686. The construction of one-dimensional Daubechies wavelet-based finite elements for structural response analysis

Bing Li [a], **Cao Hongrui** [b], **Zhengjia He** [c]
State Key Laboratory for Manufacturing Systems Engineering, Xi'an Jiaotong University
Xi'an 710049, China
E-mail: [a]*bli@mail.xjtu.edu.cn*, [b]*chr@mail.xjtu.edu.cn*, [c]*hzj@mail.xjtu.edu.cn*

(Received 3 November 2011; accepted 4 December 2011)

Abstract. The objective of this paper is to develop a family of wavelet-based finite elements for structural response analysis. First, independent wavelet bases are used to approximate displacement functions, unknown coefficients are determined through imposing the continuity, linear independence, completeness, and essential boundary conditions. A family of Daubechies wavelet-based shape functions are then developed, which are hierarchical due to multiresolution property of wavelet. Secondly, to construct wavelet-based finite elements, derivation of the shape functions for a subdomain is employed. Thus, the wavelet-based finite elements being presented are embodied with properties in adaptivity as well as locality. By wavelet preconditioning technology, the two difficulties involving imposition of boundary conditions and compatibility with the traditional finite element methods, which are gathered in the experiments of wavelet-Galerkin context, are well overcome. Numerical examples are used to illustrate the characteristics of the current elements and to assess their accuracy and efficiency.

Keywords: Daubechies wavelets, multiresolution, shape functions, wavelet-based finite elements.

1. Introduction

The finite element method (FEM) is a piecewise application of a variational method [1, 2]. It can be classified into two groups, namely narrow FEM, and a generalized FEM in terms of approximate spaces being adopted. The narrow FEM uses the low order polynomials as approximation functions, while the generalized FEM employs many other trial functions. Usually, the development of FEM is deeply related to the extension of approximation spaces. Wavelet-based finite element (WFM) is a vivid example of this development.

Since independent wavelet bases have the ability to accurately represent fairly general functions with a small number of wavelet coefficients, as well as to characterize the smoothness of such functions from the numerical behavior of these coefficients [3], wavelet theory provides a powerful mathematical tool for function approximation and multiresolution analysis. Typical applications of wavelet analysis include data compression, signal and image de-noising, data communication, and function approximation. A wavelet-based approach can also be used for the numerical solution of partial differential equations (PDEs). Dahmen [4] has reviewed the recent developments of wavelet-based schemes for PDEs. Here, a few of the notable results are briefly reviewed.

Jaffard [5] introduced wavelet scheme to the numerical solution of PDE with Dirichlet boundary condition. The certain key features of the wavelet-based method can be found in the work of Jaffard [5]. Amaratunga, Williams, Qian, and Weiss [6] represented wavelet-Galerkin solution for the one-dimensional Helmholtz boundary value problem with periodic boundary condition, and reported that their approach was superior to the finite difference method. Dumont and Lebon [7] presented a wavelet-Galerkin formulation for periodic composite elastic materials. The distribution of the different material properties is assumed to be periodic. Their work showed the usefulness of localization property of wavelet analysis in effectively modeling the local variation in material properties. Using a similar approach, Dumont and Lebon [8] developed the wavelet representation of plane elastostatic operators by means of the orthogonal Daubechies wavelet system. Ko, Kurdila, and Dilamt [9] proposed the concept of fictitious domain and the numerical boundary measure technique to handle general boundary conditions.

