626. Higher-order approximation of cubic–quintic duffing model

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Abstract. We apply an Artificial Parameter Lindstedt-Poincaré Method (APL-PM) to find improved approximate solutions for strongly nonlinear Duffing oscillators with cubic–quintic nonlinear restoring force. This approach yields simple linear algebraic equations instead of nonlinear algebraic equations without analytical solution which makes it a unique solution. It is demonstrated that this method works very well for the whole range of parameters in the case of the cubic-quintic oscillator, and excellent agreement of the approximate frequencies with the exact one has been observed and discussed. Moreover, it is not limited to the small parameter such as in the classical perturbation method. Interestingly, This study revealed that the relative error percentage in the second-order approximate analytical period is less than 0.042% for the whole parameter values. In addition, we compared this analytical solution with the Newton–Harmonic Balancing Approach. Results indicate that this technique is very effective and convenient for solving conservative truly nonlinear oscillatory systems. Utter simplicity of the solution procedure confirms that this method can be easily extended to other kinds of nonlinear evolution equations.

Keywords: Artificial Parameter Lindstedt-Poincaré Method (APL-PM); Nonlinear Cubic-Quintic Oscillation, Duffing equation.

1. Introduction

Since most phenomena in our world are essentially nonlinear and are described by nonlinear equations, the study of nonlinear vibrations and oscillations is of crucial importance in all areas of engineering sciences. Therefore, the investigation of approximate solutions of nonlinear equations can play an important role in the study of nonlinear physical phenomena. Recently, many analytical and numerical methods have emerged for solving complicated nonlinear systems. Some of these problems which are related to the cubic–quintic Duffing equation include: the nonlinear dynamics of a beam on an elastic substrate [1], the generalized Pochhammer–Chree (PC) equations [2], and the compound Korteweg–de Vries (KdV) equation [3] in nonlinear wave systems and the propagation of a short electromagnetic pulse in a nonlinear medium [4]. More recently, many effective methods [5–55] have been presented to solve these complicated nonlinear oscillation systems including: Homotopy Perturbation [6–10], Parameter-Expanding (Expansion) [11], Multiple Scale [12–14], Harmonic Balance and

Duffing Harmonic Balance [15-18], Incremental Harmonic Balance [19, 20], Variational Iteration [21-22], Variational Approach [23–26], Max-Min [27, 28], Amplitude-Frequency Formulation [29], Linearized Perturbation [30], Energy Balance [25, 31–32], Power Series [33], homotopy analysis [34], Finite Element [35], Iteration Procedures [36, 37], Newton–Harmonic Balancing [38], Lindstedt– Poincaré [39, 40], Improved Lindstedt– Poincaré [41, 42], as well as other powerful methods which are available in the literature [43-52].

Ramos [53-55] proposed an Artificial Parameter Lindstedt–Poincaré Method (APL-PM), to obtain periodic solutions. Applications of this method can be found in [53-55] for solving nonlinear evolution equations arising in mathematical fields.

The main motivation of the present work is to extend the APL-PM to a generalized cubic–quintic Duffing with variable coefficients.

2. Artificial Parameter Lindstedt– Poincaré Method

Because of this fact that many important equations raised in practical engineering systems [1, 4] are in the form of Duffing equation, it seems to be more fundamental to consider equations presented in the following general form:

$$\frac{d^2u}{dt}+f(u)=0 \tag{1}$$

With initial conditions

$$u(0)=A, \quad \frac{du}{dt}(0)=0 \tag{2}$$

Where $f(u)$ is an odd function, and u and t are generalized dimensionless displacement and time variables. By defining a new independent variable replacing the time variable, $t=\theta/\omega$, Eq. (1) can be can be written as [53]:

$$\omega^2 u''+u=p\left[u-f(u)\right], \qquad u(0)=A, \quad u'(0)=0 \tag{3}$$

Eq. (3) coincides with Eq. (2) for $p=1$. Applying Artificial Parameter Lindstedt–Poincaré procedure, the displacement and angular frequency can be expressed as Eqs. (5) and (6), respectively:

$$u=u_0+pu_1+p^2u_2+\ldots, \tag{4}$$

$$\omega=\sqrt{\omega_0^2+p\omega_1^2+p^2\omega_2^2+\ldots}. \tag{5}$$

Substituting Eqs. (4) and (5) into Eq. (3) results in:

$$\begin{aligned}&\left(\omega_0^2+\sum_{i=1}^{+\infty}p^i\omega_i^2\right)\left(u_0''^2+\sum_{i=1}^{+\infty}p^iu_i''^2\right)+\left(u_0^2+\sum_{i=1}^{+\infty}p^iu_i^2\right)\\&=p\left[\left(u_0^2+\sum_{i=1}^{+\infty}p^iu_i^2\right)-f\left(u_0^2+\sum_{i=1}^{+\infty}p^iu_i^2\right)\right]\end{aligned} \tag{6}$$

