861. Iterative method for frequency updating of simple vibrating system

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Abstract. Iterative methods for modification of vibratory characteristics of dynamic systems have attracted a lot of attention as a convenient and more economical way when compared to the traditional and costly structural dynamic optimization processes. Many complicated structures, such as telecommunication towers, chimneys and tall buildings, may be modeled as simple spring-mass systems. This paper presents an iterative method for modification of the frequencies of simple vibrating system consisting of springs and masses. The proposed algorithm may be used to adjust any of the vibration frequencies of a simple vibrating system to the target values within the desired level of accuracy. The method based on the variation of mass and/or stiffness properties of the system is simple yet efficient and needs less computational effort. The efficiency of the method is demonstrated using a numerical example. It is demonstrated that there is a faster convergence for adjustment of the lower frequencies and for the case with stiffness variation of the system rather than mass variation.

Keywords: iterative, frequency updating, simple, vibrating system.

1. Introduction

There are several techniques for frequency updating of dynamic systems that can be used for different purposes, including optimal design of structures. First-order sensitivity analysis of eigenvalues and eigenvectors was discussed by Fox and Kapoor [1]. A scheme for determining the rate of changes of eigenvalues and eigenvectors was developed. It can be helpful in the automated optimum design of structures under dynamic response constraints. Furthermore, a simplified procedure for finding the derivative of the eigenvectors of symmetric or non-symmetric systems was developed by Nelson [2].

Extensive literature review on the development of the higher-order approximation methods could be found in different studies, including the one provided by Rudisill and Bhatia [3]. Based on Nelson's method and generalized inverse techniques, two procedures were developed by Cao and Mlejnek for computation of the second order sensitivities of eigenvalues and eigenvectors [4]. However, it should be pointed out that these methods have their own limitations. The truncated Taylor series converge for small modification. However, it could lead to some degree of error depending on size of the "small modification". In addition, the computation of the higher-order eigenvalue and eigenvector derivatives is time consuming and complicated.

Iterative methods were developed due to the limitations of the aforementioned procedures in handling larger modifications in structural parameters. Kim has presented a simple formulation for the calculation of the modal design sensitivities [5]. He used the first-order perturbation equation to provide the starting point for the iterations. This method is useful for the systems with distinct and/or multiple-valued eigenvalues. To and Ewins proposed an alternative structural modification method based on the Rayleigh quotient iteration [6]. To and Ewins presented a procedure for computing the modified eigenvalues and eigenvectors. The method is based on expressing the eigenvectors of modified structure as a linear combination of the eigenvectors of the original system. They solved the problem of "small modification limitation" by contributing the nonlinear effects of all higher-order terms [7].

This paper presents a new procedure for updating the frequencies of a simple vibrating system. A special feature of this method is that the exact frequencies of the modified system can be determined without solving the eigenvalue problem for the modified structure. Considering the properties of simple vibrating system, this method requires less computational time for extracting the eigenvalues of the modified structure. Therefore it could be efficiently used in a dynamic structural optimization process. The present method is based on the formulation derived by Tsuei and Yee for modifying the dynamic properties of the undamped mechanical systems [8]. They determined the exact amount of mechanical parameters (mass or stiffness) required for shifting each frequency to a desired value. Using their procedure, the aim is to solve the alternative problem of "updating the eigenvalues after modifying the structure". Jankovic calculated the second and higher-order derivatives of the general eigenproblem [9]. Friswell extended Nelson's method for the calculation of the first-order derivatives of eigenvectors, or sensitivities, to the second- and higher-order derivatives of eigenvectors [10].

Methods based on inverse sensitivity of complex valued eigensolutions and sensitivity of frequency response functions have been developed for problems of structural system identification and vibration-based damage detection [11-15].

