

# 884. Self-synchronization theory of a dual mass vibrating system driven by two coupled exciters.

## Part 1: Theoretical analysis

Li He<sup>1</sup>, Fu Shibo<sup>2</sup>, Li Ye<sup>3</sup>, Zhao Chunyu<sup>4</sup>, Wen Bangchun<sup>5</sup>

School of Mechanical Engineering and Automation, Northeastern University  
Shenyang, 110004, China

E-mail: <sup>1</sup>hli@mail.neu.edu.cn, <sup>2</sup>fushibop9q@sohu.com, <sup>3</sup>neuyezi\_2007@126.com,  
<sup>4</sup>chyzhao@mail.neu.edu.cn, <sup>5</sup>bcwen1930@vip.sina.com

(Received 15 May 2012; accepted 4 December 2012)

**Abstract.** A vibration model is proposed and analyzed dynamically to study the self-synchronization theory of dual-mass vibration system. The differential equations of systematic motion are derived by applying Lagrange's equations. As two uncertain parameters, the coefficients of instantaneous change of average angular velocity and the phase difference of two exciters are introduced to derive the coupling equations of angular velocity of the two exciters. The conditions of synchronous implementation and stability are derived by utilizing the modified small parameter average method treated as non-dimension to the parameters. The swing of the vibration model plays a major role in the self-synchronization of two motors. The mass ratio of two eccentric blocks has an effect on the stability of synchronous operation.

**Keywords:** self-synchronization, stability, vibrating system.

### Introduction

A phenomenon of self-synchronization of unbalanced rotors was first discovered by Blekhman I. I. [1] in the 1950s. The earliest detailed research on the theory of self-synchronization of vibrating machinery with double exciters was also carried out by Blekhman I. I. [1-5] in the USSR, who established the conditions of existence and stability of self-synchronization of the exciters in vibrating systems. Achievements of vibrational technology and partial cases of the self-synchronization problem were widely reflected in volume 4 of a reference book "Vibrations in Engineering" written by Blekhman I. I. in 1981 [4]. Then the type of the unbalanced rotor was generalized [6]. Based on the self-synchronization phenomenon discovered by Blekhman I. I., a new class of vibrational machines: crushers, mills, screens, feeders were developed and successfully used in mineral processing industry. Recently, Blekhman I. I. added the definition of dynamic [7] to the traditional theory of the self-synchronization, which reduced the requirements for system. The methods of small parameter and the averaging methods were applied later by some researchers Ragulskis K. M. [8, 9], Khodzhaev K. Sh. [10], Sperling [11, 12] and Nagaev R. F. [13] from Russia, Lithuania, Germany and other countries to a number of problems. It has been proven to be useful and descriptive leading to better understanding and theoretical explanation of the mechanism of self-synchronization. In the 1980s, investigations of the problem of self-synchronization were developed in China. Wen [14-16] selected the phase difference between two exciters as the variable to simplify the analytical method for establishing the conditions of existence and stability of self-synchronization of two identical exciters in a vibrating system. Subsequently, Yamapi R. [17] introduced dynamic characteristics of two motors to the self-synchronization theory. The self-synchronization of a vibrating system from the effect of electric-mechanic motors coupling is dependent on the dynamic parameters of two induction motors [18, 19].

In this paper, an analytical approach is employed to study the self-synchronization theory of a vibrating system with two non-identical coupled exciters. By introducing two variable perturbation parameters to average angular velocity of two exciters and their phase difference, the problem of synchronization is converted into that of existence and stability of zero solution

for the equation of frequency capture. The rest of the paper is organized as follows: Section 2 is devoted to description of the dynamic model of the vibrating system. In Section 3, we analyze vibratory response of the model and obtain the conditions of synchronous implementation and that of stabilizing synchronous operation of the two exciters. The conclusions of theoretical analysis are discussed in Section 4.

### The dynamic model of the vibrating system

The dynamic model of the vibrating system is presented in Fig. 1, which is taken as an example to demonstrate the self-synchronization theory of the double mass vibration system. The system is made up of two vibratory bodies and two eccentric blocks which are driven by two induction motors. Vibratory body 1 is supported by two elastic foundations, which consist of four symmetrical isolation springs. Vibratory body 2 is connected to body 1 through master vibration spring F. The two bodies can only move relatively along the  $x$  direction. When the motors rotate, the exciting force produced by the eccentric block makes the machine vibrate. The system has four degrees of freedom: the horizontal direction  $x$ , the vertical direction  $y$ , the relative position of two bodies  $z$  and swing direction  $\psi$ . The eccentric blocks respectively rotate about their spin axes, which are denoted by  $\varphi_1$  and  $\varphi_2$ , respectively.