Expanding Eqs. (6), gives:

$$p^0: \quad u_0'' + \omega_0^2 u_0 = 0\,, \tag{7}$$

$$p^1: \quad u_1'' + \omega_1^2 u_1 - K(u_0, u_0', u_0'', \omega_0, \omega_1) = 0\,, \tag{8}$$

$$p^2: \quad u_2'' + \omega_2^2 u_2 - G(u_i, u_i', u_i'', \omega_j) = 0\,, \quad i = 0,1 \quad , \quad j = 0,1,2 \,, \tag{9}$$

$\vdots$

Where $K(u_0, u_0', u_0'', \omega_j)$ and $G(u_i, u_i', u_i'', \omega_j)$ are linear differential term. These equations can be solved stage by stage. For the primary stage, the solution of Eq. (7) is:

$$u_0 = A\cos(\theta) \tag{10}$$

By substituting Eq. (10) into Eq. (8):

$$p^1: \quad u_1'' + \omega^2 u_1 - K(A\cos(\theta), -A\sin(\theta), -A\cos(\theta), \omega_0, \omega_1) = 0 \tag{11}$$

Using Fourier expansion series, we can rewrite the right hand of Eq. (11) in the following form:

$$K(A\cos(\theta), -A\sin(\theta), -A\cos(\theta), \omega_0, \omega_1)$$
$$= \sum_{n=0}^{\infty} b_{2n+1}\cos\left[(2n+1)\omega t\right] = \sum_{n=0}^{\infty} b_{2n+1}\cos\left[(2n+1)\omega t\right] \approx b_1\cos(\omega t) \tag{12}$$

By setting $b_1 = 0$ and solving it, we can achieve ω_0. The solution of Eq. (7) using ω_0 gives u_1. Then, solving Eq. (8) with ω_0 and u_1 yields ω_1. These stages can continue for better results.

3. Implementation of APLPM to cubic–quintic Duffing model

We governed the cubic–quintic Duffing model by a nonlinear differential equation with all real and positive coefficients. In this regard, the general form of cubic–quintic Duffing equation, Eq. (13), is considered.

$$\frac{du}{dt} + \alpha u + \beta u^3 + \gamma u^5 = 0, \qquad u(0) = A, \quad \frac{du}{dt}(0) = 0 \tag{13}$$

In order to use the APLPM, by applying new variable, $t = \theta/\omega$, we have:

$$\omega^2 u'' + u = p\left([1-\alpha]u - \beta u^3 - \gamma u^5\right) = 0, \qquad u(0) = A, \quad u'(0) = 0 \tag{14}$$

Where $u'' = du/d\theta$. Substituting Eqs. (5) and (6) into Eq. (14) and equating the terms with the identical powers of p, yields:

$$p^0: \quad u_0'' + u_0 = 0\,, \tag{15}$$

$$p^1: \quad u_1'' + u_1 = -\frac{\omega_1^2}{\omega_0^2}u_0'' - \frac{\omega_1^2}{\omega_0^2}u_0 - \frac{\gamma}{\omega_0^2}u_0^5 - \frac{\beta}{\omega_0^2}u_0^3 - \frac{\alpha}{\omega_0^2}u_0 + u_0\,, \tag{16}$$

$$p^2: \quad u_2'' + u_2$$
$$= -\frac{5\gamma u_0^4 u_1}{\omega_0^2} - \frac{\omega_1^2 u_0}{\omega_0^2} + u_1 - \frac{3\beta u_0^2 u_1}{\omega_0^2} - \frac{\omega_1^2 u_1''}{\omega_0^2} + \frac{\omega_1^2 u_0}{\omega_0^2} - \frac{\omega_2^2 u_0''}{\omega_0^2} - \frac{\omega_1^2 u_1}{\omega_0^2} - \frac{\alpha u_1}{\omega_0^2}, \tag{17}$$

⋮

3.1. First-order analytical approximation

At this step, we solve Eq. (15) with the initial values $u(0) = A$ and $u'(0) = 0$, which is leads to $u_0 = A\cos\theta$. Substituting u_0 into Eq. (16) and simplifying the result, we obtain:

$$u_1'' + u_1 = A\cos t - \frac{1}{\omega_0^2}\left(\alpha A\cos\theta + \beta A^3\cos^3\theta + \gamma A^5\cos^5\theta\right) \tag{18}$$

It is possible to apply the following Fourier series expansion:

$$u_1'' + u_1 = \left(A + b_1\right)\cos\theta + \sum_{n=1}^{\infty} b_{2n+1}\cos\left[\left(2n+1\right)\theta\right] \tag{19}$$

where b_1 is as follow:

$$b_1 = \frac{4}{\pi\omega_0^2}\cdot\int_0^{\pi/2}\left(\cos\varphi\left[\alpha A\cos\varphi + \beta A^3\cos^3\varphi + \gamma A^5\cos^5\varphi\right]\right)d\varphi$$
$$= -\frac{A\left(18A^2\beta + 24\alpha + 15A^4\gamma\right)}{24\omega_0^2} \tag{20}$$