For inexact inverse iteration the costs of the inner solves using Krylov methods has been investigated in [16] and [17] for the symmetric solvers CG/MINRES and in [18] for the nonsymmetric solver GMRES. In these papers it was shown that, for the standard eigenvalue problem, the number of inner iterations remains approximately constant as the outer iteration proceeds if no preconditioner is used but increases if a standard preconditioner is applied. A so-called tuned preconditioner has been introduced in [18] for the standard Hermitian eigenproblem and in [17] for the generalized eigenproblem (though no analysis of the inner solves was provided in [17]).

This paper provides an iterative method for modification of the frequencies of simple vibrating system consisting of springs and masses respectively.

2. Parameter updating resulted from shifting frequency

In this section the formulation developed by Tsuei and Yee is briefly reviewed. The undamped equation of motion of a dynamic system under force vector $\{f(t)\}$ is represented by:

$$[M]\{\ddot{x}\}+[K]\{x\}=\{f(t)\} \tag{1}$$

where $[M]$ and $[K]$ are the mass and stiffness matrices respectively and $\{x\}$ and $\{\ddot{x}\}$ are the relative displacement and absolute acceleration vectors respectively. For harmonic excitation at frequency ω, ($\{\ddot{x}(t)\}=\omega^2\{x(t)\}$), the response of the system can be determined from the system as:

$$\{x(t)\}=[\Phi]\left([\Lambda]-\omega^2[I]\right)^{-1}[\Phi]^T\ \{f(t)\}$$

or:

$$\{x\}=[H(\omega)]\{f\} \tag{2}$$

where $[\Phi]$ is matrix of mode shapes of the system. $[H(\omega)]$ is the frequency transfer function matrix. The natural vibration of a modified dynamic system is:

$$([M]+[\delta M])\{\ddot{x}(t)\}+([K]+[\delta K])\{x(t)\}=\{0\} \tag{3}$$

where $[\delta M]$ and $[\delta K]$ are the modification in mass and stiffness matrices respectively. Since, the response $\{x\}$ is harmonic ($\{\ddot{x}(t)\}=\omega^2\{x(t)\}$), the following expression can be obtained:

$$\{x\}=[\Phi]\left([\Lambda]-\omega^2[I]\right)^{-1}[\Phi]^T\left(\omega^2[\delta M]-[\delta K]\right)\{x\}$$

or:

$$\{x\} = [H(\omega)]\left(\omega^2[\delta M] - [\delta K]\right)\{x\} \tag{4}$$

where $[I]$ is identity matrix. The natural vibration frequencies of the modified system can be obtained from Eq. (4) directly, instead of solving Eq. (3). Also in case of change in mass parameters m_q only, the above equation can be simplified. Assuming the eigenfrequency ω_s for the modified system, then the variation δm_q must satisfy the following equation:

$$\delta m_q = \frac{1}{\omega_s^2 H_{qq}(\omega_s)} \tag{5}$$

Therefore, by changing the amount of mass m_q to $m_q + \delta m_q$, the system will have the eigenfrequency ω_s. Similarly, if only the change in stiffness parameter, δk_p is considered, the modified dynamic system with an eigenfrequency ω_s has the following characteristic equation:

$$\det\begin{bmatrix} \delta k_p^{-1} + H_{ii}(\omega_s)a_{ii,p} + H_{ij}(\omega_s)a_{ji,p} & H_{ii}(\omega_s)a_{ij,p} + H_{ij}(\omega_s)a_{jj,p} \\ H_{ji}(\omega_s)a_{ii,p} + H_{ji}(\omega_s)a_{ji,p} & \delta k_p^{-1} + H_{ji}(\omega_s)a_{ij,p} + H_{jj}(\omega_s)a_{jj,p} \end{bmatrix} = 0 \tag{6}$$

3. New iterative method for frequency updating

Other frequencies (and mode shapes) must be updated after determination of the required modification for the mass or stiffness parameters. This section presents a new method for frequency updating of simple vibrating system that is based on the above formulation.