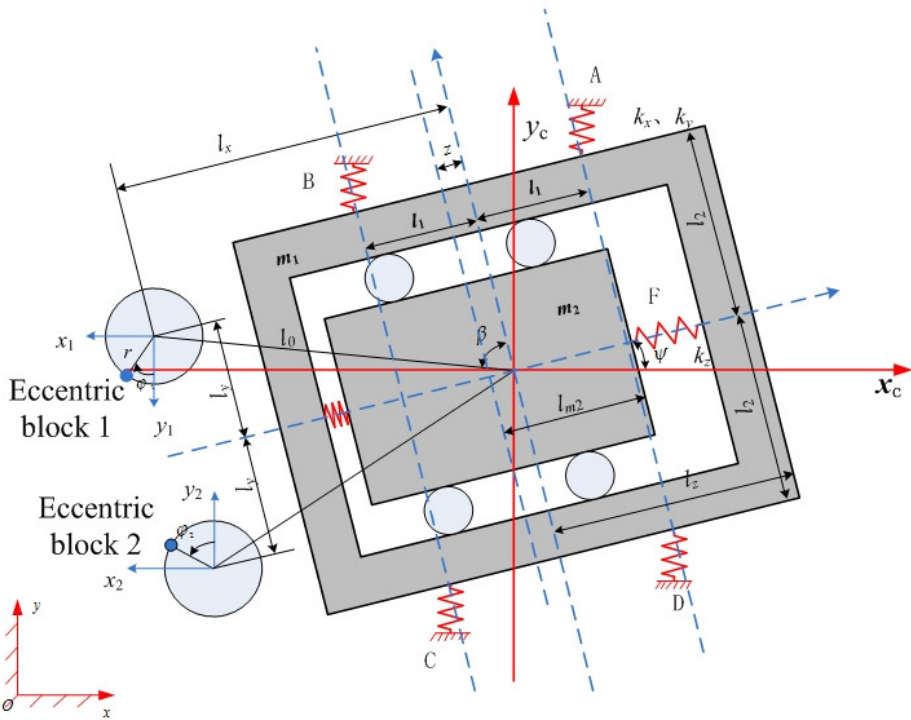


Fig. 1. The dynamic model of the vibrating system

In the coordinate system,  $x_1$  and  $x_2$  are defined as the position coordinates of body 1 and body 2, and represented as follows:

$$\mathbf{x}_1 = (x, y), \quad \mathbf{x}_2 = (x - z \cos \psi, y - z \sin \psi).$$

The eccentric blocks 1 and 2 in the reference frame have coordinates:

$$\mathbf{x}_{01} = \begin{pmatrix} x - l_y \sin \psi - l_x \cos \psi - r \sin \varphi_1 \\ y + l_y \cos \psi - l_x \sin \psi - r \cos \varphi_1 \end{pmatrix}, \quad \mathbf{x}_{02} = \begin{pmatrix} x - l_x \cos \psi + l_y \sin \psi - r \sin \varphi_2 \\ y - l_x \sin \psi - l_y \cos \psi + r \cos \varphi_2 \end{pmatrix}.$$

As illustrated in Fig. 1, four springs are symmetrically installed on the foundations and have the same linear stiffness and the same damping coefficient. The deformation of springs in the operated state of the system can be represented as:

$$\mathbf{x}_a = \begin{pmatrix} x + l_1 \cos \psi - l_2 \sin \psi - l_1 \\ y + l_1 \sin \psi + l_2 \cos \psi - l_2 \end{pmatrix}, \quad \mathbf{x}_b = \begin{pmatrix} x - l_2 \sin \psi - l_1 \cos \psi + l_1 \\ y + l_2 \cos \psi - l_1 \sin \psi - l_2 \end{pmatrix},$$

$$\mathbf{x}_c = \begin{pmatrix} x - l_1 \cos \psi + l_2 \sin \psi + l_1 \\ y - l_1 \sin \psi - l_2 \cos \psi + l_2 \end{pmatrix}, \quad \mathbf{x}_d = \begin{pmatrix} x + l_1 \cos \psi + l_2 \sin \psi - l_1 \\ y + l_1 \sin \psi - l_2 \cos \psi + l_2 \end{pmatrix}.$$

The deformation of spring F is only related to the relative position of two bodies.

The kinetic energy  $T$  of the vibrating system can be expressed as:

$$T = \frac{1}{2} m_1 \dot{\mathbf{x}}_1^T \dot{\mathbf{x}}_1 + \frac{1}{2} J_{m1} \dot{\psi}^2 + \frac{1}{2} m_2 \dot{\mathbf{x}}_2^T \dot{\mathbf{x}}_2 + \frac{1}{2} J_{m2} \dot{\psi}^2$$

$$+ \frac{1}{2} J_{m3} \dot{\varphi}_1^2 + \frac{1}{2} J_{m4} \dot{\varphi}_2^2 + \frac{1}{2} m_{01} \dot{\mathbf{x}}_{01}^T \dot{\mathbf{x}}_{01} + \frac{1}{2} m_{02} \dot{\mathbf{x}}_{02}^T \dot{\mathbf{x}}_{02} \quad (1)$$

where  $m_1$  and  $m_2$  is the mass of body 1 and body 2;  $m_{01}$  and  $m_{02}$  are the mass of exciter 1 and exciter 2, respectively;  $J_{m1}$ ,  $J_{m2}$ ,  $J_{m3}$  and  $J_{m4}$  are the moments of inertia of body 1, body 2 and two motors' rotors in terms of  $\psi$  - direction, respectively; the top dot ( $\dot{\bullet}$ ) denotes  $d(\bullet)/dt$ .

The potential energy  $V$  of the system is expressed as:

$$V = \frac{1}{2} \mathbf{x}_a^T \mathbf{k}_a \mathbf{x}_a + \frac{1}{2} \mathbf{x}_b^T \mathbf{k}_b \mathbf{x}_b + \frac{1}{2} \mathbf{x}_c^T \mathbf{k}_c \mathbf{x}_c + \frac{1}{2} \mathbf{x}_d^T \mathbf{k}_d \mathbf{x}_d + \frac{1}{2} k_f z^2 \quad (2)$$

where  $\mathbf{k}_a$ ,  $\mathbf{k}_b$ ,  $\mathbf{k}_c$ ,  $\mathbf{k}_d$  and  $k_f$  are the stiffness coefficients of springs A, B, C, D and F,  $\mathbf{k}_a = \mathbf{k}_b = \mathbf{k}_c = \mathbf{k}_d = \text{diag}(k_x/4, k_y/4)$ .

The viscous dissipation function  $D$  of the system can be described as:

$$D = \frac{1}{2} \dot{\mathbf{x}}_a^T \mathbf{f}_a \dot{\mathbf{x}}_a + \frac{1}{2} \dot{\mathbf{x}}_b^T \mathbf{f}_b \dot{\mathbf{x}}_b + \frac{1}{2} \dot{\mathbf{x}}_c^T \mathbf{f}_c \dot{\mathbf{x}}_c + \frac{1}{2} \dot{\mathbf{x}}_d^T \mathbf{f}_d \dot{\mathbf{x}}_d + \frac{1}{2} f_f \dot{z}^2 \quad (3)$$

where  $\mathbf{f}_a$ ,  $\mathbf{f}_b$ ,  $\mathbf{f}_c$ ,  $\mathbf{f}_d$  and  $f_f$  are the damping coefficients of springs A, B, C, D and F,  $\mathbf{f}_a = \mathbf{f}_b = \mathbf{f}_c = \mathbf{f}_d = \text{diag}(f_x/4, f_y/4)$ .

