757. The multivariable finite elements based on B-spline wavelet on the interval for 1D structural mechanics

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Abstract: Wavelet finite elements with two kinds of variables for 1D structural mechanics are constructed based on B-spline wavelet on the interval (BSWI) and the generalized variational principle. In contrast to the traditional method, the BSWI element with two kinds of variables (TBSWI) can improve the solution accuracy of the generalized stress apparently, because generalized displacement and stress are interpolated separately. Another superiority of the elements constructed is the interpolating function BSWI, which has very good approximation property, further guarantees solution accuracy. Euler beam, Timoshenko beam and Elastic foundation beam are studied providing several numerical examples to verify the efficiency.

Keywords: B-spline wavelet on the interval, multivariable, structural mechanics.

1. Introduction

The wavelet method can be viewed as a method in which the approximating function is defined by use of a multiresolution technique based on scaling or wavelet functions, similar to those used in signal and image processing [1]. Wavelet finite element method (WFEM) is a new numerical method, which takes scaling and wavelet functions to substitute polynomial in traditional method. Based on the superior properties (multiresolution, orthogonality etc.) of wavelet, WFEM hold many superior properties, such as multi-resolution property and various basis functions for structural analysis, so WFEM is widely used by many researchers both in numerical analysis domains [2-6] as well as in structural analysis field [7-13]. The wavelet method was proved to converge for a wide class of elliptic operator equations including, in particular, differential operators as well as singular integral operators by Dahlke and S. Dahlke et al. [6]. Based on Daubechies wavelet, Li B. constructed Daubechies wavelet element and adaptive scheme for structural response analysis [7-8]. References [9-11] proposed a crack identification method for beam, I-beam and running rotor system based on wavelet finite element model and determinant transformation method. You Q. et al. [12] studied the simulations on the boundaries of the simply supported and a continuous bridge. Zhong Y. T. et al. [13] proposed a new wavelet-based element and support vector regression for pipe crack detection.

To the above wavelet elements for structural analysis the finite element formulations are all derived from generalized potential energy functional, which has only one generalized field function, generalized displacement field function. They should calculate moment by differentiation of displacement and calculation error would be brought into the results. Multivariable finite element method (MFEM) can solve this problem. Based on multivariable generalized variational principle [14-17], generalized displacement, stress and strain field functions are all treated as independent variables, so they are interpolated separately in MFEM. Shen P. C. did a lot of work on MFEM. Based on multivariable variational principle, he derived the formulation of multivariable potential energy functional for many structures in his book [18].

Using spline function to substitute polynomial as interpolating function, Shen P. C. analyzed the bending, vibration and stability problems of beam, plate and shell etc. by MFEM [19-22]. Han J. G. introduced wavelet into MFEM and constructed multivariable wavelet finite element method (MWFEM) for a thick plate [23]. However, there is a deficiency of using spline wavelet as interpolating function in Han's research. Spline wavelet does not have explicit expressions, which will bring many troubles to the integration and differentiation of wavelet coefficients.

BSWI has explicit expressions, and is the best one among all existing wavelets in approximation of numerical calculation [2]. Therefore, taking BSWI as interpolating function, the BSWI element with two kinds of variables is constructed in this paper. Firstly, the multivariable formulations are derived from generalized potential energy functional with two kinds of variables, then taking BSWI as interpolating function to discrete generalized displacement and stress field functions. The structures of Euler beam, Timoshenko beam and Elastic foundation beam are analyzed and the results are compared with BSWI element and traditional method to verify the efficiency.

2. B-spline wavelet on the interval [0, 1]

Chui and Quak constructed B-spline wavelet on the interval [24], and gave its decomposition and reconstruction algorithm in 1994 [25]. In practical numerical calculation, BSWI of even order is frequently chosen, to have at least one inner wavelet on the interval [0, 1], the following condition must be satisfied

$$2^j \geq 2m-1 \tag{1}$$

where m and j are the order and scale of BSWI respectively. While 0 scale mth order B-spline functions and the corresponding wavelets are given by Goswami J. C. in Ref. [26], j scale mth order BSWI (simply denoted as BSWIm_j) scaling functions $\phi^j_{m,k}(\xi)$ and the corresponding wavelets $\psi^j_{m,k}(\xi)$ can be evaluated by the following formulas

$$\varphi^j_{m,k} = \begin{cases} \varphi^l_{m,k}(2^{j-l}\xi), & k=-m+1,\dots,-1 \quad \text{(0 boundary scaling functions)} \\ \varphi^l_{m,2^j-m-k}(1-2^{j-l}\xi), & k=2^j-m+1,\dots,2^j-1 \quad \text{(1 boundary scaling functions)} \\ \varphi^l_{m,0}(2^{j-l}\xi-2^{-l}k), & k=0,\dots,2^j-m \quad \text{(inner scaling functions)} \end{cases} \tag{2}$$

$$\psi^j_{m,k}(\xi) = \begin{cases} \psi^l_{m,k}(2^{j-l}\xi), & k=-m+1,\dots,-1 \quad \text{(0 boundary wavelets)} \\ \psi^l_{m,2^j-2m-k+1}(1-2^{j-l}\xi), & k=2^j-2m+2,\dots,2^j-m \quad \text{(1 boundary wavelets)} \\ \psi^l_{m,0}(2^{j-l}\xi-2^{-l}k), & k=0,\dots,2^j-2m+1 \quad \text{(inner wavelets)} \end{cases} \tag{3}$$

Therefore, one-dimensional scaling functions $\boldsymbol{\Phi}$ at the lower resolution approximation space V_j are given by

$$\boldsymbol{\Phi} = \left[\varphi^j_{m,-m+1}(\xi) \quad \varphi^j_{m,-m+2}(\xi) \quad \dots \quad \varphi^j_{m,2^j-1}(\xi)\right] \tag{4}$$

Semi-orthonormal wavelets $\boldsymbol{\Psi}$ at detail space W_j are

$$\boldsymbol{\Psi} = \left[\psi^j_{m,-m+1}(\xi) \quad \psi^j_{m,-m+2}(\xi) \quad \dots \quad \psi^j_{m,2^j-m}(\xi)\right] \tag{5}$$

Let j_0 be the scale for which the condition Eq. (1) is satisfied. Then for each $j > j_0$, let

$l=0$, we can get the scaling and wavelet functions easily through Eq. (2) and Eq. (3). There are $m-1$ boundary scaling functions and wavelets at 0 and 1, 2^j-m+1 inner scaling functions, and 2^j-2m+2 inner wavelets.

