725. A new solution for the determination of the generalized coupling coefficient for piezoelectric systems

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Abstract. Recently, novel damping devices based on shunted piezoceramics have been investigated. Piezoceramics are therefore embedded into the mechanical structure and convert some part of the kinetic vibration energy into electrical energy. Subsequently, this energy is dissipated in the electrical network that is connected at the electrodes of the piezoceramics. The network is designed with the aim to maximize the dissipation, which results in a damping effect on the mechanical system. Alternatively, the converted energy can be stored and utilized to power electronic devices like sensors for health monitoring, called Energy Harvesting.
In both cases, the converted energy and the damping performance depend on the so called generalized electromechanical coupling coefficient *K*. It is therefore crucial to maximize this factor. Besider the piezoelectric material properties, the coupling coefficient also depends on the vibration mode of the piezoceramics. Only for a constant mechanical strain distribution in the whole volume the generalized coupling coefficient *K* is equal to the material coupling *k*. In all other cases, *K* is smaller than *k*.

This publication presents a general derivation of the generalized coupling coefficient *K* for an arbitrary, uniaxial deformation of the piezoceramics, which is based on the potential energy stored in the piezoceramics. The general result is applied to a piezoelectric bending bimorph and verified by a finite element model.

Keywords: piezoceramics, shunt damping, energy harvesting, coupling coefficient.

Introduction

Mechanical vibrations are typically unwanted, as they reduce life-time and endurance, increase wear and might lead to a sound radiation. Especially modern light-weight systems are prone to mechanical vibrations. Piezoceramics are frequently used for vibration damping, because they can be operated in the whole acoustic frequency range. Piezoelectric shunt damping is a well known technique to damp the vibrations of mechanical structures [1]. This technique relies on the piezoelectric effect that converts mechanical energy into electrical energy. A damping effect on the host structure is observed when the electrical energy is dissipated. The design of the electrical shunt aims to maximize the energy dissipation. Different networks have been developed, namely inductance-resistance networks (LR) [2], negative capacitance (LRC) [3, 4], and synchronized switch damping on inductor (SSDI) [5] techniques.

In order to maximize the damping performance, the dissipated energy - and therefore also the transferred energy - must be as high as possible. The transferred energy depends on the piezoelectric constants as well as the vibration mode and the location of the piezoceramics within the structure. While higher piezoelectric constants generally increase the amount of transferred energy, the location can typically only be optimized for one specific eigenform of the structure at the same time. One measure for the coupling of the piezoceramics is the generalized coupling coefficient *K*. It is defined by the energies in the system:

$$K^2 = \frac{U_{\mathrm{conv}}}{U + U_{\mathrm{conv}}} \tag{1}$$

with the piezoelectrically converted energy U_{conv} and the overall potential energy U, stored in the piezoceramics and the mechanical host structure [6]. In order to maximize the coupling and therefore the damping performance, the energy of the system must be calculated precisely.

Alternatively, the generalized coupling coefficient can be obtained by the shift in resonance frequency from short circuit to open electrodes, [7]:

$$K^2 = \frac{f_{\mathrm{oc}}^2 - f_{\mathrm{sc}}^2}{f_{\mathrm{oc}}^2} \tag{2}$$

This is a very convenient way to measure the coupling coefficient experimentally. In many cases, the system, especially the piezoceramics, is modeled with some simplifications. Typically, a homogeneous electric field within the ceramics is assumed. This might be an eligible approximation for many cases, for example a bimorph with a thin piezoelectric layer, [8]. But in other cases it leads to errors in the calculation of the potential energy, and therefore the calculated eigenfrequencies.

This paper presents a detailed calculation of the converted and stored energy of a piezoceramics, highlighting the influence of the generalized coupling coefficient *K*. In many cases, the deformation of the piezoceramics occurs mainly in one direction. Therefore, a uniaxial deformation is assumed for the calculations. Based on the energies the generalized coupling coefficient is derived. A piezoelectric trimorph is studied as an example, and the results are compared with a Finite element model. The comparisons prove that the proposed solution gives a much better result than the typical approximations.

