415. On evolution of libration points similar to Eulerian in the model problem of the binary-asteroids dynamics

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Abstract. The binary asteroids are of current interest in the modern dynamics as there have been up to 50 discoveries of binaries. Estimates are that about 20% of near-Earth asteroids may be binary asteroids. Nevertheless the known asteroids pairs are rather rare objects in the Solar System.
There are a number of papers studying the various aspects of asteroid pair dynamics. In this paper we study some stationary motions in the system of binary asteroid. Using the model for the first time suggested in [1], we approximate the bigger asteroid by the dumbbell-shaped rigid body. Moreover we assume that the smaller asteroid has mass close to zero.
It was shown in [1] that the motion equations for the considered system have the stationary solutions corresponding to the smaller asteroid's equilibria relative to the axis of the regular precession and the dumbbell rod These equilibria similar to the libration points in the Restricted Circular Problem of Three Bodies (RCP3B). There are two types of such equilibria. The equal distances from the dumbbell endpoints characterize the equilibria of the first type, therefore we shall name them 'the libration points similar to Lagrangian' or 'the triangular libration points' (TLPs) by analogy to a classical problem. At difference with RCP3B there are two or one TLPs or they do not exist.
Equilibria of the second type in something are similar to the Eulerian libration points. They belong to the plane containing the bigger asteroid's angular momentum and the dumbbell rod. Moreover, these equilibria belong to the strip bounded by straight lines crossing the dumbbell endpoints and being perpendicular to angular momentum. Therefore we shall name them 'the coplanar libration points'(CLP). The CLPs coordinates are computed by following procedure. Two algebraic equations are deduced. One of these equations determines the curve containing all CLPs. We say that this curve is 'the Geometrical locus of CLPs' (GL). The second equation allows to locate CLPs in GL. Studying evolution of CLPs in GL it can be proved that the number of CLPs varies from 3 up to 7, but if the dumbbell consist of equal spheres then only the odd number of CLPs is possible. However, if the dumbbell is asymmetric then the number of CLPs can be equal 4 or 6 for some rare situations.

Key words: Three Bodies Problem, Binary asteroid, Eulerian libration point, Lagrangian libration point.

Introduction.

The relative motion in the binary-asteroid system is one of the most interesting topics in astrodynamics. Discover more than fifty asteroids pairs and the estimation that up to 20% of asteroids are binaries explain such interest [2,3].

It is necessary to note some common features of the binary-asteroid system. First, the external influences (in particular the Sun gravitation) upon the relative motions of the pair's components are rather small. Secondly, the mass of one asteroid in the pair is small in comparison with mass of another. Thirdly, the bigger asteroid in the majority of known binaries has a very complicated shape. Note, however, that lots of asteroids are quasi-axial-symmetric ((243) Ida, (762) Pulcova, etc.) or dumbbell-shaped ((216) Kleopatra, (4769) Castalia, (433) Eros, etc..

The motion of a particle or a spacecraft about asteroid has been studied in many different papers. ([4-14], etc. Certainly, this list is not full). In these papers different models of an asteroid pair dynamics have been investigated. In this paper we consider another model based on the following simple assumptions [1]. We neglect the external influences upon relative motion of the pair. We consider the smaller asteroid as a particle with mass close to zero that moves under gravitation of the bigger asteroid. We approximate the bigger asteroid as a dumbbell-shaped body, i.e. as two massive spheres joint by the weightless rod. Here we follow the papers [15], where the high-precision approximation of the Earth gravitational potential was constructed by using the integrable problem of two immovable centers. But in the considered case the attracting centers are moving.

From the assumptions made it follows, that the motion of the dumbbell-shaped asteroid is the regular precession. The equations of a smaller asteroid's motion depend on three parameters. Two parameters define the precession of the bigger asteroid. The third parameter characterizes the difference between masses of the spheres on the ends of the dumbbell. These equations are reduced to the Restricted Circular Three-Body Problem (RCP3B) at a right angle of nutation and one special value for the precession angular velocity. Thus two-parametrical generalization of RCP3B takes place. Note also that the same equations describe the spacecraft rotation around a single asteroid.

The formulated generalized problem has stationary solutions corresponding to the smaller asteroid's equilibria relative to axises of the bigger asteroid's precession and nutation. There exist two types of such equilibria.

The equal distances from the dumbbell endpoints characterize the points of the first type, therefore we will name them 'the libration points similar to Lagrangian' or 'the triangular libration points' (TLP) by analogy to the classical problem. The coordinates of TLP were defined by explicit formulae having a simple geometrical interpretation. These points belong to some circle. The plane of the circle is perpendicular to the angular momentum vector. The TLP belong to the plane crossing the dumbbell in its midpoint and being perpendicular to the rod. Clearly, there are two or one TLP or they don't exist. Unlike a classical problem the triangles formed by such points and the dumbbell will be not equilateral and not coplanar but only isosceles in general.