Substituting Eq. (20) into Eq. (19) gives:

$$u_1'' + u_1 = \left(\frac{24\omega_0^2 A - A\left(18A^2\beta + 24\alpha + 15A^4\gamma\right)}{24\omega_0^2}\right)\cos\theta$$
$$+\sum_{n=1}^{\infty} b_{2n+1}\cos\left[\left(2n+1\right)\theta\right] \tag{21}$$

No secular term in u_1 requires that:

$$\frac{24\omega_0^2 A - A\left(18A^2\beta + 24\alpha + 15A^4\gamma\right)}{24\omega_0^2} = 0 \tag{22}$$

Solving Eq. (22), we obtain the first order approximate solution of Eq. (1) as follow:

$$\omega_{1th} = \omega_0 = \sqrt{\alpha + \frac{3\beta A^2}{4} + \frac{5\gamma A^4}{8}}, \tag{23}$$

Where angular frequency ω_0 is the first-order analytical approximation. Eq. (23) gives the same frequency resulted in by the applications of the harmonic balance and the first order approximation of Newton–harmonic balancing approach [38]. Therefore, the corresponding approximate analytical periodic solution u_1 can then be achieved by substituting Eq. (23) into Eq. (16) as:

$$u_1 = -\frac{A^3 \sin^2\theta \cos\theta \left(2\gamma A^2 \cos^2\theta + 6\beta + 7\gamma A^2\right)}{6\left(8\alpha + 6\beta A^2 - 5\gamma A^4\right)} \tag{24}$$

3.1. Second-order analytical approximation

To determine the second-order approximate solution, it is necessary to substitute $u_0 = A\cos\theta$ and Eqs. (23) and (24) into Eq. (17). So, we can obtain:

$$\begin{aligned} & u_2'' + u_2 = \frac{A\psi\cos\theta}{6\left(6A^2\beta + 8\alpha + 5A^4\gamma\right)^2}, \\ & \psi = -80\gamma^2 A^8 \cos^8\theta - 200\gamma^2 A^8 \cos^6\theta - 288\gamma\beta A^6 \cos^6\theta + 290\gamma^2 A^8 \cos^4\theta + \\ & \quad 132\gamma\beta A^6 \cos^4\theta + 384\gamma\omega_1^2 A^4 \cos^4\theta - 144\beta^2 A^4 \cos^4\theta + 25\gamma^2 A^8 \cos^2\theta + \\ & \quad 228\gamma\beta A^6 \cos^2\theta + 180\beta^2 A^4 \cos^2\theta + 384\beta\omega_1^2 A^2 \cos^2\theta - 35\gamma^2 A^8 - 72\gamma\beta A^6 + \\ & \quad -36A^4\beta^2 + 384\alpha\omega_1^2 \end{aligned} \tag{25}$$

Similar to the first step, using the Fourier series, the right hand of Eq. (25) will become as:

$$\begin{aligned} & A\psi\cos\theta = \sum_{n=0}^{\infty} b_{2n+1} \cos\left[(2n+1)\theta\right] = \left(\frac{4}{\pi}\int_0^{\pi/2} \left(\psi\cos^2\varphi\right) d\varphi\right)\cos\theta \\ & + \sum_{n=1}^{\infty} b_{2n+1} \cos\left[(2n+1)\theta\right] \\ & = \left[0.25A\begin{pmatrix} 65A^8\gamma^2 + 36A^4\beta^2 + 1536\alpha\omega_1^2 \\ +96A^6\beta\gamma + 960A^4\gamma\omega_1^2 + 1152\omega_1^2 A^2\beta \end{pmatrix}\right]\cos(\theta) \\ & + \sum_{n=1}^{\infty} b_{2n+1} \cos\left[(2n+1)\theta\right] \end{aligned} \tag{26}$$

From Eqs. (25) and (26) we can obtain the secular term as:

$$\left[0.25A\begin{pmatrix} 65A^8\gamma^2 + 36A^4\beta^2 + 1536\alpha\omega_1^2 \\ +96A^6\beta\gamma + 960A^4\gamma\omega_1^2 + 1152\omega_1^2 A^2\beta \end{pmatrix}\right] \tag{27}$$

No secular term in u_2 requires that:

$$\omega_1 = \frac{A^2\sqrt{-\left(18A^2\beta - 24\alpha + 15\gamma A^4\right)\left(65\gamma^2 A^4 + 96A^2\beta\gamma + 36\beta^2\right)}}{24\left(6A^2\beta + 8\alpha + 5A^4\gamma\right)^2} \tag{28}$$