The equations of motion are derived by using Lagrange's equations:

$$\frac{d}{dt} \frac{\partial(T-V)}{\partial \dot{q}_i} - \frac{\partial(T-V)}{\partial q} + \frac{\partial D}{\partial \dot{q}_i} = Q_i \quad (4)$$

where  $q_i$  is the generalized coordinate of the system,  $Q_i$  is the generalized force of the system.

If  $q = [x, y, z, \psi, \varphi_1, \varphi_2]^T$  is chosen as the generalized coordinates, the generalized forces are  $Q_x = Q_y = Q_z = Q_\psi = 0$ ,  $Q_{\varphi_1} = T_{e1}$  and  $Q_{\varphi_2} = T_{e2}$ , in which  $T_{e1}$  and  $T_{e2}$  are the electromagnetic torques of the two motors.

Substitute Eqs. (1), (2) and (3) into (4), respectively. The differential equations of motion of the system are:

$$\begin{aligned}
 M\ddot{x} - m_2\ddot{z} + f_x\dot{x} + k_x x &= m_{01}r(\dot{\varphi}_1 \cos \varphi_1 - \dot{\varphi}_1^2 \sin \varphi_1) + m_{02}r(\dot{\varphi}_2 \cos \varphi_2 - \dot{\varphi}_2^2 \sin \varphi_2) \\
 M\ddot{y} + f_y\dot{y} + k_y y &= -m_{01}r(\dot{\varphi}_1 \sin \varphi_1 + \dot{\varphi}_1^2 \cos \varphi_1) + m_{02}r(\dot{\varphi}_2 \sin \varphi_2 + \dot{\varphi}_2^2 \cos \varphi_2) \\
 m_2(\ddot{z} - \ddot{x}) + f_f\dot{z} + k_f z &= 0 \\
 J\ddot{\psi} + f_\psi\dot{\psi} + k_\psi\psi &= m_{01}r \cos \varphi_1 l_x \dot{\varphi}_1^2 + m_{01}r \sin \varphi_1 l_y \dot{\varphi}_1^2 - m_{02}r \cos \varphi_2 l_x \dot{\varphi}_2^2 \\
 &\quad - m_{02}r \sin \varphi_2 l_y \dot{\varphi}_2^2 + m_{01}r \sin \varphi_1 l_x \ddot{\varphi}_1 - m_{01}r \cos \varphi_1 l_y \ddot{\varphi}_1 - m_{02}r \sin \varphi_2 l_x \ddot{\varphi}_2 + m_{02}r \cos \varphi_2 l_y \ddot{\varphi}_2 \\
 (J_{m3} + m_{01}r^2)\ddot{\varphi}_1 + f_{d1}\dot{\varphi}_1 &= T_{e1} + m_{01}r(\ddot{x} \cos \varphi_1 - \ddot{y} \sin \varphi_1 + l_x \ddot{\psi} \sin \varphi_1 - l_y \ddot{\psi} \cos \varphi_1) \\
 (J_{m4} + m_{02}r^2)\ddot{\varphi}_2 + f_{d2}\dot{\varphi}_2 &= T_{e2} + m_{02}r(\ddot{x} \cos \varphi_2 + \ddot{y} \sin \varphi_2 - l_x \ddot{\psi} \sin \varphi_2 + l_y \ddot{\psi} \cos \varphi_2)
 \end{aligned} \tag{5}$$

where  $M$  is the mass of the system,  $M = m_1 + m_2 + m_{01} + m_{02}$ ,  $J$  is the moments of inertia of the system rotating in terms of  $\psi$ -direction,  $J = J_{m1} + J_{m2} + m_{01}l_x^2 + m_{02}l_x^2 + m_{01}l_y^2 + m_{02}l_y^2$ ,  $k_\psi$  is the constants of the springs in  $\psi$ -direction,  $k_\psi = k_y l_1^2 + k_x l_2^2$ ,  $f_\psi$  is the damping constants in  $\psi$ -direction,  $f_\psi = f_y l_1^2 + f_x l_2^2$ ,  $(\ddot{\bullet})$  denotes  $d^2(\bullet)/dt^2$ .

### Dynamic analysis of the vibrating system

As illustrated in Fig. 1, assume that:

$$\varphi_1 = \varphi + \alpha, \quad \varphi_2 = \varphi - \alpha, \tag{6}$$

where  $\varphi$  is the average phase of the two eccentric rotors,  $\dot{\varphi}$  is the average angular velocity of the two eccentric rotors. When the system is in the steady state, we assume that the angular velocity of the two motors is  $\omega_{m0}$  (constant). Introducing the coefficient of instantaneous change of average angular velocity and the phase difference between the two exciters:  $\varepsilon_1, \varepsilon_2$ , then we have:

$$\dot{\varphi} = (1 + \varepsilon_1)\omega_{m0}, \quad \dot{\alpha} = \varepsilon_2\omega_{m0}, \tag{7}$$

$$\dot{\varphi}_1 = (1 + \varepsilon_1 + \varepsilon_2)\omega_{m0}, \quad \dot{\varphi}_2 = (1 + \varepsilon_1 - \varepsilon_2)\omega_{m0}. \tag{8}$$