Equilibria of the second type in something are similar to the Eulerian libration points. They belong to the plane containing the bigger asteroid's angular momentum and the dumbbell rod. Moreover, these equilibria belong to the strip bounded by straight lines crossing the dumbbell endpoints and being perpendicular to angular momentum. Therefore we will name them 'the coplanar libration points'(CLP). Two algebraic equations determine the CLPs coordinates. One of these equations is quadratic and does not depend on the regular precession's angular velocity. Roots of this equation define two branches of the curve that can be called 'the Geometrical locus of CLPs' (GL). Substituting these roots in the second equation we get two new equations that allow locating CLPs in GL. These equations are solved numerically. Varying the precession's angular velocity, we see that the number of CLPs is not less than 3 and is not more than 7. It is easy to prove that if the dumbbell consist of spheres having equal mass, i.e. the dumbbell is symmetric, then only the odd number of CLPs is possible. However, if the dumbbell is asymmetric then the number of CLPs can be equal 4 or 6. The last situation is rather rare. In particular, it takes place if two of CLPs merge.

Formulation of the problem.

Suppose an asteroid consisting of two homogeneous spheres with masses m_1 and m_2 rotates about its angular momentum vector $\mathbf{L}=\mathbf{L}_0$ (Fig.1). Without loss of a generality, $m_2 \le m_1$. Let ϑ be the angle of nutation, ω be the precession angular velocity. (From our assumptions it follows that $\vartheta = const$, $\omega = const$). By O denote the mass center of the asteroid. Let also l_1 and l_2 be the distances from O to the centers of the spheres.

Suppose $Oxyz$ is the right-handed coordinate system with the origin O such that Oz and $\mathbf{L}$ are equally directed, the axis of the dumbbell belong to the plane Oxz and the first coordinate of the center of the smaller sphere is positive. $Oxyz$ rotates about $\mathbf{L}$ with the angular velocity ω.

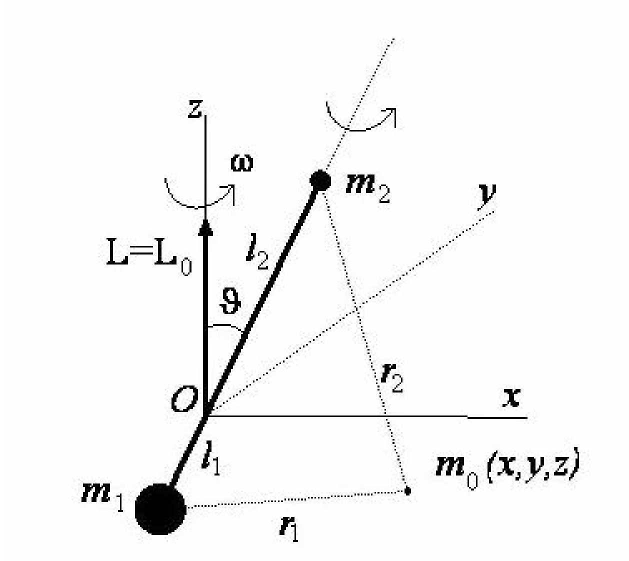

Fig. 1.

Let $m_0(x, y, z)$ be a particle of the mass close to zero. The distances r_1 and r_2 from m_0 to the dumbbell endpoints are defined by formulae

$$r_1 = \left[\left(x + \frac{m_2}{m_1+m_2} l \sin\vartheta\right)^2 + y^2 + \left(z + \frac{m_2}{m_1+m_2} l \cos\vartheta\right)^2\right]^{1/2} \quad (1)$$

$$r_2 = \left[\left(x - \frac{m_1}{m_1+m_2} l \sin\vartheta\right)^2 + y^2 + \left(z - \frac{m_1}{m_1+m_2} l \cos\vartheta\right)^2\right]^{1/2}$$

Equation of motion for the particle m_0 can be written as

$$W = \frac{U}{m_0},\ U = Gm_0\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right),\ [G] = g^{-1}cm^3c^{-2}$$

$$\begin{cases} \bar{x} - 2\omega\dot{y} - \omega^2 x = \dfrac{\partial W}{\partial x}, \\ \bar{y} + 2\omega\dot{x} - \omega^2 y = \dfrac{\partial W}{\partial y}, \\ \bar{z} = \dfrac{\partial W}{\partial z}, \end{cases}$$

where G is the Gravity constant.