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From Eqs. (23 and 28) into Eq. (5), we can obtain the angular frequency ω_{2th} as a more accurate and higher-order approximation compared to ω_0 :

$$\omega_{2th} = \sqrt{\omega_0^2 + \omega_1^2} \tag{29}$$

4. Illustrative examples of cubic–quintic Duffing oscillators

In order to verify the effectiveness of the proposed higher-order analytical approximate method, the exact solutions and another approximate solution, namely, Newton–harmonic balancing approach is used to compute the angular frequencies of the cubic–quintic oscillator for different parameters. The various approximate solutions are presented in Figs. 1–4. Tables 1–4 list the corresponding numerical results of angular frequencies. These figures and tables correspond to small and large amplitudes of oscillation for different parameters of α, β and γ. For reference, the exact frequency ω_e is obtained by direct integration of governing nonlinear differential Eq. (1) of the dynamical system. Imposing the initial conditions, the solution is [38]:

$$\begin{aligned}
\omega_e(A) &= \frac{\pi k_1}{2\int_0^{\pi/2}\left(1+k_2\sin^2 t+k_3\sin^4 t\right)^{-1/2} dt}, \\
k_1 &= \sqrt{\alpha + \frac{\beta A^2}{2} + \frac{\gamma A^4}{3}}, \\
k_2 &= \frac{3\beta A^2 + 2\gamma A^4}{6\alpha + 3\beta A^2 + 2\gamma A^4}, \\
k_3 &= \frac{2\gamma A^4}{6\alpha + 3\beta A^2 + 2\gamma A^4}.
\end{aligned} \tag{30}$$

The approach presented herein is suitable for small as well as large amplitudes of oscillation. The various limiting approximations for $A \to \infty$ can be derived based on Eqs. (23) and (29) as follows:

$$\lim_{A\to\infty} \frac{\omega_{1th}(A)}{\omega_e(A)} = \lim_{A\to\infty} \frac{T_{1th}(A)}{T_e(A)} = 1.05856 \tag{31}$$

$$\lim_{A\to\infty} \frac{\omega_{2th}(A)}{\omega_e(A)} = \lim_{A\to\infty} \frac{T_{2th}(A)}{T_e(A)} = 0.99958 \tag{32}$$

From Eqs. (31) and (32), it is obvious that the relative errors of the first-order and second-order approximations of APL-PM as compared to the exact solution are less than 5.86 % and 0.042 %, respectively. It is noted that the maximum errors of the third-order approximation of Newton–harmonic balancing approach to the exact solution is 0.23. As it can be seen through Tables 1-4, the first-order solutions of two approximate methods are equal and relatively inaccurate. Although the third-order solution of NHBA is good, the second-order solution of APL-PM is excellent.

The frequency ratio and amplitude relationship of the cubic–quintic Duffing oscillator governed by $u''+u+u^3+u^5=0$ is shown in Fig. 1. As observed, the second-order approximation of the proposed method has satisfactory agreement with the exact solution for $A\in[0.1,1000]$. Figs. (2-4) are also related to, $u''+2u+u^3+u^5=0$, $u''+5u+3u^3+u^5=0$ and $u''+u+10u^3+100u^5=0$, respectively.

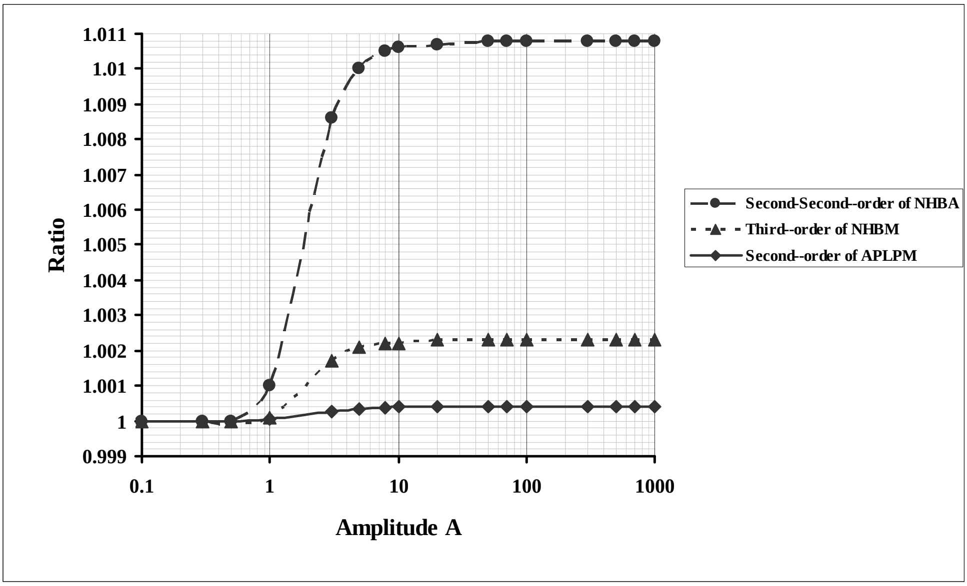

Fig. 1. Comparison of approximate frequencies with the exact frequency for cubic–quintic Duffing oscillator for $\alpha=\beta=\gamma=1$