If the average values of  $\varepsilon_1$  and  $\varepsilon_2$  are zero, the system will implement frequency capture and achieve self-synchronization. Herein we focus the attention on the non-resonant vibrating system.  $\dot{\varepsilon}_1$  and  $\dot{\varepsilon}_2$  are much smaller than 1 during the steady-state operation, so  $\ddot{\varphi}_1$  and  $\ddot{\varphi}_2$  can be neglected in Eq. (5). Assume  $m_{01} = m_0$  and  $m_{02} = \eta m_0$  ( $0 < \eta < 1$ ). For this vibrating system the damping coefficients of the springs are so small that they can be ignored. The responses of the system can be given by:

$$\begin{aligned}
 x &= r\mu_x[\sin(\varphi + \alpha + \gamma_x) + \eta \sin(\varphi - \alpha + \gamma_x)] \\
 y &= r\mu_y[\cos(\varphi + \alpha + \gamma_y) - \eta \cos(\varphi - \alpha + \gamma_y)] \\
 z &= r\mu_z[\sin(\varphi + \alpha + \gamma_z) + \eta \sin(\varphi - \alpha + \gamma_z)] \\
 \psi &= r\mu_\psi[l_x \cos(\varphi + \alpha + \gamma_x) + l_y \sin(\varphi + \alpha + \gamma_x) - l_x \eta \cos(\varphi - \alpha + \gamma_x) - l_y \eta \sin(\varphi - \alpha + \gamma_x)]
 \end{aligned} \tag{9}$$

where:

$$\begin{aligned}
 \mu_x &= \frac{(-\omega_{m0}^2 m_2 + k_f)(-\omega_{m0}^2 m_{01})}{(-\omega_{m0}^2 M + k_x)(-\omega_{m0}^2 m_2 + k_f) - (\omega_{m0}^2 m_2)^2}, & \mu_y &= \frac{-\omega_{m0}^2 m_{01}}{k_y - M \omega_{m0}^2}, \\
 \mu_z &= \frac{(-\omega_{m0}^2 m_2)(-\omega_{m0}^2 m_{01})}{(-\omega_{m0}^2 M + k_x)(-\omega_{m0}^2 m_2 + k_f) - (\omega_{m0}^2 m_2)^2}, & \mu_\psi &= \frac{\omega_{m0}^2 m_{01}}{k_\psi - J \omega_{m0}^2},
 \end{aligned}$$

$\pi - \gamma_x, \pi - \gamma_y, \pi - \gamma_z$  and  $\pi - \gamma_\psi$  denote the phase angles in the direction of  $x, y, z, \psi$ , respectively.

We differentiate Eq. (9) by using the chain rule on each component of  $\alpha$  and  $\varphi$ , and obtain  $\dot{x}, \dot{y}, \dot{z}$  and  $\dot{\psi}$  from Eq. (8). Substituting  $\dot{x}, \dot{y}, \dot{z}$  and  $\dot{\psi}$  into the last two equations of Eq. (5), we integrate each of them  $\varphi = 0 \sim 2\pi$ . With neglect of the high-order terms of  $\dot{\epsilon}_1$  and  $\dot{\epsilon}_2$ , we obtain:

$$\begin{aligned}
 (J_{m3} + m_{01}r^2)\omega_{m0}(\dot{\bar{\epsilon}}_1 + \dot{\bar{\epsilon}}_2) + f_{d1}\omega_{m0}(\bar{\epsilon}_1 + \bar{\epsilon}_2 + 1) &= T_{e1} - T_{L1} \\
 (J_{m4} + m_{02}r^2)\omega_{m0}(\dot{\bar{\epsilon}}_1 - \dot{\bar{\epsilon}}_2) + f_{d2}\omega_{m0}(\bar{\epsilon}_1 - \bar{\epsilon}_2 + 1) &= T_{e2} - T_{L2}
 \end{aligned} \tag{10}$$

and:

$$\begin{aligned}
 T_{L1} &= \chi'_{11}\dot{\bar{\epsilon}}_1 + \chi'_{12}\dot{\bar{\epsilon}}_2 + \chi_{11}\bar{\epsilon}_1 + \chi_{12}\bar{\epsilon}_2 + \chi_a + \chi_{f1} \\
 T_{L2} &= \chi'_{21}\dot{\bar{\epsilon}}_1 + \chi'_{22}\dot{\bar{\epsilon}}_2 + \chi_{21}\bar{\epsilon}_1 + \chi_{22}\bar{\epsilon}_2 - \chi_a + \chi_{f2}
 \end{aligned} \tag{11}$$

Dimensionless parameters are as follows:

$$\begin{aligned}
 \chi_a &= \frac{1}{2}m_{01}r^2\omega_{m0}^2W_c \sin 2\alpha \\
 \chi_{f1} &= \frac{1}{2}m_{01}r^2\omega_{m0}^2W_{s0} + \frac{1}{2}m_{01}r^2\omega_{m0}^2W_s \sin 2\alpha \\
 \chi_{11} &= m_{01}r^2\omega_{m0}^2(W_{s0} + W_s \cos 2\alpha + W_c \sin 2\alpha) \\
 \chi_{12} &= m_{01}r^2\omega_{m0}^2(W_{s0} - W_s \cos 2\alpha - W_c \sin 2\alpha) \\
 \chi'_{11} &= -\frac{1}{2}m_{01}r^2\omega_{m0}(W_{c0} - W_c \cos 2\alpha + W_s \sin 2\alpha) \\
 \chi'_{12} &= -\frac{1}{2}m_{01}r^2\omega_{m0}(W_{c0} + W_c \cos 2\alpha - W_s \sin 2\alpha) \\
 \chi_{f2} &= \frac{1}{2}\eta^2m_{01}r^2\omega_{m0}^2W_{s0} + \frac{1}{2}m_{01}r\omega_{m0}^2W_s \cos 2\alpha
 \end{aligned} \tag{12}$$

$$\begin{aligned} \chi_{f2} &= \frac{1}{2} \eta^2 m_{01} r^2 \omega_{m0}^2 W_{s0} + \frac{1}{2} m_{01} r \omega_{m0}^2 W_s \cos 2\alpha \\ \chi_{21} &= m_{01} r^2 \omega_{m0}^2 (\eta^2 W_{s0} + W_s \cos 2\alpha - W_c \sin 2\alpha) \\ \chi_{22} &= m_{01} r^2 \omega_{m0}^2 (-\eta^2 W_{s0} + W_s \cos 2\alpha - W_c \sin 2\alpha) \\ \chi_{21}' &= \frac{1}{2} m_{01} r^2 \omega_{m0} (-\eta^2 W_{c0} + W_c \cos 2\alpha + W_s \sin 2\alpha) \\ \chi_{22}' &= \frac{1}{2} m_{01} r^2 \omega_{m0} (\eta^2 W_{c0} + W_c \cos 2\alpha + W_s \sin 2\alpha) \\ W_c &= -\eta(\mu_x \cos \gamma_x - \mu_y \cos \gamma_y + (l_x^2 + l_y^2) \mu_\psi \cos \gamma_\psi) \\ W_{s0} &= \mu_x \sin \gamma_x + \mu_y \sin \gamma_y - (l_x^2 + l_y^2) \mu_\psi \sin \gamma_\psi \\ W_s &= -\eta(-\mu_x \sin \gamma_x + \mu_y \sin \gamma_y - (l_x^2 + l_y^2) \mu_\psi \sin \gamma_\psi) \\ W_{c0} &= \mu_x \cos \gamma_x + \mu_y \cos \gamma_y - (l_x^2 + l_y^2) \mu_\psi \cos \gamma_\psi \end{aligned}$$

