866. Multiple inverse problem

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Abstract. The problem of providing the required level of quality of products and/or technological processes often becomes difficult due to the fact that there is no general theory for determining optimal sets of values of primary factors, i.e. output parameters of parts and units comprising an object as well as ensuring the equivalence of object parameters to the quality requirements. This is the main reason why the development of complex systems and objects of vital importance takes several years. To create this theory, one has to overcome a number of difficulties and solve the following tasks: creation of reliable and stable mathematical models that indicate the influence of primary factors on output parameters; determination of accurate solutions when mathematical models are poorly stipulated; and creation of a method of assigning nominal values and tolerances for primary factors with regard to economical, technological and other criteria. All of the above-listed tasks are equally important. The present work is an attempt to describe a solution for this problem. The mathematically formalized aspect of the problem of providing the required level of quality has been called the “multiple inverse problem”.

Keywords: optimization, multiple inverse problem, parameter, deviation.

1. Introduction

Creation of any new machinery, mechanism, technological, medical and other systems and processes begins with establishing technical specifications (TS) for output parameters.

These technical specifications are expressed as nominal values and tolerances for output parameters. The next step for developers is to design, develop, manufacture and adjust a system (or an object) that is capable of performing the specified functions with output parameters being within a range of values determined by the TS, thus providing the required level of quality.

The values of complex object output parameters usually depend on nominal values and tolerances for thousands of components and units which comprise an object (these nominal values and tolerances are further referred to as input parameters or primary factors). Interaction of these units and components provides functional purpose of an object.

It is well known that the implementation of complex objects, especially those that do not have analogues in the industry, is hard and time-consuming. And, of course, the most urgent question during development stage is whether the given unit or component meets the specified requirements under the proposed technology; if not, then what kind of component and technology would satisfy the requirements?

Without the ability to analyze all solutions accomplished in the past, it is necessary to note that the ongoing development of science and technology makes the problem ever-changing and more complex, but always up-to-date. Its solution is defined by the extent of use of the latest scientific achievements. A formal description of the problem, as well as possible method of solution, is described below.

2. Statement of the problem

Let the quality of any object to be rated by values of its output parameters (specifications) which are represented by the vector $\overline{Y} = \{Y_1, Y_2, ..., Y_m\}$.

To provide the required level of quality means that the following expressions are always true:

$$[y_i] < Y_i < [Y_i],\ i = 1, 2, ...m, \tag{1}$$

where $[y_i]$ and $[Y_i]$ are lower and upper limits (deviations) of the parameter Y_i specified in TS, accordingly.

We are going to look for a solution as a set of values of primary factors. This set can be expressed by the following inequalities:

$$[x_i] < X_i < [X_i],\ i = 1, 2, ...n. \tag{2}$$

Any solution in this set must satisfy expressions (1) imposed on values of output parameters.

We will refer to the stated problem as "multiple inverse problem". This term emphasizes that the solution of the problem suggests the determination of a set of points (region) in the space of the primary factors. This is what makes it different from "point inverse problems", which are traditionally solved in many technical branches. In the point inverse problem, only one vector of the primary factors and (or) one set of input parameters has to be determined, if the vector $\overline{Y}$ is given.

3. Reduction to the problem of optimization

The above-mentioned vector $\overline{Y}$ is completely defined by a set of values of primary factors $\overline{X} = \{X_1, X_2, ..., X_n\}$ with operator:

$$\overline{Y} = \overline{f}(X_1, X_2, ..., X_n; B_1, B_2, ..., B_k) \tag{3}$$

which establishes an interconnection between these vectors. The structure of $\overline{f}$ and the vector of parameters of the mathematical model $B_1,\ B_2,\ ...,\ B_k$ corresponds to the physical nature of an object and its functional purpose.

As a rule, industrial, physical, economical, and other considerations allow to set very wide boundaries for the range of possible values of primary factors. Considering these boundaries and also the type of coordinates of the operator (3), the system (1) can be written as:

$$\begin{aligned} &Y_i = f_i(X_1, X_2, ..., X_n; B_1, B_2, ..., B_k),\ i = 1,\ 2,\ ...,\ m \\ &C_i < X_i < D_i,\ i = 1,\ 2,\ ...,\ n \end{aligned} \tag{4}$$

It should be noted that the structure of functions or sets of values of primary factors may vary. For example, one of primary factors X_i can have discrete or even finite set of values. Then, this fact must be reflected in (4) by such notion as $X_i = 1,\ 2,\ ...,\ N$.

The boundary system (4) determines some curve-shaped region in the n-dimensional space of primary factors. The geometrical meaning of finding within the region some sets in the form of (2) is inscribing n-dimensional parallelepipeds into this region.