Noting that in the simple vibrating system the mass matrix is diagonal, Eq. (5) (that has a simple form) is used. On the other hand, since in simple vibrating system the stiffness matrix is tri-diagonal, Eq. (6) can be transformed to a simple form. A typical simple vibrating system is shown in Fig. (1). If the stiffness parameter k_1 is the only parameter to be changed, then Eq. (4) changes to the following form:

$$x_1 = -\delta k_1 H_{11}(\omega) x_1 \tag{7}$$

The modification δk_1 at the desired frequency ω_s can be evaluated from the following simple scalar equation:

$$\delta k_1 = \frac{-1}{H_{11}(\omega_s)} \tag{8}$$

In case of any change for the stiffness parameter k_q ($q > 1$), one has:

$$\delta k_q = \frac{1}{H_{(q-1)q} + H_{q(q-1)} - H_{(q-1)(q-1)} - H_{qq}} \tag{9}$$

where the elements of H matrix are at the desired frequency ω_s.

3. 1. Modification in parameter m_q of the mass matrix

First, the element H_{qq} must be calculated. Note that $[J] = \left([\Lambda] - \omega^2[I]\right)^{-1}$ is a diagonal matrix and:

$$J_{ii} = \frac{1}{\lambda_i - \lambda}, \quad \lambda = \omega^2 \tag{10}$$

From definition of H matrix, the element H_{qq} becomes:

$$H_{qq} = \sum_{j=1}^{N} \frac{\varphi_{qj}^2}{\lambda_j - \lambda} \tag{11}$$

Eq. (5) can be rewritten to:

$$\lambda = \frac{1}{\delta m_q H_{qq}} \tag{12}$$

Substituting Eq. (11) into Eq. (12) with some rearrangement would lead to the following recursive equation for updating *i*-th eigenfrequency:

$$\lambda_i^{(n+1)} = \frac{\lambda_i + \lambda_i^{(n)} \left(\lambda_i^{(n)} - \lambda_i\right) \delta m_q S_{qi}^{(n)}}{1 + \varphi_{qi}^2 \, \delta m_q}, \quad S_{qi}^{(n)} = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{\varphi_{qj}^2}{\lambda_j - \lambda_i^{(n)}} \tag{13}$$

The λ_i is the *i*-th eigenvalue of the original system, φ_{qi} is an element of modal matrix $[\Phi]$, n is the order of iteration and $\lambda_i^{(0)}$ is the starting point. For both $\lambda_i^{(0)} = 0$ and $\lambda_i^{(0)} = \lambda_i$, the amount of $\lambda^{(1)}$ becomes $\lambda^{(1)} = \lambda_i / (1 + \varphi_{qi}^2 \, \delta m_q)$. In this study the starting point $\lambda^{(0)}$ is considered to be equal to λ_i.

3. 2. Modification in parameter k_q of stiffness matrix

Substituting Eq. (11) with $q = 1$ into Eq. (8), the following equation is obtained:

$$\sum_{j=1}^{N} \frac{\varphi_{1j}^2}{\lambda_j - \lambda} + \frac{1}{\delta k_1} = 0 \tag{14}$$

Also, substituting Eq. (13) for $q = 1$ into Eq. (14), the following recursive equation is obtained:

$$\lambda_i^{(n+1)} = \lambda_i + \frac{\varphi_{1i}^2}{\delta k_1^{-1} + S_{1i}^{(n)}} \tag{15}$$

In the case of modifying the parameter k_q, ($q > 1$) of stiffness matrix, from Eq. (11) the following equation is obtained:

$$\lambda_i^{(n+1)} = \frac{\lambda_i + \left(\varphi_{(q-1)j} - \varphi_{qj}\right)^2}{\delta k_q^{-1} + S_{qi}^{\prime(n)}}, \quad S_{qi}^{\prime(n)} = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{\left(\varphi_{(q-1)j} - \varphi_{qj}\right)^2}{\lambda_j - \lambda_i^{(n)}} \tag{16}$$

4. Case study

As shown in Fig. 1, many complicated structures such as telecommunication towers, chimneys and tall buildings can be modeled as simple spring-mass systems. The important parameter in evaluating the iterative methods is speed of convergence of the recursive equation. For the proposed algorithm, this will be verified using an example. The example is a three-spring-mass model shown in Fig. 1. A computer program for frequency updating was written using MATLAB according to that algorithm.

