agent links to a single neighbor at each scale and consequently the connections will scale like the agents,  $d_k = d_B$ , yielding  $\gamma = 3$ . In the other limit case, agents should connect to all neighbors at a each scale, i.e. for each set of  $N_p$  agents we find  $N_p^2 - N_p \sim N_p^2$  links and thus  $d_k = 2d_B$ , i.e.  $\gamma = 2$ . Since both limit cases yield a relation between  $d_k$  and  $d_B$ , the same conclusion should hold even in non-fractal networks similar to what is reported in Ref.<sup>129</sup> Not all power-law degree distributions are invariant under renormalization. Still, it is reasonable to expect that even in the case they are not strictly invariant, the exponent characterizing their power-law degree distribution should lie between the two limit cases of (invariant) degree distributions. It is therefore a general result as shown next for observed financial indices.

In a real network, at each scale each agent should have a typical number of connections between these two extremes, namely one and  $N$ , resulting in an exponent  $\gamma$  between 2 and 3 and in the corresponding two bounding values for  $m$  in Eq. (4.83),  $m_{min} = 2$  and  $m_{max} = 7/2$ . Figure 5.22 plots the cumulative distributions for all indices together with the boundaries  $P_{min}(dx/x) \sim (dx/x)^{-m_{min}}$  and  $P_{max}(dx/x) \sim (dx/x)^{-m_{max}}$ . This is an important result since, independently of the network complexity, the return distributions are characterized by heavy-tails in a limited range of frequencies. Consequently, the amplitude of the associated risk measure is also limited.

To finish this section we discuss possible applications from these findings. Having such bounding values an important application deals with risk evaluation. The ability to measure risk is fundamental when we talk about any economic activity. When for instance banks lend people money to buy houses, they must have a way of estimating the risk of those activities. In short, what is the most one can lose on a particular investment? The financial property Value at Risk, or simply  $VaR$ , provides an answer. Value at Risk evaluates the percentile of the predictive probability distribution for the size of a future financial loss. Mathematically, for a prescribe  $\alpha$  degree of confidence and within a time horizon  $\Delta t$  the value at risk is defined as the value  $x^*$  such that

$$VaR_\alpha(x^*, \Delta t) \equiv \int_{x^*}^{\infty} p(x) dx = 1 - \alpha \quad (5.39)$$

where  $p(x)$  is the PDF for the loss or the negative return of a economic variable relevant for the intended investment. If one is dealing with shares portfolio, the relevant variable would be one such as the ones in Fig. 4.4, symbolized here as  $x$ .

The confidence of the estimate given for  $VaR_\alpha(x^*, \Delta t)$  depends therefore on the choice of



the PDF  $p(x)$  for the losses or returns. Since under the assumptions above we can take the PDF  $p(x)$  as a Pareto distribution and since we can bound the exponent value defining such a Pareto distribution, we get a straightforward way for bounding any estimate of  $VaR$ . While  $VaR$  is a measure of risk, i.e. a risk model for estimating how much one can lose in a specific investment based in some functional form of  $p(x)$ , by bounding its value through the two bounding exponent values deduced above we are able to evaluate how “risky” are such risk models and risk measures. By “risky” we refer more specifically to the choice of  $p(x)$  when evaluating the risk measure, in this case  $VaR_\alpha$ . In other words, our bounding exponent values can provide us with a way to evaluate the “model risk” of a particular model for risk evaluation.

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| <b>5.2.3.3</b> | Conclusion: towards a risk model |
|----------------|----------------------------------|

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These findings in this section help to solve the controversy about Mandelbrot hypothesis<sup>90</sup> that the distribution of financial returns are explained by Lévy distributions, and therefore would yield an exponent smaller than 3. Some authors have argued against Mandelbrot hypothesis, basing their positions<sup>20</sup> in empirical measures of the return distributions which yield exponents larger than 3. In this paper we presented the analytical and empirical evidence that both statements are in fact correct. Each one is considering a different effect of the same phenomenon: though what we measure in the time series of returns are links between agents, corresponding to exponents larger than 3, the random variables behind Lévy distributions are the agents, which corresponding to the exponent  $\gamma$  which is limited by  $2 < \gamma < 3$ .