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This kind of problem has more than one solution due to the fact that there are countless sets of such parallelepipeds that may be inscribed into the above-mentioned region. Every one of those parallelepipeds can be defined by the point $\overline{X}_0 = \{X_{10}, X_{20}, ..., X_{n0}\}$, which belongs to the region and corresponds to one of the object base versions and the set of lower δ_i and upper Ω_i deviations of primary factors from their nominal values corresponding to the boundaries of the tolerance zones of primary factors, i.e. chosen technology.

The following relations are evident here:

$$X_{i0} - \delta_i < X_i < X_{i0} + \Omega_i, \; i = 1, \; 2, \; ..., \; n. \tag{5}$$

But not every solution (5) of the problem formulated above can be realized in practice because of various design, technological, economical and other considerations such as high cost of production or the absence of necessary equipment, components, materials, personnel with required qualification and peculiarities of an object.

These considerations can be analytically formulated as criteria expressed through deviation of primary factors from their nominal values:

$$F_i = F_i\left(\delta_1, \delta_2, ..., \delta_n; \Omega_1, \Omega_2, ..., \Omega_n\right), \; i = 1, \; 2, \; ..., \; L. \tag{6}$$

It is evident that of the above mentioned parallelepipeds the most acceptable for practical implementation are those, in which criteria (6) or some of them are optimized and the rest are added to the restrictions (4).

Various criteria are possible for optimization of tolerances of primary factors. Cost minimization function is the most important. Since the dependence of this function on current values of each of primary factor tolerances is usually unknown, it might be plausible to replace it by equivalent criteria. For example, it might be possible to demand maximization of all or some tolerances. Then criteria (6) will be as follows:

$\max_i\left(\delta_1, \delta_2, ..., \delta_n; \Omega_1, \Omega_2, ..., \Omega_n\right) \to \max$.

Thus, the problem of providing the required level of object quality is reduced to the multi-criterion optimization problem with certain restrictions. Deviations δ_i and Ω_i have to be determined so that restrictions (1) are satisfied within region (5).

4. Necessary points of the solution

There are some difficulties in solving the formulated problem. Resolving each of these difficulties has to be the part of the solution.

4.1. Mathematical modeling

In practice there are cases when a model describing the functional part of restrictions (1) is known. But as a rule, complicated objects can have either unknown (inaccurate) parameters or unknown structure, or both. Therefore, it is necessary to develop an easily implemented approach to the task of determining functional dependences of primary factors and output parameters, which have reliable coefficients adjusted to the object model.

An active controlled influence on an object can become the ground for algorithmization of operative mathematical modeling.

Let the model of the experimental sample be represented as:

$$Y_i = f_i\left(X_1, X_2, ..., X_n; B_1, B_2, ..., B_k\right),\ i = 1,\ 2,\ ...,\ L. \qquad (7)$$

First, let us suppose that the structure of functions f_i is known, but parameters B_i are unknown. If we substitute the values of output parameters which were measured during real-life operating conditions and values of certain primary factors into (7), then the values of k unknown primary factors and j coefficients of the model should also satisfy (7). As a rule, system (7) is not completely defined (that is $k + j > L$) and has countless sets of solutions. However, keeping in mind that the object operates and really exists, it would be natural to find a solution which completely corresponds to the given object. Hence, it is necessary to redefine the task by conducting additional experiments. Method of test parameters suggested here is a practical way to provide the aforementioned redefinition. To implement the method, $k + j - L$ additional elements are either entered or their values are varied and (or) the object is put into test modes of operation which were specified in TS (technical specifications), i.e. object operation is actively controlled. The influence of test components (or modes) along with components whose parameters are being identified allow the measurement of missing values of output parameters, make the system (7) complete, and, in case of independence of system equations, identify the target factors and coefficients of the model, i.e. to find a solution of inverse point problem.

When structure of functions f_i is unknown, it is recommended to disintegrate them by series according to any complete system of functions, for example, the Taylor series:

$$Y_i = B_{i0} + \sum_j B_{ij} X_j + \sum_{j,k} B_{ijk} X_j X_k + ...,\ i = 1,\ 2,\ ...,\ L \qquad (8)$$

and identify coefficients in several sequential stages. First, we suggest the determination of coefficients of linear approximation using the required quantity of test parameters. Then, after employing additional test parameters, functions Y_i which are adequate to an object are selected. The square and other approximations are considered for the rest of the functions. Convergence of this process is easily proven. In most frequent practical situations the number of stages usually does not exceed two or three.

When putting the method into practice, one has to start with considering the characteristics of large units as primary factors and to sort out those which have major influence on functionality of an object. Then, the dependence of those large unit characteristics on smaller units can be determined in the same manner and so on. This hierarchical approach allows operative adjustment of the model to the object under examination with regard to the degree of its idealization and functioning conditions, and eliminates the necessity of registration and analysis of inessential primary factors.

