691. Dynamic analysis of preload nonlinearity in nonlinear beam vibration

Hamid M. Sedighi [1, a], **Arash Reza** [1, b], **Jamal Zare** [2, c]
[1]Department of Mechanical Engineering, Shahid Chamran University, Ahvaz, Iran
[2]Department of Drilling Engineering, Petroleum University of Technology, Ahvaz, Iran
E-mail: *[a]hmsedighi@gmail.com, [b]arashreza@gmail.com, [c]jamal.zare@hotmail.com*

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Abstract. The objective of this paper is to propose a novel exact equivalent function (EF) for preload nonlinearity. The nonlinear preloaded spring force, which is exerted on the cantilever beam has been rewritten with a definite force-displacement relationship using new EF. This approach permits us to overcome severe computational issues that are encountered in the analytical investigations of nonlinear problems. Highly nonlinear equation of beam vibration under the influence of preloaded spring at its end with cubic nonlinearity is considered and the related analytical solution is obtained through Parameter-expansion Method (PEM). Finally, the soundness of the introduced EF would be verified by comparison of the results with the obtained solutions using numerical method.

Keywords: exact equivalent function, preload nonlinearity, nonlinear vibration of beam, parameter - expansion method.

Introduction

For many years, the nonlinear vibrations of straight beams have been studied by several investigators [1-6]. The source of nonlinearity of vibration systems are generally considered as due to the following aspects: (1) the physical nonlinearity, (2) the geometric nonlinearity and, (3) the nonlinearity of boundary conditions. In the case of discontinuous nonlinear boundary condition, the analytical solution of such problems becomes very complex. Preload nonlinearity, as a discontinuous nonlinear boundary condition, due to its inherent difficulty, has not been modeled exactly by researchers, till present.

Preloaded spring elements are encountered in many practical mechanical and structural systems either due to intentional pre-compression, unintended manufacturing or heat treatment process. However, approximation of this nonlinear condition in order to obtain the analytical solution of mentioned systems behavior has been always the major difficulty of engineering computations. Rogers et al. [7] studied the joystick dynamics where the preload stiffness (as a stiff spring) was based on measured force-displacement profile. Aktiirk et al. [8] performed an approximated theoretical investigation about the effect of varying preload on the vibration characteristics of a shaft bearing system. Dynamics of a mechanical oscillator with preload nonlinearity was investigated by Chengwu and Rajendra [9]. They smoothen the preload nonlinearity with arctan function. A rotary piezoelectric motor design using a preloaded beam stator was investigated by Wajchman et al. [10]. They approximated the preload nonlinearity empirically, in order to achieve the optimum efficiency of the motor performance.

Recently, considerable attention has been directed towards analytical solutions for nonlinear equations without small parameters. There have been several classical approaches employed to solve the governing nonlinear differential equations to study the nonlinear vibrations including perturbation methods [11], frequency amplitude formulation [12], Artificial Parameter Lindstedt-Poincaré Method [13], Min-Max method [5, 14], Multiple Scales method [15], Variational Iteration Method [16], HAM [17-19], Semi-analytical finite element [20] and Homotopy Perturbation Method (HPM) [21] are used to solve nonlinear problems.

Parameter-expansion Method (PEM) is one of the most effective methods for analytical solution of nonlinear differential equations. PEM has been shown to effectively, easily and accurately solve large nonlinear problems with components that converge rapidly to accurate solutions. Sweilam and Al-Bar [22] implemented the Parameter-expansion Method to the coupled Van der Pol oscillators. Shin et al. [23] applied PEM to approximate the solution of the coupled nonlinear self-excited oscillators and achieved the frequency of mentioned systems. The nonlinear vibrations analysis of inextensible beams investigated by Kimiaeifar et al. [24] using PEM. They also studied the influence of different parameters on the system response stability. Sweilam and Khader [25] investigated the application of PEM to the coupled system of nonlinear partial differential equation and showed the solution accuracy by focusing on Manakov systems.

