404. A novel dry friction modeling and its impact on differential equations computation and Lyapunov exponents estimation

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Abstract. A novel dry friction modeling and its impact on differential equations computation and Lyapunov exponents estimation are studied in this paper. A brief review of some existing standard friction laws are presented and novel continuous friction model is proposed, which takes into account some elements of the mentioned friction models. We show that our continuous friction model is suitable for analysis of stick-slip vibrations caused by dry friction and is more efficient from a computational point of view in comparison with the other presented friction models. Its advantages are illustrated and discussed using a two degree-of-freedom model. Although there are numerous works in the scientific literature dedicated to stick-slip vibrations, we consider a rigid body lying on a belt which moves at non-constant velocity that is less investigated. The behavior of the system is monitored via standard motion analysis. Time series, phase portraits, bifurcation diagram as well as the Lyapunov exponents are reported.

Keywords: dry friction, two degree-of-freedom, bifurcation diagram, Lyapunov exponents.

1. Introduction

Computing Lyapunov exponents is one of the most known and most important identification tools in nonlinear dynamics allowing for identification and investigation of chaotic dynamics in nonlinear dynamical systems. Although the Lyapunov's theory for estimation chaotic dynamics was used by Oseledec as early as forty years ego and there are numerous works in the scientific literature dedicated to chaotic vibrations, in general there are not many effective methods available for the Lyapunov exponents estimation. The classical Lyapunov exponents definition can be used with success only in systems governed by differential equations with smooth right-hand sides. On the other hand, methods commonly used for the Lyapunov exponents computation require smooth vector fields as a necessary condition to be satisfied.
There are many discontinuous systems arising due to physical discontinuities such as dry friction, impact, and backlash in mechanical systems or diode elements in electrical circuits. For these cases another numerical/analytical approaches may be also applied, i.e. phase portraits, bifurcation diagrams, the Melnikov-type methods, Fourier spectra, as well as various methods experimentally and numerically oriented. Our investigations in this paper will be concern with systems governed by differential equations with non-smooth right-hand sides due to dry friction phenomenon. The response of a mechanical system with dry friction may include several nonlinear effects such as stick-slip vibrations, self-sustained oscillations and other nonlinear instability phenomena [8, 16].
Although dry friction belongs to one of the known phenomenon exhibited by mechanical systems, its proper mathematical modeling does not belong to easy tasks. Friction force between sliding surfaces is a complex process and depends on several parameters, i.e. relative velocity of sliding surfaces, normal load, time, temperature. Publications from a mechanical point of view are mainly concerned with dry friction stick-slip oscillations with various models of friction. An extensive literature review on dry friction models can be found in the works of [1, 2, 3, 10, 11, 14].
Even in the last decade stick-slip vibrations were the aim of research of many authors, for example in the works [7, 9, 13]. In these works, stick-slip induced vibrations are studied for cases where body or bodies are riding on a driving belt as a foundation that moves at a constant velocity. In our work, as the example of a mechanical system which exhibits stick-slip vibrations, the mechanical model with non-constant belt (foundation) velocity is studied.
In the next Section 2 a brief introduction and review of some existing standard friction laws is given first and then our proposed novel continuous friction model is presented. Several various friction models will be used to facilitate the comparison of obtained results. These models and their impact on differential equations computation and Lyapunov exponents estimation are studied using 2-dof model with dry friction (Section 3). This approach can be used for all problems involving dry friction, e.g. for Filippov systems described by a set of first-order ordinary differential equations with a discontinuous right-hand side. Numerical methods to calculate periodic and non-periodic solutions in the system's phase space, bifurcations diagram as well as the Lyapunov exponents can be found in Section 4. Results and conclusions of our study are presented in Section 5 and Section 6, respectively.

2. Several Dry Friction Formulations and Proposed Continuous Friction Model

Several dry friction formulations have been proposed based on the classical (discontinuous) Coulomb model. One of the simplest form of the friction law is the classical Amontons-Coulomb law, in which friction force F_{fr} is defined (in dimensionless form) as a function of the relative velocity V_r of sliding surfaces in the slip phase and as a function of the externally applied force F_{ex} (all forces acting in the system excluding friction force) in the stick phase. The constitutive relation for F_{fr} is known as the signum model with static friction point and describe dry friction phenomenon in the correct and accurate way. Note that during numerical simulation an exact value of zero is rather rarely computed. For this reason the mentioned signum model from a numerical point of view is equivalent to the classical Coulomb model.