For these equations there is Jacobi integral expressed as $\frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{1}{2}\left(x^2 + y^2\right) - \left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right) = h$.

Suppose the dimensionless distance between the gravitating spheres equal to 1. Using the dimensionless variables

$x = l\xi$, $y = l\eta$, $z = l\zeta$, $\omega dt = d\tau$, $l = l_1 + l_2$.

We can rewrite the motion equations as

$$\begin{cases} \xi'' - 2\eta' - \xi = \dfrac{\partial \tilde{W}}{\partial \xi} \\ \eta'' - 2\xi' - \eta = \dfrac{\partial \tilde{W}}{\partial \eta} \\ \zeta'' = \dfrac{\partial \tilde{W}}{\partial \zeta} \end{cases}, \quad (2)$$

where the prime $(\ )'$ means a derivative w.r.t. dimensionless time τ,

$$\tilde{W} = \alpha\left[\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}\right]; \ \rho_1 = \frac{r_1}{l}, \ \rho_2 = \frac{r_2}{l}, \ (3)$$

Parameters μ and α are defined by the formulae

$$\mu = \frac{m_2}{m_1+m_2}, \left(1-\mu = \frac{m_1}{m_1+m_2}\right),$$

$$\alpha = \frac{G(m_1+m_2)}{\omega^2 l^3} \neq 1$$

Substituting (1) and (3) in (2) we obtain

$$\xi'' - 2y' - \xi =$$
$$= \alpha\mu(1-\mu)\sin\vartheta \cdot \left(\frac{1}{\rho_2^3} - \frac{1}{\rho_1^3}\right) - \alpha\mu\left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}\right);$$
$$\eta'' + 2\xi' - \eta = -\alpha\eta\left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}\right);$$
$$\zeta'' =$$
$$= \alpha\mu(1-\mu)\cos\vartheta \cdot \left(\frac{1}{\rho_2^3} - \frac{1}{\rho_1^3}\right) - \alpha\mu\left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}\right);$$
$$\rho_1 = \left[(\xi + \mu\sin\vartheta)^2 + \eta^2 + (\zeta + \mu\cos\vartheta)^2\right]^{1/2}$$
$$\rho_1 = \left[(\xi - (1-\mu)\sin\vartheta)^2 + \right.$$
$$\left. + \eta^2 + (\zeta - (1-\mu)\cos\vartheta)^2 \ \right]^{1/2}$$

If α = 1 and $\vartheta = \pi/2$ then the equations (4) are reduced to the equations of motion for the RCP3B. Consequently, ***two-parametric generalization for the classic problem takes place***, namely $\alpha \neq 1$and $\vartheta \neq 1$.

Note that

1. Clearly, $0 \le \vartheta \le \pi$. Hence, inequality $\sin\vartheta \ge 0$ holds.
2. Obviously, $0 \le \mu \le 1/2$ because of $m_2 \le m_1$. $\mu = 1/2$ if the dumbbell is symmetric, i.e. $m_1 = m_2$.

It is well-known that there exist five stationary motions of the RCP3B for which the distances between the bodies do not vary [16, 17]. The corresponding points of relative equilibria for the smallest particle are called the libration points. There are two Lagrangian libration points and three Eulerian libration points. The formulated generalized problem has the similar points for the relative equilibria of the smaller asteroid in the rotating frame of references *Oxyz.*

The equations

$$\alpha\left[\mu(1-\mu)\sin\vartheta \cdot \left(\frac{1}{\rho_2^3} - \frac{1}{\rho_1^3}\right) - \xi\left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}\right)\right] + \xi = 0;$$
$$\left[1 - \alpha\left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}\right)\right]\eta = 0; \quad (5)$$
$$\mu(1-\mu)\cos\vartheta \cdot \left(\frac{1}{\rho_2^3} - \frac{1}{\rho_1^3}\right) - \zeta\left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}\right) = 0$$

determine the coordinates of such points. Only two cases are possible.

1.The distances from the particle m_0 to the dumbbell endpoints are equal, i.e. $\rho_1 = \rho_2 \ne l/2$. Such points can be named 'the libration points similar to Lagrangian' or 'triangular librtation points'(TLP). From (5) it follows that in this case $\zeta = 0$. Hence, TLPs belong to the plane *Oxy*.

2. $\eta = 0$, i.e. the considered points belong to the plane *Oxz* containing the vector **L** and the rod of the dumbbell. Such points can be named 'the libration points similar to Eulerian' or 'coplanar libration points'(CLP).

The triangular libration points.

The following statements are fulfilled [1, 18].

