741. Analytical study on the non-linear vibration of Euler-Bernoulli beams

Iman Pakar, Mahmoud Bayat
Department of Civil Engineering, Torbateheydarieh Branch, Islamic Azad University
Torbateheydarieh, Iran
Email: *mbayat14@yahoo.com*

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Abstract. In this study, He's Variational Approach Method (VAM) is used to obtain an accurate analytical solution for the nonlinear vibrations of Euler-Bernoulli beams subjected to axial loads. It is demonstrated that the method works very well for the whole range of initial amplitudes and does not need small perturbation. It is sufficiently accurate in the case of both linear and nonlinear physics and engineering problems. Finally, the accuracy of the solution obtained with the approximate VAM method is shown graphically and compared with that of the numerical solution.

Keywords: non-linear vibration, Euler-Bernoulli beam, variational approach method.

Introduction

The investigation of beam vibrations is an important issue in mechanical and civil engineering. Linear and nonlinear partial differential equations in space and time are presented - the governing equation of the nonlinear vibration of beams in space and time. Finding exact solutions for nonlinear equations are very difficult, therefore many researchers have worked on the asymptotic methods for nonlinear equations. We can reduce some nonlinear equations to ordinary equations by using the Galerkin method and then apply the direct techniques to solve them such as perturbation methods in time domain. In the recent years, many approximate analytical methods have been proposed for studying nonlinear vibration equations of beams and shells such as Homotopy perturbation [1], energy balance [2-5], variational approach [6-7], Iteration perturbation method [8], max-min approach [9] and other analytical and numerical methods [10-17].

The vibration problems of uniform Euler- Bernoulli beams have been considered a lot in the last decades. Biondi and Caddemi [18] studied on the flexural stiffness and slope discontinuities for uniform Euler–Bernoulli beam and applied a close form solution for the governing equation. Lai et al. [19] considered the nonlinear vibration of Euler-Bernoulli beam with different supporting conditions by applying the Adomian decomposition method (ADM). Naguleswaran [20] developed the work on the changes of cross section of an Euler–Bernoulli beam resting on elastic end supports. Pirbodaghi et al. [21] used homotopy analysis method (HAM) for analyzing the free vibration of Euler–Bernoulli beam. They illustrated that the amplitude of the vibration has a great effect on the nonlinear frequency and buckling load of the beams. Liu et al. [22] applied He's variational iteration method to obtain an analytical solution for an Euler-Bernoulli beam with different supporting conditions. In this study, we have applied Variational Approach Method (VAM) to solve the nonlinear vibration of Euler-Bernoulli beams. The paper is organized as follows:

In section 1 we consider the mathematical formulation of the problem. The basic idea of variational approach method is presented in section 2. Then the application of the method for solving the nonlinear governing equation is provided in section 3. To show the applicability and accuracy of the proposed method, some comparisons between analytical and numerical solutions are presented in section 4. Finally, we show that VAM can converge to a precise cyclic solution for high nonlinear systems.

Mathematical Formulation

Consider a straight Euler-Bernoulli beam of length L, a cross-sectional area A, the mass per unit length of the beam *m*, a moment of inertia I, and a modulus of elasticity E that is subjected to an axial force of magnitude P as shown in Fig. 1.

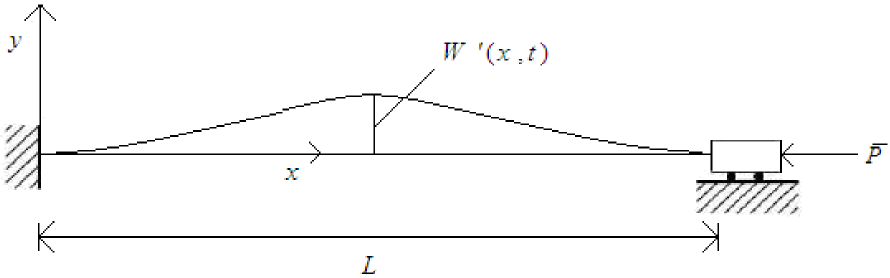

Fig. 1. A schematic of an Euler-Bernoulli beam subjected to an axial load

The equation of motion including the effects of mid-plane stretching is given by:

$$m\frac{\partial^2 w'}{\partial t'^2}+EI\frac{\partial^4 w'}{\partial x'^2}+\bar{P}\frac{\partial^2 w'}{\partial x'^2}-\frac{EA}{2L}\frac{\partial^2 w'}{\partial x'^2}\int_0^L\left(\frac{\partial^2 w'}{\partial x'^2}\right)^2 dx'=0 \quad (1)$$