Observe that the dependence of friction force on the relative velocity based on the signum model is non-continuous function for $V_r = 0$ and standard numerical procedures devoted for solving differential equations cannot be used. For this reason the friction curve is therefore often approximated by a continuous or smooth function. Usually, friction curve approximated by these functions are continuous or even smooth, but for $V_r = 0$ we have always $F_{fr} = 0$, too. The friction force depends on V_r but not depends on F_{ex} in the stick phase.

In one of the recent paper [13] devoted to mathematical modeling of dry friction, the so called switch model is proposed and used in order to match the obtained numerically simulation results with those given by the experimental investigation of the mechanical bodies exhibiting stick–slip vibrations. Switch model (from a mathematical point of view) is governed by three systems of nonlinear ordinary differential equations: one for the slip phase, a second for the stick phase and a third for the transition from stick to slip. In the work [13] it has been shown that from a computational point of view the smoothing methods are more expensive than the switch model based methods.

Below we presented (in non-dimensional form) an alternative continuous friction model taking into account some elements of the mentioned friction models. First we divided the space $F_{ex} - V_r$ into the following four regions as follows

$$
\begin{aligned}
&V_1 : |V_r| > \varepsilon,\\
&V_2 : \left[(0 \le V_r \le \varepsilon) \cap (F_{ex} > F_s)\right] \cup \left[(-\varepsilon \le V_r \le 0) \cap (F_{ex} < -F_s)\right],\\
&V_3 : \left[(0 < V_r \le \varepsilon) \cap (F_{ex} < -F_s)\right] \cup \left[(-\varepsilon \le V_r < 0) \cap (F_{ex} > F_s)\right],\\
&V_4 : (|V_r| \le \varepsilon) \cap (|F_{ex}| \le F_s),
\end{aligned}
$$

where F_s denotes maximum static friction force. The proposed continuous friction model has the following form

$$
F_{fr}(V_r, F_{ex}) = \begin{cases} F(|V_r|)\operatorname{sgn} V_r, & \text{for } V_1 \\ F_s \operatorname{sgn} F_{ex}, & \text{for } V_2 \\ (2A_3 - 1)F_s \operatorname{sgn} V_r, & \text{for } V_3 \\ A_3(-F_{ex} + F_s \operatorname{sgn} V_r) + F_{ex}, & \text{for } V_4 \end{cases}, \quad A_3 = \frac{V_r^2}{\varepsilon^2}\left(3 - 2\frac{|V_r|}{\varepsilon}\right). \tag{1}
$$

Figure 1 show friction force defined by above formula (1) as a function of V_r and F_{ex}.

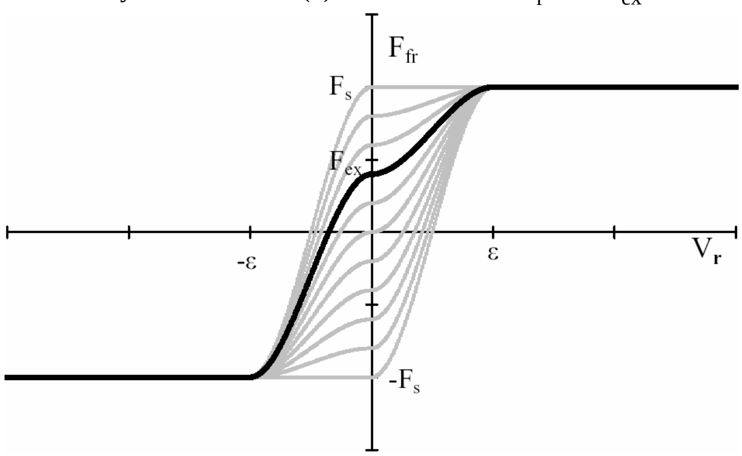


Fig. 1. The friction force as a function of relative velocity V_r in stick phase for several fixed externally forces F_{ex}

In our model friction force is a continuous function on V_r (like in smoothing methods) and for $V_r = 0$ friction force is equal to externally applied force F_{ex} (like in signum model). This model of friction has been already used by the authors of this work in studies [5, 6, 15]. In this work we take $F(|V_r|) = F_s/(1+\delta(|V_r|-\varepsilon))$.

3. Two Degree-of-Freedom Model

Advantages of the proposed friction model and its impact on differential equations computation are illustrated and discussed using two degree-of-freedom model with dry friction, which exhibit both regular and non-regular dynamics. Although there are numerous works in the scientific literature dedicated to stick-slip vibrations, a rigid body lying on a belt which moves at non-constant velocity and Lyapunov exponents estimation in these systems are less investigated. In this work the mechanical model with both constant and non-constant belt (foundation) velocity is studied. In addition, we introduce external harmonic excitation, too. We consider a two degree-of-freedom mechanical system (model) with dry friction consisting of (block of) small mass m riding on a driving belt as a foundation that is moving at constant or non-constant velocity v_{dr} and attached to inertial great mass M by spring k. Between mass m and belt dry friction occurs with a friction T_{fr} (with maximum static friction force T_s). The relative velocity of the mass m with respect to the belt as a foundation is denoted by $v_r = v_{dr} - \dot{x}$. In addition, harmonic excitation with amplitude T_0 and circular frequency ω_1 is added to the small mass m. The mentioned system is shown in Figure 2. This model possesses stick-slip periodic and non-periodic solutions. Velocity of the belt is denoted by $v_{dr} = v_0 \cos\omega t$.