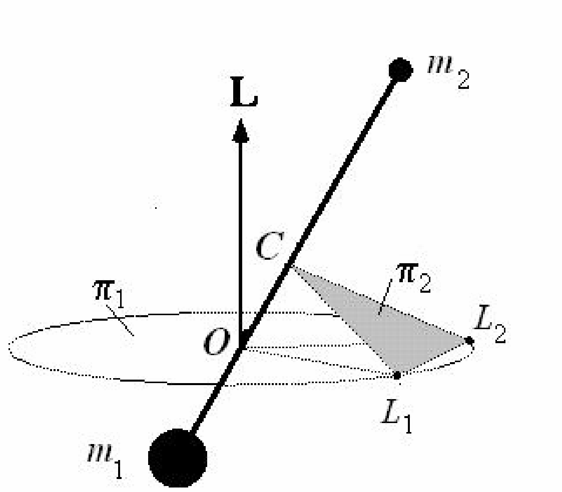

Fig. 2.

1. TLPs belong to the plane intersecting the dumbbell's axis in its midpoint and being perpendicular to the rod (plane π_2 in Fig. 2).

2. TLPs belong to the plane intersecting the dumbbell in its mass center and being perpendicular to its angular momentum (plane π_1 in Fig. 2).

3. TLPs belong to the cylinder determined by the equation $\eta^2 + \xi^2 = \alpha^{\frac{2}{3}} - \mu(1-\mu)$.

4. Coordinates of TLPs are determined by formulae

$$\xi = \frac{1-2\mu}{2\sin\vartheta}, \ \eta = \pm\sqrt{\alpha^{\frac{2}{3}} - \frac{1-4\mu(1-\mu)\cos^2\vartheta}{4\sin^2\vartheta}}.$$

5. If $\mu(1-\mu)<1/36$ then TLPs are stable. If $\mu(1-\mu)>1/36$ then both stability and instability are possible.

From 1-4 it follows that there are two or one TLPs or TLPs do not exist.

Geometrical locus and coordinates computation of coplanar libration points.

From equality $\eta=0$ it follows that the coordinates of CLPs satisfy equations

$$\zeta\left\{\frac{1-\mu}{\rho_1^3}+\frac{\mu}{\rho_2^3}\right\}+\left\{(1-\mu)\mu\left[\frac{1}{\rho_1^3}-\frac{1}{\rho_2^3}\right]\cos\vartheta\right\}=0 \qquad (6)$$

$$\begin{aligned}&\xi\left\{1-\alpha\frac{1-\mu}{\rho_1^3}-\alpha\frac{\mu}{\rho_2^3}\right\}-\\&-\alpha\left\{(1-\mu)\mu\left[\frac{1}{\rho_1^3}-\frac{1}{\rho_2^3}\right]\sin\vartheta\right\}=0\end{aligned} \qquad (7)$$

Using (6, 7) it is easy to prove that the CLPs belong to the strip bounded by straight lines crossing the dumbbell endpoints and being perpendicular to angular momentum, i.e. $\mu\cos\vartheta<\zeta<(1-\mu)\cos\vartheta$.

Note that (6) does not depend of α. Factually this equation is quadratic equation of ξ. It can be reduced to a form

$$\begin{aligned}&\Phi(\zeta)\left\{[\xi+\mu\sin\vartheta]^2+[\zeta+\mu\cos\vartheta]^2\right\}=\\&=[\xi-(1-\mu)\sin\vartheta]^2+[\zeta-(1-\mu)\cos\vartheta]^2\end{aligned}, \qquad (8)$$

where

$$\Phi(\zeta)=\left[\frac{\mu}{1-\mu}\cdot\frac{(1-\mu)\cos\vartheta-\zeta}{\mu\cos\vartheta+\zeta}\right].$$

The roots

$$\xi_{1,2}=-\frac{\mu\Phi(\zeta)+1-\mu}{\Phi(\zeta)-1}\sin\vartheta\pm$$
$$\pm\sqrt{\left(\frac{\mu\Phi(\zeta)+1-\mu}{\Phi(\zeta)-1}\right)^2\sin^2\vartheta-F(\zeta)}$$
$$\left(F(\zeta)=\frac{\Phi(\zeta)\mu^2\sin^2\vartheta+(1-2\mu)\cos^2\vartheta+2\zeta\cos\vartheta}{\Phi(\zeta)-1}\right)$$

of (8) determine in the plane Oξζ two branches of a curve that can be called a Geometrical Locus (GL) of CLPs. Typical forms of GL are depicted in Fig. 3abc.