Although the wavelet transform with its space-scale localization is an attractive technique to apply to the solution of problems with localized structures, traditional, biorthogonal wavelet transforms have difficulties dealing with boundaries [10]. In order to cope with this problem, the idea of the combination of wavelet-Galerkin method and piecewise variation in FEM context has been presented. Chen and Wu [11, 12] constructed spline wavelets elements, and successfully solved the problems of frame structures vibration and membrane vibration respectively. Castro and Freitas [13] applied wavelet analysis in the implementation of stress model of the hybrid-mixed FEM. The two-dimensional wavelet-based hybrid-mixed stress elements were adopted to solve the displacement and stress field of square plate and thick cylinder under particular load. In the view of numerical integration with respect to wavelet terms, Newton-Lotes quadrature rules were applied in [13] because the wavelet are defined only at dyadic points. However, the irregularity of the wavelet functions required the use of a large number of control points, in addition, realization of these integrals are difficult since they are highly oscillatory. Using the two-scale relation of wavelet, Ma and Xue et al. [14], and Li and Chen et al. [15] employed a more efficient integral method, which was based on the fact that an integral problem can be transformed to solution of linear scaling equations [16]. The beam bending problem was solved successfully in [14, 15]. To end of satisfying interelement continuity conditions, the transform matrix that realizes the transform between wavelet spaces and physical spaces was constructed in [14, 15]. However, ill-condition of the transform matrix for higher order wavelet or higher resolution space is the price paid to ensure interelement continuity condition. Due to the lack of shape functions in WFE, another difficulty in [11-16] is the compatibility between the WFE and the traditional finite elements.

This paper aims at solving above difficulties by constructing shape functions, which are similar to those in traditional finite element context except for the point of employing wavelet bases as approximate functions. When the scaling functions with higher order or in higher resolution space are used, wavelet preconditioning based on column and row balance theory of matrix presented can greatly decrease the condition number of transform matrix. The relation between the order of the approximation function used for dependent variable u and the number of nodes in the wavelet element is derived. The family of WFM with locality and hierarchical property are built.

The outline of this paper is sketched as follows. The properties of multiresolution analysis and Daubechies wavelet are briefly reviewed in Section 2. In Section 3 we describe how to construct a class of wavelet-based shape functions and give element stiffness matrixes and load vectors. Numerical examples are given in Section 4.

2. Multiresolution analysis and Daubechies wavelets

2.1. Multiresolution analysis

In this section, a brief review of multiresolution analysis is given. More details can be found in [17].

Let $L^2(R)$ denotes the vector space of measurable, square-integrable one-dimensional real numbersA multiresolution analysis of $L^2(R)$ is defined as a sequence of closed subspaces V_j with the following properties:

1. $V_j \subset V_{j+1}$,

2. $f(x) \in \mathrm{V_j} \Leftrightarrow f(2x) \in \mathrm{V_{j+1}}$, and $f(x) \in \mathrm{V_0} \Leftrightarrow f(x+1) \in \mathrm{V_0}$,
3. $\bigcup_{j \in \mathrm{Z}} \mathrm{V}_j$ is dense in $\mathrm{L}^2(\mathrm{R})$; $\bigcap_{j \in \mathrm{Z}} \mathrm{V}_j = \{0\}$,
4. A scaling function $\phi(x) \in \mathrm{V_0}$ exists such that the set $\{\phi(x-k) \mid k \in \mathrm{Z}\}$ is a Riesz basis of $\mathrm{V_0}$.

Consequently, a sequence $\mathbf{p}_k \in \ell^2(\mathrm{Z})$ exists, $\ell^2(\mathrm{Z})$ denotes the integer space of all square-summable bi-infinite sequences, such that the scaling function $\phi(x)$ satisfies a refinement equation

$$\phi(x) = \sum_k p_k \phi(2x-k), \quad k \in \mathrm{Z}. \tag{1}$$

The set of functions $\{\phi_{j,k}(\mathrm{x}) \mid k \in \mathrm{Z}\}$ with $\phi_{j,k}(x) = 2^{\frac{j}{2}} \phi(2^j x - k)$ is a Riesz basis of V_j. Let W_j denote a subspace complementing the subspace V_j in V_{j+1}, i.e., $\mathrm{V}_{j+1} = \mathrm{V}_j \oplus \mathrm{W}_j$. Each element of V_{j+1} can be uniquely written as the orthogonal sum of an element in V_j and an element in W_j that contains the details needed to pass from an approximation at level j to an approximation at level $j+1$. A function $\psi(x)$ is a mother wavelet if the set of functions $\{\psi(x-k) \mid k \in \mathrm{Z}\}$ is a Riesz basis of W_0. Since the mother wavelet is also an element of V_1, a sequence $\mathbf{q}_l \in \ell^2(\mathrm{Z})$ exists such that the wavelet function $\psi(x)$ satisfies