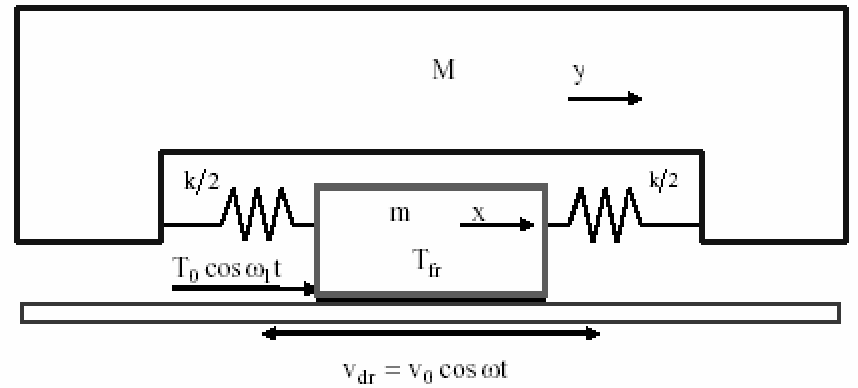


Fig. 2. Two degree-of-freedom model with dry friction and harmonic excitation

The following second order differential equations govern the system dynamics

$$\begin{cases} m\ddot{x} = -k(x-y) + T_{fr}(v_r(\omega t), T_{ex}) + T_0 \cos\omega_1 t \\ M\ddot{y} = k(x-y) \end{cases}, \quad (2)$$

where the dot $(\cdot)$ denotes differentiation with respect to dimension time t. In our model $T_{ex} = k(x-y) - T_0 \cos\omega_1 t$.

This dynamical system can be expressed as a set of first-order ordinary differential equations. The governing equations read

$$\begin{cases} \dot{x} = v \\ \dot{v} = -(k/m)(x-y) + T_{fr}(v_r(z), T_{ex})/m + T_0 \cos\varphi/m \\ \dot{y} = w \\ \dot{w} = (k/M)(x-y) \\ \dot{z} = \omega \\ \dot{\varphi} = \omega_1 \end{cases}. \quad (3)$$

Let us introduce similarity coefficients t_*, x_*, $v_* = x_*/t_*$ and the following dimensionless parameters: $\tau = t/t_*$, $X = x/x_*$, $Y = y/x_*$, $V = v/v_*$, $W = w/v_*$, $Z = z$, $\phi = \varphi$, $\omega_0^2 = kt_*^2/m$, $\Omega_0^2 = kt_*^2/M$, $\Omega = \omega t_*$, $\Omega_1 = \omega_1 t_*$, $F_s = T_s t_*/(mv_*)$, $F_0 = T_0 t_*/(mv_*)$, $V_0 = v_0/v_*$, $V_{dr} = V_0 \cos Z$, $V_r = V_{dr} - \dot{X}$, $F_{ex} = \omega_0^2(X-Y) - F_0 \cos\phi$. In our calculations we take $t_* = \sqrt{m/k}[s]$ and $x_* = 1[m]$. Then, vibrations of the masses m and M are governed by the following non-dimensional second order equations

$$\begin{cases} \ddot{X} = -(X-Y) + F_{fr}(V_r(Z), F_{ex}) + F_0 \cos\phi \\ \ddot{Y} = \Omega_0^2(X-Y) \end{cases}, \quad (4)$$

which are further recast to the following form

$$\begin{cases} \dot{X} = V \\ \dot{V} = -(X - Y) + F_{fr}(V_r(Z), F_{ex}) + F_0 \cos\phi \\ \dot{Y} = W \\ \dot{W} = \Omega_0^2 (X - Y) \\ \dot{Z} = \Omega \\ \dot{\phi} = \Omega_1 \end{cases}, \qquad (5)$$

where a dot $(\cdot)$ denotes now the differentiation with respect to non-dimensional time τ.