It can be observed from the above formulas that the system exerts resisting moment  $\chi_a$  on the motor at a higher speed to slow it down. In the meantime, the system exerts driving moment  $\chi_a$  on the motor at a lower speed to speed it up. And ultimately two motors reach the same speed. As the mainly influencing factor of  $\chi_a$ ,  $W_c$  depends primarily on the swing of the system. The maximum value of  $\chi_a$  is  $m_{01} r^2 \omega_{m0}^2 W_c / 2$ .

Compared with the change of  $\varphi$  in terms of time  $t$ ,  $\alpha$ ,  $\varepsilon_1$ ,  $\varepsilon_2$ ,  $\dot{\varepsilon}_1$  and  $\dot{\varepsilon}_2$  are very small and they can be regarded as slow-varying parameters. In one circle of motion,  $\alpha$ ,  $\varepsilon_1$ ,  $\varepsilon_2$ ,  $\dot{\varepsilon}_1$  and  $\dot{\varepsilon}_2$  are assumed to be the middle values of their integration  $\bar{\alpha}$ ,  $\bar{\varepsilon}_1$ ,  $\bar{\varepsilon}_2$ ,  $\dot{\bar{\varepsilon}}_1$  and  $\dot{\bar{\varepsilon}}_2$ , respectively [18, 19].

When the two motors operate near the  $\omega_{m0}$ , their electromagnetic torques can be given by:

$$\begin{aligned} T_{e1} &= T_{e01} - k_{e01} (\bar{\varepsilon}_1 + \bar{\varepsilon}_2) \\ T_{e2} &= T_{e02} - k_{e02} (\bar{\varepsilon}_1 - \bar{\varepsilon}_2) \end{aligned} \tag{13}$$

with:

$$\begin{aligned} T_{e01} &= \frac{n_p L_{m1}^2 U_{s0}^2}{L_{s1}^2 R_{r1}} \frac{\omega_s - n_p \omega_{m0}}{1 + \sigma_1^2 \tau_{r1}^2 (\omega_s - n_p \omega_{m0})^2} \\ T_{e02} &= \frac{n_p L_{m2}^2 U_{s0}^2}{L_{s2}^2 R_{r2}} \frac{\omega_s - n_p \omega_{m0}}{1 + \sigma_2^2 \tau_{r2}^2 (\omega_s - n_p \omega_{m0})^2} \\ k_{e01} &= \frac{n_p^2 L_{m1}^2 U_{10}^2}{L_{s1}^2 R_{r1} \omega_s} \frac{1 - \sigma_1^2 \tau_{r1}^2 (\omega_s - n_p \omega_{m0})^2}{[1 + \sigma_1^2 \tau_{r1}^2 (\omega_s - n_p \omega_{m0})^2]^2} \frac{\omega_{m0}}{\omega_s} \\ k_{e02} &= \frac{n_p^2 L_{m2}^2 U_{10}^2}{L_{s2}^2 R_{r2} \omega_s} \frac{1 - \sigma_2^2 \tau_{r2}^2 (\omega_s - n_p \omega_{m0})^2}{[1 + \sigma_2^2 \tau_{r2}^2 (\omega_s - n_p \omega_{m0})^2]^2} \frac{\omega_{m0}}{\omega_s} \end{aligned}$$

where  $R_{si}$  and  $R_{ri}$  are the stator and rotor resistances;  $L_{si}$  and  $L_{ri}$  are the stator and rotor inductances;  $L_{mi}$  is the mutual inductance;  $n_p$  denotes the number of pole pairs;  $\tau_{ri}$  is the rotor time constant,  $\tau_r = L_r / R_r$ ;  $\sigma_i$  is the leakage coefficient,  $\sigma = 1 - L_m^2 / L_s L_r$ ,  $i = 1, 2$ .

In Eq. (10), the moments of inertia of the rotors of the two motors are much less than  $m_{01}r^2$  and  $m_{02}r^2$ . Hence,  $J_{m3}$  and  $J_{m4}$  can be neglected. Introducing the following non-dimensional parameters into them:

$$\begin{aligned} \rho_1 &= 1 - \frac{W_{c0}}{2}, \quad \rho_2 = \eta \left( 1 - \frac{W_{c0}}{2} \right), \\ \kappa_1 &= \frac{k_{e01}}{m_0 r^2 \omega_{m0}} + \frac{f_{d1}}{m_0 r^2 \omega_{m0}} + W_{s0}, \quad \kappa_2 = \frac{k_{e02}}{m_0 r^2 \omega_{m0}} + \frac{f_{d2}}{m_0 r^2 \omega_{m0}} + \eta^2 W_{s0}, \\ v_1 &= \bar{\varepsilon}_1 + \bar{\varepsilon}_2, \quad v_2 = \bar{\varepsilon}_1 - \bar{\varepsilon}_2, \\ u_1 &= \frac{T_{e01}}{m_0 r^2 \omega_{m0}} - \frac{f_{d1}}{m_0 r^2} - \frac{\omega_{m0}}{2} W_{s0} - \frac{\omega_{m0}}{2} W_s \cos 2\alpha - \frac{\omega_{m0}}{2} W_c \sin 2\alpha, \\ u_2 &= \frac{T_{e02}}{m_0 r^2 \omega_{m0}} - \frac{f_{d2}}{m_0 r^2} - \frac{\omega_{m0}}{2} \eta^2 W_{s0} - \frac{\omega_{m0}}{2} W_s \cos 2\alpha + \frac{\omega_{m0}}{2} W_c \sin 2\alpha. \end{aligned} \tag{14}$$