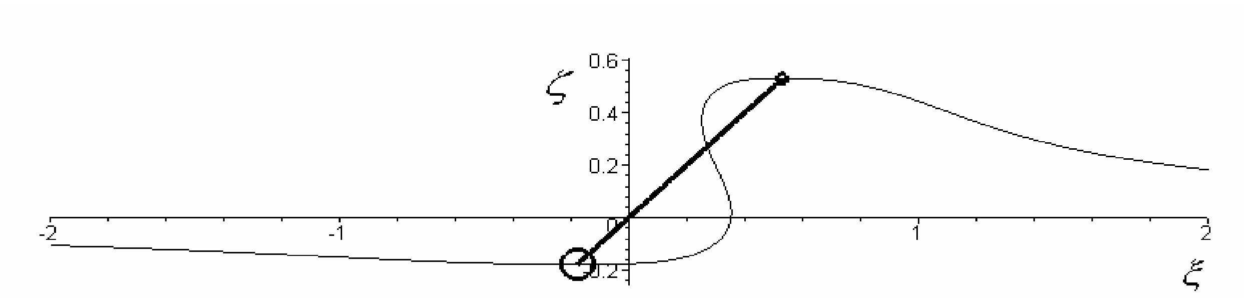

Fig. 3a.

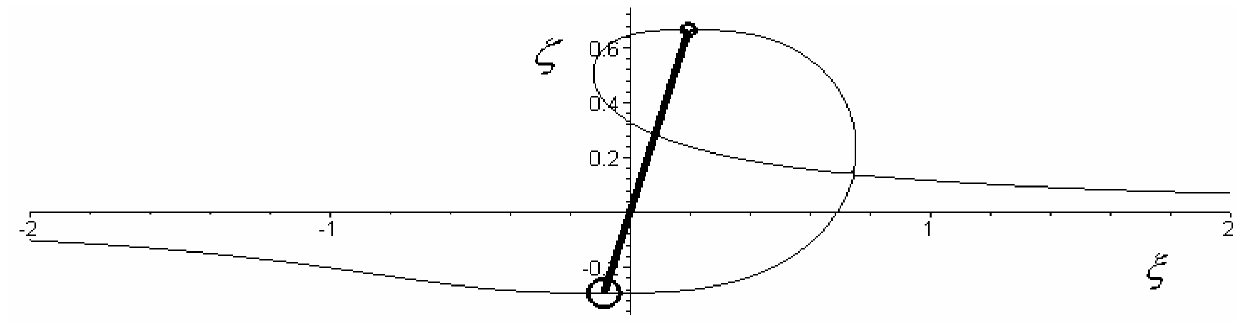

Fig. 3b.

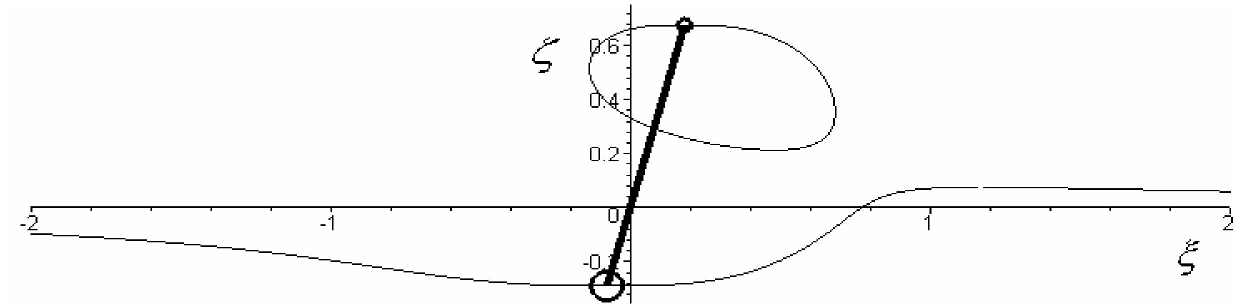

Fig. 3c.