4. Numerical Computational Methods

The methods commonly used to compute the Lyapunov exponents require smooth vector fields as a necessary condition. In signum model and switch model friction force is non-continuous function of relative velocity and therefore a continuous friction model is proposed in this paper, which does not posses this disadvantage and can be used during analysis of the systems, where the Lyapunov exponents are computed by the standard procedures [4, 12]. Note only, that while computing Lyapunov exponents, besides six equations (5) also six additional systems of equations ($n = 1,2,3,4,5,6$)

$$\begin{cases} \dot{\tilde{X}}^{(n)} = \tilde{V}^{(n)} \\ \dot{\tilde{Y}}^{(n)} = -(\tilde{X}^{(n)} - \tilde{Y}^{(n)}) + dF_{fr}^{(n)} - (F_0 \sin\phi)\tilde{\phi}^{(n)} \\ \dot{\tilde{Y}}^{(n)} = \tilde{W}^{(n)} \\ \dot{\tilde{W}}^{(n)} = \Omega_0^2 (\tilde{X}^{(n)} - \tilde{Y}^{(n)}) \\ \dot{\tilde{Z}}^{(n)} = 0 \\ \dot{\tilde{\phi}}^{(n)} = 0 \end{cases}, \qquad (6)$$

with respect to perturbations are solved, where

$$dF_{fr}^{(n)} = (\partial F_{fr}/\partial X)\tilde{X}^{(n)} + (\partial F_{fr}/\partial Y)\tilde{Y}^{(n)} + (\partial F_{fr}/\partial V)\tilde{V}^{(n)} + \\ + (\partial F_{fr}/\partial W)\tilde{W}^{(n)} + (\partial F_{fr}/\partial Z)\tilde{Z}^{(n)} + (\partial F_{fr}/\partial \phi)\tilde{\phi}^{(n)} . \qquad (7)$$

Finally, forty two equations are solved. The differential equations of motion are solved via the Runge-Kutta-Fehlberg (RKF 45) method with varied time step h ($h_{min} = 10^{-5}, h_{max} = 10^{-1}$) and with a Runge-Kutta-Fehlberg tolerance of $\eta = 10^{-6}$ and steepness parameter $\varepsilon = 10^{-3}$ and the Gramm-Schmidt ortonormalization technique [4]. The behavior of the system is monitored via standard motion analysis like time series, phase portraits, bifurcations diagram as well as the Lyapunov exponents.

5. Results

Our numerical computations are carried out for the particular case $M >> m$. The initial non-dimensional parameters of our model are $F_s = 1$, $\delta = 3$, $\Omega_0^2 = 0.00002$, $V_0 = 0.2$. Let us consider first dynamics of the system without harmonic excitation, i.e. $F_0 = 0$ and for $\Omega = 0$, i.e. when the belt is driving with the constant velocity $V_{dr} = V_0 = \text{const}$. For this case, the phase portraits obtained with signum model, smoothing method (hyperbolic tangent approximation), switch model and our continuous friction model are shown in Figure 3.

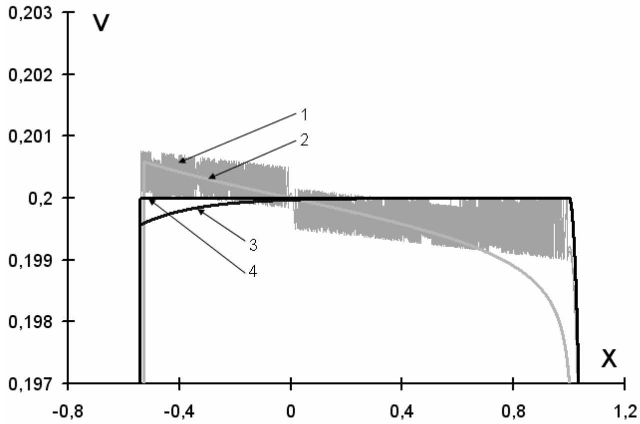


Fig. 3. Phase portraits of the analyzed for $V_{dr} = V_0 = \text{const.}$ for different models of friction: signum model (curve 1), smoothing method (curve 2), switch model (curve 3) and our continuous friction model (curve 4)

The periodic stick-slip oscillations have the sliding velocity almost the same at each model in slip phase but is visible, that in the sticking phase some differences are observed. The differences occur in result of another approximating friction force application in near-zero relative velocity neighborhood. Contrary to the other models results, our results are better (almost exact solutions). In Figure 4 are shown the phase portraits obtained using both switch model and proposed continuous friction model for our mechanical system. Periodic stick-slip oscillations occur in this case for $V_{dr} = \text{const.}$, too.