The eleven scaling functions $\phi^3_{4,k}(\xi)$ for $m = 4$ at scale $j = 3$ are given below, among them, 0 boundary functions are $\phi^3_{4,-3}(\xi)$, $\phi^3_{4,-2}(\xi)$, and $\phi^3_{4,-1}(\xi)$; 1 boundary functions are $\phi^3_{4,5}(\xi)$, $\phi^3_{4,6}(\xi)$, and $\phi^3_{4,7}(\xi)$; inner functions are $\phi^3_{4,0}(\xi)$, $\phi^3_{4,1}(\xi)$, $\phi^3_{4,2}(\xi)$, $\phi^3_{4,3}(\xi)$, and $\phi^3_{4,4}(\xi)$. Fig. 1 shows the scaling functions and wavelets:

$$\phi^3_{4,-3}(\xi)=\frac{1}{6}\times\begin{cases}6-18\times(2^3\xi)+18\times(2^3\xi)^2-6\times(2^3\xi)^3, \xi\in[0,0.125]\\ 0, others\end{cases}$$

$$\phi^3_{4,-2}=\frac{1}{6}\times\begin{cases}18\times(2^3\xi)-27\times(2^3\xi)^2+\frac{21}{2}(2^3\xi)^3, \xi\in[0,0.125]\\ 12-18\times(2^3\xi)+9\times(2^3\xi)^2-\frac{3}{2}\times(2^3\xi)^3, \xi\in[0.125,0.25]\\ 0, others\end{cases}$$

$$\phi^3_{4,-1}=\frac{1}{6}\times\begin{cases}9\times(2^3\xi)^2-\frac{11}{2}(2^3\xi)^3, \xi\in[0,0.125]\\ -9+27\times(2^3\xi)-18\times(2^3\xi)^2+\frac{7}{2}\times(2^3\xi)^3, \xi\in[0.125,0.25]\\ 27-27\times(2^3\xi)+9\times(2^3\xi)^2-2\times(2^3\xi)^3, \xi\in[0.25,0.375]\\ 0, others\end{cases}$$

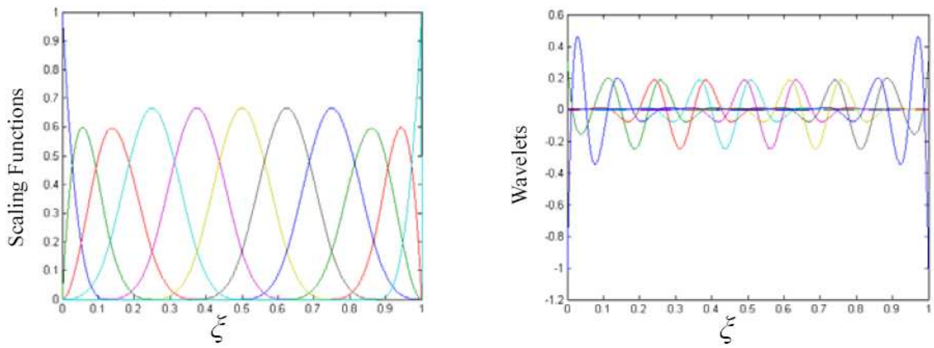
$$\phi^3_{4,0}=\frac{1}{6}\times\begin{cases}(2^3\xi)^3, \xi\in[0,0.125]\\ -4-12\times(2^3\xi)+12\times(2^3\xi)^2-3\times(2^3\xi)^3, \xi\in[0.125,0.25]\\ -44+60\times(2^3\xi)-24\times(2^3\xi)^2+3\times(2^3\xi)^3, \xi\in[0.25,0.375]\\ 64-48\times(2^3\xi)+12\times(2^3\xi)^2-(2^3\xi)^3, \xi\in[0.375,0.5]\end{cases}$$

$$\varphi^3_{4,1}(\xi)=\varphi^3_{4,0}(\xi-0.125),\quad \varphi^3_{4,2}(\xi)=\varphi^3_{4,0}(\xi-0.25),\quad \phi^3_{4,3}(\xi)=\phi^3_{4,0}(\xi-0.375)$$
$$\varphi^3_{4,4}(\xi)=\varphi^3_{4,0}(\xi-0.5),\quad \varphi^3_{4,5}(\xi)=\varphi^3_{4,-1}(1-\xi),\quad \varphi^3_{4,6}(\xi)=\varphi^3_{4,-2}(1-\xi),\quad \phi^3_{4,7}(\xi)=\phi^3_{4,-3}(1-\xi)$$

3. TBSWI elements for one-dimensional structures

3. 1 Euler beam

As shown in Fig. 2, there are n+1 nodes on the standard solving domain, and two degrees of freedom (DOF) at each node. In order to construct the TBSWI element, we should first translate the solving domain $\Omega=\{x|x\in[a,b]\}$ to standard solving domain $\Omega_s=\{\xi|\xi\in[0,1]\}$.



(a) (b)

Fig. 1. BSWI4$_3$ on the interval [0,1] (a) scaling functions (b)wavelets

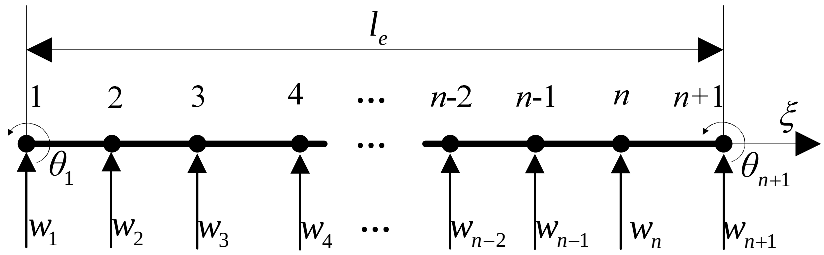


Fig. 2. The standard solving domain and placement of nodes and DOFs of Euler beam

Assuming the coordinate values are:

$$x_h \in [x_1, x_{n+1}] \qquad (1 \le h \le n+1) \tag{6}$$

we define transformation formula:

$$\xi = \frac{x - x_1}{l_e} \qquad (0 \le \xi \le 1) \tag{7}$$

Therefore, using Eq. (7), we can map x to standard solving domain.

Substituting Eq. (6) to Eq. (7), we can obtain the mapping value ξ_h of each node

$$\xi_h = \frac{x_h - x_1}{l_e} \qquad (0 \le \xi_h \le 1, 1 \le h \le n+1) \tag{8}$$

Based on the generalized potential energy functional with two kinds of variables, there are two DOFs at each node. Displacement and moment field function are treated as independent variables, the generalized potential energy functional with two kinds of variables for Euler beam bending is [18]:

$$\boldsymbol{\Pi}_{2p}(w, M) = -\int_a^b M \frac{d^2 w}{dx^2} dx - \int_a^b \frac{M^2}{2EI} dx - \int_a^b q(x) w dx - \sum_i p_i w(x_i) - \int_a^b \frac{1}{2} \lambda \bar{m} w^2 dx \tag{9}$$

where E is elastic modulus; $I = BH^3/12$ is inertia moment, and B is the beam width, H is the beam height; $q(x)$ is distributed load; P_i is concentrated load and x_i is acting position; λ is the vibration eigenvalue, and $\bar{m}$ is density; $w(x)$ is the transverse displacement function; $M(x)$ is moment function.