However, the hierarchical principle of modeling can only be employed if output parameters of separate units can be measured at every stage. If there is no such possibility due to the design, the multi-cascade modeling method can be used. Let the interconnection between output parameters of separate units (cascades) and output characteristic be determined, that is the function $Y = f\left(\varphi_1, \varphi_2, ..., \varphi_s\right)$, where $\varphi_i = \varphi_i\left(X_{i1}, X_{i2}, ..., X_{it}\right)$ is output characteristic of i-th unit, is specified. Then, by fixing the values of primary factors of all the cascades except one and measuring the output parameter Y, we can create a model of every cascade and combine these models within one common model with variation of primary factors of all units [1].

We would like to emphasize that this method of mathematical modeling is a special case of the inverse problem solution.

4.2. Providing the model stability

However, models are practically important only if experimental input information errors are not causing unacceptably large errors of the values being determined, i.e. models are stable. It is shown in (2) how the stability of a model is determined with regard to all or some group of factors, as well as an estimation of the relative error of parameters which are identified by linear model:

$$\frac{\|\Delta\bar{X}\|}{\|\bar{X}\|}\le C(A)\frac{\|\Delta\bar{Y}\|}{\|\bar{Y}\|}+\left[C(A)\right]^2\frac{\|\Delta A\|}{\|A\|}. \tag{9}$$

This estimation is expressed through the number of stipulations $C(A)$ of matrix A and errors of characteristics and elements of A being measured. Estimation (9) allows to explain the decrease in stability of a model with increase of the degree of A. In other words, it shows the necessity to search for compromise between giving thorough description of an object using large number of factors and ensuring the stability of the model. It is evident from (9) that a model can be regularized not only by influencing the operator of A, which is not always possible in real-life production environment for various reasons. Same level of regularization, based on its statistical nature, can be achieved with influencing the vector of measured parameters Y. To achieve this regularization, the realization of each parameter being identified is calculated from experimental model with the value of the vector of output parameters Y measured multiple times under the same conditions. Mathematical expectations calculated on the base of these realizations are assumed as true values for parameters being identified. The derived estimation of the number of realizations sufficient to ensure the accuracy of the method:

$$n\ge t^2\left[C(A)\frac{\sigma_1}{\|\bar{Y}\|}+\left[C(A)\right]^2\frac{\sigma_2}{\|A\|}\right]\Big/\delta^2 \tag{10}$$

(where σ_1, σ_2 - mean quadratic deviations of the vector $\bar{Y}$ elements and matrix A correspondingly, t - Student's coefficient) shows that the described method of statistical solution is mostly efficient when coupled with methods of influencing the operator of A [2]. Estimation (9) is also interesting in a purely practical aspect, since it establishes a functional interdependence between economical factors (accuracy of a method and accuracy of measuring equipment) and theoretical factors (accuracy of a model); therefore, it allows choosing one of these factors to satisfy the other two which are set a priory.

5. Problems associated with optimization

A particular method of optimization for a given task is chosen from sufficiently large collection of already-developed in detail algorithms of optimization.

It is evident that the results of the criteria optimization greatly depend on the choice of object's base version, i.e. on the point. Here we offer some recommendations associated with it.

5.1. Construction of regions

It is suggested to implement the algorithm with construction of regions of primary factors spreading from a base point in the optimum way along with ensuring that restrictions (1) are valid at every step. This base point can often be determined out of physical or practical considerations. But there are cases when this point is unknown, and the problem of determining one becomes very difficult.

5.2. On choice of base points

Recommendations for a choice of base point are based on the following considerations. The points belonging to the space of primary factors should be chosen as base points in such a way that one of mathematical expectations of random variable describing the distribution of the first primary factor serves as a first coordinate, second mathematical expectation – as a second coordinate and so on.

Because of a great number of random and unpredictable situations that may occur during manufacturing process and operation of an object, as well as non-stability of the properties of construction materials, characteristics of an object may be treated as random variables. Then we can estimate the true values using the method of confidence intervals [3], provided that laws of distribution are known.