For convenience, the following non-dimensional variables are used:

$x=x'/L,\ w=w'/\rho,\ t=t'(EI/ml^4)^{1/2},\ P=\bar{P}L^2/EI$

where $\rho=(I/A)^{1/2}$ is the radius of gyration of the cross-section. As a result Eq. (1) can be written as follows:

$$\frac{\partial^2 w}{\partial t^2}+\frac{\partial^4 w}{\partial x^2}+P\frac{\partial^2 w}{\partial x^2}-\frac{1}{2}\frac{\partial^2 w}{\partial x^2}\int_0^L\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx=0 \quad (2)$$

Assuming $w(x,t)=W(t)\phi(x)$, where $\phi(x)$ is the first eigenmode of the beam [23] and applying the Galerkin method, the equation of motion is obtained as follows:

$$\frac{d^2W(t)}{dt^2}+(\alpha_1+P\alpha_2)W(t)+\alpha_3W^3(t)=0 \quad (3)$$

The Eq. (3) is the differential equation of motion governing the non-linear vibration of Euler-Bernoulli beams. The center of the beam is subjected to the following initial conditions:

$$W(0)=W_{\max},\ \frac{dW(0)}{dt}=0 \quad (4)$$

where $W_{\max}$ denotes the non-dimensional maximum amplitude of oscillation and α_1,α_2 and α_3 are as follows:

$$\alpha_1=\left(\int_0^1\left(\frac{\partial^4\phi(x)}{\partial x^4}\right)\phi(x)dx\right)\Big/\int_0^1\phi^2(x)dx \quad (5.a)$$

$$\alpha_2=\left(\int_0^1\left(\frac{\partial^2\phi(x)}{\partial x^2}\right)\phi(x)dx\right)\Big/\int_0^1\phi^2(x)dx \quad (5.b)$$

$$\alpha_3=\left(\left(-\frac{1}{2}\right)\int_0^1\left(\frac{\partial^2\phi(x)}{\partial x^2}\int_0^1\left(\frac{\partial^2\phi(x)}{\partial x^2}\right)^2dx\right)\phi(x)dx\right)\Big/\int_0^1\phi^2(x)dx \quad (5.c)$$

Post-buckling load–deflection relation for the problem can be obtained from Eq. (3) as:

$$P = (-\alpha_1 - \alpha_3 W^2)/\alpha_2 \tag{6}$$

Neglecting the contribution of W in Eq. (6), the buckling load can be determined as:

$$P_c = P_L = -\alpha_1/\alpha_2 \tag{7}$$

Basic Concept of Variational Approach Method

He [24] suggested a variational approach, which is different from the known variational methods in open literature. Hereby we give a brief introduction of the method:

$$\ddot{u} + f(u) = 0 \tag{8}$$

Its variational principle can be easily established utilizing the semi-inverse method [24]:

$$J(u) = \int_0^{T/4} \left(-\frac{1}{2}\dot{u}^2 + F(u) \right) dt \tag{9}$$

where T is period of the nonlinear oscillator, $\partial F / \partial u = f$. Assume that its solution can be expressed as:

$$u(t) = \Delta \cos(\omega t) \tag{10}$$

where Δ and ω are the amplitude and frequency of the oscillator, respectively. Substituting Eq. (10) into Eq. (9) results in:

$$\begin{aligned} J(\Delta, \omega) &= \int_0^{T/4} \left(-\frac{1}{2}\Delta^2\omega^2 \sin^2 \omega t + F(\Delta \cos \omega t) \right) dt \\ &= \frac{1}{\omega} \int_0^{\pi/2} \left(-\frac{1}{2}\Delta^2\omega^2 \sin^2 t + F(\Delta \cos t) \right) dt \\ &= -\frac{1}{2}A^2\omega \int_0^{\pi/2} \sin^2 t \, dt + \frac{1}{\omega} \int_0^{\pi/2} F(A \cos t) dt \end{aligned} \tag{11}$$

Applying the Ritz method, we require:

$$\frac{\partial J}{\partial \Delta} = 0 \tag{12}$$

$$\frac{\partial J}{\partial \omega} = 0 \tag{13}$$

But with a careful inspection, for most cases we find that:

$$\frac{\partial J}{\partial \omega} = -\frac{1}{2}\Delta^2 \int_0^{\pi/2} \sin^2 t \, dt - \frac{1}{\omega^2} \int_0^{\pi/2} F(\Delta \cos t) dt < 0 \tag{14}$$

Thus, we modify conditions Eq. (12) and Eq. (13) into a simpler form:

$$\frac{\partial J}{\partial \omega} = 0 \tag{15}$$

From which the relationship between the amplitude and frequency of the oscillator can be obtained.