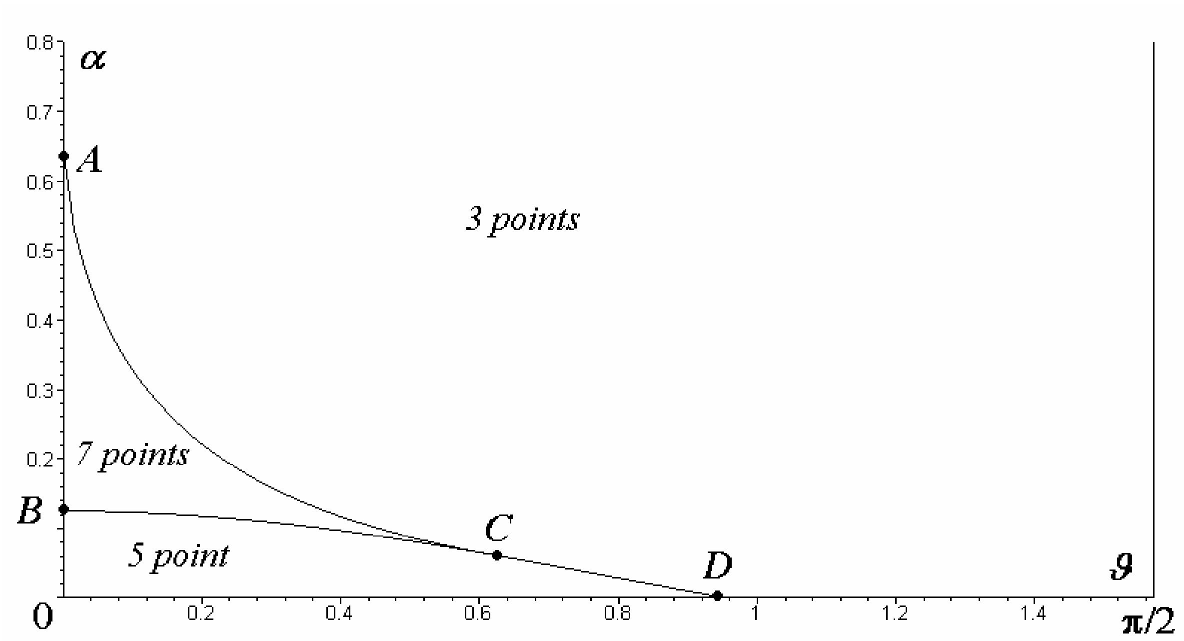

Fig. 4.

From (7) it follows that

$$\alpha^{-1} = \varphi(\xi,\zeta;\mu,\vartheta) = \frac{1-\mu}{\xi\rho_1^3}(\xi + \mu\sin\vartheta) + \frac{\mu}{\xi\rho_2^3}(\xi - (1-\mu)\sin\vartheta) \qquad (9)$$

Substituting $\xi_{1,2}$ for ξ in (9), we get two equations for coordinate ζ of CLPs

$$\alpha^{-1} = \varphi_{1,2}(\zeta;\mu,\vartheta), \qquad (10)$$

It can be proved that the number of real roots of (10) varies from 3 to 7. These equations can be solved numerically.

The diagram for the number of CLPs if the dumbbell is symmetric.

If the dumbbell is symmetric $(\mu = 1/2 \Leftrightarrow m_1 = m_2)$ then two statements are fulfilled.

1.The origin $\xi = \zeta = 0$ is the CLP.

2.If the point with coordinates , $\xi = \xi_0$, $\zeta = \zeta_0$ is CLP then the point with coordinates $\xi = -\xi_0$, $\zeta = -\zeta_0$ also is CLP.

From these statements it follows that if the dumbbell is symmetric then the number of CLPs is odd. Hence in this case there exist 3, 5 or 7 CLPs. Corresponding areas in the plane of parameters ϑ and α is depicted in fig.4. In this figure the equation of the curve *BCD* is

$$\alpha = \frac{2-3\sin^2\vartheta}{16}$$

the curve *AC* was defined numerically. Coordinates of the mentioned points are *A*(0;), *B*(0;1/8), *C*(0.626016 $=35^o52'5''$,0.60653), *D*($\arctan\sqrt{2}$;0). Note also that there exist 5 CLPs for the points belonging to the curves *AC* and *BC*, there exist 3 CLPs for the points belonging to *CD*.

The CLPs evolution for the symmetric dumbbell.

From the diagram in Fig. 4 it follows that if the dumbbell is symmetric then there exist three types of CLPs evolution at the increasing of α.

First type takes place if $\arctan\sqrt{2} \le \vartheta \le \pi/2$ (Fig. 5a). In this case there exist 3 points for any α. Points L_1, L_2 move from attracting centers to infinity.

Second type takes place if (Fig.5b). In this case there exist 3 or 5 points. Points L_1, L_2 move from attracting centers to infinity, points L_3, L_4 move from 'vertical' axis to the dumbbell midpoint. Points L_0, L_3, L_4 merge at L_0 if $\alpha = 2 - 3\sin^2\vartheta/16$.