$$\psi(x) = \sum_k q_k \phi(2x-k), \quad k \in \mathrm{Z}. \tag{2}$$

The set of wavelet functions $\{\psi_{j,k}(x) \mid k \in \mathrm{Z}\}$, with $\psi_{j,k}(x) = 2^{\frac{j}{2}} \psi(2^j x - k)$, is now a Riesz basis of $L^2(\mathbf{R})$.

All wavelet bases are associated with multiresolution analysis, which is a framework in which function $f(x) \in \mathrm{L}^2(\mathrm{R})$ can be considered as a limit of successive approximations

$$f(x) = \lim_{j \to \infty} \mathbf{P}_j f(x), \quad j \in \mathrm{Z}, \tag{3}$$

where the different $\mathrm{P}_j f(x)$ corresponds to smooth versions of $f(x)$ with a "smoothing out action radius" of the resolution of 2^j.

2.2. Daubechies wavelets

As an example of multiresolution analysis, a family of orthogonal Daubechies wavelets with compactly supported property have been constructed by Daubechies in [18] with the following properties.

1) Compact support

A family of Daubechies wavelets are generated by scaling function $\phi_N(x)$ and wavelet function $\psi_N(x)$. Both of them have nonzero values over a small portion of the domain. Note that D*Nj* denotes the Daubechies wavelet with order N in the resolution space j, the supports for $\phi_N(x)$ and $\psi_N(x)$ are given below.

$$\mathrm{supp}\,\phi_N = [0, 2N-1], \tag{4}$$

$$\mathrm{supp}\,\psi_N = [1-N, N]. \tag{5}$$

2) Cancellation property

Since D*Nj* is orthogonal to polynomials up to $N-1$ order, the scaling function $\phi_N(x)$ and wavelet function $\psi_N(x)$ have $N-1$ order of vanishing moments

$$\int_{-\infty}^{\infty} x^k \phi_N(x)dx = 0, \; k = 0,1,\cdots,N-1, \tag{6}$$

$$\int_{-\infty}^{\infty} x^k \psi_N(x)dx = 0, \; k = 0,1,\cdots,N-1. \tag{7}$$

3) Orthogonal property

The scaling function $\phi_N(x)$ and wavelet function $\psi_N(x)$ of D*Nj* satisfy the following orthogonal conditions:

$$\int_{-\infty}^{\infty} \phi_N(x-j)\phi_N(x-m)dx = \delta_{j,m}, \tag{8}$$

$$\int_{-\infty}^{\infty} \phi_N(x)\psi_N(x-m)dx = 0 \quad m \in \mathrm{Z}, \tag{9}$$

where $\delta_{j,m} = \begin{cases} 1 & j = m \\ 0 & j \neq m \end{cases}$.

In addition to above three main properties, the scaling function $\phi_N(x)$ satisfies the normalized condition

$$\int_{-\infty}^{\infty} \phi_N(x)dx = 1. \tag{10}$$

When scaling functions are employed as approximate functions in the procedure of construction of WFE, the first property proves to be more effective in application of using minimum degrees of freedom over an element to approximate displacement functions. Moreover, sparseness of the matrix is a result of the scaling functions, which have compactly supported property. Property 2 allows one to perfectly interpolate polynomials of degree up to $N-1$ by the scaling function with order N. The experiments gathered in wavelet-Galerkin context indicate that property 3 satisfies that the matrix is sparse as well as banded if the global nodes are numbered sequentially.