Application of the Variational Approach Method

Consider the Eqs. (3) and (4) for the vibration of an Euler-Bernoulli beam. Its variational formulation can be readily obtained as follows:

$$J(W) = \int_0^{T/4} \left(-\frac{1}{2}\left(\frac{dW(t)}{dt}\right)^2 + \frac{1}{2}(\alpha_1 + P\alpha_2) W^2(t) + \alpha_3 W^4(t) \right) dt \, . \tag{16}$$

Assume that its solution can be expressed as:

$$W(t) = W_{\max}\cos(\omega t) \tag{17}$$

By substituting Eq. (17) into Eq. (16) we obtain:

$$J(W) = \int_0^{T/4}\left(-\frac{1}{2}(W_{\max}\omega\sin\omega t)^2 + \frac{1}{2}(\alpha_1 + P\alpha_2)(W_{\max}\cos\omega t)^2 + \alpha_3(W_{\max}\cos\omega t)^4\right)dt \tag{18}$$

The stationary condition with respect to $W_{\max}$ leads to:

$$\frac{\partial J}{\partial W_{\max}} = \int_0^{T/4}\left(-W_{\max}\,\omega^2\sin^2\omega t + (\alpha_1 + P\,\alpha_2)\left(W_{\max}\cos^2\omega t\right) + \alpha_3 W_{\max}{}^3\cos^4\omega t\right)dt = 0 \tag{19}$$

or

$$\frac{\partial J}{\partial W_{\max}} = \int_0^{\pi/2}\left(-W_{\max}\,\omega^2\sin^2 t + (\alpha_1 + P\,\alpha_2)\left(W_{\max}\cos^2 t\right) + \alpha_3 W_{\max}{}^3\cos^4 t\right)dt = 0 \tag{20}$$

We have:

$$\omega^2 = \frac{\int_0^{\pi/2}\left(\alpha_1 W_{\max}\cos^2 t + p\alpha_2 W_{\max} cos^2 t + \alpha_3 W_{\max}{}^3 cos^4 t\right)dt}{\int_0^{\pi/2}\left(W_{\max}\sin^2 t\right)dt} \tag{21}$$

Solving Eq. (21), according to ω , the non-linear frequency is:

$$\omega_{NL} = \frac{1}{2}\sqrt{4(\alpha_1 + P\,\alpha_2) + 3\alpha_3 W_{\max}{}^2} \tag{22}$$

It's period can be written in the form:

$$T_{VAM} = \frac{4\pi}{\sqrt{4(\alpha_1 + P\alpha_2) + 3\alpha_3 W_{\max}{}^2}} \tag{23}$$

Non-linear to linear frequency ratio is:

$$\frac{\omega_{NL}}{\omega_L} = \frac{1}{2}\frac{\sqrt{4\left(\alpha_1 + p\alpha_2\right) + 3\alpha_3 W_{\max}{}^2}}{\sqrt{\alpha_1 + p\alpha_2}} \tag{24}$$

According to Eq. (17) and Eq. (22), we can obtain the following approximate solution:

$$W(t) = W_{\max}\cos\left(\frac{1}{2}\sqrt{4(\alpha_1 + P\,\alpha_2) + 3\alpha_3 W_{\max}{}^2}\;t\right) \tag{25}$$

Results and Discussions

To illustrate and verify the accuracy of the Variational Approach Method (VAM), comparison with published data and exact solutions is presented. The exact frequency ω_{Exact} for a dynamic system governed by Eq. (3) can be derived, as shown in Eq. (26), as follows:

$$\omega_{Exact} = 2\pi \Bigg/ \left(4\sqrt{2}W_{\max}\int_0^{\pi/2}\frac{csgn\ sin(t)}{\sqrt{W_{\max}{}^2 sin^2(t)\left(W_{\max}{}^2\alpha_3 cos^2(t) + 2p\alpha_2 + 2\alpha_1 + W_{\max}{}^2\alpha_3\right)}}\,dt\right) \tag{26}$$

Table 1 represents the comparison of nonlinear to linear frequency ratio (ω_{NL}/ω_L) with those reported by Azrar et al. [25] and the exact one. The maximum relative error of the analytical approaches is 2.004109 % for the first order analytical approximations as it is shown in the Table 1.