By using scaling functions in Eq. (4) as the interpolating function to form field function, and translating the corresponding coordinate to standard solving interval, the displacement field function and moment field function can be obtained as following:

$$w(\xi) = \boldsymbol{\Phi}\mathbf{T}^e\boldsymbol{w}^e \tag{10}$$

$$M(\xi) = \boldsymbol{\Phi}\mathbf{T}^e\boldsymbol{M}^e \tag{11}$$

where $\mathbf{T}^e = \left[\boldsymbol{\Phi}^T(\xi_1) \quad \frac{1}{l_e}\frac{d\boldsymbol{\Phi}^T(\xi_1)}{d\xi} \quad \boldsymbol{\Phi}^T(\xi_2) \quad \ldots \quad \boldsymbol{\Phi}^T(\xi_n) \quad \boldsymbol{\Phi}^T(\xi_{n+1}) \quad \frac{1}{l_e}\frac{d\boldsymbol{\Phi}^T(\xi_{n+1})}{d\xi}\right]^{-1}$ is the transformation matrix; $\boldsymbol{w}^e = \{w_1\ w_2 \ldots w_{n+1}\}^{\mathrm{T}}$, $\boldsymbol{M}^e = \{M_1\ M_2 \ldots M_{n+1}\}^{\mathrm{T}}$.

Substituting Eqs. (10) - (11) into Eq. (9), according to the generalized variational principle $\frac{\partial \Pi_{2b}}{\partial \boldsymbol{M}^e} = 0$ and $\frac{\partial \Pi_{2b}}{\partial \boldsymbol{w}^e} = 0$, we can obtain TBSWI formulation for Euler beam as follows:

$$\begin{bmatrix} -\frac{1}{EI}\boldsymbol{\Gamma}^{00} & -\boldsymbol{\Gamma}^{02} \\ -\boldsymbol{\Gamma}^{20} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{M}^e \\ \boldsymbol{w}^e \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{P}^e \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \lambda\overline{m}\boldsymbol{\Gamma}^{00} \end{bmatrix}\begin{bmatrix} \boldsymbol{M}^e \\ \boldsymbol{w}^e \end{bmatrix} \tag{12}$$

So the TBSWI formulation for Euler beam bending is:

$$\begin{bmatrix} -\frac{1}{EI}\boldsymbol{\Gamma}^{00} & -\boldsymbol{\Gamma}^{02} \\ -\boldsymbol{\Gamma}^{20} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{M}^e \\ \boldsymbol{w}^e \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{P}^e \end{bmatrix} \tag{13}$$

The TBSWI formulation for Euler beam vibration is:

$$\begin{bmatrix} -\frac{1}{EI}\boldsymbol{\Gamma}^{00} & -\boldsymbol{\Gamma}^{02} \\ -\boldsymbol{\Gamma}^{20} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{M}^e \\ \boldsymbol{w}^e \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \lambda\overline{m}\boldsymbol{\Gamma}^{00} \end{bmatrix}\begin{bmatrix} \boldsymbol{M}^e \\ \boldsymbol{w}^e \end{bmatrix} \tag{14}$$

To distributed load, $\boldsymbol{P}^e = l_e\int_0^1 q(\xi)\boldsymbol{\Phi}^T d\xi$, while to concentrated load, $\boldsymbol{P}^e = \sum_i p_i\boldsymbol{\Phi}^T(\xi_i)$.

where the integral terms are:

$$\boldsymbol{\Gamma}^{0,0} = \left(\mathbf{T}^e\right)^T l_e\int_0^1 \boldsymbol{\Phi}^T\boldsymbol{\Phi}d\xi(\boldsymbol{T}^e)$$

$$\boldsymbol{\Gamma}^{2,0} = \left(\mathbf{T}^e\right)^T \frac{1}{l_e}\int_0^1 \frac{d^2\boldsymbol{\Phi}^T}{d\xi^2}\boldsymbol{\Phi}d\xi(\boldsymbol{T}^e)$$

$$\boldsymbol{\Gamma}^{0,2} = \left(\mathbf{T}^{2,0}\right)^T$$

Following Eq. (12) and (13), we can solve the bending and vibration problems of Euler beam as traditional method.

3. 2 Timoshenko beam

Considering the influence of shear deformation, Timoshenko proposed the beam theory with two generalized displacement in 1921 [27]. Fig. 3 shows the placement of nodes and DOFs on the standard solving domain. There are two generalized displacement on each node, displacement and slope. The transformation formula between solving domain and the standard solving domain is the same as to Euler beam.

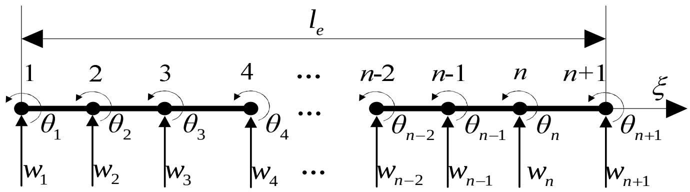


Fig. 3. The standard solving domain and placement of nodes and DOFs of Timoshenko beam

The generalized potential energy functional with two kinds of variables of Timoshenko beam is [18]:

$$\begin{aligned}\boldsymbol{\Pi}_{2p}\left(w,\theta,M\right) = &-\int_a^b M\frac{d\theta}{dx}dx + \frac{k_\tau GA}{2}\int_a^b\left(\frac{dw}{dx}-\theta\right)^2 dx - \int_a^b\left(\frac{M^2}{2EI}\right)dx - \\ &-\int_a^b qwdx - \sum_i p_i w(x_i) - \int_a^b \frac{1}{2}\lambda\bar{m}w^2 dx\end{aligned} \tag{15}$$

where k_τ is the shear deformation coefficient which can be evaluated numerically [28]; G is the shear modulus; A is cross-section area. Other symbols are the same as in Eq. (9).

By taking scaling functions in Eq. (4) as the interpolating function to form the field function, and translating the corresponding coordinate to standard solving interval, displacement field function and moment field function can be obtained as following:

$$w(\xi) = \boldsymbol{\Phi}\mathbf{T}^e\boldsymbol{w}^e \tag{16}$$

$$\theta(\xi) = \boldsymbol{\Phi}\mathbf{T}^e\boldsymbol{\theta}^e \tag{17}$$

$$M(\xi) = \boldsymbol{\Phi}\mathbf{T}^e\boldsymbol{M}^e \tag{18}$$

where $\mathbf{T}^e = \left[\boldsymbol{\Phi}^T(\xi_1)\quad \boldsymbol{\Phi}^T(\xi_2)\quad ...\quad \boldsymbol{\Phi}^T(\xi_n)\quad \boldsymbol{\Phi}^T(\xi_{n+1})\right]^{-1}$ is the transformation matrix; $\boldsymbol{w}^e = \left\{w_1\ w_2 ... w_{n+1}\right\}^{\mathrm{T}}$, $\boldsymbol{\theta}^e = \left\{\theta_1\ \theta_2 ... \theta_{n+1}\right\}^{\mathrm{T}}$, $\boldsymbol{M}^e = \left\{M_1\ M_2 ... M_{n+1}\right\}^{\mathrm{T}}$.