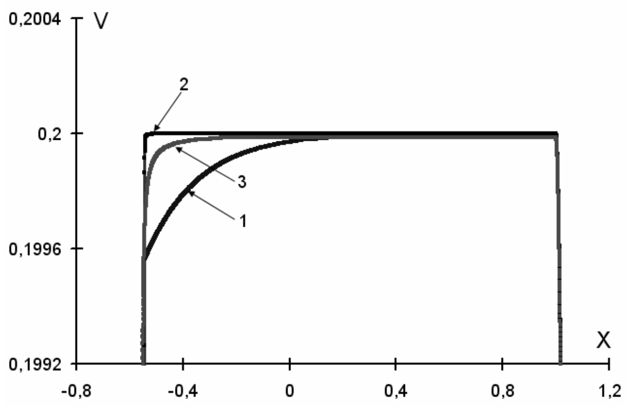
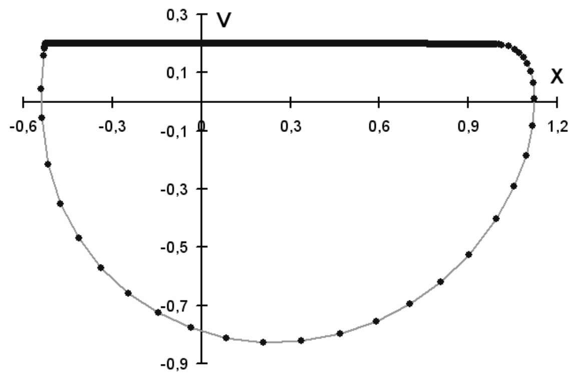


Fig. 4. Phase portraits: switch model (curve 1), continuous friction model (curve 2) and continuous friction model for $\varepsilon = 10^{-2}$ (curve 3)

The sliding velocity is almost the same as above (for both switch and continuous friction models), but in the sticking phase some differences are observed, too. Contrary to the switch model results (curve 1), our results (curve 2) using the applied continuous friction model are better. Namely, we have obtained almost exact (high precision numerical computations) even for larger (10 times) parameter ε (curve 3). It allows to obtain the same accuracy as in the switch model, but for larger time steps and steepness parameter ε. Finally we see that the switch model is more expensive than the continuous friction model from the computational point of view in this case.

Figure 5 shows phase portraits obtained with the two compared friction models (the standard approximation using a signum function modeled by the hyperbolic tangent function in the stick phase and our friction model). For the first friction model the computation took 3353 integrations points to obtain the orbit with the non-dimensional period time 12. Small time steps are not necessary near the transitions, but during the whole stick phase, as Figure 5a shows. For our friction model the computation took only 104 integrations points to obtain the same orbit (Figure 5b).

a)



b)

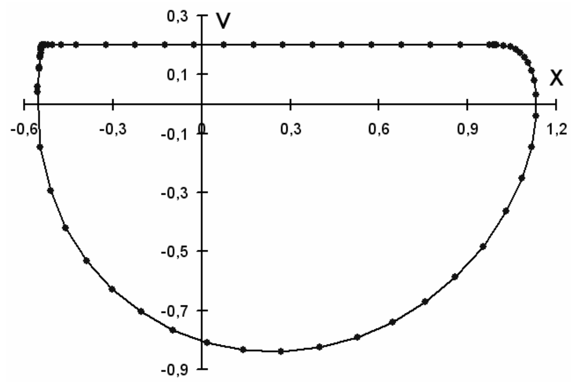


Fig. 5. Points of trajectories of motion in the system's phase space for different models of friction: a) smoothing hyperbolic tangent approximation and b) the proposed model

In our calculations small time steps are taken only near the transitions between stick and slip phases and time step in the stick phase is bounded by maximum time step h_{max}. Consequently, we show that in this case smoothing approximation of classical signum function is clearly more expensive than the proposed model. The differential in the first friction model is extremely large for relative velocity equal to zero, whereas in our model it is equal to zero and this is an advantage for numerical computations.

Our proposed friction model is continuous function of relative velocity and therefore it can be used during analysis of the systems, where the Lyapunov exponents are computed by the standard procedures. In our model with harmonic excitation ($F_0 = 0.5, \Omega_1 = 2$) the dependences of the largest Lyapunov exponent λ with V_{dr} and F_0 as control parameters and displacement X of the mass m on the vertical axis are reported in Figure 6 using proposed continuous friction model during computations.