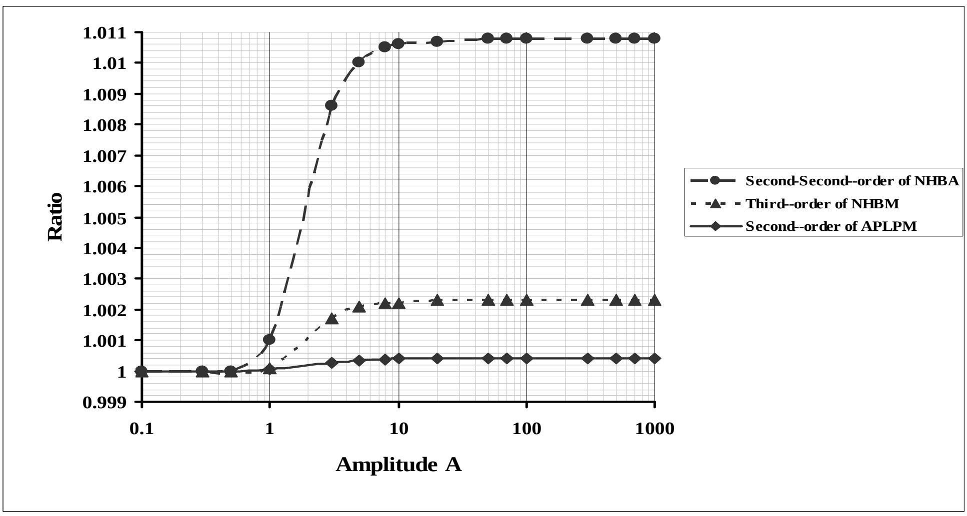

Fig. 2. Comparison of approximate frequencies with exact frequency for the cubic–quintic Duffing oscillator for $\alpha=2$ and $\beta=\gamma=1$

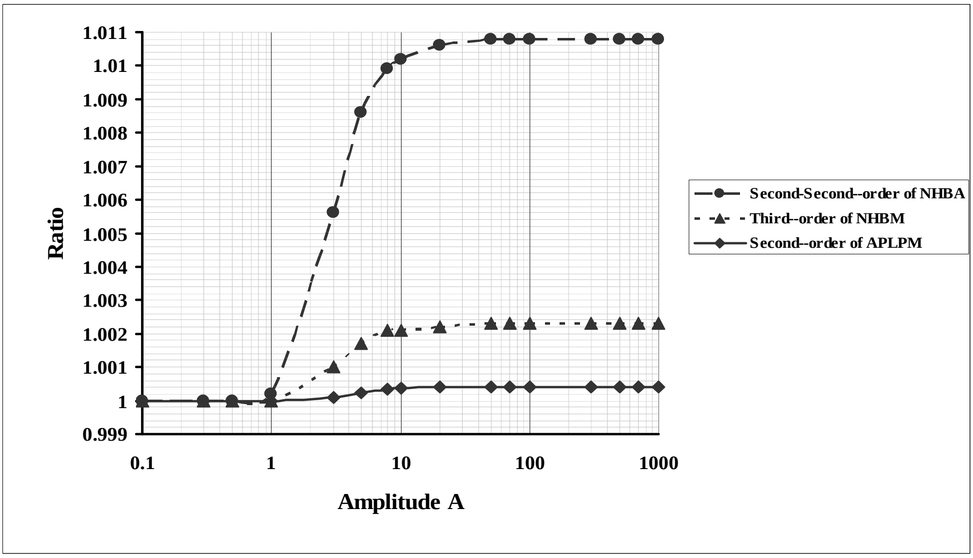

Fig. 3. Comparison of approximate frequencies with the exact frequency for cubic–quintic Duffing oscillator for $\alpha = 5,\ \beta = 3$ and $\gamma = 1$