Substituting Eqs. (16)-(18) into Eq. (15), according to generalized variational principle $\frac{\partial\Pi_{2b}}{\partial\boldsymbol{w}^e}=0$, $\frac{\partial\Pi_{2b}}{\partial\boldsymbol{\theta}^e}=0$ and $\frac{\partial\Pi_{2b}}{\partial\boldsymbol{M}^e}=0$, we can obtain TBSWI formulation for Timoshenko beam bending as follows:

$$\begin{bmatrix} -\frac{1}{EI}\boldsymbol{\Gamma}^{00} & -\boldsymbol{\Gamma}^{01} & 0 \\ -\boldsymbol{\Gamma}^{10} & k_\tau GA\boldsymbol{\Gamma}^{00} & -k_\tau GA\boldsymbol{\Gamma}^{01} \\ 0 & -k_\tau GA\boldsymbol{\Gamma}^{10} & k_\tau GA\boldsymbol{\Gamma}^{11} \end{bmatrix}\begin{bmatrix}\boldsymbol{M}^e \\ \boldsymbol{\theta}^e \\ \boldsymbol{w}^e\end{bmatrix} = \begin{bmatrix}0 \\ 0 \\ \boldsymbol{P}^e\end{bmatrix} + \begin{bmatrix}0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda\bar{m}\boldsymbol{\Gamma}^{00}\end{bmatrix}\begin{bmatrix}\boldsymbol{M}^e \\ \boldsymbol{\theta}^e \\ \boldsymbol{w}^e\end{bmatrix} \tag{19}$$

Therefore, the TBSWI formulation for Timoshenko beam bending is:

$$\begin{bmatrix} -\frac{1}{EI}\mathbf{\Gamma}^{00} & -\mathbf{\Gamma}^{01} & 0 \\ -\mathbf{\Gamma}^{10} & k_\tau GA\mathbf{\Gamma}^{00} & -k_\tau GA\mathbf{\Gamma}^{01} \\ 0 & -k_\tau GA\mathbf{\Gamma}^{10} & k_\tau GA\mathbf{\Gamma}^{11} \end{bmatrix} \begin{bmatrix} \boldsymbol{M}^e \\ \boldsymbol{\theta}^e \\ \boldsymbol{w}^e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \boldsymbol{P}^e \end{bmatrix} \tag{20}$$

The TBSWI vibration formulation for Timoshenko beam is:

$$\begin{bmatrix} -\frac{1}{EI}\mathbf{\Gamma}^{00} & -\mathbf{\Gamma}^{01} & 0 \\ -\mathbf{\Gamma}^{10} & k_\tau GA\mathbf{\Gamma}^{00} & -k_\tau GA\mathbf{\Gamma}^{01} \\ 0 & -k_\tau GA\mathbf{\Gamma}^{10} & k_\tau GA\mathbf{\Gamma}^{11} \end{bmatrix} \begin{bmatrix} \boldsymbol{M}^e \\ \boldsymbol{\theta}^e \\ \boldsymbol{w}^e \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda\overline{m}\mathbf{\Gamma}^{00} \end{bmatrix} \begin{bmatrix} \boldsymbol{M}^e \\ \boldsymbol{\theta}^e \\ \boldsymbol{w}^e \end{bmatrix} \tag{21}$$

To distributed load, $\boldsymbol{P}^e = l_e \int_0^1 q(\xi)\boldsymbol{\Phi}^T d\xi$, while to concentrated load, $\boldsymbol{P}^e = \sum_i p_i \boldsymbol{\Phi}^T(\xi_i)$, where the integral terms are:

$$\mathbf{\Gamma}^{0,0} = \left(\mathbf{T}^e\right)^T l_e \int_0^1 \boldsymbol{\Phi}^T \boldsymbol{\Phi} d\xi (\boldsymbol{T}^e)$$

$$\mathbf{\Gamma}^{1,0} = \left(\mathbf{T}^e\right)^T \frac{1}{l_e} \int_0^1 \frac{d\boldsymbol{\Phi}^T}{d\xi} \boldsymbol{\Phi} d\xi (\boldsymbol{T}^e)$$

$$\mathbf{\Gamma}^{1,1} = \left(\mathbf{T}^e\right)^T \int_0^1 \frac{d\boldsymbol{\Phi}^T}{d\xi} \frac{d\boldsymbol{\Phi}}{d\xi} d\xi (\boldsymbol{T}^e)$$

$$\mathbf{\Gamma}^{0,1} = \left(\mathbf{T}^{1,0}\right)^T$$

3. 3 Elastic foundation beam

Generally, there are three typical computational models for Elastic foundation beam: Winkler foundation model, elastic semi-infinite foundation model and layered foundation model, while Winkler foundation model is commonly used.

According to Winkler assumption, the generalized potential energy functional with two kinds of variables for bending and vibration of Elastic foundation beam is [18]:

$$\boldsymbol{\Pi}_{2p}\left(w, M\right) = -\int_a^b M \frac{d^2 w}{dx^2} dx - \int_a^b \frac{M^2}{2EI} dx + \frac{k_e}{2} \int_a^b w^2 dx - \int_a^b \frac{1}{2} \lambda \overline{m} w^2 dx - \int_a^b q(x) w dx - \sum_i p_i w(x_i) \tag{22}$$

where k_e is the Winkler foundation coefficient, other symbols are the same as in Eq. (9) and Eq. (15).

Taking BSWI scaling functions in Eq. (4) as interpolating function to discrete solving domain, and according to generalized variational principle, the TBSWI formulation for bending of Elastic foundation beam is:

$$\begin{bmatrix} -\dfrac{\mathbf{\Gamma}^{00}}{EI} & -\mathbf{\Gamma}^{02} \\ -\mathbf{\Gamma}^{20} & k_e\mathbf{\Gamma}^{00} \end{bmatrix} \begin{bmatrix} \boldsymbol{M}^e \\ \boldsymbol{w}^e \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{P}^e \end{bmatrix} \tag{23}$$

The TBSWI formulation for vibration problem is:

$$\begin{bmatrix} -\dfrac{\mathbf{\Gamma}^{00}}{EI} & -\mathbf{\Gamma}^{02} \\ -\mathbf{\Gamma}^{20} & k_e\mathbf{\Gamma}^{00} \end{bmatrix} \begin{bmatrix} \boldsymbol{M}^e \\ \boldsymbol{w}^e \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \lambda\overline{m}\mathbf{\Gamma}^{00} \end{bmatrix} \begin{bmatrix} \boldsymbol{M}^e \\ \boldsymbol{w}^e \end{bmatrix} \tag{24}$$

To distributed load, $\boldsymbol{P}^e = l_e \int_0^1 q(\xi)\boldsymbol{\Phi}^T d\xi$, while to concentrated load, $\boldsymbol{P}^e = \sum_i p_i \boldsymbol{\Phi}^T(\xi_i)$, where the integral terms are:

$$\mathbf{\Gamma}^{0,0} = \left(\mathbf{T}^e\right)^T l_e \int_0^1 \boldsymbol{\Phi}^T \boldsymbol{\Phi} d\xi (\boldsymbol{T}^e)$$

$$\mathbf{\Gamma}^{2,0} = \left(\mathbf{T}^e\right)^T \frac{1}{l_e} \int_0^1 \frac{d^2\boldsymbol{\Phi}^T}{d\xi^2} \boldsymbol{\Phi} d\xi (\boldsymbol{T}^e)$$

$$\mathbf{\Gamma}^{0,2} = \left(\mathbf{T}^{2,0}\right)^T$$

4. Numerical examples

In order to verify the correctness and efficiency of the TBSWI element of Euler beam, Timoshenko beam and Elastic foundation beam constructed in section 3, several numerical examples of corresponding beam under different boundary conditions and load are provided here. The BSWI scaling functions at scale $j = 3$, order $m = 4$ are chosen to discrete the solving domain and establish the TBSWI elements (simply denoted as TBSWI4_3).