3. Derivation of element equations

The basic idea of wavelet-based finite element method, which is similar to the traditional FEM, is to discretize a body into an assemble of discrete finite elements that are interconnected at nodal points on element boundary. The displacement field is approximated over each WFE in terms of the nodal displacements. The procedure of the derivation of element equations is given below.

3.1. Construction of shape function

Consider the problem of finding the function $u(x)$ that satisfies the differential equation

$$-\frac{d}{dx}(AE\frac{du}{dx}) - f = 0, \quad 0 < x < L, \tag{11}$$

and the boundary conditions

$$u(0) = 0, \quad (AE\frac{du}{dx})\bigg|_{x=L} = P, \tag{12}$$

where $f = f(x)$, the cross-section area A, Young's modulus E and P are the date of the problem.

The domain $\Omega \equiv (0, L)$ of the problem is divided into a set of WFEs. A typical element

$\Omega^e = (x_A, x_B)$ is isolated from the mesh (Fig. 1). Assume that the displacement $u(x)$ is approximated by

$$u = \sum_{i=2-2N}^{2^j-1} c_{i+2N-1}\phi_N(2^j\xi - i), \tag{13}$$

in matrix form yields

$$\mathbf{u} = \boldsymbol{\varphi}\mathbf{c}, \tag{14}$$

where $\mathbf{c}$ is the unknown coefficient vector, and ξ is the local coordinate. The transformation between the global coordinate system x to the local coordinate system ξ, which has the origin at the left end node of the element, is achieved by the linear 'stretch' transformation given by

$$\xi = \frac{x - x_A}{L_e}, \tag{15}$$

where x_A is the global coordinate of the left end node of the element Ω^e and L_e denotes the element length (see Fig. 1). Obviously, the value of ξ is always between 0 and 1. The element number of $\mathbf{c}$ is $2^j - 2 + 2N$ with respect to equations (4, 5).

According to the basic idea in the traditional FEM book [2], shape functions require to satisfy the continuity, linear independence, completeness, and essential boundary conditions [2]. Here, taking D30 as an example, the continuity is obviously satisfied, the scaling functions $\phi_3(\xi)$ are linear independent and complete. To satisfy the remaining requirement, we require u to satisfy the essential boundary conditions of the element

$$\mathbf{u}^* = \mathbf{Rc}, \tag{16}$$

where $\mathbf{u}^*$ is the node displacement vector, $\mathbf{R}$ stands for the transform matrix, and its explicit form is

$$\mathbf{R} = \begin{bmatrix} \phi_3(4) & \phi_3(3) & \phi_3(2) & \phi_3(1) & \phi_3(0) \\ \phi_3(4+\frac{1}{4}) & \phi_3(3+\frac{1}{4}) & \phi_3(2+\frac{1}{4}) & \phi_3(1+\frac{1}{4}) & \phi_3(\frac{1}{4}) \\ \phi_3(4+\frac{1}{2}) & \phi_3(3+\frac{1}{2}) & \phi_3(2+\frac{1}{2}) & \phi_3(1+\frac{1}{2}) & \phi_3(\frac{1}{2}) \\ \phi_3(4+\frac{3}{4}) & \phi_3(3+\frac{3}{4}) & \phi_3(2+\frac{3}{4}) & \phi_3(1+\frac{3}{4}) & \phi_3(\frac{3}{4}) \\ \phi_3(5) & \phi_3(4) & \phi_3(3) & \phi_3(2) & \phi_3(1) \end{bmatrix}. \tag{17}$$

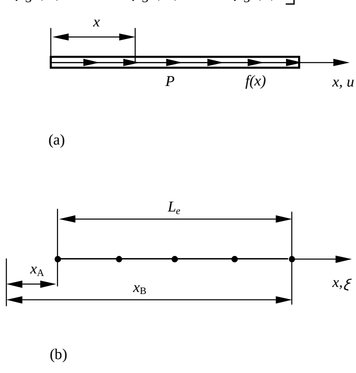


Fig. 1. WFE discretization of a one-dimensional domain of equation (11)
(a) Physical problem; (b) A typical wavelet finite element for D30