First, the required modifications in structural parameters for shifting some of the frequencies to the desired values are investigated. Equations (5), (8) and (9) are used to determine the required change in the stiffness and mass matrix of the system for the desired frequency shifting. Figs. 2-3 demonstrate the structural modification versus frequency for structural parameters.

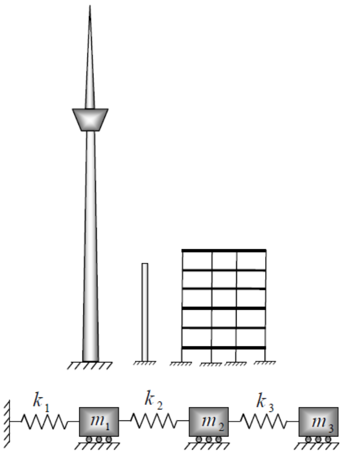

Fig. 1. Structural model as a simple spring-mass system

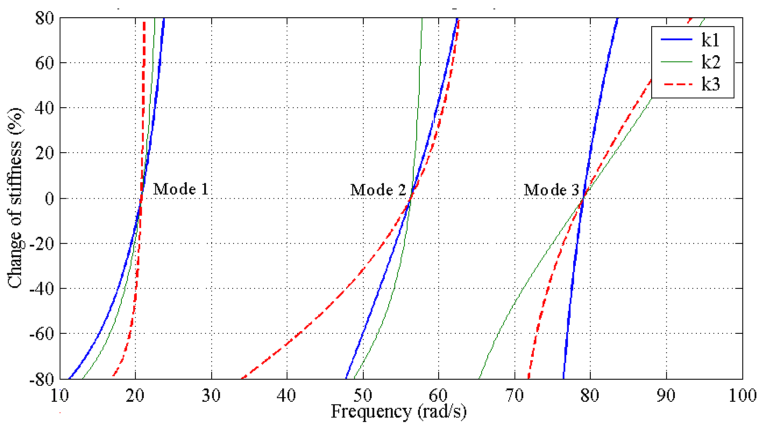

Fig. 2. Required change in stiffness for frequency shifting

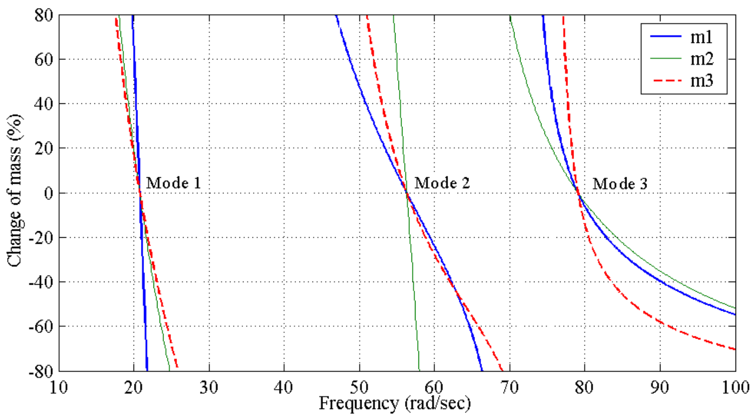

Fig. 3. Required change in mass for frequency shifting

After modifying the structural parameters to achieve the target frequency, the change in other frequencies is determined. The effects of 50 % modification in k_2 and m_3 on the frequencies of the system are provided in Figs. 4-7. It is demonstrated that the proposed method is converging very fast. It takes less number of iteration to achieve the target frequency with stiffness modification compared to mass modification as well. The performance of the algorithm has been studied for the case with large modifications in structural parameters through many appropriate examples demonstrating the efficiency of the method. The results are the same as those presented herein.