They are written in matrix form, i.e.:

$$A\dot{\mathbf{v}} = B\mathbf{v} + \mathbf{u} \tag{15}$$

where:

$$\begin{aligned} A &= \begin{pmatrix} \rho_1 & a_{12} \\ a_{21} & \rho_2 \end{pmatrix}, \quad \mathbf{v} = \{v_1 \ v_2\}^T, \quad \mathbf{u} = \{u_1 \ u_2\}^T, \quad B = -\omega_{m0} \begin{pmatrix} \kappa_1 & b_{12} \\ b_{21} & \kappa_2 \end{pmatrix}, \\ a_{12} &= \frac{1}{2} W_c \cos 2\alpha - \frac{1}{2} W_s \sin 2\alpha, \quad a_{21} = \frac{1}{2} W_s \sin 2\alpha + \frac{1}{2} W_c \cos 2\alpha, \\ b_{12} &= W_s \cos 2\alpha + W_c \sin 2\alpha, \quad b_{21} = W_s \cos 2\alpha - W_c \sin 2\alpha. \end{aligned}$$

Equation (15) is called the equation of frequency capture of the vibrating system.

### The condition of synchronous implementation

It can be seen from Eq. (15), when  $\mathbf{v} = \mathbf{0}$  the system achieves self-synchronization. Substituting  $\mathbf{v} = \mathbf{0}$  into Eq. (15), we obtain  $u_1 = u_2 = 0$ :

$$\begin{aligned} &\frac{T_{e01}}{m_0 r^2 \omega_{m0}} - \frac{f_{d1}}{m_0 r^2} - \frac{\omega_{m0}}{2} W_{s0} - \frac{\omega_{m0}}{2} W_s \cos 2\alpha - \frac{\omega_{m0}}{2} W_c \sin 2\alpha \\ &= \frac{T_{e02}}{m_0 r^2 \omega_{m0}} - \frac{f_{d2}}{m_0 r^2} - \frac{\omega_{m0}}{2} \eta^2 W_{s0} - \frac{\omega_{m0}}{2} W_s \cos 2\alpha + \frac{\omega_{m0}}{2} W_c \sin 2\alpha = 0 \end{aligned} \tag{16}$$

Rearranging  $u_1$  and  $u_2$ ,  $u_1 = u_2 = 0$ , we have:

$$(T_{e01} - T_{e02}) - (f_{d1}\omega_{m0} - f_{d2}\omega_{m0}) - \frac{1}{2}(1 - \eta^2)m_0r^2\omega_{m0}^2W_{s0} = m_0r^2\omega_{m0}^2W_c \sin 2\alpha \quad (17)$$

Because of  $\sin 2\alpha \leq 1$ , the condition of synchronous implementation is:

$$m_0r^2\omega_{m0}^2W_c > (T_{e01} - T_{e02}) - (f_{d1}\omega_{m0} - f_{d2}\omega_{m0}) - \frac{1}{2}(1 - \eta^2)m_0r^2\omega_{m0}^2W_{s0} \quad (18)$$

### Stability of the synchronization

Linearizing Eq. (15) around  $\alpha = \alpha_0$ , the three first order differential equations of the system are derived in the following manner that we add the two rows as the first row and then subtract the second row from the first as the second one. Then we obtain:

$$\alpha = \alpha_0 + \Delta\alpha, \quad \Delta\dot{\alpha} = \omega_{m0}\bar{\varepsilon}_2, \quad (19)$$

$$\begin{aligned} (\rho_1 + \rho_2 + W_c \cos 2\alpha_0)\varepsilon_1' + (\rho_1 - \rho_2 + W_s \sin 2\alpha_0)\varepsilon_2' &= -\omega_{m0}[(\kappa_1 + \kappa_2 + 2W_s \cos 2\alpha_0)\varepsilon_1 \\ &+ (\kappa_1 - \kappa_2 - 2W_c \sin 2\alpha_0)\varepsilon_2] + u_{10} + u_{20} + 2\omega_{m0}W_s \sin 2\alpha_0(\alpha - \alpha_0) \\ (\rho_1 - \rho_2 - W_s \sin 2\alpha_0)\varepsilon_1' + (\rho_1 + \rho_2 - W_c \cos 2\alpha_0)\varepsilon_2' &= -\omega_{m0}[(\kappa_1 - \kappa_2 + 2W_c \sin 2\alpha_0)\varepsilon_1 \\ &+ (\kappa_1 + \kappa_2 - 2W_c \cos 2\alpha_0)\varepsilon_2] + u_{10} - u_{20} - 2\omega_{m0}W_c \cos 2\alpha_0(\alpha - \alpha_0) \end{aligned} \quad (20)$$

Adding  $\Delta\dot{\alpha} = \omega_{m0}\bar{\varepsilon}_2$  as the third new row, Eq. (20) can be rewritten as:

$$\dot{z} = Cz \quad (21)$$

where:

$$C = E^{-1}D, \quad z = \{\bar{\varepsilon}_1 \ \bar{\varepsilon}_2 \ \bar{\alpha} - \bar{\alpha}_0\}^T, \\ E = \begin{pmatrix} e_{11} & e_{12} & 0 \\ e_{21} & e_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D = -\omega_{m0} \begin{pmatrix} d_{11} & d_{12} & -2W_s \sin 2\alpha_0 \\ d_{21} & d_{22} & 2W_c \cos 2\alpha_0 \\ 0 & -1 & 0 \end{pmatrix},$$

$$\begin{aligned} e_{11} &= \rho_1 + \rho_2 + W_c \cos 2\alpha_0, & e_{12} &= \rho_1 - \rho_2 + W_s \sin 2\alpha_0, \\ e_{21} &= \rho_1 - \rho_2 - W_s \sin 2\alpha_0, & e_{22} &= \rho_1 + \rho_2 - W_c \cos 2\alpha_0, \\ d_{11} &= \kappa_1 + \kappa_2 + 2W_s \cos 2\alpha_0, & d_{12} &= \kappa_1 - \kappa_2 - 2W_c \sin 2\alpha_0, \\ d_{21} &= \kappa_1 - \kappa_2 + 2W_c \sin 2\alpha_0, & d_{22} &= \kappa_1 + \kappa_2 - 2W_s \cos 2\alpha_0. \end{aligned}$$