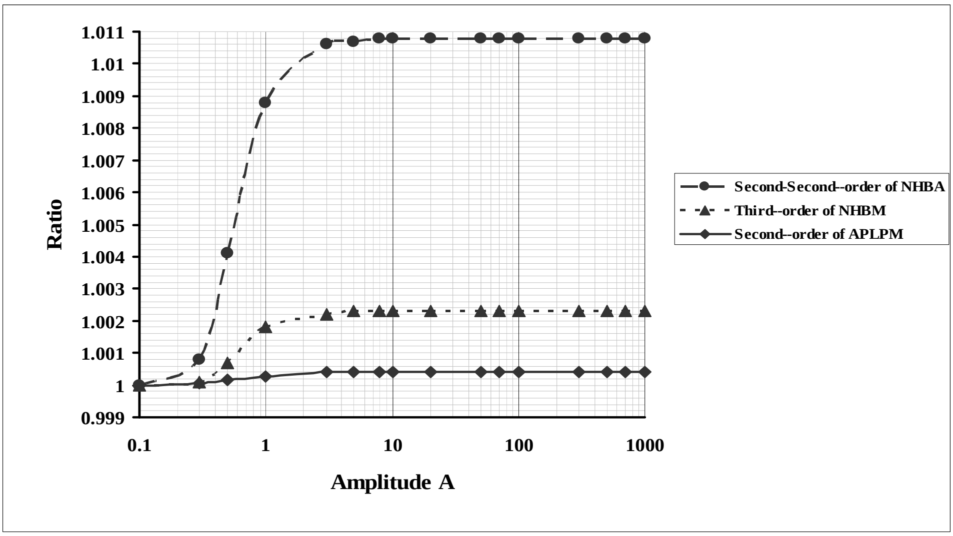

Fig. 4. Comparison of approximate frequencies with the exact frequency for cubic–quintic Duffing oscillator for $\alpha = 1,\ \beta = 10$ and $\gamma = 100$

The proposed APL-PM for the second-order analytical approximation demonstrates noticeable improvement as compared with the lower-order analytical approximation. It is also observed that the method is effective for solving highly nonlinear oscillators with cubic and/or quintic nonlinearity and it has clear advantage over the classical perturbation method, which is restricted by the presence of a small parameter in the governing differential equation.

Table 1. Percentage of errors for comparison of approximate frequencies with exact frequency for $\alpha = \beta = \gamma = 1$

		Newton–Harmonic Balancing Approach			Artificial Parameter Lindstedt– Poincaré	
A	ω_e	ω_{1th}	ω_{2th}	ω_{3th}	ω_{1th}	ω_{2th}
0.1	1.00377	1.00377 (0.00)	1.00377 (0)	1.00377 (0)	1.00377 (0.00)	1.00377 (0.00)
0.3	1.03554	1.03565 (0.01)	1.03554 (0)	1.03554 (0)	1.03565 (0.01)	1.03554 (0.00)
0.5	1.10654	1.10750 (0.09)	1.10658 (0)	1.10655 (0)	1.10750 (0.09)	1.10654 (0.00)
1	1.52359	1.54110 (1.15)	1.52507 (0. 10)	1.52375 (0. 01)	1.54110 (1.15)	1.52348 (0. 007)
3	7.26863	7.64035 (5.11)	7.33114 (0. 86)	7.28115 (0. 17)	7.64035 (5.11)	7.26675 (0. 026)
5	19.1815	20.2577 (5.61)	19.3735 (1.00)	19.2215 (0.21)	20.2577 (5.61)	20.2577 (0. 034)
8	48.2946	51.0784 (5.76)	48.8010 (1.05)	48.4011 (0.22)	51.0784 (5.76)	51.0784 (0. 039)
10	75.1774	79.5362 (5.80)	75.9738 (1.06)	75.3454 (0.22)	79.5362 (5.80)	75.1474 (0.040)
20	299.223	316.703 (5.84)	302.435 (1.07)	299.903 (0.23)	316.703 (5.84)	299.099 (0.041)
50	1867.57	1976.90 (5.85)	1887.69 (1.08)	1871.84 (0.23)	1976.90 (5.85)	1866.79 (0.042)
70	3659.98	3874.26 (5.86)	3699.42 (1.08)	3668.33 (0.23)	3874.26 (5.86)	3658.44 (0.042)
100	7468.83	7906.17 (5.86)	7549.34 (1.08)	7485.89 (0.23)	7906.17 (5.86)	7465.69 (0.042)
300	67215.57	71151.72 (5.86)	67940.22 (1.08)	67369.12 (0.23)	71151.72 (5.86)	67187.26 (0.042)
500	186709.04	197642.83 (5.86)	188721.99 (1.08)	187135.59 (0.23)	197642.83 (5.86)	186630.42 (0.042)
700	365949.25	387379.49 (5.86)	369894.64 (1.08)	366785.29 (0.23)	387379.49 (5.86)	365795.14 (0.042)
1000	746834.69	790569.89 (5.86)	754886.52 (1.08)	748540.91 (0.23)	790569.89 (5.86)	746520.19 (0.042)

Table 2. Percentage of errors for comparison of approximate frequencies with exact frequency for $\alpha = 2$ and $\beta = \gamma = 1$