These results obtained with geometrically reasoning are coherent with the ones found in Chapter 3, since  $\gamma = 2$  means that agents grow independently from each other and  $\gamma = 3$  that agents cannot grow without exchanging with others with no losses and no donations.

Finally, since the exponent is bounded, the total risk associated with the process being observed is also bounded between one lower and one upper boundary values, enabling one to actually measure the “risk” of a particular model for risk evaluation. We described the particular case of risk measure, namely the Value at Risk, but other approaches could be also taken, for example the expected shortfall,<sup>1,2</sup> which considers the average Value at Risk with respect to its confidence level  $\alpha$ .





# 6

## CONCLUSION AND DISCUSSION

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### Take home messages:

- There is an economic equilibrium that derives directly from the definition of economy and it can be found applying the statistical mechanics framework to a gas of correlated multiplicative processes;
- There are several empirical evidences of such an equilibrium and it possible to model canonical and microcanonical ensembles. The exponent of the weight distribution of the economic agent network characterizes the system the same way temperature characterizes a physical isolated system equilibrium;
- Complex networks are one example of the materialization of such gas of correlated multiplicative processes;
- It is possible to model fluctuations around such equilibrium with a Fokker-Plank equation. The result is similar to model the system as a Self-Organized Critical system;
- There are several empirical evidences that, as predicted analytically and numerically, the distributions of fluctuations are also power-law distributions with an exponent that is a magnification of the underlying economic network, which denies Mandelbrot's claim that market fluctuations are  $\alpha$ -stable Lévy distributions;
- Raising minimum capital levels in banks does not favor the stability of the financial system, because the equilibrium configuration of banks is a complex network and not a fluid in physical equilibrium;
- Since fluctuations depend on the underlying economic network it is possible to frame the amplitude of the fluctuations between a minimum and a maximum value and, with that, overcome the limitation of fitting models to empirical data which we show to be inefficient.

This thesis is devoted to study a framework in which economic phenomena can be treated in the same way physical phenomena are treated: even if a deterministic law is not possible to be derived, the laws defining random variables behavior or probability functions do not change in time.

We deduced from the theoretical definition of economy that it can be looked as a set of correlated multiplicative processes. Each of the multiplicative processes  $dx_i/x_i = \beta$  are divergent processes when looked individually. When looked as a gas they can be transformed into critical multiplicative processes  $d\bar{x}_i/\bar{x}_i = d\log(\bar{x}_i) = 0$ , i.e., processes where the expected quantity in one generation  $x_i(t)$  is equal to the previous one  $x_i(t-1)$ .

This is fundamental in our approach, since we prove that such a set shows a power-law distribution of the multiplicative quantity  $x_i$  with exponent 2 when the individual multiplicative processes are independent from each other. If the quantity  $x_i$  in one multiplicative process is fully dependent on the others, then the exponent takes the value 3. Also, we have showed that this set of critical multiplicative processes has an invariant quantity, the sum of normalized logarithms. This invariance of the sample space leads us to a parallelism with the physical notion of equilibrium. In this case, with energy and mass invariance, the sample space is fixed. Thus a fixed ensemble of possible states is formed in which the full set of statistical physics tools can be applied. This is the case in both thermal equilibrium and in this set of multiplicative processes. The invariance of the set of multiplicative process in which  $d\bar{x}_i/\bar{x}_i = 0$  we call logarithmic invariance.

Using scale-free networks as particular cases of sets of multiplicative processes, the weight of the connections between nodes is the multiplied quantity  $x_i$ . We can derive an equivalent to a temperature and a heat capacity from such a system, building microcanonical and canonical ensembles, under the assumption that the system in the first case and the set system-bath are invariant under exchanges of logarithms of  $x_i$ .