Table 1. Comparison of non-linear to linear frequency ratio (ω_{NL}/ω_L) for the simply supported beam

$W_{\max}$	Present Study (VAM)	Exact solution	Pade approximate P{4,2} [25]	Pade approximate P{6,4} [25]	Error % $(\omega_{VAM}-\omega_{ex})/\omega_{ex}$
0.2	1.044031	1.0438823	1.0438824	1.0438823	0.014211
0.4	1.16619	1.1644832	1.1644868	1.1644832	0.146604
0.6	1.345362	1.3397037	1.3397374	1.3397039	0.422385
0.8	1.56205	1.5505542	1.5506741	1.5505555	0.741395
1	1.802776	1.7844191	1.7846838	1.7844228	1.028712
1.5	2.462214	2.4254023	2.4261814	2.4254185	1.517775
2	3.162278	3.1070933	3.1084562	3.1071263	1.776077
2.5	3.881044	3.8079693	3.8099164	3.8080203	1.918985
3	4.609772	4.5192025	4.5217205	4.5192713	2.004109

To further illustrate and verify the accuracy of this approximate analytical approach, comparisons of the time history oscillatory displacement response for Euler-Bernoulli beams with exact solutions are presented in Figs. 2-3. From Figs. 2-3 it is observed that the motion of the system is a periodic one and the amplitude of vibration is a function of the initial conditions. Fig. 4 represents the phase plane for this problem obtained from VAM for $\alpha_3 = 0.5$ to $\alpha_3 = 2.5$. It is periodic with a center at (0, 0). This situation also occurs in the unforced, undamped cubic Duffing oscillators. The Influence of α_3 on nonlinear to linear frequency and of α_1 are presented in Figs. 5-6. The effect of different parameters α_3 and α_1 are studied in Fig. 7 simultaneously. It illustrates that VAM is a very simple method that quickly converges and is valid for a wide range of vibration amplitudes and initial conditions. The accuracy of the results shows that the method can be potentiality used for the accurate analysis of strongly nonlinear oscillation problems.

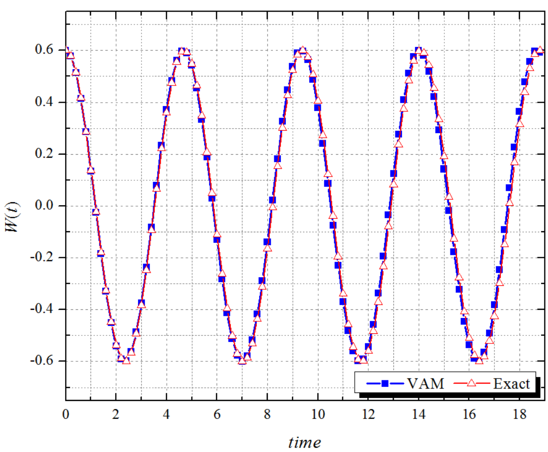

Fig. 2. Comparison of analytical solution of $W(t)$ based on time with the exact solution for simply supported beam, $W_{\max} = 0.6$, $\alpha_1 = 1$, $\alpha_2 = 0$, $\alpha_3 = 3$

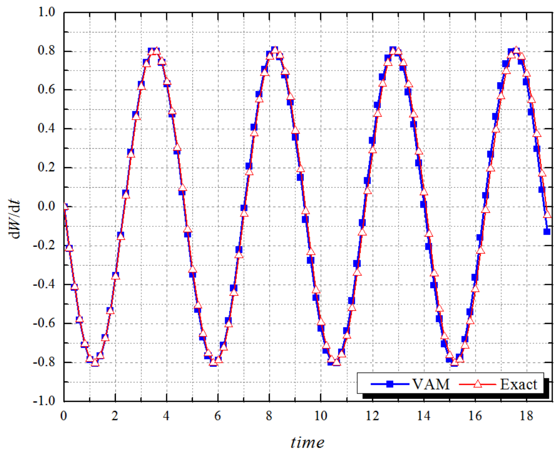

Fig. 3. Comparison of analytical solution of dW/dt based on time with the exact solution for a simply supported beam, $W_{max} = 0.6,\ \ \alpha_1 = 1,\ \ \alpha_2 = 0,\ \ \alpha_3 = 3$