Solving the determinant equation  $\det(C - \lambda I) = 0$ , the characteristic equation for the eigenvalue  $\lambda$  is given by:

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0 \quad (22)$$

where  $c_1 = 4\omega_{m0} h_1/h_2$ ,  $c_2 = 2\omega_{m0} h_2/h_0$ ,  $c_3 = 2\omega_{m0} h_3/h_0$ ,

$$h_0 = 4\rho_1\rho_2 - W_c^2 \cos^2 2\alpha_0 + W_s^2 \sin^2 2\alpha_0$$



$$\begin{aligned}
 h_1 &= \kappa_2 \rho_1 + \kappa_1 \rho_2 - W_c W_s \\
 h_2 &= 2\kappa_1 \kappa_2 + W_c (\rho_1 + \rho_2) \cos 2\alpha_0 + W_c^2 - 2W_s^2 \cos^2 2\alpha_0 \\
 &\quad + W_s (\rho_1 - \rho_2) \sin 2\alpha_0 - W_s^2 + W_c^2 \sin^2 2\alpha_0 \\
 h_3 &= W_c (\kappa_1 + \kappa_2) \cos 2\alpha_0 + 2W_c W_s + W_s (\kappa_1 - \kappa_2) \sin 2\alpha_0
 \end{aligned} \tag{23}$$

In engineering,  $\gamma$  is so small that  $\sin \gamma \approx 0$ . Hence, compared with  $W_c$  in the expression of  $c_1$ ,  $c_2$  and  $c_3$ ,  $W_s$  can be neglected. Then  $h_0$ ,  $h_1$ ,  $h_2$  and  $h_3$  can be simplified as follows:

$$\begin{aligned}
 h_{01} &= 4\rho_1 \rho_2 - W_c^2 \cos^2 2\alpha_0 \\
 h_{11} &= \kappa_2 \rho_1 + \kappa_1 \rho_2 \\
 h_{21} &= 2\kappa_1 \kappa_2 + W_c (\rho_1 + \rho_2) \cos 2\alpha_0 + W_c^2 + W_c^2 \sin^2 2\alpha_0 \\
 h_{31} &= W_c (\kappa_1 + \kappa_2) \cos 2\alpha_0
 \end{aligned} \tag{24}$$

Use the Routh-Hurwitz criterion to determine the stability of the system. If the trivial solution  $z_i = 0$  is stable, it must satisfy the following conditions:

$$c_1 > 0, \quad c_3 > 0, \quad c_1 \cdot c_2 > c_3. \tag{25}$$

(1) If  $h_{01} > 0$ ,

$$\begin{aligned}
 4\rho_1 \rho_2 - W_c^2 \cos^2 2\alpha_0 &> 0 \\
 \kappa_2 \rho_1 + \kappa_1 \rho_2 &> 0 \\
 2\kappa_1 \kappa_2 + W_c (\rho_1 + \rho_2) \cos 2\alpha_0 + W_c^2 &> 0 \\
 W_c (\kappa_1 + \kappa_2) \cos 2\alpha_0 &> 0 \\
 4(\kappa_2 \rho_1 + \kappa_1 \rho_2)(2\kappa_1 \kappa_2 + W_c^2 + W_c^2 \sin^2 2\alpha_0) &> -W_c \cos^2 2\alpha_0 (W_c^2 \cos^2 2\alpha_0 + 4\kappa_2 \rho_1^2 + 4\kappa_1 \rho_2^2)
 \end{aligned} \tag{26}$$

From the first four formulas of Eq. (26), we can deduce that:

$$W_c \cos 2\alpha_0 > 0, \quad \rho_1 > 0, \quad \rho_2 > 0, \quad 4\rho_1 \rho_2 - W_c^2 \cos^2 2\alpha_0 > 0. \tag{27}$$

When these conditions are met, the left side of the fifth formula of Eq. (26) is greater than zero, while the right side is less than zero. Hence, Eq. (26) can satisfy Eq. (25).

(2) If  $h_{01} < 0$ ,

$$\begin{aligned}
 4\rho_1 \rho_2 - W_c^2 \cos^2 2\alpha_0 &< 0 \\
 \kappa_2 \rho_1 + \kappa_1 \rho_2 &< 0 \\
 2\kappa_1 \kappa_2 + W_c (\rho_1 + \rho_2) \cos 2\alpha_0 + W_c^2 &< 0 \\
 W_c (\kappa_1 + \kappa_2) \cos 2\alpha_0 &< 0 \\
 4(\kappa_2 \rho_1 + \kappa_1 \rho_2)(2\kappa_1 \kappa_2 + W_c^2 + W_c^2 \sin^2 2\alpha_0) &< -W_c \cos^2 2\alpha_0 (W_c^2 \cos^2 2\alpha_0 + 4\kappa_2 \rho_1^2 + 4\kappa_1 \rho_2^2)
 \end{aligned} \tag{28}$$

From the first four formulas of Eq. (28), we obtain:

$$W_c \cos 2\alpha_0 < 0, \rho_1 < 0, \rho_2 < 0, 4\rho_1\rho_2 - W_c^2 \cos^2 2\alpha_0 < 0. \quad (29)$$

When these conditions are satisfied, the left side of the fifth formula of Eq. (28) is less than zero, while the right side is greater than zero. Hence, Eq. (28) cannot satisfy Eq. (25).

Therefore, Eq. (26) is the condition of the synchronous stability.