With the above assumptions applied to an economic system, we argue that the above results present an immediate explanation for the emergence of a natural inflation and for the resilience of wealth inequalities that were observed in the Romanian society. Furthermore, we tested our analytical and computational results against data from the US economy. This is the largest data set in the world for economic metrics and correcting economic indices for inflation and number of economic agents the results agree with the expected from the theoretical developments: an economy like the US seems to fluctuate around an apparent stationary level.

Knowing that the logarithmic invariance is a statistically equivalent to physical equilibrium, this implies that we can deal with fluctuations around that equilibrium, in the same way it is done with Statistical Physics deal with Markovian processes, i.e., by Fokker-Planck equations or Langevin equations. We made a parallelism with the principle of self-organized criticality (SOC) to show that fluctuations in multiplicative process gas can be taken as a SOC system and we deduced the Fokker-Plank equation showing that fluctuations in such a system follow a power-law whose exponent is proportional to the ratio between the drift and the diffusion coefficients.

We then make the parallelism between the economic network and a scale-free network, supported by data from several authors, and introduced a SOC mechanism in that network

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to promote fluctuations and showed, analytically, computationally and empirically that fluctuations in principal stock market indices follow the same power-law deduced from the Fokker-Planck equation. Moreover, is in agreement with the stationary solution of the Fokker-Planck equation, which is the power-law distribution that characterizes the economic network.

To show that the SOC threshold theoretically assume is in fact present in the economy, we took data from an average of  $7 \times 10^5$  Portuguese companies during ten years to show that such a threshold is formed naturally in the economy, making use of a new technique to find localized correlations between Pareto distribute random variables.

In the scope of the thesis we applied the above theoretical results to address financial risk problems, one in the so called 'market risk' and one in 'credit risk'. In the first, we have used geometrical arguments to show that even if it is not possible to determine correctly the market risk of a financial asset, it is possible to determine the risk of being wrong measuring risk, which can be important for regulation purposes.

In credit risk we address the regulatory problem of raising minimum levels of capital for banks as a consequence of the third version of the Basel accords. We showed that, contrary of what is claimed by the regulator committee, raising minimum levels of capital does not improve the resilience of the financial system but, in conditions of normal economic growing, the system becomes more fragile.

To have proper data from an economic system is not an easy task due to all confidentiality issues involved, both personal and corporative. Moreover, economic agents are persons, not particles. We recognize that the existence of a different volume of empirical data would improve considerably the thesis. But we have the conscience that getting data from banks is forbidden by law and getting data from government institutions is practically impossible<sup>1</sup>.

Of course, the assumption that a person is, on average, a non-intelligent process can be object of discussion, not only in the scope of this thesis, but also in the scope of Economics where models are built in the same principle. It is our believe that the archaeological findings that show contemporary forms of hominids, in which the one that appears to have an economy, winning of the natural selection of the fittest, are coherent with the idea of a person as a dummy economic process. But, obviously, there are no proofs of that and probably never will be. In this sense, we are not faraway from economists that need the same assumption.

The physics for the sets of multiplicative processes is, nevertheless, developed and consistent with the empirical data from US economy, the principal stock markets in the world, the data from Romanian social security and the data from the Portuguese companies in the last 10 years. Also, since we cannot build an economic lab where we can introduce humans, simulate economies in the sense we understand it and see if those simulations were in agreement with theoretical findings.

The spreading of these findings in the Economics community is another difficult task. They introduce a considerably different way of looking at modeling in Economics. Nevertheless, this does not impact on future work. We have developed the necessary tools to address financial modeling in several ways, both from the apparent equilibrium for stationary modeling and

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<sup>1</sup>Though considerable effort was done to it!

from fluctuations for market relating issues. The fact that we could find the Fokker-Planck equation for the fluctuations allows us to link to the full set of tools that have been developed through the years in Financial Mathematics. Also, the established fact that economic variables related to wealth follows a Pareto distribution because that is the equilibrium configuration is of considerable importance to solve a machine learning problem in finance related with this type of variables which contribution, by definition, can not be learned. And the scientific future work will make its path through this link.

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