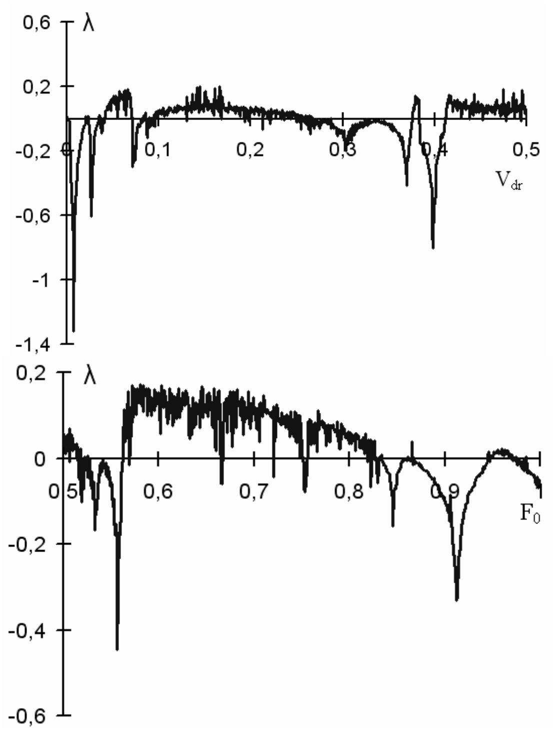


Fig. 6. Dependences of the largest Lyapunov exponent λ with V_{dr} and F_0 as control parameters

The periodic and non-periodic solutions are detected using bifurcation diagram in the velocity V_{dr} and the displacement X plane, too, in Figure 7.

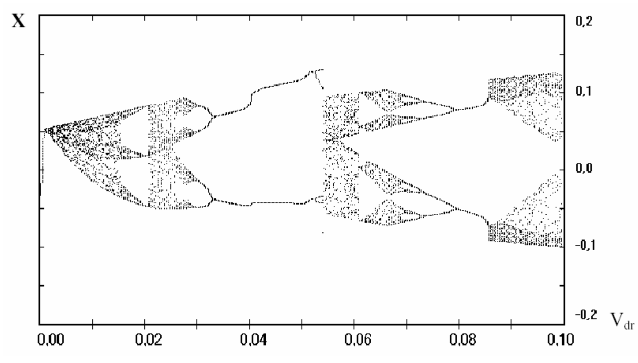
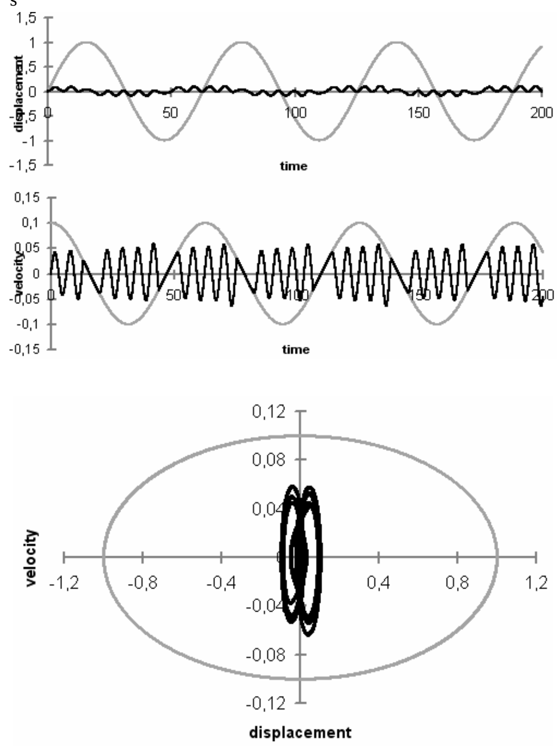


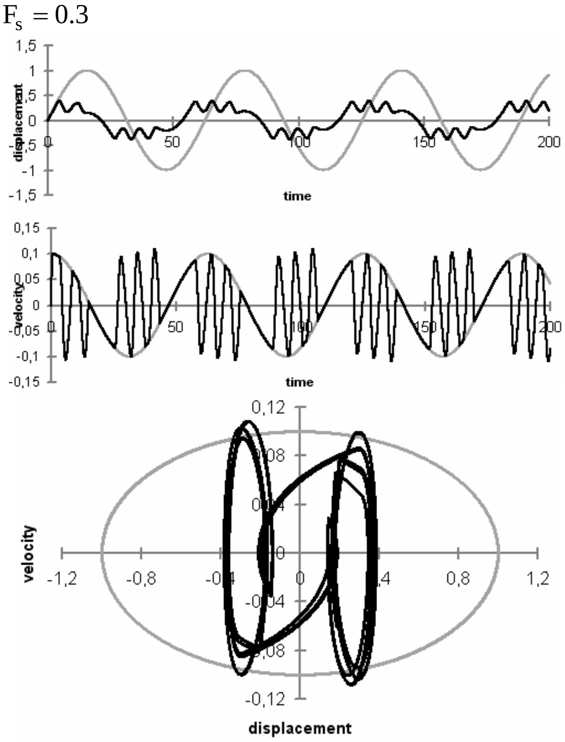
Fig. 7. Bifurcation diagram of the analyzed system with V_{dr} as the control parameter and X on the vertical axis

Let us consider now dynamics of the system without harmonic excitation, but for $\Omega \neq 0$, i.e. when the belt is driving with the non-constant (harmonic oscillation) velocity $V_{dr} = V_0 \cos \Omega\tau$, where $V_0 = 0.1$. The studied mechanical system possesses various interesting solutions include stick-slip vibrations, as show below. Figures 7 and 8 present different behaviors of analyzed mechanical system using our proposed model.