The presence of preload nonlinearity in dynamical systems gives rise to an appreciable complexity in the analytical solution procedure. This is the main reason that no exact analytical definition has been proposed in the earlier research works. As mentioned above, this nonlinearity has been approximated by trigonometric functions or has been solved numerically. Moreover, the PEM has not been developed in the field of beam dynamical behavior, until now. The objective of this paper is to introduce the innovative exact EF for preload nonlinearity as a boundary condition and to implement the PEM in the nonlinear beam vibrations.

In this work, based on the Galerkin theory, nonlinear ordinary differential equation of beam vibration is extracted from partial differential equation with first mode approximation. Then preload nonlinear boundary condition of beam is modeled using new introduced EF. The results presented in this paper demonstrate that the proposed EF is very effective and convenient for nonlinear oscillators where the preload nonlinear boundary condition exists. The exact analytical solution of mentioned system is obtained using PEM and demonstrates that one term in series expansions is sufficient to result in a highly accurate solution of the problem.

Governing equation

Consider the system that is shown in figure 1, where the cantilever beam is subjected to preloaded spring at its end. Neglecting the shearing deformations and out-of-plane motion of the beam, governing partial differential equation for the nonlinear flexural vibration of the beam is, as follows [26]:

$$m\ddot{v} + EIv^{iv} + EI\left[v'\left(v'v''\right)'\right]' + \frac{1}{2}m\left\{v'\int_L^x\left[\frac{\partial^2}{\partial t^2}\int_0^x v'^2 dx\right]dx\right\}' = 0 \tag{1}$$

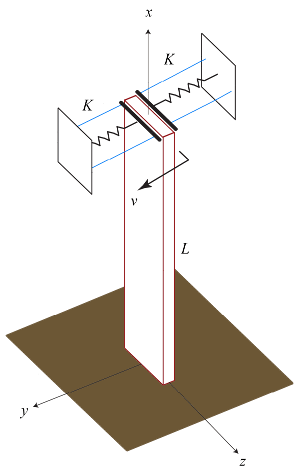

Fig. 1. Cantilever beam with preload nonlinear boundary condition

Here x is the axial coordinate, which is measured from the origin, v denotes the lateral vibration in y direction, m is the mass per unit length of the beam, E is Young's modulus and I is the area moment of inertia. The boundary conditions for the beam of length L are:

$$v(0,t)=\frac{\partial v}{\partial x}(0,t)=0,\ \frac{\partial^2 v}{\partial x^2}(L,t)=0,\ EI\frac{\partial^3 v}{\partial x^3}(L,t)=F_{pl}(L,t) \tag{2}$$

where $F_{pl}(L,t)$ is boundary condition at its end and is described by the following nonlinear preload formula with cubic nonlinearity:

$$F_{pl}\left(v(L,t)\right)=\begin{cases} F_0+K\left(v(L,t)\right)^3 & v(L,t)>0 \\ -F_0+K\left(v(L,t)\right)^3 & v(L,t)<0 \end{cases} \tag{3}$$

where K is the constant of nonlinear spring. Assuming $v(x,t)=q(t)\varphi(x)$, where $\varphi(x)$ is the first eigenmode of the clamped-free beam and can be expressed as:

$$\varphi(x)=\cosh(\lambda x)-\cos(\lambda x)-\frac{\cosh(\lambda L)+\cos(\lambda L)}{\sinh(\lambda L)+\sin(\lambda L)}\left(\sinh(\lambda x)-\sin(\lambda x)\right) \tag{4}$$

where $\lambda=1.875$ is the root of the characteristic equation for first eigenmode. Applying the weighted residual Bubnov-Galerkin method yields:

$$\int_0^L\left(m\ddot{v}+EIv^{iv}+EI\left[v'\left(v'v''\right)'\right]'+\frac{1}{2}m\left\{v'\int_L^x\left[\frac{\partial^2}{\partial t^2}\int_0^x v'^2dx\right]dx\right\}'\right)\varphi(x)\,dx=0 \tag{5}$$

to implement the end nonlinear boundary condition, applying integration by part on equation (5), it is converted to the following:

$$\int_0^L\left(m\ddot{v}+EI\left[v'\left(v'v''\right)'\right]'+\frac{1}{2}m\left\{v'\int_L^x\left[\frac{\partial^2}{\partial t^2}\int_0^x v'^2dx\right]dx\right\}\right)\varphi(x)\,dx+\int_0^L EIv^{iv}\varphi(x)\,dx=0 \tag{6}$$

$$\begin{aligned}&\int_0^L\left(m\ddot{v}+EI\left[v'\left(v'v''\right)'\right]'+\frac{1}{2}m\left\{v'\int_L^x\left[\frac{\partial^2}{\partial t^2}\int_0^x v'^2dx\right]dx\right\}\right)\varphi(x)\,dx\\&+EIv'''\varphi(x)\Big|_0^L-\int_0^L EIv'''d\left(\varphi(x)\right)=0\end{aligned} \tag{7}$$

In the above equation the boundary condition term $EIv'''(L,t)$ is replaced by $F_{pl}(L,t)$. So, we can obtain the nonlinear equation in terms of the time-dependent variables as:

$$\ddot{q}+\beta_1 q+\beta_2 q^3+\beta_4 q\dot{q}^2+\beta_5 q^2\ddot{q}+F_{pl}(L,t)=0 \tag{8}$$

where:

$$\beta_1=12.3624EI/mL^4,\ \beta_2=40.44\,EI/mL^6+16K/mL,\ \beta_4=\beta_5=4.6/L^2 \tag{9}$$

To solve nonlinear ordinary equation (8) analytically, the preload condition F_{pl}, must be formulated, properly. We introduce suitable and novel exact equivalent function for this nonlinearity as:

$$F_{pl}(u) = \left(\frac{1}{2} + \frac{1}{2}\frac{|u|}{u}\right)\left(F_0 + Ku^3\right) + \left(\frac{1}{2} - \frac{1}{2}\frac{|u|}{u}\right)\left(-F_0 + Ku^3\right) \tag{10}$$

Figure 2 shows the equivalent function for F_{pl} graphically. Using this new definition of F_{pl}, equation (9) is written as follows:

$$\ddot{q} + \beta_1 q + 1.\left[\beta_2 q^3 + \beta_3 |q|/q + \beta_4 q\dot{q}^2 + \beta_5 q^2 \ddot{q}\right] = 0 \tag{11}$$

where:

$$\beta_3 = 2F_0/mL \tag{12}$$

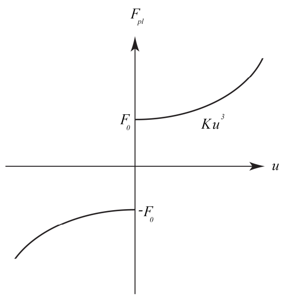

Fig. 2. Plot of EF preload nonlinearity

Analytical solution procedure

Consider the equation (11) for the vibration of a cantilever Euler-Bernoulli beam with the following general initial conditions:

$$q(0) = A\,, \quad \dot{q}(0) = 0 \tag{13}$$

The limit-cycles of oscillating systems are periodic motions with the period $T = 2\pi/\omega$, and thus $q(t)$ can be expressed by such a set of base functions:

$$\cos(m\omega t), \ m = 1, 2, 3, \ldots \tag{14}$$

We denote the angular frequency of oscillation by ω and note that one of our major tasks is to determine $\omega(A)$, i.e., the functional behavior of ω as a function of the initial amplitude A. In the PEM, an artificial perturbation equation is constructed by embedding an artificial parameter $p \in [0,1]$ which is used as an expanding parameter.