## Conclusions

A vibration model is developed for the self-synchronization theory of a dual-mass vibration system, in which two unbalanced rotors are installed symmetrically about the mass center of the rigid frame. Lagrange's equations are utilized to establish the differential equations of motion for dynamic analysis of the model. To obtain the coupling equations of angular velocities of two exciters, two uncertain parameters, including coefficient of instantaneous change of average angular velocity and the phase difference between two exciters, are introduced.

Dimensionless parameters are obtained by adopting the modified small parameter average method to derive the conditions of synchronous implementation and stability. If Eq. (18) is satisfied, the system can implement the self-synchronization.  $\chi_a$  is the coupling torque of the system. It acts on the faster motor as resisting moment and acts on the slower motor as driving moment. Ultimately two motors reach the same speed. When  $2\alpha = 90^\circ$ ,  $\chi_a$  is the maximum.

If the speeds of two motors are identical and mass ratio of two eccentric blocks is 1, the synchronous speed is equal to the rated speed. Hence the phase difference of two motors is zero, and the mechanism can only implement the horizontal movement, this phenomenon is called complete symmetry. Otherwise, it can implement the elliptical motion in  $xy$ -plane.

When the non-dimensional moments of inertia of the two exciters are all greater than zero and Eq. (26) is satisfied, synchronization is at the stable state on which the mass ratio of two eccentric blocks and their eccentric radius etc, have effects. To guarantee the synchronous stability, the phase difference must be in the interval of  $(-\pi/2, \pi/2)$ , and masses of two eccentric blocks should be made as small as possible.

## Acknowledgements

We appreciate the reviewer's warm work earnestly for helping us review manuscripts and providing us with useful reference list. Those comments are all valuable and very helpful for understanding and recognizing the problem of self-synchronization, as well as of the important guiding significance for our research.

This work is supported by National Science Foundation of China (Grant No. 51175071, 51075063), NCET of MOE (Grant No. NCET-10-0271), Fundamental Research Funds for the Central Universities (Grant No. N100503002) and Science Foundation of Liaoning Province (Grant No. 201102072).

## References

- [1] **Blekhman I. I.** Self-synchronization of vibrators in some types of vibrational machines. *Inzhenerny Sbornik*, Vol. 16, 1953, p. 49-72, (in Russian).
- [2] **Belkhman I. I.** Synchronization of Dynamical Systems. Nauka, Moscow, 1971, (in Russian).
- [3] **Blekhman I. I.** Synchronization in Science and Technology. New York: ASME Press, 1988, p. 255, (Engl. transl. of the Russian issue: Moscow: Nauka, 1981).
- [4] **Blekhman I. I.** Vibrations in Engineering. *Mashinostroeniye*, Moscow, Vol. 4, 1981, p. 478, (in Russian).
- [5] **Blekhman I. I.** Vibrational Mechanics. Fizmatlit, Moscow: Nauka, 1994, p. 394, (in Russian, Engl. transl.: Singapore et al., World Scientific Publishing Co., 2000, p. 510).

- [6] **Blekhman I. I.** Selected Topics in Vibrational Mechanics. World Scientific, Singapore, 2004.
- [7] **Blekhman I. I., Fradkov A. L., Tomchina O. P., etc.** Self-synchronization and controlled synchronization: general definition and example design. *Mathematics and Computers in Simulation*, Vol. 58, 2002, p. 367-384.
- [8] **Ragulskis K. M.** Mechanisms on the Vibrating Basis. Kaunas: Acad. of Sciences of Lithuanian SSR, 1963, p. 232, (in Russian).
- [9] **Ragulskis K. M., Vitkus I. I., Ragulskiene V. L.** Self-synchronization of Systems. Self-synchronous and Vibro-shock Systems. Vilnius: Mintis, 1965, p. 186, (in Russian).
- [10] **Khodzhaev K. Sh.** Synchronization of the mechanical vibrators related to the linear oscillatory system. *Izv. AN SSSR, MTT*, Vol. 4, 1967, p. 14-24, (in Russian).
- [11] **Sperling L.** Selbstsynchronisation statisch und dynamisch ungewichtiger vibratoren. Teil I Grundlagen. *Technische Mechanik*, Vol. 14, Issue 1, 1994, p. 61-76, (in German).
- [12] **Sperling L.** Selbstsynchronisation statisch und dynamisch ungewichtiger vibratoren. Teil II Ausführung und Beispiele. *Technische Mechanik*, Vol. 14, Issue 2, 1994, p. 85-96, (in German).
- [13] **Nagaev R. F.** Dynamics of Synchronizing Systems. Berlin, Heidelberg: Spriger, 2003, p. 326, (Engl. transl. of the Russian issue: Quasiconservative Synchronizing Systems, St. Petersburg: Nauka, 1996).
- [14] **Wen Bangchun** Some new results to research on vibration synchronization theory with its application. *Journal of Vibration and Shock*, Vol. 3, 1983, p. 1-10.
- [15] **Wen Bangchun, Guan Lizhang** Research concerning the theory of synchronization on a self-synchronous vibrating machine with single and dual mass in space motion. *Northeast. Univ. (Nat. Sci.)*, Vol. 19, 1980, p. 54-63.
- [16] **Wen Bangchun, Zhao Chunyu, Fan Jian** The application and development of synchronization theory in mechanical system. *Journal of Vibration Engineering*, Vol. 3, 1997, p. 264-272.
- [17] **Yamapi R., Wofo P.** Dynamics and synchronization of coupled self-sustained electromechanical devices. *Journal of Sound and Vibration*, Vol. 5, 2005, p. 1151-1170.
- [18] **Zhao Chunyu, Liu Zhan, Ye Xiaofen et al.** Self-synchronization theory of a vibrating system with two-motor drives rotating in opposite directions. *Journal of Mechanical Engineering*, Vol. 9, 2009, p. 24-30.
- [19] **Wang Degang, Zhao Qinghua, Zhao Chunyu et al.** Self-synchronous feature of a vibrating system driven by two motors with the same rotation direction. *Journal of Vibration, Measurement & Diagnosis*, Vol. 30, 2010, p. 218-222.