		Newton–Harmonic Balancing Approach			Artificial Parameter Lindstedt– Poincaré	
A	ω_e	ω_{1th}	ω_{2th}	ω_{3th}	ω_{1th}	ω_{2th}
0.1	1.41688	1.41688 (0.00)	1.41688 (0.00)	1.41688 (0.00)	1.41688 (0.00)	1.41688 (0.00)
0.3	1.43960	1.43964 (0.003)	1.43960 (0.00)	1.43960 (0.00)	1.43964 (0.003)	1.43960 (0.00)
0.5	1.49177	1.49217 (0.03)	1.49179 (0.00)	1.49178 (0.00)	1.49217 (0.03)	1.49177 (0.00)
1	1.82682	1.83712 (0.56)	1.82763 (0. 04)	1.82689 (0.004)	1.83712 (0.56)	1.82674 (0.004)
3	7.34386	7.70552 (4.92)	7.40403 (0. 82)	7.35578 (0. 16)	7.70552 (4.92)	7.34163 (0.030)
5	19.2104	20.2824 (5.58)	19.4014 (0.99)	19.2501 (0.21)	20.2824 (5.58)	19.2037 (0.039)
8	48.3061	51.0882 (5.76)	48.8120 (1.05)	48.4125 (0.22)	51.0882 (5.76)	48.2873 (0.040)
10	75.1848	79.5424 (5.80)	75.9809 (1.06)	75.3527 (0.22)	79.5424 (5.80)	75.1548 (0.040)
20	299.225	316.705 (5.84)	302.437 (1.07)	299.905 (0.23)	316.705 (5.84)	299.101 (0.041)
50	1867.57	1976.90 (5.85)	1887.70 (1.08)	1871.84 (0.23)	1976.90 (5.85)	1866.79 (0.042)
70	3659.98	3874.26 (5.85)	3699.42 (1.08)	3668.33 (0.23)	3874.26 (5.85)	3658.44 (0.042)
100	7468.83	7906.17 (5.86)	7549.34 (1.08)	7485.89 (0.23)	7906.17 (5.86)	7465.69 (0.042)
300	67215.57	71151.72 (5.86)	67940.22 (1.08)	67369.12 (0.23)	71151.72 (5.86)	67187.26 (0.042)
500	186709.04	197642.83 (5.86)	188721.99 (1.08)	187135.59 (0.23)	197642.83 (5.86)	186630.42 (0.042)
700	365949.25	387379.49 (5.86)	369894.64 (1.08)	366785.29 (0.23)	387379.49 (5.86)	365795.14 (0.042)
1000	746834.69	790569.89 (5.86)	754886.52 (1.08)	748540.91 (0.23)	790569.89 (5.86)	746520.19 (0.042)

5. Conclusion

The Artificial Parameter Lindstedt–Poincaré method has been applied to obtain periodic solution for truly cubic-quintic nonlinear oscillator. The major conclusion is that this approach provides excellent approximation to the solution of these nonlinear systems with high accuracy for the whole solution domain. The analytical representations obtained using the Artificial Parameter Lindstedt–Poincaré technique give excellent approximations to the exact solutions for the whole range of amplitude values.

These approximate solutions are better than the approximate solutions obtained using the Newton–harmonic balancing approach. For the second order approximation, the maximum relative error of the analytical approximate frequency obtained using the APL–PM for the cubic-quintic oscillator is 0.04%, while the relative error is 1.08% when the second order approximation is considered for the Newton–harmonic balancing approach. An interesting feature considered in this paper is to show that the second order approximation result of APL–PM is better than the third order approximation of Newton–harmonic balancing approach, which constitutes 0.23%.

Table 3. Percentage of errors for comparison of approximate frequencies with exact frequency for $\alpha = 5,\ \beta = 3$ and $\gamma = 1$