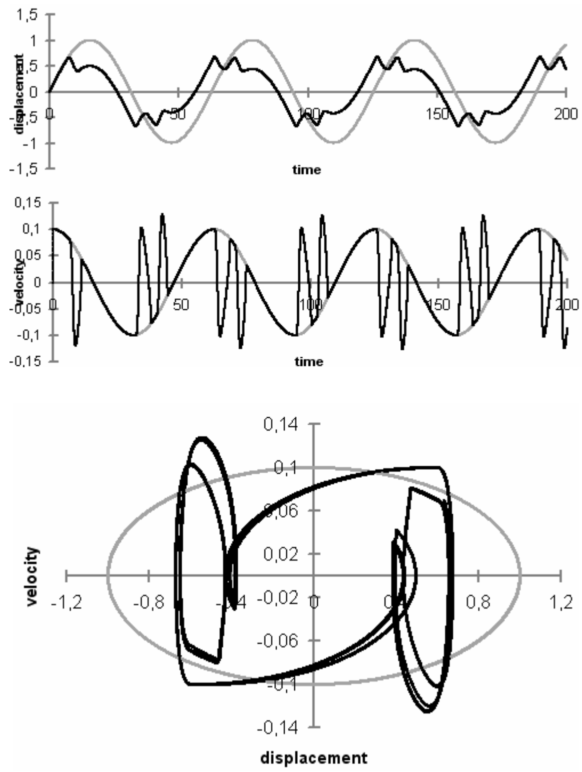
a) $F_s = 0.05$



b) $F_s = 0.3$



c) $F_s = 0.6$



d) $F_s = 0.95$

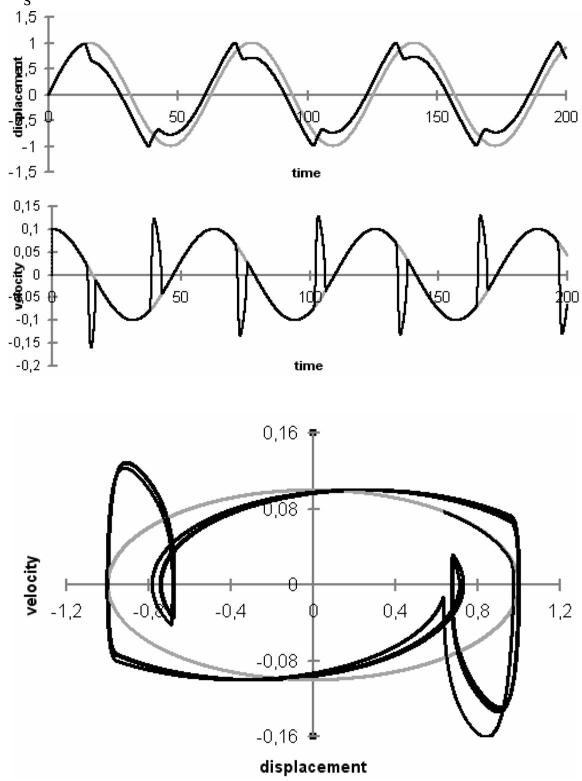


Fig. 7. Time series and phase portraits of the analyzed system for $\Omega = 0.1$ and various static friction force F_s of the belt (grey line) and the mass m (black line)