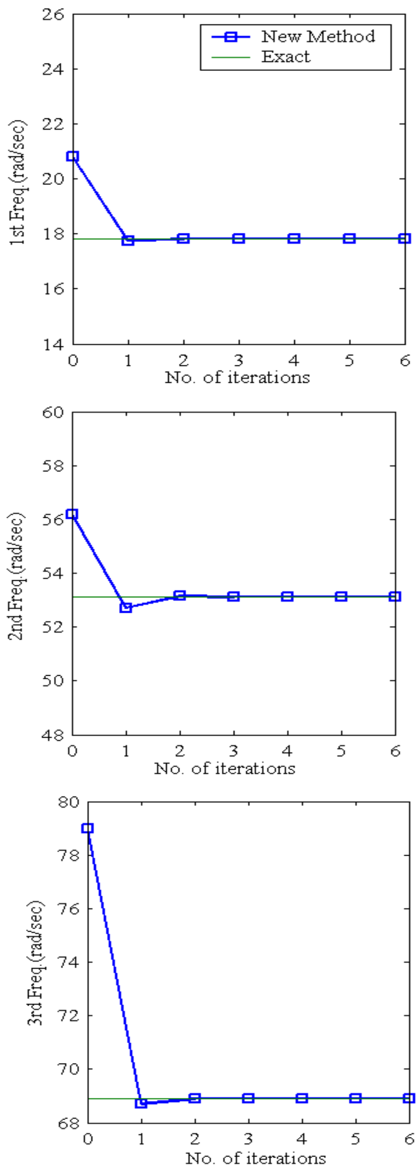

Fig. 4. Required steps for frequency updating (50 % decrease in k_2)

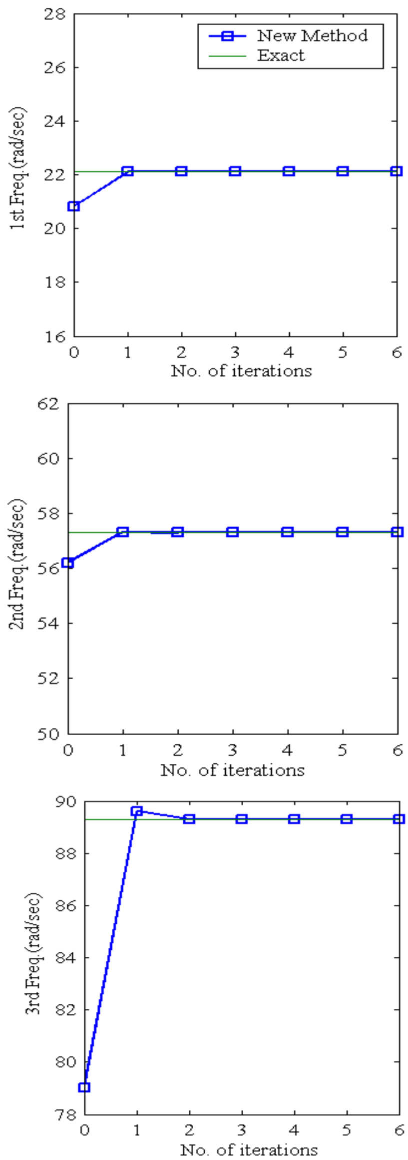

Fig. 5. Required steps for frequency updating (50 % increase in k_2)