Constitutive piezoelectric equations

In many practical applications, mechanical stress and strain mainly act in one direction. The linear constitutive piezoelectric equations based on IEEE standard 176 [9] then read

$$\begin{bmatrix} S_i \\ D_3 \end{bmatrix} = \begin{bmatrix} s_{ii}^{\mathrm{E}} & d_{3i} \\ d_{3i} & \varepsilon_{33}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} T_i \\ E_3 \end{bmatrix} \tag{3}$$

The indices denote the axis directions, with the 3-axis being axis of polarization. Two cases can be distinguished. $i = 3$ is the longitudinal effect, where the mechanical stress acts in the same direction as the polarization. For $i = 1$, the mechanical stress is orthogonal to the direction of the polarization and the electrical field, which is the transversal effect. The mechanical strain and stress are termed *S* and *T*, the dielectric charge displacement *D* and the electrical field strength *E*. The compliance under the condition of a constant electric field is termed s^{E} and *d* is the piezoelectric charge constant. Further on, ε^{T} denotes the absolute permittivity under constant stress.

An important derivative parameter for a piezoceramics is the material coupling coefficient *k*, which is defined as

$$k_{3i}^2 = \frac{d_{3i}^2}{s_{ii}^{\mathrm{E}} \varepsilon_{33}^{\mathrm{T}}} \tag{4}$$

Typical values are in the range $0.3 < k < 0.7$.

Calculation of the potential energy and the converted energy

For the calculation of *K*, the potential energy as well as the converted energy must be calculated, cp. (1). In the following we will assume a unidirectional mechanical strain distribution $S_i = f(x_1, x_2, x_3)$, whose direction is denoted by the *i*. Also the electrical field is unidirectional in direction of polarization P (3-axis). Generally, the energy stored in the piezoceramics then reads

$$U = \frac{1}{2} \int_V \left(T_i S_i + D_3 E_3 \right) \mathrm{d}V \tag{5}$$

The geometry of the piezoceramics under investigation is described by the length ℓ_3 in x_3-direction, which is also the distance between the electrodes, and a cross-section area in the $x_2 - x_3$ plane of A_{el}, which is therefore also the area of the electrodes, cp. Fig. 1.

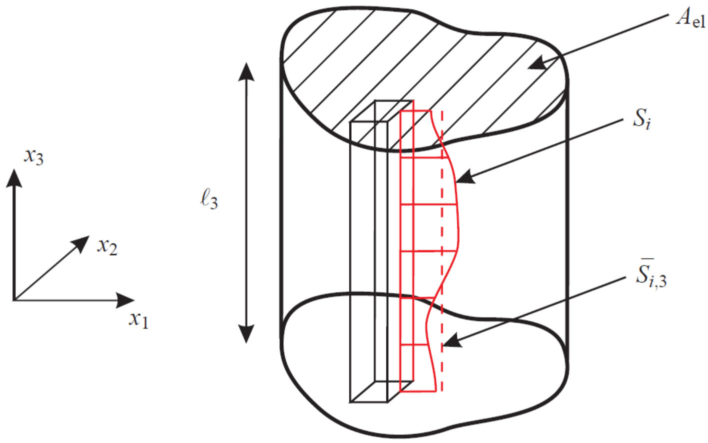

Fig. 1. Piezoceramics under uniaxial strain distribution

The energies will be calculated based on an assumed deformation, described by the mechanical strain, and the electrical voltage u_{p} across the electrodes. After some mathematical calculations [10] this energy can finally be written as

$$U = \frac{1}{2} C_{\mathrm{p}} u_{\mathrm{p}}^2 + \frac{1}{2} \frac{1}{s_{ii}^{\mathrm{E}}} \int_V \left[S_i^2 + \frac{k_{3i}^2}{1-k_{3i}^2} \Delta S_{i,3}^2 \right] \mathrm{d}V \tag{6}$$

Herein, C_{p} denotes the blocked capacitance of the piezoceramics. Further of, definitions for the mean strain $\bar{S}_{i,3}$ across the length ℓ_3 and the difference strain $\Delta S_{i,3}$ between the actual strain S_i and the mean strain are used,

$$\bar{S}_{i,3}(x_1,x_2) = \frac{\int_0^{\ell_3} S_i(x_1,x_2,x_3)\mathrm{d}x_3}{\ell_3}, \quad \Delta S_{i,3}(x_1,x_2,x_3) = S_i(x_1,x_2,x_3) - \bar{S}_{i,3}(x_1,x_2) \tag{7}$$

From (6) it can be concluded that the energy stored in the piezoceramics consists of an electrical part stored in the capacitance C_{p}, and a mechanical part described by the compliance s_{ii}^{E} and the strain. Additionally, an uneven strain distribution ($\Delta S_{i,3} > 0$) adds further energy. This only occurs for a piezoelectric medium, where $k_{3i} > 0$.