According to PEM the solution of equation (11) is expanded into a series of p in the form:

$$q(t) = q_0(t) + pq_1(t) + p^2 q_2(t) + \ldots \tag{15}$$

The coefficients 1 and β_1' in the equation (11) are expanded in a similar way:

$$\begin{aligned} &1 = 1 + pa_1 + p^2 a_2 + \dots \\ &\beta_1 = \omega^2 - pb_1 - p^2 b_2 + \dots \\ &1 = pc_1 + p^2 c_2 + \dots \end{aligned} \tag{16}$$

where a_i, b_i, c_i $(i = 1, 2, 3, \dots)$ are to be determined. When $p = 0$, equation (11) becomes a linear differential equation for which an exact solution can be calculated for $p = 1$. Substituting equations (15) and (16) into equation (11):

$$\begin{aligned} &(1 + pa_1)(\ddot{q}_0 + p\ddot{q}_1) + (\omega^2 - pb_1)(q_0 + pq_1) + (pc_1 + p^2 c_2)\cdot \\ &\cdot\Big[\beta_2 (q_0 + pq_1)^3 + \beta_4 (q_0 + pq_1)(\dot{q}_0 + p\dot{q}_1)^2 + \dots \\ &\dots + \beta_5 (q_0 + pq_1)^2 (\ddot{q}_0 + p\ddot{q}_1) + \beta_3 f_{pl} (q_0 + pq_1)\Big] = 0 \end{aligned} \tag{17}$$

where:

$$f_{pl}(q) = |q|/q \tag{18}$$

in equation (18) we have taken into account the following expression:

$$f_{pl}(q) = f_{pl}(q_0 + pq_1 + p^2 q_2 + \dots) = f_{pl}(q_0) + pq_1 f'_{pl}(q_0) + O(p^2) \tag{19a}$$

where:

$$f'_{pl}(q) = \frac{df_{pl}}{dq} = f''_{pl}(q) = f'''_{pl}(q) = \dots = 0 \tag{19b}$$

therefore:

$$f_{pl}(q) = f_{pl}(q_0 + pq_1 + p^2 q_2 + \dots) = f_{pl}(q_0) \tag{20}$$

Collecting the terms of the same power of p in equation (17), we obtain a series of linear equations of which the first equation is:

$$\ddot{q}_0(t) + \omega^2 q_0(t) = 0, \qquad q_0(0) = A, \quad \dot{q}_0(0) = 0 \tag{21}$$

with the solution:

$$q_0(t) = A\cos(\omega t), \tag{22}$$

substitution of this result into the right-hand side of second equation gives:

$$\begin{aligned} &\ddot{q}_1(t) + \omega^2 q_1(t) = \\ &= \left(b_1 A - \frac{8}{\pi} c_1 \beta_3 F_0 - \frac{3}{4} c_1 \beta_2 A^3 + \frac{3}{4} c_1 \beta_5 A^3 \omega^2 - \frac{1}{4} c_1 \beta_4 A^3 \omega^2 + a_1 A \omega^2 \right) \cos(\omega t) \\ &+ \frac{1}{4} c_1 A^3 (\beta_4 \omega^2 + \beta_5 \omega^2 - \beta_2) \cos(3\omega t). \end{aligned} \tag{23}$$

In the above equation, the possible following Fourier series expansion has been accomplished:

$$f_{pl}\left(q_0\right)= f_{pl}\left(A\cos\left(\omega t\right)\right)=\sum_{n=1}^{\infty} h_n\cos\left(n\omega t\right)=h_1\cos\left(\omega t\right)+h_2\cos\left(2\omega t\right)+... \quad (24)$$

where:

$$h_n=\frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f_{pl}\left(A\cos\theta\right)\cos\left(n\theta\right)d\theta, \quad (25)$$

and the functions f_{pl} are substituted from equations (18) and (20). The first terms of the expansion in equations (25) are given by:

$$h_1=\frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f_{pl}\left(A\cos\theta\right)\cos\left(\theta\right)d\theta=\frac{8F_0}{\pi} \quad (26)$$