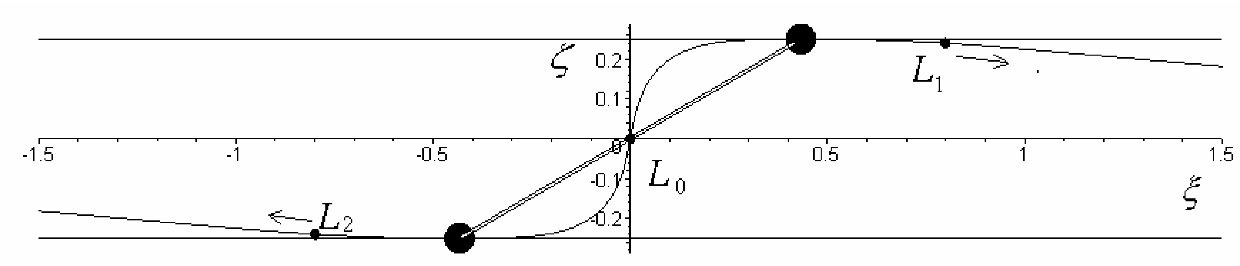

Fig. 5a.

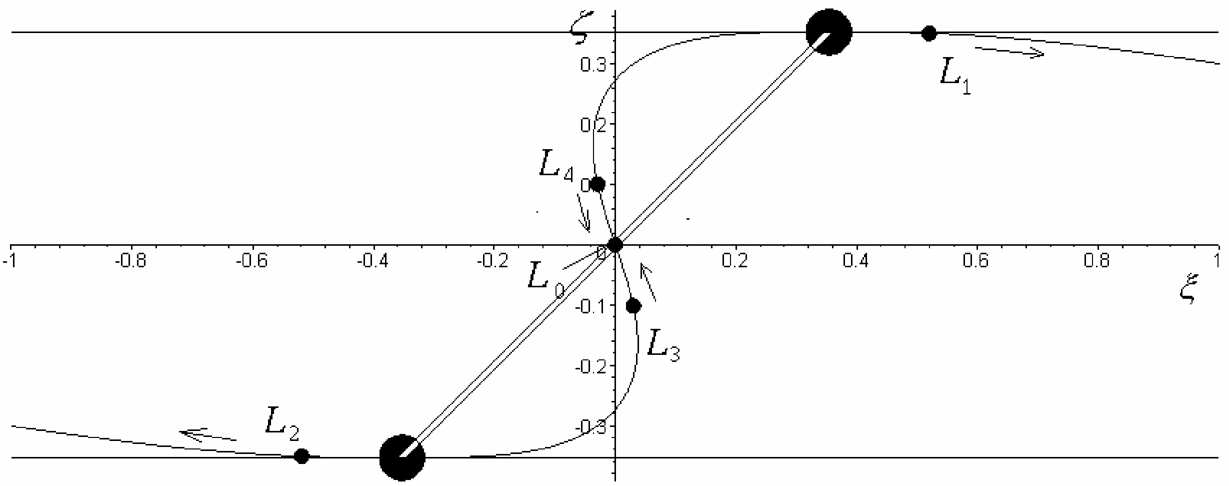

Fig. 5b.