Determination of the generalized piezoelectric coupling coefficient

For the calculation of *K*, also the converted energy U_{conv} must be obtained. The linear dependency between the electrical voltage u_{p} and strain can be determined as

$$u_{\mathrm{p}} = \frac{k_{3i}^2}{1-k_{3i}^2} \frac{1}{d_{3i}} \frac{\ell_3}{A_{\mathrm{el}}} \int_{A_{\mathrm{el}}} \bar{S}_{i,3} \mathrm{d}A_{\mathrm{el}} \tag{8}$$

This result directly allows to calculate the converted energy:

$$U_{\text{conv}} = \frac{1}{2}\frac{1}{s_{ii}^{\text{E}}}\frac{k_{3i}^2}{1-k_{3i}^2}\left(\int_{A_{\text{el}}} \bar{S}_{i,3}\mathrm{d}A_{\text{el}}\right)^2 \frac{\ell_3}{A_{\text{el}}} \tag{9}$$

and finally the generalized electromechanical coupling coefficient:

$$K^2 = k_{3i}^2 \frac{1}{(1-k_{3i}^2)\dfrac{V\int_V S_i^2 \mathrm{d}V}{\left(\int_V S_i \mathrm{d}V\right)^2} + k_{3i}^2\left(1+\dfrac{V\int_V \Delta S_{i,3}^2 \mathrm{d}V}{\left(\int_V S_i \mathrm{d}V\right)^2}\right)} \tag{10}$$

A discussion of this term proves that the generalized coupling coefficient is always smaller or equal to the material coupling, $K \le k_{3i}$. For a constant mechanical strain ($\Delta S_{i,3} = 0$) it is equal, in all other cases it is smaller.

Application: Bending Trimorph

Let us now utilize the obtained results for the calculation of a bending trimorph. The piezoelectric layers are polarized in thickness-direction (x_3). Using the Euler-Bernoulli-assumptions, the strain due to a bending occurs in x_1 direction,

$$S_1(x_1,x_3) = -\left(x_3 + \frac{\ell_{3,\text{s}}}{2}\right) w''(x_1) \tag{11}$$

The bending deformation is described by the deflection curve $w(x_1)$. Consequently, the mean mechanical strain and the difference strain read

$$\bar{S}_{1,3} = \frac{\int_0^{\ell_3} S_1 \mathrm{d}x_3}{\ell_3} = -\left(e + \frac{\ell_3}{2}\right) w''(x_1), \quad \Delta S_{1,3} = S_1 - \bar{S}_{1,3} = \left(\frac{\ell_3}{2} - x_3\right) w''(x_1) \tag{12}$$

As the mechanical strain is not constant in the piezoelectric layer, there is a local electric field due to the deformation, even though the voltage at the electrodes u_p is zero. For the electric field it follows

$$E_3 = -\frac{k_{31}^2}{1-k_{31}^2}\frac{1}{d_{31}}\Delta S_{1,3} - \frac{u_\text{p}}{\ell_3} = \frac{k_{31}^2}{1-k_{31}^2}\frac{1}{d_{31}}\left(x_3 - \frac{\ell_3}{2}\right) w''(x_1) \tag{13}$$

The electric field changes linearly in thickness direction x_3. The electric potential, which is the integral of E_3 with respect to x_3, is therefore quadratic.

Validation with Finite Element Model

We now want to validate the results for the bending trimorph by finite element calculations. First a finite element model of a clamped-free beam without piezoelectric elements is set up and compared to the analytical results obtained from the Euler-Bernoulli-beam. We choose a long and thin beam to satisfy the Euler-Bernoulli assumptions: length $\ell_1 = 100\text{mm}$ and height $\ell_3 = 1\text{mm}$. Further material properties are summarized in Table 1.

The finite element model is build from PLANE183 planar 8-node elements with quadratic displacement behavior. In total 6400 elements are used. Now, the 1st eigenfrequency is calculated for both models:

$$f_{1,\text{FEM}} = 8.35551102\,\text{Hz}, \quad f_{1,\text{analyt}} = 8.35516595\,\text{Hz} \tag{14}$$

A slight difference in frequencies is observed, due to the Euler-Bernoulli assumptions where lateral contraction and shear stress are neglected. Thus, the analytical calculations are performed with a tuned Young's modulus, calculated as:

$$E_c = \frac{f_{1,\mathrm{FEM}}^2}{f_{1,\mathrm{analy}}^2} E = 2.1001734610^9 [\mathrm{N/m^2}] \tag{15}$$