Solving for $\mathbf{c}$ in terms of $\mathbf{u}^*$, we have

© VIBROENGINEERING. JOURNAL OF VIBROENGINEERING. DECEMBER 2011. VOLUME 13, ISSUE 4. ISSN 1392-8716

$$\mathbf{c} = \mathbf{T}\mathbf{u}^*, \tag{18}$$

where $\mathbf{T} = \mathbf{R}^{-1}$, and '-1' denotes the inverse. Substituting equation (18) for $\mathbf{c}$ in equation (14), and collecting the coefficients of $\mathbf{u}^*$, we get

$$\mathbf{u} = \mathbf{N}\mathbf{u}^*, \tag{19}$$

where $\mathbf{N} = \boldsymbol{\varphi}\mathbf{T}$, is shape function collection. Since the shape functions are derived from equation (19) in such a way that $\mathbf{u}$ is equal to $\mathbf{u}^*$ at node. In addition to satisfy the property that $N_i(\xi) = 0$ outside the element Ω^e, they have the following properties:

1. $N_i(\xi_j) = \delta_{ij}$,
2. $\sum_{i=1}^{5} N_i(\xi) = 1$.

When wavelet bases are used as approximation functions, an obvious advantage over polynomial interpolation is that two alternative *p*-refinement strategies, in which the shape function order of all (some) elements is increased in order to improve the solution accuracy, may be adopted. The first consists in always using the same degree of refinement j, or resolution level, while increasing the wavelet order, N, associated with the selected basis functions. In the second alternative, the wavelet family used in the analysis is always the same while the degree of refinement is successively increased.

The family of WFE of D*Nj* is developed and listed in Table 1.

Note that the number of node s, over an element should match the number of unknown coefficients c_i in order to ensure the transform matrix to be square matrix. The relation between the wavelet order N, the degree of refinement j, and the number of node s is derived below

$$s = 2^j - 2 + 2N. \tag{20}$$

3.2. Wavelet preconditioning

Due to the compactly supported property of wavelet bases, the transform matrix being constructed by the scaling functions is almost sparse. When the wavelet order N or the degree of refinement j is lifted, the condition number κ of the transform matrix $\mathbf{R}$ is grown exponentially. The ill-conditioned matrix always leads to the numerical instability during the procedure of matrix inverse. For the purpose of improving the condition number, a diagonal preconditioning in the wavelet bases is presented here, which is based on the column and row balance preconditioning of matrix.

Assume r_{ij} is the *i*th row and *j*th column element of the transform matrix $\mathbf{R}$, which is $n \times n$ square matrix, the row pivot h_i and column pivot g_j are respectively calculated by

$$h_i = \max_{1 \le j \le n} |r_{ij}|, \quad g_j = \max_{1 \le i \le n} |r_{ij}|. \tag{21}$$

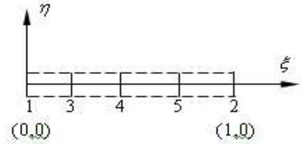
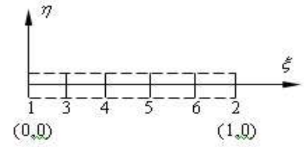
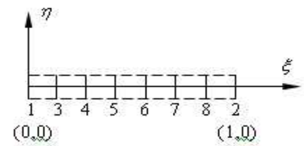
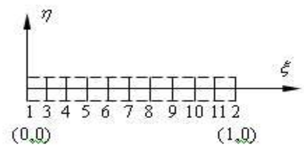
Then we obtain the diagonal preconditioners $\mathbf{h}$ and $\mathbf{P}$