		Newton–Harmonic Balancing Approach			Artificial Parameter Lindstedt–Poincaré	
A	ω_e	ω_{1th}	ω_{2th}	ω_{3th}	ω_{1th}	ω_{2th}
0.1	2.24111	2.24111 (0.00)	2.24111 (0.00)	2.24111 (0.00)	2.24111 (0.00)	2.24111 (0.00)
0.3	2.28193	2.28201 (0.004)	2.28193 (0.00)	2.28193 (0.00)	2.28201 (0.004)	2.28193 (0.00)
0.5	2.36615	2.36676 (0.03)	2.36616 (0.00)	2.36615 (0.00)	2.36676 (0.03)	2.36615 (0.00)
1	2.79627	2.80624 (0.36)	2.79670 (0. 02)	2.79630 (0.00)	2.80624 (0.36)	2.79625 (0.00)
3	8.37877	8.71063 (3.96)	8.42601 (0. 56)	8.38730 (0. 10)	8.71063 (3.96)	8.37789 (0. 01)
5	20.2164	21.2574 (5.15)	20.3911 (0.86)	20.2514 (0.17)	21.2574 (5.15)	20.2118 (0.023)
8	49.2955	52.0481 (5.58)	49.7844 (0.99)	49.3969 (0.21)	52.0481 (5.58)	49.2794 (0.033)
10	76.1698	80.4984 (5.68)	76.9487 (1.02)	76.3326 (0.21)	80.4984 (5.68)	76.1425 (0.036)
20	300.204	317.655 (5.81)	303.399 (1.06)	300.879 (0.22)	317.655 (5.81)	300.083 (0.040)
50	1868.55	1977.85 (5.85)	1888.65 (1.08)	1872.81 (0.23)	1977.85 (5.85)	1867.77 (0.042)
70	3660.95	3875.21 (5.85)	3700.38 (1.08)	3669.31 (0.23)	3875.21 (5.85)	3659.42 (0.042)
100	7469.81	7907.12 (5.85)	7550.30 (1.08)	7486.86 (0.23)	7907.12 (5.85)	7466.66 (0.042)
300	67216.54	71152.67 (5.86)	67941.18 (1.08)	67370.09 (0.23)	71152.67 (5.86)	67188.24 (0.042)
500	186710.01	197643.78 (5.86)	188722.95 (1.08)	187136.56 (0.23)	197643.78 (5.86)	186631.40 (0.042)
700	365950.22	387380.44 (5.86)	369895.60 (1.08)	366786.26 (0.23)	387380.44 (5.86)	365796.12 (0.042)
1000	746835.66	790570.84 (5.86)	754887.48 (1.08)	748541.88 (0.23)	790570.84 (5.86)	746521.17 (0.042)

Table 4. Percentage of errors for comparison of approximate frequencies with exact frequency for $\alpha = 1,\ \beta = 10$ and $\gamma = 100$

		Newton–Harmonic Balancing Approach			Artificial Parameter Lindstedt–Poincaré	
A	ω_e	ω_{1th}	ω_{2th}	ω_{3th}	ω_{1th}	ω_{2th}
0.1	1.03970	1.03983 (0.01)	1.03970 (0.00)	1.03970 (0.00)	1.03983 (0.01)	1.03970 (0.00)
0.3	1.46259	1.47691 (0.98)	1.46373 (0.08)	1.46271 (0.01)	1.47691 (0.98)	1.46251 (0.006)
0.5	2.52469	2.60408 (3.14)	2.53505 (0.41)	2.52642 (0.07)	2.60408 (3.14)	2.52426 (0.017)
1	8.01005	8.42615 (5.19)	8.08069 (0. 88)	8.02429 (0.18)	8.42615 (5.19)	8.00790 (0.027)
3	67.7097	71.6310 (5.79)	68.4255 (1.06)	67.8606 (0. 22)	71.6310 (5.79)	67.6828 (0. 040)
5	187.199	198.119 (5.83)	189.203 (1.07)	187.623 (0. 23)	198.119 (5.83)	187.122 (0. 041)
8	478.463	506.440 (5.85)	483.608 (1.08)	479.552 (0. 23)	506.440 (5.85)	478.263 (0. 042)
10	747.323	791.044 (5.85)	755.366 (1.08)	749.027 (0. 23)	791.044 (5.85)	747.010 (0. 042)
20	2987.83	3162.75 (5.85)	3020.02 (1.08)	2994.65 (0. 23)	3162.75 (5.85)	2986.57 (0. 042)
50	18671.34	19764.71 (5. 86)	18872.63 (1.08)	18714.00 (0.23)	19764.71 (5. 86)	18663.48 (0.042)
70	36595.36	38738.38 (5. 86)	36989.90 (1.08)	36678.97 (0.23)	38738.38 (5. 86)	36579.95 (0.042)
100	74683.91	79057.42 (5. 86)	75489.08 (1.08)	74854.53 (0.23)	79057.42 (5. 86)	74652.46 (0.042)
300	672151.27	711512.95 (5.86)	679397.92 (1.08)	673686.86 (0.23)	711512.95 (5.86)	671868.22 (0.042)
500	1867085.99	1976424.01 (5.86)	1887215.59 (1.08)	1871351.54 (0.23)	1976424.01 (5.86)	1866299.75 (0.042)
700	3659488.07	3873790.61 (5.86)	3698942.10 (1.08)	3667848.54 (0.23)	3873790.61 (5.86)	3657947.03 (0.042)
1000	7468342.49	7905694.62 (5.86)	7548860.93 (1.08)	7485404.69 (0.23)	7905694.62 (5.86)	7465197.52 (0.042)

In general, the first-order periodic solution of APL–PM is generally acceptable as compared to the exact solution while the second-order periodic solution is in good and excellent agreement with the exact solution. In summary, proposed method is simple in its principle, and can be used to solve other conservative truly nonlinear oscillators with complex nonlinearities.

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