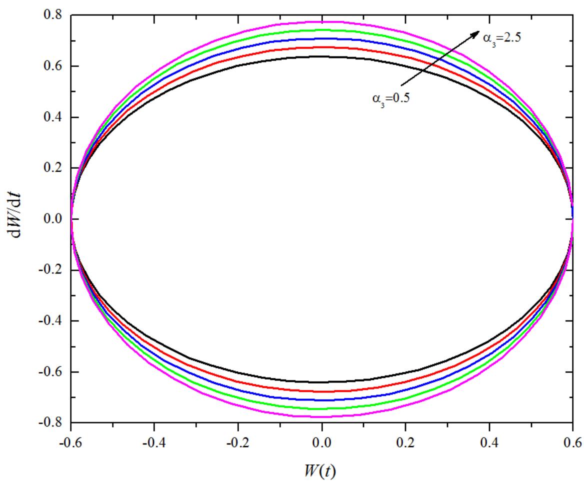

Fig. 4. The phase plane for $W_{max} = 0.6,\ \ \alpha_1 = 1,\ \ \alpha_2 = 0,\ \ p = 3$

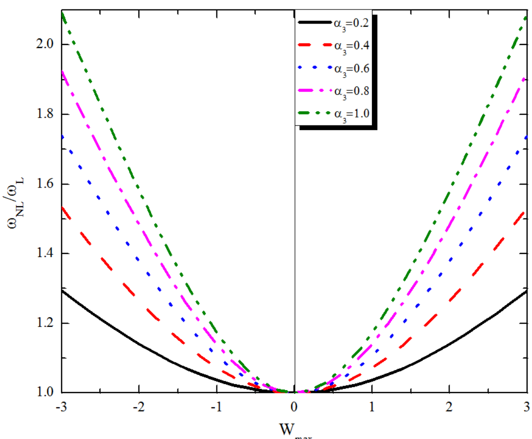

Fig. 5. Influence of α_3 on the nonlinear to linear frequency base on W_{max} for $\alpha_1 = 1,\ \ \alpha_2 = 0.5,\ \ p = 2$

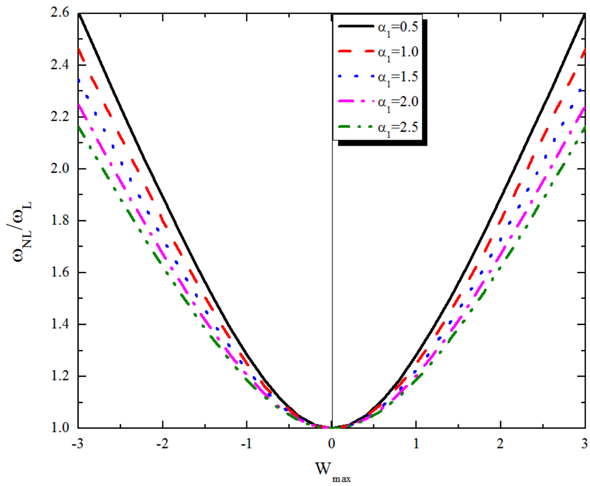

Fig. 6. Influence of α_1 on the nonlinear to linear frequency base on $W_{\max}$ for $\alpha_2 = 1, \quad \alpha_3 = 3, \quad p = 3$

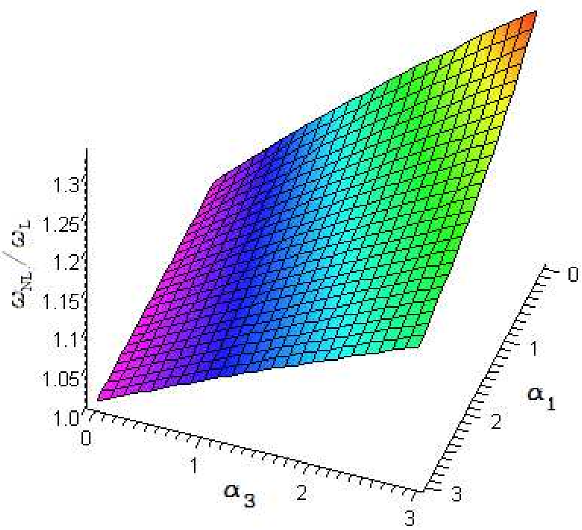

Fig. 7. Sensitivity analysis of nonlinear to linear frequency for $W_{\max} = 1, \quad \alpha_2 = 3, \quad p = 3$

Conclusion

A fairly uncomplicated but effective method for non-natural oscillators – He's Variational Approach Method has been applied for analysis of the nonlinear vibration of Euler-Bernoulli beams. It has been established that VAM is clearly effective, convenient and does not require any linearization or small perturbation, and is adequately accurate in the case of both linear and nonlinear problems in physics and engineering. It can be observed that the results of VAM require smaller computational effort and already a first-order approximation leads to accurate solutions. Variational Approach Method can be simply extended to any nonlinear equation for the analysis of nonlinear systems. The obtained results from the approximate analytical solutions are in excellent agreement with the corresponding exact solutions.

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