$$\mathbf{h} = \mathrm{diag}(\frac{1}{h_1} \quad \frac{1}{h_2} \quad \cdots \quad \frac{1}{h_n}), \quad \mathbf{g} = \mathrm{diag}(\frac{1}{g_1} \quad \frac{1}{g_2} \quad \cdots \quad \frac{1}{g_n}), \tag{22}$$

where 'diag' produces a diagonal matrix. The diagonal preconditioner arises naturally from the wavelet bases and leads to well-conditioned matrix $\widehat{\mathbf{R}}$ ($\widehat{\mathbf{R}} = \mathbf{hRg}$) and efficient numerical implementations

$$\mathbf{R}^{-1} = \mathbf{g}\widehat{\mathbf{R}}^{-1}\mathbf{h}. \tag{23}$$

Table 1. One-dimensional wavelet-based finite elements
($t_{i,j}$ is the *i*th row and *j*th column element of $\mathbf{T}$)

Serial no.	Element name	Configuration	Shape functions
1	D30		$N_i = \sum_{k=-4}^{0} t_{k+5,i}\phi_3(\xi - k)$, $i = 1,2,\ldots,5$, $i \in Z$
2	D31		$N_i = \sum_{k=-4}^{1} t_{k+5,i}\phi_3(2\xi - k)$, $i = 1,2,\ldots,6$, $i \in Z$
3	D32		$N_i = \sum_{k=-4}^{3} t_{k+5,i}\phi_3(2^2\xi - k)$, $i = 1,2,\ldots,8$, $i \in Z$
4	D60		$N_i = \sum_{k=-10}^{0} t_{k+11,i}\phi_6(\xi - k)$, $i = 1,2,\ldots,11$, $i \in Z$

3.3. Evaluation of element stiffness matrix and load vector

After constructing wavelet-based shape functions, the procedure of derivation of element equations can be achieved as done in the traditional FEM. For DNj, the stiffness matrix and load vector are given below

$$\mathbf{K_e} = \frac{EA}{Le}\mathbf{\Gamma}^{11}_{i\,j}, \tag{24}$$

$$\mathbf{P_e} = Le\int_0^1 \phi_N(2^j\xi - i)f(\xi)d\xi, \tag{25}$$

where,

$$\mathbf{\Gamma}^{11}_{i\,j} = \int_0^1 \phi'_N(2^j\xi - i)\phi'_N(2^j\xi - k)d\xi, \tag{26}$$

are the connection coefficients described by Ma et al. [14].

4. Numerical examples

To demonstrate the characteristics of the current elements and to assess its accuracy and

© VIBROENGINEERING. JOURNAL OF VIBROENGINEERING. DECEMBER 2011. VOLUME 13, ISSUE 4. ISSN 1392-8716

efficiency, two straight bar structures with uniform load, a fixed-fixed and a fixed-free, are considered respectively.

4.1. Fixed-fixed bar

Fig. 2 shows a fixed-fixed straight bar with uniform load. The rectangular cross-section area A, the length L, Young's modulus E, and uniform load $f(x)=1$ are the data of the problem.

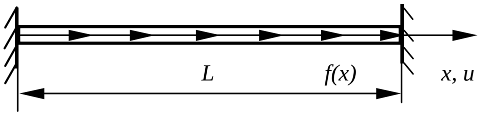


Fig. 2. A fixed-fixed straight bar

To calculate the displacement field of above problem, traditional FEM at least adopts two bar elements even if there is no point load or variable cross-section, while since WFE is placed with certain number internal nodes, one WFE is enough to deal with this problem. Here, D30, D31, D32, and D60 WFEs in Table 1 are used respectively. The condition numbers κ and $\hat{\kappa}$ for the transform matrix $\mathbf{R}$ and $\hat{\mathbf{R}}$ are given in Table 2. The results for longitudinal displacement are represented in Table 3. The results using analytical method and D60 WFE are shown in Fig. 3 respectively.