4. 1 Euler beam

Example 1: Simply supported Euler beam with distributed load. As shown in Fig. 4, the corresponding parameters are: elastic modulus E = 1.2×10^6 N/m^2, beam width B = 0.1 m, beam height H = 0.05 m, beam length L = 1 m, distributed load q = 1 N/m, respectively.

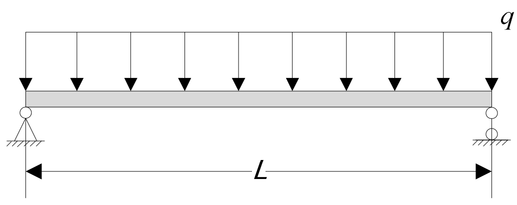


Fig. 4. The Euler beam with distributed load and simply supported on two ends

Adopting BSWI4_3 as trial function, the bending problem of Euler beam shown in Fig. 4 is analyzed. The corresponding results are shown in Table 1 and Table 2. Table 1 gives the solving results of displacement and moment at every node, and compared the results of TBSWI4_3 (22 DOFs) with BSWI4_3 (11 DOFs), BEAM3 element (66 DOFs) and the theoretical solution [29]. With respect to displacement solution, all elements can get the solution with nearly the same accuracy, while in terms of moment solution it can be observed that the results of TBSWI4_3 are the same of BEAM3 with more DOFs. So with less DOFs, TBSWI4_3 can get the results with good precision both of displacement and moment. In Table 2, the central displacement and moment of TBSWI4_3 are compared with Spline element with two kinds of variables and BSWI4_3 element. From the results, it can be easily noticed that the displacement and moment results of TBSWI are equal to the exact solution, which is better than the other two elements in moment solving.

Table 1. Displacement and moment of Euler beam simply supported with distributed load

x	Transverse displacement				Moment		
	TBSWI4_3 (22 DOFs)	BSWI4_3 (11 DOFs)	Beam3 (66 DOFs)	Exact [29]	TBSWI4_3 (22 DOFs)	BSWI4_3 (11 DOFs)	Beam3 (66 DOFs)
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
L/10	-0.003270	-0.003270	0.003269	-0.003270	-0.045000	-0.045052	-0.045000
2L/10	-0.006186	-0.006186	0.006185	-0.006187	-0.080000	-0.079427	-0.080000
3L/10	-0.008470	-0.008470	0.008467	-0.008470	-0.105000	-0.104427	-0.105000
4L/10	-0.009920	-0.009920	0.009917	-0.009920	-0.120000	-0.120052	-0.120000
5L/10	-0.010417	-0.001042	0.001041	-0.010417	-0.125000	-0.126302	-0.125000
6L/10	-0.009920	-0.009920	0.009917	-0.009920	-0.120000	-0.120052	-0.120000
7L/10	-0.008470	-0.008470	0.008467	-0.008470	-0.105000	-0.104427	-0.105000
8L/10	-0.006186	-0.006186	0.006185	-0.006187	-0.080000	-0.079427	-0.080000
9L/10	-0.003270	-0.003270	0.003269	-0.003270	-0.045000	-0.045052	-0.045000
10L/10	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

Table 2. Central displacement and moment of simply supported Euler beam with distributed load

Methods	DOFs	EIw/qL^4	M/qL^2
TBSWI	22	0.01302	0.12500
Spline FEM with two kinds of variables [18]	14	0.01302	0.13021
	28	0.01302	0.12630
	42	0.01302	0.12558
	56	0.01302	0.12532
BSWI	11	0.01302	0.12630
Exact		0.01302	0.12500

Example 2: Cantilever Euler beam with sinusoidal load. As shown in Fig. 5, the corresponding parameters are: bending rigidity EI = 1 Nm^2, beam length L = 1 m, distributed sinusoidal load $q(x) = \sin(\pi x/L)$.

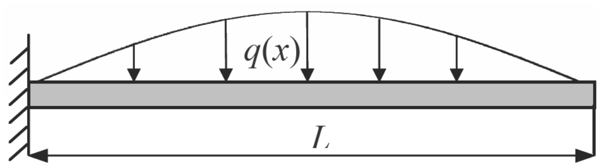


Fig. 5. Equal cross-section cantilever beam with sinusoidal load

Table 3 lists the displacement, slope and moment results of TBSWI4_3, BSWI4_3 element and theoretical solution. The relative errors of TBSWI4_3 (22 DOFs), 8 BEAM3 elements (24 DOFs) and 64 BEAM3 elements (192 DOFs) etc. are shown in Fig. 6. By comparing the results, we can observe that TBSWI4_3 can provide results with great precision. With 22 DOFs, TBSWI4_3 can provide results even better than 64 BEAM3 elements (192 DOFs). Transverse displacement, slope and moment results of TBSWI4_3 are all very close to the exact solution, especially to moments, TBSWI4_3 provides great results better than BSWI4_3. So TBSWI4_3 is very effective at solving the displacement, slope and moment. Furthermore, it is better than other methods in terms of moment solving.

Table 3. Displacement, slope and moment of cantilever beam with sinusoidal load

x	Transverse			Slope			Moment		
	TBSWI4_3 (22 DOFs)	BSWI4_3 (11 DOFs)	Exact	TBSWI4_3 (22 DOFs)	BSWI4_3 (11 DOFs)	Exact	TBSWI4_3 (22 DOFs)	BSWI4_3 (11 DOF)	Exact
0	0.0000	0.0000	0.0000	-0.0000	0.0000	-0.0000	0.3183	0.3182	0.3193
L/8	-0.0023	-0.0023	-0.0023	-0.0348	-0.0348	-0.0349	0.2398	0.2393	0.2398
2L/8	-0.0083	-0.0083	-0.0083	-0.0602	-0.0602	-0.0602	0.1671	0.1662	0.1671
3L/8	-0.0170	-0.0170	-0.0170	-0.0771	-0.0771	-0.0771	0.1053	0.1041	0.1053
4L/8	-0.0273	-0.0273	-0.0273	-0.0871	-0.0871	-0.0871	0.0578	0.0565	0.0578
5L/8	-0.0386	-0.0386	-0.0386	-0.0922	-0.0922	-0.0922	0.0258	0.0246	0.0258
6L/8	-0.0502	-0.0502	-0.0502	-0.0942	-0.0942	-0.0942	0.0079	0.0070	0.0079
7L/8	-0.0620	-0.0620	-0.0620	-0.0946	-0.0946	-0.0946	0.0010	0.0006	0.0010
8L/8	-0.0739	-0.0739	-0.0739	-0.0947	-0.0947	-0.0947	0.0000	0.0001	0.0000