Solution of equation (23) should not contain the so-called secular term $\cos\left(\omega t\right)$. To ensure so, the right-hand side of this equation should not contain the terms cos, i.e. the coefficients of cos must be zero:

$$b_1A-\frac{8}{\pi}c_1\beta_3F_0-\frac{3}{4}c_1\beta_2A^3+\frac{3}{4}c_1\beta_5A^3\omega^2-\frac{1}{4}c_1\beta_4A^3\omega^2+a_1A\omega^2=0 \quad (27)$$

equation (16) for one term approximation of series respect to p and for $p=1$ yields:

$$a_1=0,\ b_1=\omega^2-\beta_1,\ c_1=1 \quad (28)$$

From equations (27) and (28) we can easily find that the solution ω is:

$$\omega\left(A\right)=\pm\sqrt{\frac{\beta_1+\frac{8}{\pi A}\beta_3F_0+\frac{3}{4}\beta_2A^2-4\beta_3}{1+\frac{3}{4}\beta_5A^2-\frac{1}{4}\beta_4A^2}} \quad (29)$$

Replacing ω from equation (29) into equation (22) yields:

$$q\left(t\right)\approx q_0\left(t\right)=A\cos\left(\sqrt{\frac{\beta_1+\frac{8}{\pi A}\beta_3F_0+\frac{3}{4}\beta_2A^2-4\beta_3}{1+\frac{3}{4}\beta_5A^2-\frac{1}{4}\beta_4A^2}}\ t\right) \quad (30)$$

Results and discussion

To show the soundness of the obtained analytical solution, the authors also calculate the variation of non-dimensional amplitude A/L versus normalized time $\tau=\omega t$, numerically. As can be observed in the figure 3, the first order approximation of $q\left(t\right)$, which is obtained through PEM and new EF, has an excellent agreement with numerical results using fourth-order Runge–Kutta method.

To indicate the effect of preload force on the response of beam vibration, the nondimentional preload parameter $g = F_0/KA$ is introduced. As depicted in figure 4, for the same normalized amplitude, the frequency of beam vibration increases, when the preload parameter gets larger. Also, the phase plane of the problem that is obtained from PEM has been shown in figure 5. It is evident that the solution converges rapidly and is valid for a wide range of preload parameter and initial conditions.

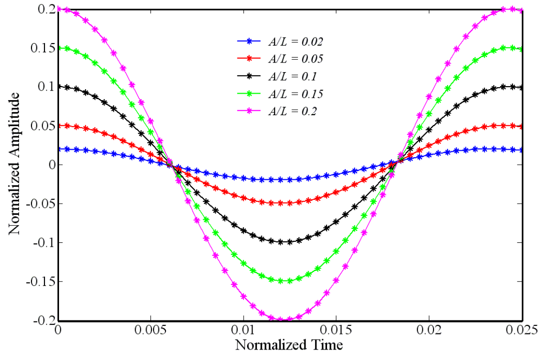

Fig. 3. Comparison of the approximated first order periodic solution (continuous line) with the numerical results (stars)