Table 2. Condition numbers κ and $\hat{\kappa}$ of the transform matrixes $\mathbf{R}$ and $\hat{\mathbf{R}}$.

Element name	κ	$\hat{\kappa}$
D30	5.7e2	3.0e1
D31	1.7e6	1.2e4
D32	2.5e5	1.4e3
D60	6.4e10	1.1e6

Table 3. The displacement of the center point of the bar (EA/L^2)

Element name	WFE	Exact	Error %
D30	0.124986561	0.125	0.011
D31	0.124990314		0.008
D32	0.124994215		0.005
D60	0.124996428		0.003

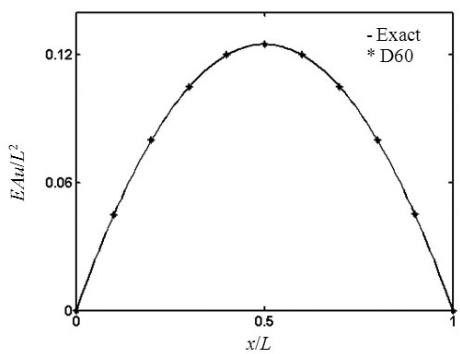


Fig. 3. The displacement of the fixed-fixed straight bar

4.2. Fixed-free bar

A fixed-free straight bar with uniform load is analyzed, the input parameters, except for boundary condition, are the same as the former example. The D30, D31, D32, and D60 WFEs are also used in this problem respectively. The results for longitudinal stress are presented in Table 4. The results of the displacement using analytical method and D60 WFE are shown in Fig. 4 respectively.

Table 4. The stress of the center point of the bar

Element name	WFE	Exact	Error %
D30	0.501004 E	0.5 E	0.201
D31	0.500845 E		0.169
D32	0.500537 E		0.107
D60	0.500435 E		0.009

From the experimental results and plots, the following assessments are made for the two boundary cases.

1. Considering the variable boundary conditions, WFE has the high accuracy for the solution of the displacement field and stress field. For Example 1, using D60 WFE the error for the center point displacement of the bar is 0.003%, while for Example 2, the stress error for the center point is only 0.009%. Of cause, taking the tradition bar element, the results at nodes are also enough accurate, however, the error for the internal point of the element is high due to the linear assumption made prior to the analysis. In the view of stress analysis, Table 4 illustrates the results of WFE are in good agreement with the exact solutions. That is because the internal strain of WFE is the weight sum of the derivative of scaling function, which is not constant but variable to the displacement. While the traditional bar element is characterized with constant strain. Thus, its accuracy in stress and strain analysis is lower than one in displacement analysis. In order to get desirable accuracy, the refined mesh is required. (See Tables 3, 4).
2. Table 2 illustrates that the diagonal preconditioner leads to well-conditioned matrix, which can contribute to numerical implementation. (See Table 2).

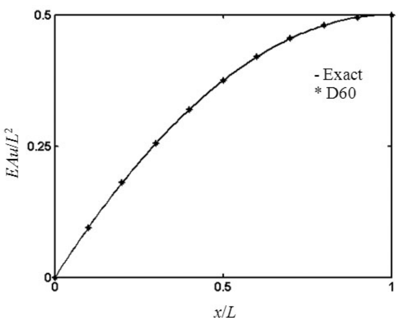


Fig. 4. The displacement of the fixed-free straight bar

The solution is consistently refined either by increasing the degree of refinement j, or the wavelet order N. For a given finite element mesh, the two alternative forms of *p*-refinement produce the same estimate for the displacement field and stress field (see Tables 3, 4).

5. Conclusions

A family of WFEs is developed by wavelet preconditioning technology. The Daubechies wavelets are used to approximate the displacement in the domain, unknown coefficients are determined through imposing the essential boundary condition. A family of shape functions based on the wavelet bases is then constructed, which is hierarchical due to multiresolution property of wavelet. To construct WFE, derivation of these shape functions for a subdomain is employed. Thus, the wavelet finite elements being presented are embodied with properties in locality and adaptivity. Numerical examples illustrate that the results of the displacement and stress are in good agreement with the analytical ones. It is believed that the current work not only extends the library of finite elements, also provides a powerful tool for modeling singularities in tension of crack, damage etc.