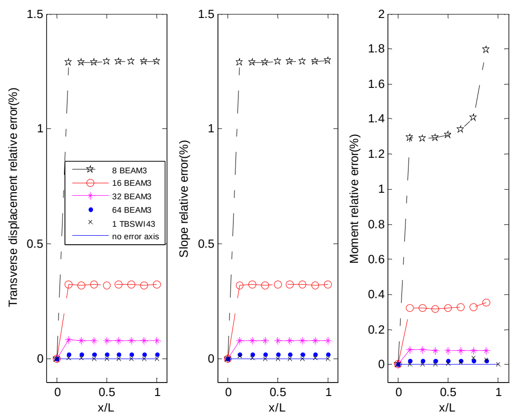


Fig. 6. Relative error of TBSWI4_3 element and BEAM3 element for the cantilever beam with sinusoidal load

Example 3: Vibration of Euler beam. The corresponding parameters are: elastic modulus E = 2.06×10^{11} N/m^2, beam width B = 0.012 m, beam height H = 0.02 m, beam length L = 0.565 m, density $\bar{m} = 7890\ \mathrm{kg/m^3}$.

The vibration problems of Euler beam under six different boundary conditions are analyzed. The results of the first three circular frequencies are given in Table 4, and the corresponding mode shapes are compared with theoretical solution in Fig. 7. Through comparison with BSWI4_3 and theoretical solution in ref. [30], we can observe that TBSWI4_3 performs well in terms of vibration analysis. With respect to circular frequency, TBSWI4_3 yields results that are better than BSWI4_3. As for the mode shapes in Fig. 7, the results of TBSWI4_3 almost coincide with the theoretical solution, confirming that TBSWI4_3 is efficient in vibration analysis.

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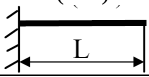
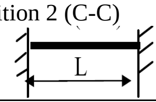
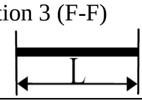
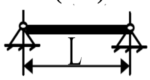
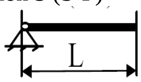
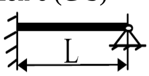
Table 4. The first three frequencies of Euler beam under different boundary conditions

Euler beam	Methods	ω_1 /rad·s^{-1}	ω_2 /rad·s^{-1}	ω_3 /rad·s^{-1}
Condition 1 (C-F)	TBSWI4$_3$ (44 DOFs)	324.929	2036.297	5702.712
	BSWI4$_3$ (22 DOFs)	324.929	2036.305	5701.927
	Exact [30]	324.893	2036.216	5702.036
Condition 2 (C-C)	TBSWI4$_3$ (44 DOFs)	2067.605	5699.432	11173.422
	BSWI4$_3$ (22 DOFs)	2067.615	5699.668	11175.059
	Exact [30]	2067.604	5699.430	11173.162
Condition 3 (F-F)	TBSWI4$_3$ (44 DOFs)	2067.607	5699.483	11173.815
	BSWI4$_3$ (22 DOFs)	2067.615	5699.677	11175.140
	Exact [30]	2067.604	5699.430	11173.162
Condition 4 (S-S)	TBSWI4$_3$ (44 DOFs)	912.089	3648.357	8208.900
	BSWI4$_3$ (22 DOFs)	912.090	3648.419	8209.543
	Exact [30]	912.089	3648.357	8208.803
Condition 5 (S-F)	TBSWI4$_3$ (44 DOFs)	1424.858	4617.467	9634.209
	BSWI4$_3$ (22 DOFs)	1424.861	4617.579	9635.171
	Exact [30]	1424.857	4617.451	9633.942
Condition 6 (C-S)	TBSWI4$_3$ (44 DOFs)	1424.858	4617.454	9634.083
	BSWI4$_3$ (22 DOFs)	1424.861	4617.577	9635.146
	Exact [30]	1424.857	4617.451	9633.942

C: clamp supported; F: free; S: simply supported.

4. 2 Timoshenko beam

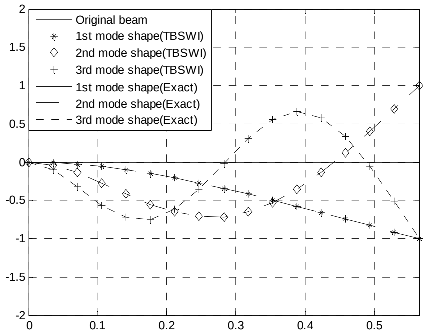
Example 1: Timoshenko beam clamped on two sides with distributed load. As shown in Fig. 8, the corresponding parameters are: bending rigidity EI = 13/6×10^{10} Nm^2, shear rigidity GA = 10^{11} N, shear coefficient $k_\tau = 6/5$, beam length L = 10 m, distributed load q = 10^5 N/m.

Displacement, slope and moment results of TBSWI4$_3$ and traditional element are compared in Fig. 9. It can be observed from the results that with 33 DOFs, TBSWI4$_3$ can provide results with the same accuracy of 100 traditional elements (202 DOFs), confirming that the TBSWI4$_3$ is an efficient method.

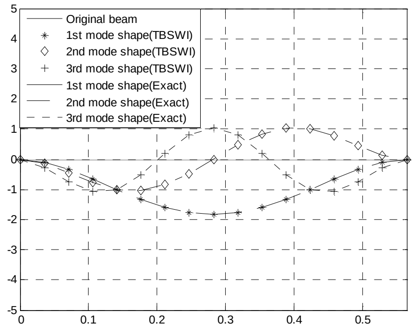
Example 2: Timoshenko beam simply supported on two sides with linear distributed variable load. As shown in Fig. 10, the corresponding parameters are: bending rigidity EI = 1 Nm^2, shear rigidity GA = 1 N, shear coefficient $k_\tau = 6/5$, beam length L = 1 m, distributed load q = 1 N/m.

Solution for transverse displacement, slope and moment is listed in Table 5, and the results of TBSWI4$_3$ are compared with 100 traditional elements and theoretical solution. The attractive point here is the moment solution, even compared with 100 traditional elements (202 DOFs), TBSWI4$_3$ (33 DOFs) can also get the results better than traditional elements. So TBSWI4$_3$ element can do excellent job in analysis of Timshenko beam with variable load, especially in terms of moment solution.