For a long time, the normal distribution or its modifications were considered the best approximation for which the majority of statistical criteria and estimations can be applied. However, a lot of practical problems indicating that the normal distribution is not as universal as it was thought have turned up lately. Situations emerging during the study of real-life processes show that many of object parameters have probability density functions that are different from normal, and often are even not single-summit. Therefore, the physical essence and new technical schemes of such probability processes are considered here [4]. The schemes are based on representation of each selection of realization of random variable in the form of a set of sub-selections, united by some dominant causes for dispersion of values of this variable. Such situations can often be described by linear combination of Gauss functions with some weight coefficients P_i defining the significance of each sub-selection in the total selection of realization:

$$f\left(X, a_1, a_2, \ldots, a_n; S_1, S_2, \ldots, S_n; P_1, P_2, \ldots, P_n\right)=\sum_{i=1}^{n} \frac{P_i}{S_i \sqrt{2\pi}} \exp\left(-\frac{\left(x-a_i\right)^2}{2S_i^2}\right). \quad (11)$$

Various methods of finding unknown parameters of function (11) depended on selected criterion of histograms' approximation and the required accuracy of calculations are described in [4]. The established values allow writing down the integral function which, in turn, enables to write down the equation for determination of acceptable values of [X]:

$$W=P\left\{X<[X]\right\}=\sum_{i=1}^{n} \frac{P_i}{S_i \sqrt{2\pi}} \int_{-\infty}^{[X]} \exp\left(-\frac{\left(x-a_i\right)^2}{2S_i^2}\right) dx. \quad (12)$$

A reasonable technique for histograms plotting should be employed during experimental data processing to avoid the possibility of missing a considerable part of distribution by using too large spacing intervals on one hand, and the emergence of insignificant sub-selections by using too small spacing intervals on the other hand. It is a good idea to start the plotting of histograms with the smallest possible spacing interval comparable with the measuring accuracy, and approximate them with a function of type (11) where the number of items is equal to the number of spacing intervals. The determined weight coefficients, which are less than the pre-set probability $\alpha=1-W$ give an indication of unimportant sub-selections being joined with adjacent sub-selections. The spacing interval is then gradually increased and the same procedure is repeated until each weight coefficient becomes comparable with α.

The described method of statistical data processing shows an internal structure of experimental data and also determines a methodology of processing this data, in particular, a methodology employed during the study of the production errors, standard documentation development, and development of quality control methods.

5.3. Restrictions fulfillment verification

Verification of fulfillment of restrictions (1) at every step of optimization during construction of regions spreading from a base point can be accomplished on a set of uniformly distributed points belonging to the constructed region. But in some practical situations this verification technique can be simplified. For example, when partial derivatives of the functions (3) have invariable signs, verification may be accomplished only for corner points of a region.

5.4. Optimal base version selection

As the number of base points can be more than one, it is natural to employ the optimizing algorithm for each of the possible base versions separately and to choose the most optimal of them with regard to criteria (6).

6. Possible applications

This approach, which generally formalizes a problem of ensuring the satisfaction of TS requirements for output parameters of an object or technological process, allows, firstly, to establish an interconnection between the task of an object base version selection, determined by nominal values of its primary factors, and the task of establishing design and technological tolerances for these values depended on the TS restrictions for object output parameters. In particular, this approach considers a study of the possibility of implementing the existing and well-known units, processes, and technological solutions for the use in the object or technological process being created.

This gives the ability to formulate and solve a problem of synthesizing design versions of objects that have optimal sensitivity to manufacturing and operational deviations of their primary factors, that is, directly link the selection of object base version with specific features of its practical implementation.

Secondly, this approach allows formalization of many important problems of design, manufacturing, and testing procedures regardless of technical branch of application.

On the same grounds, the recommendations for establishing tolerances for primary factors of the whole object as well as certain units during design, manufacturing, and operational development stages can be produced. The recommendations for selective assembly of equipment suggest grouping of components and materials on the base of actual values of their parameters in such a way that the most favorable combinations are produced.

In serial production, the solution of multiple inverse problem with consideration of statistical nature of parameters, allows to predict the defect yield (waste percentage) of production and determine the conditions necessary for satisfaction of TS requirements.

The same approach is effective when solving other types of problems. For instance, we can check whether or not the possibi1ity of achieving the desired result on al1 or several output parameters under given design or technological conditions exists, i.e. whether there exists a solution of corresponding multiple inverse problem. If a solution does not exist or it is out of reasonable 1imits in opinion of designer or production engineer, then a given object fails to meet the requirements of TS, and it is necessary to look for essentially different design or technological solutions.

This approach is also capable of providing a solution not only for the object as a whole but also for its components as well.

Thus, the considered approach is a natural reflection of a set of real-life situations emerging at design, manufacturing and operational stages of an object.

Conclusion

To verify the versatility of the described theory, multiple inverse problem was formulated and solved for various branches of technical engineering including:

- providing durability and impermeability for radio-electronic elements [5];
- enhancing the stability of output parameters of secondary radio-detection equipment (airplane transponders) [l];
- reduction of vibration activity of gas-turbine engines and turbo-pump assembly units to preset level [2, 6-8, 11];
- assigning well-reasoned tolerances for residual imbalance during balance and assembly of rotors [4, 9];
- developing balancing techniques for flexible rotors [10].

Each of the above-listed applications is complex enough in itself, and it would take more time and space than we have, to give their full description here. Therefore, more detailed information may be found in the provided references.

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