Acknowledgements

This work is supported by National Natural Science Foundation of China (No. 11176024).

References

[1] **B. Li, Z. J. He** "Frequency-based crack identification for static beam with rectangular cross-section", Journal of Vibroengineering, Vol. 13 (3), pp. 477-486, 2011.

[2] **J. N. Reddy** "An introduction to the finite element method", McGraw-Hill Inc., New York, 1984.

[3] **A. Cohen** "Numerical Analysis of Wavelet Methods", North-Holland Press, 2003.

[4] **W. Dahmen** "Wavelet methods for PDEs - some recent developments", Journal of Computational and Applied Mathmatics, Vol. 128, pp. 133-185, 2001.

[5] **S. Jaffard, P. Laurencot** "Orthonormal wavelets, analysis of operators and applications to numerical analysis", Wavelets: a Tutorial in Theory and Applications, Academic Press, New York, pp. 543-601, 1992.

[6] **K. Amaratunga, J. R. Williams, S. Qian, J. Weiss** "Wavelet-Galerkin solution for one dimensional partial differential equations", International Journal for Numerical Methods in Engineering, Vol. 37, pp. 2703-2716, 1994.

[7] **S. Dumont, F. Lebon** "Representation of plane elastostatics operators in Daubechies wavelets", Computers and Structures, Vol. 60, pp. 561-569, 1996.

[8] **S. Dumont, F. Lebon** "Wavelet-Galerkin method for periodic heterogeneous media", Computers and Structures, Vol. 61, pp. 55-65, 1996.

[9] **J. Ko, A. J. Kurdila, M. S. Pilant** "A class of finite element methods based on orthonormal compactly supported wavelets", Computational Mechanics, Vol. 16, pp. 235-244, 1995.

[10] **O. V. Vasilyev, C. Bowman** "Second-generation wavelet collocation method for the solution of partial differential equations", Journal of Computational Physics, Vol. 165, pp. 660-693, 2000.

[11] **W. H. Chen, C. W. Wu** "A spline wavelets element method for frame structures vibration", Computational Mechanics, Vol. 16, pp. 11-21, 1995.

[12] **C. W. Wu, W. H. Chen** "Extension of spline wavelets element method to membrane vibration analysis", Computational Mechanics, Vol. 18, pp. 46-54, 1996.

[13] **L. M. S. Castro, J. A. T. D. Freitas** "Wavelets in hybrid-mixed stress elements", Computer Methods in Applied Mechanics and Engineering, Vol. 190, pp. 3977-3998, 2001.

[14] **J. X. Ma, J. J. Xue, S. J. Yang, Z. J. He** "A study of the construction and application of a Daubechies wavelet-based beam element", Finite Elements in Analysis and Design, Vol. 39, pp. 965-975, 2003.

[15] **B. Li, X. F. Chen, Z. J. He, J. Zhuo** "Beam bending analysis using wavelet finite element", Proceedings of the International Conference on Wavelet Analysis and Its Applications, Vol. 1, pp. 448-453, 2003.

[16] **A. Latto, H. Resnikoff, E. Tenenbaum** "The evaluation of connection coefficients of compactly supported wavelets", Proc. French - U.S.A, Workshop on Wavelets and Turbulence, Princeton Univ., Springer, New York, 1992.

[17] **S. Mallet** "A Wavelet Tour of Signal Processing", 2nd Edition, Academic Press, San Diego, 1998.

[18] **I. Daubechies** "Orthonormal bases of compactly supported wavelets", Communications on Pure and Applied Mathematics, Vol. 41, pp. 909-996, 1998.