Table 1. Material properties for finite element models

	PLANE 183	PLANE 223
Young's modulus $[\mathrm{N/m^2}]$ E	$2.1 \cdot 10^9$	$2.1 \cdot 10^9$
Poisson ratio ν [-]	0.3	0.3
Shear modulus $[\mathrm{N/m^2}]$	$E/(2(1+\nu))$	$E/(2(1+\nu))$
Density $[\mathrm{kg/m^3}]$	7850	7850
Piezoelectric strain coefficients [C/N]		
d_{31}	N/A	$4 \cdot 10^{-11}$
d_{32}	N/A	0
d_{33}	N/A	0
rel. permittivity [-]		
ε_{32}^T	N/A	0
ε_{33}^T	N/A	12

Solving the analytical model with this tuned Young's modulus, we have an exact agreement of analytical and finite element eigenfrequencies. This is important in the further procedure where model errors are judged by the values of the frequencies. In the next step a finite element model of the trimorph is set up. It consists of three layers: top and bottom layer, each of height $h_{\mathrm{pzt}} = 3/8\,\mathrm{mm}$, meshed by PLANE223 planar 8-node quadratic elements with piezoelectric capabilities activated. The middle layer has a height of $h = 1/4\,\mathrm{mm}$ and is meshed with PLANE183 elements again. The material properties are summarized in Tab. 1. We can now examine the electric potential within the piezoelectric material for isolated as well as short-cut electrodes. As stated in the previous chapter, even though the electrodes might be short-cut and $u_p = 0$, the electric potential changes quadratically over the piezo thickness, which can be seen from Fig. 2. These potentials within the material are often neglected by analytical solutions.

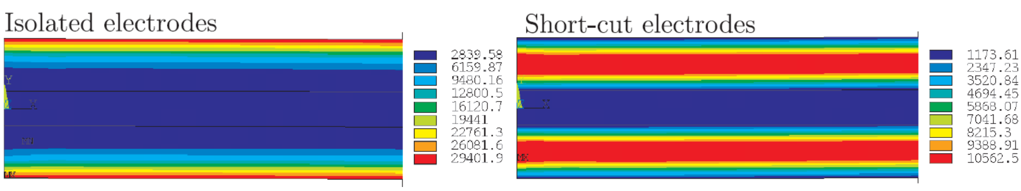

Fig. 2. Electric potential for the 1st eigenfrequency at isolated and short-cut electrodes (first 5 mm of beam shown)

Now the results obtained from our new analytical solution, the classic analytical solution and the finite element results are compared, Fig. 3. The new analytic solution matches the finite element results nearly perfectly for both short cut and isolated electrodes, while the classic solution increasingly deviates with higher material coupling factor *k*. The classic solution underestimates both resonance frequencies, because it neglects the additional term for the potential energy. From (6) it can be seen that the influence of the neglected potentials increases with the material coupling coefficient k_{3i}. Note that for short-cut electrodes the classic solution gives the same eigenfrequency for all values of the material coupling factor.

Conclusions

This paper deals with the precise calculation of the potential energy and the generalized coupling coefficient of piezoelectric systems. It is shown that the potential energy of the piezoceramics can be divided into an electrical part, stored in the piezocapacitance, and a mechanical part, which depends on the strain distribution within the ceramics. Even if the electrical voltage across the electrodes is zero, there might be an electric field within the ceramics, which contributes to the stored energy. Also the generalized coupling coefficient is influenced by this term.

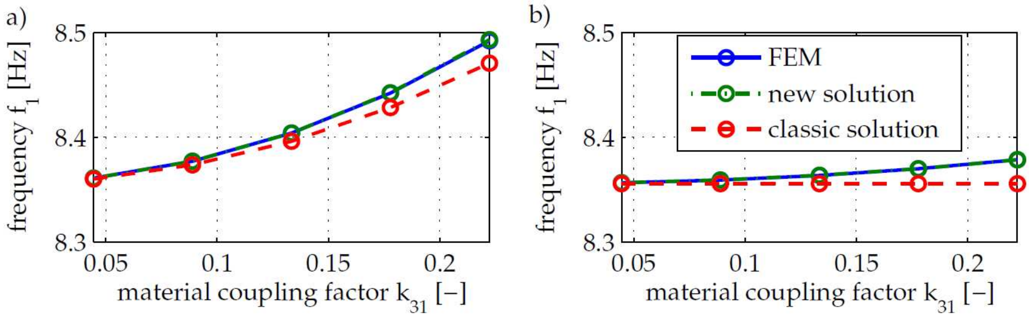

Fig. 3. a) Isolated electrodes eigenfrequencies for different material coupling factors b) short-cut electrodes eigenfrequencies

As an example, a clamped beam with piezoelectric layers is discussed. A comparison with a finite element model validates the analytical results.

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