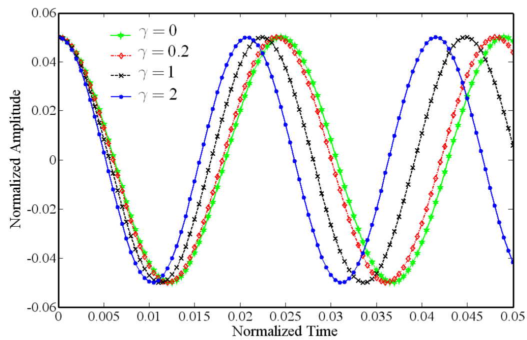

Fig. 4. The influence of preload parameter on the vibrational response

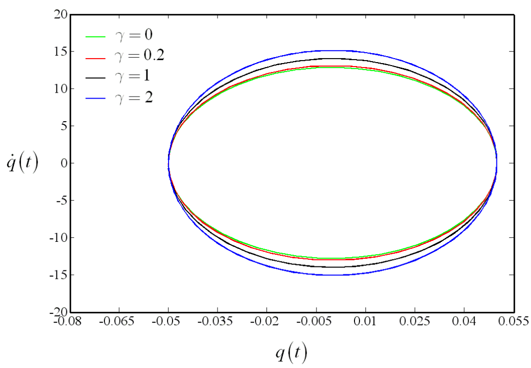

Fig. 5. The effect of preload parameter on the beam vibration phase plane

The variation of frequency versus normalized amplitude and preload forces is represented in figure 6. As can be observed, the more the preloaded force, the larger the limit cycle frequency. Regardless the preload force value, when the normalized amplitude increases, the frequency decreases. From figure 7, as the vibration amplitude shifts upwards, at first, the frequency

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reduces until reaches a minimum value and then increases continuously. In addition, decreasing the beam length leads to increasing in the response frequency. Typical amplitude of cantilever beam vibration with normalized amplitude $A/L = 0.2$ along its length is illustrated in figure 8.

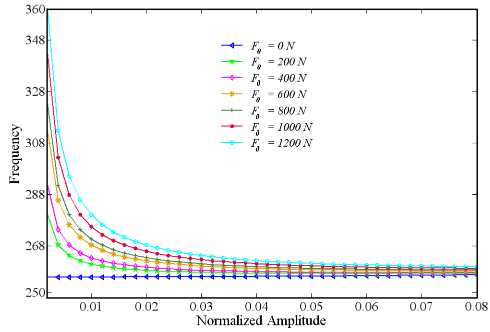

Fig. 6. Comparison of frequency vs. normalized amplitude corresponding to various preload values

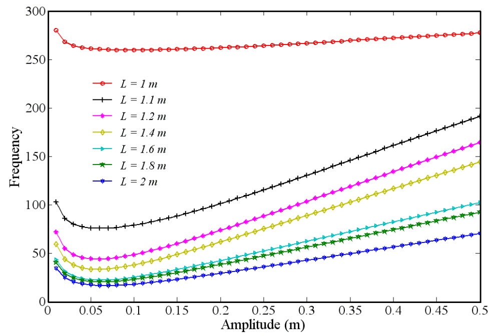

Fig. 7. Comparison of frequency vs. normalized amplitude corresponding to various beam length values

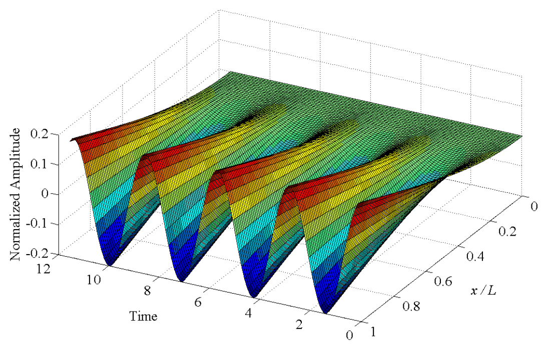

Fig. 8. Typical vibration amplitude along beam length with $A/L = 0.2$

Conclusion

Novel EF for discontinuous preload nonlinearity has been employed to predict analytical response of nonlinear cantilever beam vibration in the time domain. The preload nonlinearity, as a boundary condition of cantilever beam, redefined exactly using the continuous functions. This new EF is implemented in nonlinear vibration of cantilever beam and an excellent first-order analytical approximate solution by PEM was obtained. It appears from the present work that the introduced EF can make the analytical investigation of the nonlinear problems fairly easy. The authors believe that the introduced procedure has a special potential to be applied to other strong nonlinearities such as preload, deadzone and saturation discontinuities.

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