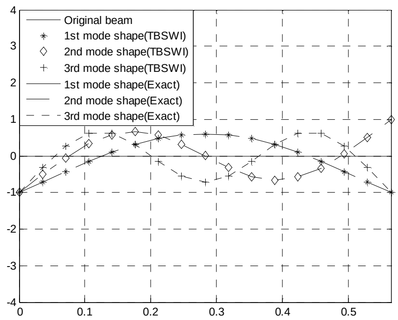
Example 3: Vibration problem of Timoshenko beam. The corresponding parameters are: elastic modulus E = 2.06×10^{11} N/m^2, shear modulus G = 5×10^{12} N/m^2, shear coefficient $k_\tau = 6/5$, beam width B = 0.01 m, beam height H = 0.2 m, beam length L = 1 m, density $\bar{m}$ = 7890 kg/m^3 .



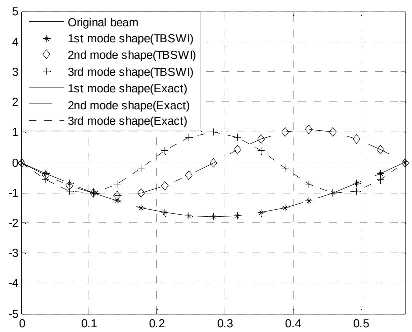
(a) Condition 1



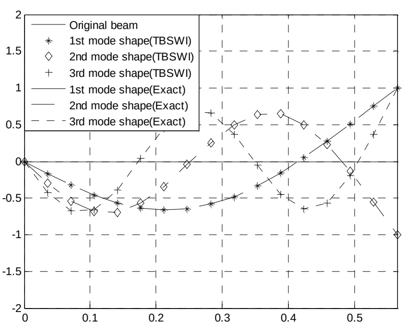
(b) Condition 2



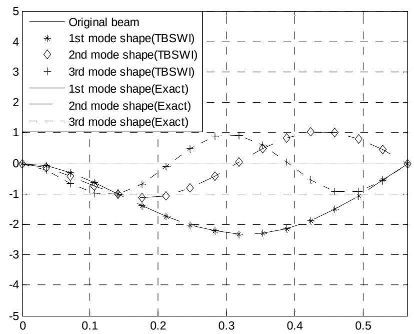
(c) Condition 3



(d) Condition 4



(e) Condition 5



(f) Condition 6

Fig. 7. The first three mode shapes of Euler beam

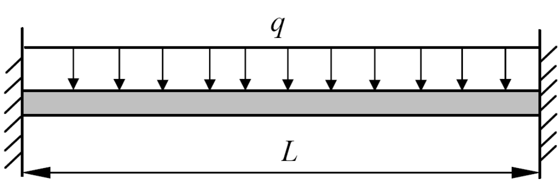


Fig. 8. Clamp supported Timoshenko beam with distributed load

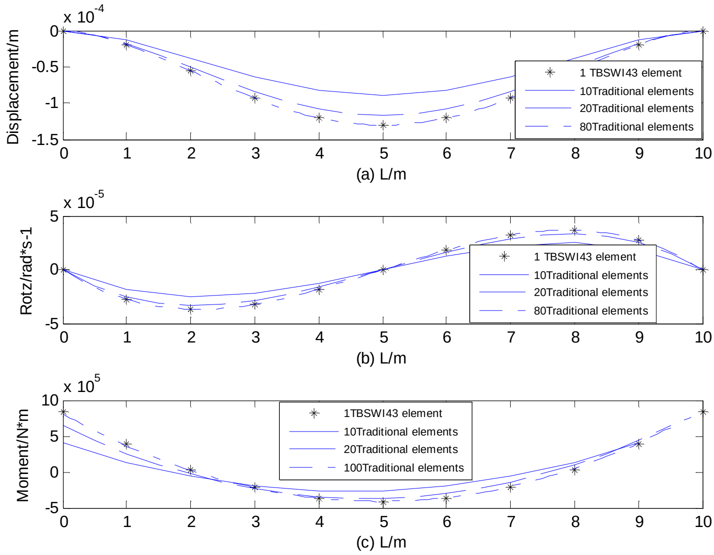


Fig. 9. Solving solution of TBSWI4$_3$ element and traditional element for a Timoshenko beam

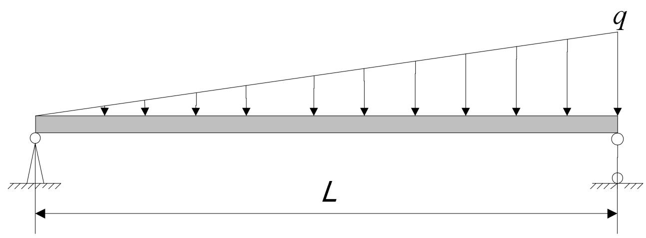


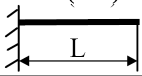
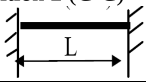
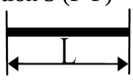
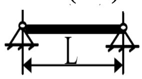
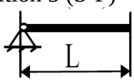
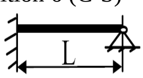
Fig. 10. Simply supported Timoshenko beam with linear distributed variable load

Table 5. Displacement, slope and moment of simply supported Timoshenko beam

x	Transverse displacement		Slope		Moment		
	1 TBSWI4$_3$	100 Traditional elements	1 TBSWI4$_3$	100 Traditional elements	1 TBSWI4$_3$	100 Traditional elements	Theoretical solution
0	0.000000	0.000000	-0.038889	-0.038886	0.000000	-0.001666	0.000000
L/10	-0.031333	-0.031333	-0.037231	-0.037227	-0.033000	-0.034611	-0.033000
2L/10	-0.060672	-0.060671	-0.032355	-0.032353	-0.064000	-0.065456	-0.064000
3L/10	-0.086040	-0.086039	-0.024563	-0.024562	-0.091000	-0.092201	-0.091000
4L/10	-0.105504	-0.105502	-0.014356	-0.014354	-0.112000	-0.112845	-0.112000
5L/10	-0.117188	-0.117185	-0.002431	-0.002430	-0.125000	-0.125390	-0.125000
6L/10	-0.119296	-0.119294	0.010311	0.010311	-0.128000	-0.127835	-0.128000
7L/10	-0.110134	-0.110132	0.022770	0.022768	-0.119000	-0.118180	-0.119000
8L/10	-0.088127	-0.088126	0.033645	0.033642	-0.096000	-0.094426	-0.096000
9L/10	-0.051842	-0.051841	0.041436	0.041432	-0.057000	-0.054571	-0.057000
10L/10	0.000000	0.000000	0.044444	0.044438	0.000000	0.000000	0.000000

As shown in Table 6, the results of vibration problem for the Timoshenko beam under six different boundary conditions are compared with theoretical solution in ref. [30] of a thin beam. The satisfactory results indicate that TBSWI Timoshenko beam element is efficient in vibration analysis. It can obtain satisfactory results for different boundary conditions. Thus, TBSWI element can perform well both in the case of bending analysis and free vibration analysis.