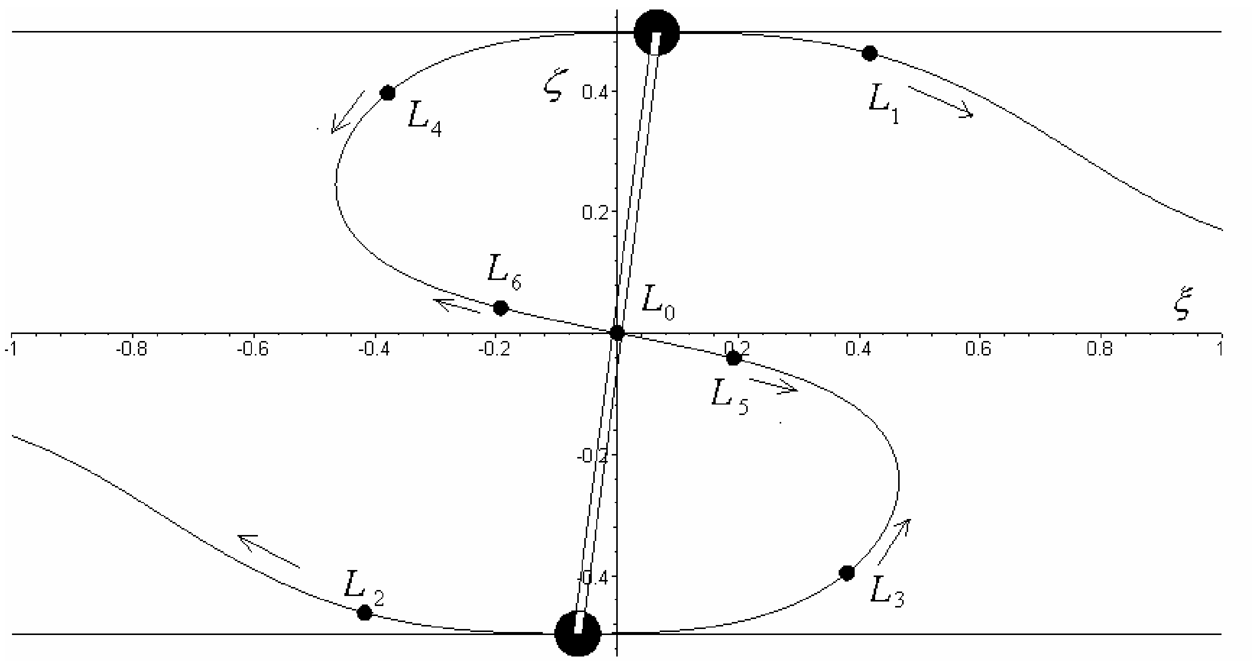

Fig. 5c.

Third type take place if $0<\vartheta<35^o 52' 5''$ (Fig. 5c). In this case there exist 3, 5 or 7 points. If $\alpha > 2-3\sin^3\vartheta/16$ then two new points L_5, L_6 are born. There exist α^* that if $\alpha=\alpha^*$ then L_5 collides with L_3 and L_6 collides with L_4. If $\alpha>\alpha^*$ then the points L_3 L_4 L_5 L_6 disappear.

Examples for the asymmetric dumbbell.

It can be shown that if the dumbbell is asymmetric $(m_1 \neq m_2)$ then there exist 3, 4, 5, 6 or 7 CLPs. So the even number of CLPs is possible. But this situation is rather rare. Factually, it take place only if two of CLPs merge. For instance, if $\mu=0.25$; $\vartheta=15^o$; $\alpha=0.046$; then there exist 4 CLPs. Here two of CLPs merge at L_4 (Fig.6). If $\mu=0.475$; $\vartheta=7^o 30'$; $\alpha=0.255$; then there exist 6 CLPs. In this case two points merge also at L_4 (Fig. 7).

Conclusions.

In this paper, stationary motions in the binary asteroid is studied. Points corresponding to the smaller asteroid's equilibria relative to the axises of precession and nutation of the dumbbell-shaped bigger asteroid are considered. Such points usually name the libration points. Using the model suggested in [1] the algebraic equations for coordinates of the libration points are deduced. These equations are reduced to a form convenient for numerical analysis. The number of the coplanar libration points (CLP), i.e. the libration points belonging to the plane composed by system angular momentum and axis of the dumbbell is counted. This number varies from 3 up to 7, but it equal to 4 or 6 only for rather rare situations.

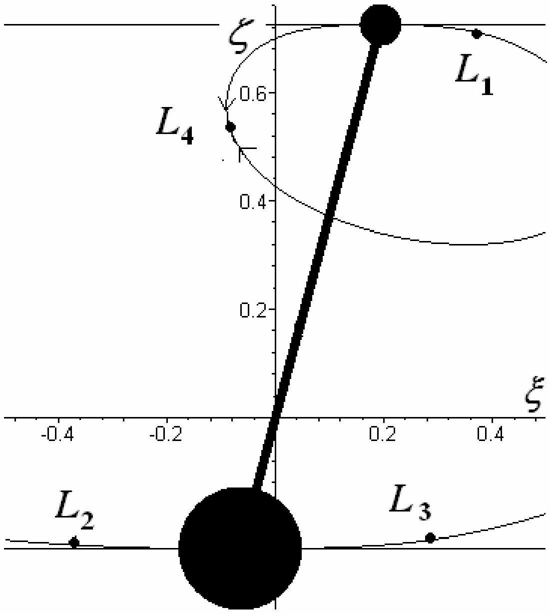

Fig. 6.

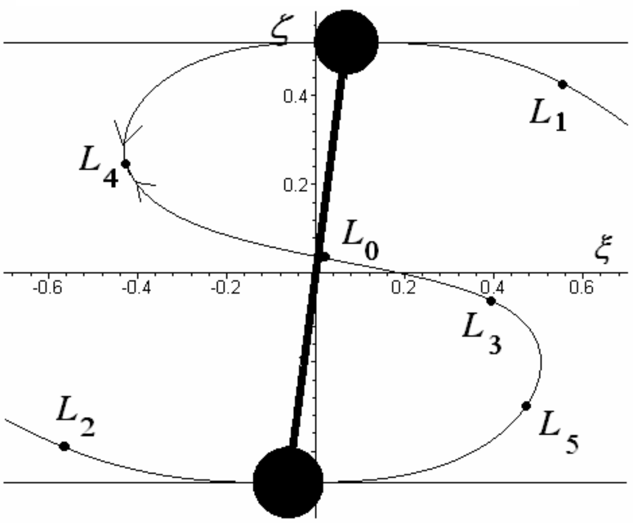

Fig. 7.

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