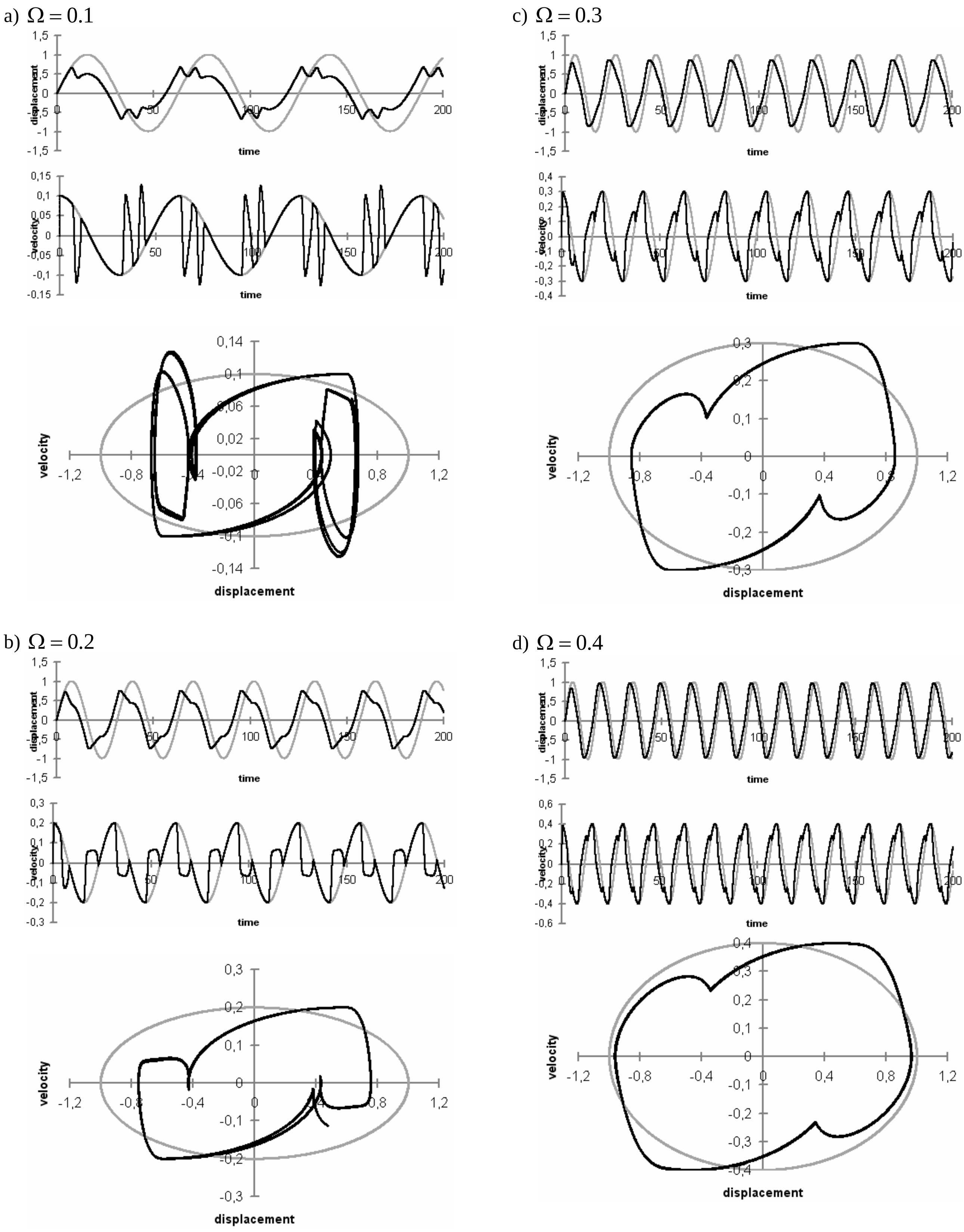


Fig. 8. Time series and phase portraits of the analyzed system for $F_s = 0.6$ and various angular velocity Ω of the belt (grey line) and the mass m (black line)

5. Conclusions

In this paper a brief review of some existing standard friction laws based on the classical (discontinuous) Coulomb model are presented and novel continuous friction model, which takes into account some elements of mentioned friction models is proposed. Their impact on differential equations computation and Lyapunov exponents estimation are studied. We show that our model is suitable for stick-slip vibration simulations. This model is validated using a two-degree-of-freedom mechanical system with dry friction. It has been observed that our continuous friction model yields engineering accepted results and possesses advantages in comparison to other friction models. The obtained results have been compared with those given by switch model application, and they indicate better numerical efficiency of our proposed continuous model. The obtained results exhibit advantages of the proposed algorithm in comparison to the algorithm using smoothing approaches, too. Continuous friction model is validated and it gives correct results and almost exact solutions (high precision numerical computations), even if the numerical steepness parameter ε is extremely large. It allows obtaining the same accuracy as in the switch model faster and for larger steepness parameter ε. For these reasons our calculations are less expensive from the computational point of view.

During the mentioned analysis we have applied the standard techniques, i.e. time series, monitoring of phase portraits, bifurcation diagrams and the Lyapunov exponents. One of the important advantages of our novel model is associated with direct application of the standard numerical procedures devoted to either solving nonlinear differential equations or to computation and estimation of the Lyapunov exponents. In addition, some interesting dynamic behaviors (including stick-slip vibrations) of the analyzed system are reported and analyzed.

The proposed continuous friction model may also be suitable for simulation of the stick-slip vibrations and it may be applied to model friction force in any other mechanical systems.

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