Table 6. First three frequencies of Timoshenko beam under different boundary conditions

Timoshenko beam	Methods	ω_1 /rad·s^{-1}	ω_2 /rad·s^{-1}	ω_3 /rad·s^{-1}
Condition 1 (C-F)	TBSWI4_3	1036.861	6484.444	18132.558
	Exact (thin beam)[30]	1037.254	6500.361	18201.199
Condition 2 (C-C)	TBSWI4_3	6575.049	18079.067	35679.436
	Exact (thin beam)[30]	6600.309	18194.007	35667.525
Condition 3 (F-F)	TBSWI4_3	6594.983	18171.464	35935.073
	Exact (thin beam)[30]	6600.309	18193.007	35667.525
Condition 4 (S-S)	TBSWI4_3	2909.373	11619.159	26192.240
	Exact (thin beam)[30]	2911.617	11646.466	26204.552
Condition 5 (S-F)	TBSWI4_3	4544.637	14710.661	30837.107
	Exact (thin beam)[30]	4548.501	14740.058	30753.952
Condition 6 (C-S)	TBSWI4_3	4538.802	14676.962	30734.660
	Exact (thin beam)[30]	4548.501	14740.058	30753.952

C: clamp supported; F: free; S: simply supported.

4. 3 Elastic foundation beam

Example 1: Bending problem of Elastic foundation beam. As shown in Fig. 11, the corresponding parameters are: elastic modulus $E = 1.2\times10^6$ N/m^2, beam width $B = 0.1$ m, beam height $H = 0.1$ m, beam length $L = 1$ m, distributed load $q_0 = 1$ N/m, Winkler foundation coefficient $k_e = 1000EI/L^4$.

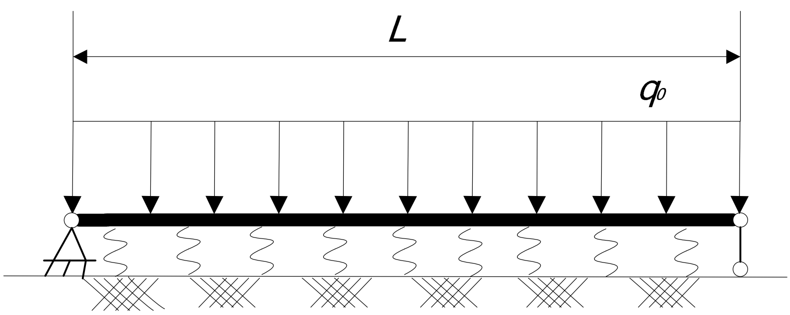


Fig. 11. Elastic foundation beam simply supported with distributed load

In Table 7, the solution of central displacement and moment are compared with Spline mixed FEM with two kinds of variables [18], BEAM3 and theoretical solution [31]. To displacement solution, the three methods can all obtain perfect results, while to moment solution TBSWI performs better than Spline mixed method with two kinds of variables. Fig. 12 shows the displacement and moment comparison of TBSWI element and BEAM3 element along beam length, the two elements are inosculated very well, which further proves that TBSWI element is an efficient element, very capable in generalized stress analysis.

Table 7. Central displacement and moment of Elastic foundation beam

Methods	DOFs	Simply supported beam with distributed load	
		$100wqL^4/EI$	$100MqL^2$
$TBSWI4_3$	22	0.1116	0.79642
Spline mixed FEM with two kinds of variables [18]	34	0.1116	0.7899
BEAM3	33	0.1116	0.79667
Theoretical solution [31]		0.1116	0.7966

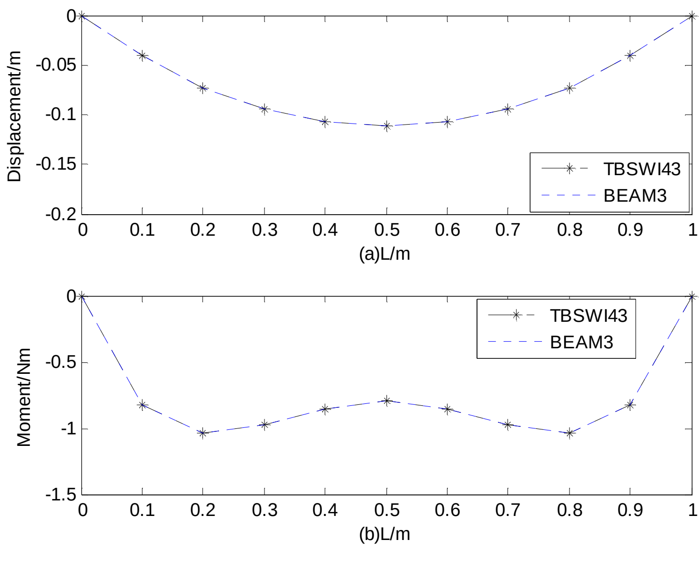


Fig. 12. Displacement and moment of Elastic foundation beam

Example 2: Vibration of Elastic foundation beam. The corresponding parameters are: elastic modulus $E = 1.2\times10^6$ N/m^2, beam width $B = 0.1$ m, beam height $H = 0.1$ m, beam length $L = 1$ m, density $\bar{m} = 7890\ \mathrm{kg/m^3}$, Winkler foundation coefficient $k_e = 1000EI/L^4$.

The first order frequency coefficient is shown in Table 8, and the results are compared with spline mixed element with two kinds of variable [32], Spline element [33] and theoretical solution [34]. We can witness that $TBSWI4_3$ element performs very well both in the case of a simply supported and clamp supported Elastic foundation beam. The solution indicates that

TBSWI element is not only efficient in static analysis of Elastic foundation beam, but also in free vibration analysis.

Table 8. First order frequency coefficient of Elastic foundation beam $\left(\alpha^* = \omega_{\min} L^2 \sqrt{\overline{m}/EI}\right)$

Methods	Simply supported	Clamp supported
$TBSWI4_3$	33.1272	38.7371
Spline mixed FEM with two kinds of variables [32]	33.1272	38.7382
Spline FEM [33]	33.1273	38.7376
Theoretical solution [34]	33.1272	38.7584

5. Conclusion

Based on B-spline wavelet on the interval and the multivariable generalized potential energy functional, we constructed BSWI element with two kinds of variables for 1D structural mechanics. The matrix formulations are derived from multivariable generalized potential energy functional, and BSWI is selected as trial function to construct the generalized displacement field function and generalized stress field function. Euler beam, Timoshenko beam and Elastic foundation beam under different boundary conditions are analyzed.

To traditional method, there is only one generalized field function, displacement, so they should calculate generalized stress and strain by differentiation of displacement, which will affect the solving precision. However, the BSWI element with two kinds of variables constructed in this paper can avoid this problem, because generalized displacement and generalized stress are treated as independent variables, which can be solved directly instead of differentiation. Besides, the semi-orthogonal, compactly supported BSWI is selected as a trial function, which is excellent in approximation of numerical calculation among all existing wavelets. In order to verify the TBSWI element, the bending and vibration problems of Euler beam, Timoshenko beam and Elastic foundation beam were analyzed. Computational results indicate that the TBSWI element is a steady and efficient method, which can perform very well not only in static analysis, but in vibration analysis as well.

Acknowledgements

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