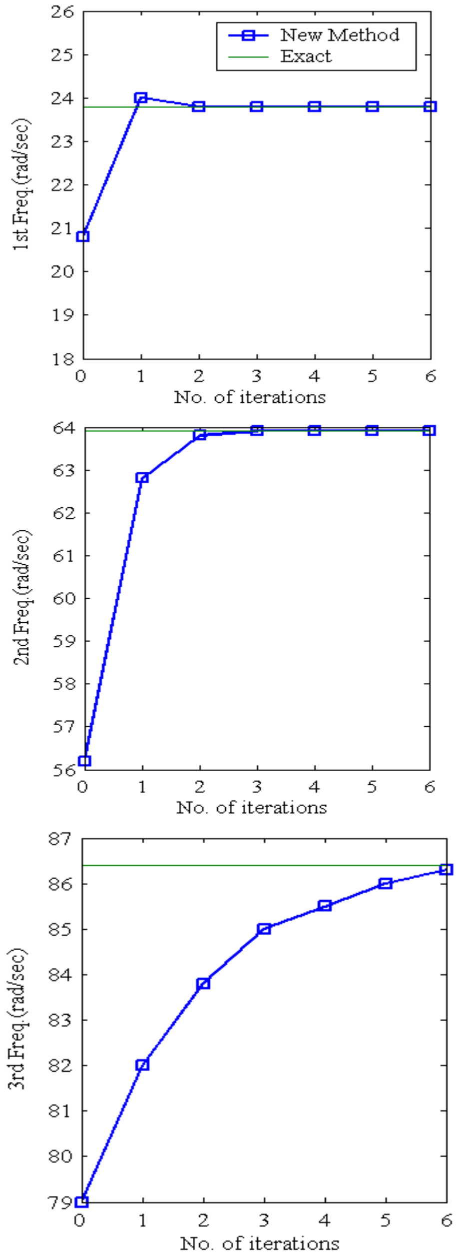

Fig. 6. Required steps for frequency updating (50 % decrease in m_3)

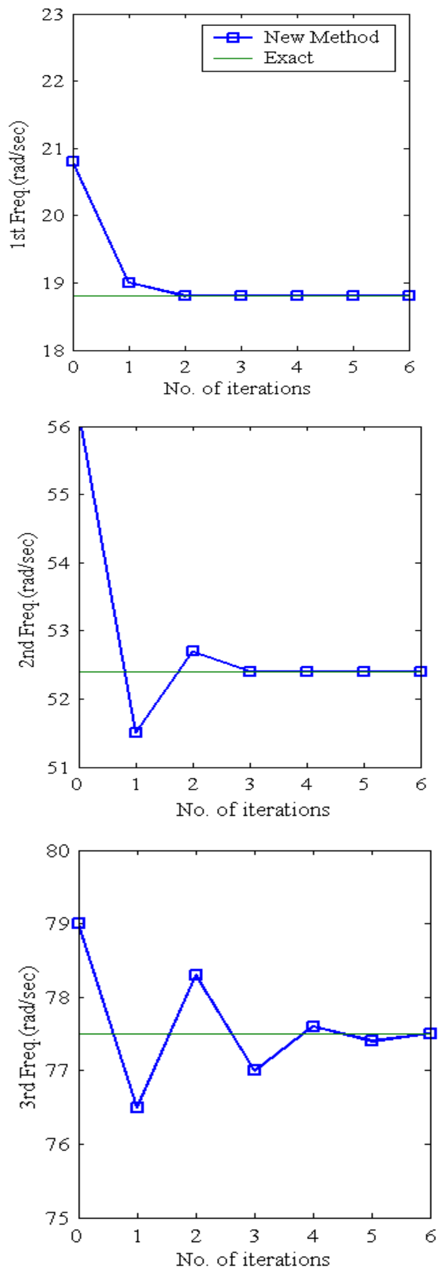

Fig. 7. Required steps for frequency updating (50 % increase in m_3)

5. Conclusion

A new iterative method was presented for frequency updating of the simple vibrating system. The efficiency of the method was examined using numerical example with three degrees of freedom. The extensive parametric study performed indicates that the method is more efficient for the stiffness modification rather than mass modification. However, in any case of stiffness or mass modification, only a few iterations are needed to obtain the desired frequencies with the required accuracy. Furthermore, the convergence rate for lower frequencies is faster than for higher frequencies. Due to extensive use of simple vibrating systems in structural analysis and importance of dynamic characteristics of those models, the proposed method can be very helpful in design process of such kind of structures.

6. References

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