358. Slow steady-state flow of viscous fluid near vibrating cylinders; analytical solution for symmetric channel

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Abstract.Oscillating cylindrical body in a viscous fluid initiates oscillating and not oscillating, slow steady-state creeping, flows. Solution of the creeping flow can be deduced if flow of the ideal fluid is determined. In this paper plane steady-state flow of the ideal fluid past a circular cylinder, placed symmetrically in a channel, is found. The theory of analytical functions and conformal mapping are applied.

Key words: Ultrasound, thrombolysis, oscillations, viscous fluid, ideal fluid, creeping or Stokes flow.

Introduction

In recent years there has been a considerable in crease in the incidence of thromboembolic complications in various diseases (obliterating atherosclerosis, obliterating thromboangiitis, etc.) For example, asmany as one million patients die of coronary occlusions alone in the United States each year[1].In addition to well-known surgital technologies, such new methods as ballon, laser, and ultrasonic angioplasty, etc. also gain wide recognition.Ultrasonic medicinal thrombolysis, i.e.,combined application of ultrasound frequency of more than 100kHz and certain drugs was found to cause a twofold increase in the rate of thrombolysis induced by some fibrinolytic preparations. Ultrasonic thrombolysis is implemented using special waveguides equipped with working tips of various shape. The oscillations freguency of the tips is 19-44 kHz; cavitation, acoustic and contact effects are major thrombolytic mechanisms[1].

The paper deals with original device [2] for a ultrasonic angioplasty, presented in Fig. 1. The specifics of this device is the ability to generate directed cavitation stream destroying the deposits on internal surfaces of blood – vessels.

When cylindrical body executes high frequency swing oscillations in a viscous fluid, flow of this fluid can be resolved into two principle parts. The first and may be the most pronounced flow is oscillations of the fluid at the same frequency as the body with progressively reduced amplitude when distance from the body increases. The second flow is less well understood and has no periodic component. It consists of slow steady-state flow of the fluid which follows the dominant periodic flow. Theoretical investigation of this secondary flow and its initiation are presented by Schlichting [3, 4]. The whole investigation of this complex flow can be resolved into three subsequent parts. In the first part of investigation the fluid is assumed to be ideal and longitudinal velocity component v_τ on the body surface is determined.

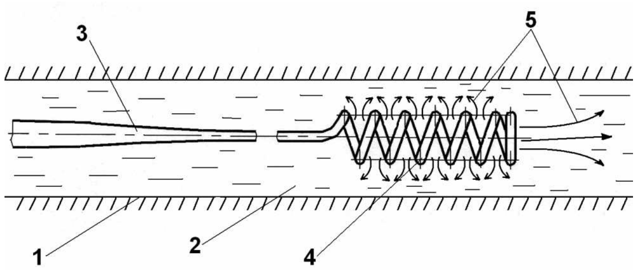

Fig. 1. Schematic of waveguide for ultrasonic angioplasty with increased ability to follow the curvatures of arteries and veins. 1- internal surface of blood-vessel; 2- liquid (blood); 3- waveguide in which longitudinal resonant oscillations are generated; 4- resonator; 5- radial cavitations streams and axial cavitations jets

Applying this velocity the boundary layer solution for the viscous fluid flow can be found, and this is the second part of the whole investigation. In this part the non-linear Navier-Stokes equations are solved approximately: the first approximation presents only oscillatory part of the flow, but the second approximation has the steady-state flow as a component, already mentioned above. Nevertheless, only the longitudinal component on the oscillating body is found from this solution. Evaluation of a steady-state flow in the vicinity of the body is the third part of the investigation. When the steady-state flow is slow then inertia forces are small and can be neglected, therefore viscosity of the fluid is the most important. So the flow, discussed in the third part of the investigation, can be named as creeping or Stokes flow. Notice that although the flow is steady-state the non-linear convective terms are in the Navier-Stokes equations.

Determination of the ideal fluid flow is a classical problem and is widely covered in the literature. However solutions in explicit form are found only for the individual cases. Analytical solution of the plane uniform potential flow past two cylinders is presented by Crowdy [5]. Any aditional obstacle in the fluid creates a considerable challegne for the researche. The problem of reconstructing the free surface of cavitating flow has been studied by Antipov, Silvestrov [6]. The steady flow of an ideal fluid in 6a channel with a free surface is investigated : two stationary plates are in the fluid. An inverse method of transformation from the physical plane (x,y) to the plane (φ,ψ), where φ is velocity potential, ψ is stream function, is presented by Borges [7].

In this paper plane steady-state flow of the ideal fluid past a circular cylinder, placed symmetrically in a channel, is found.

2. Conformal mapping

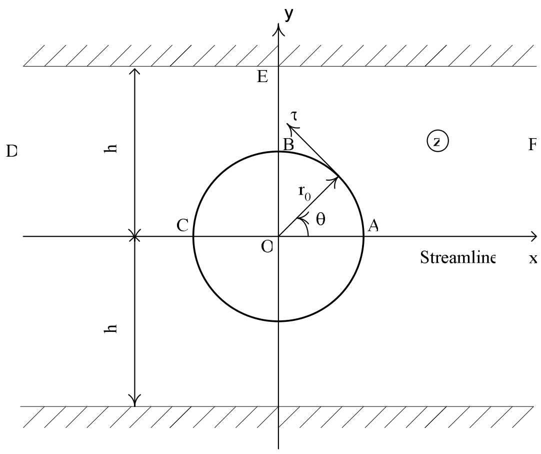

Fig. 2. The circular cylinder placed symmetrically in a channel

The circular cylinder is placed in a channel width $2h$ (Fig. 2). The stationary fluid flow is assumed from the point F in the direction of D, therefore axis x is the symmetry line and also the stream line. The upper part of the flow above the x axis can be considered. This fluid domain is singly connected. The analytic function $\varsigma = f(z)$ of the complex variable $z = x + iy$, conformally mapping the fluid flow domain to the upper half-plane $\eta \geq 0$ of the function $\varsigma = \xi + i\eta$, will be deduced.

The function

$$z_1 = e^{\pi z/h} - e^{-a}, \quad a = \pi \frac{r_o}{h}, \tag{1}$$

maps the fluid domain ABCDEF in Fig. 3 to the upper half-plane of the variable z_1 with the same part ABC eliminated (Fig. 3a).

If a point $z = r_o e^{i\theta} = r_o(\cos\theta + i \sin\theta)$ marked off on the circle in the z plan, then image of this point in z_1 plane is

$$x_1 = e^{a\cos\theta}\cos(a\sin\theta) - e^{-a},$$
$$y_1 = e^{a\cos\theta}\sin(a\sin\theta).$$

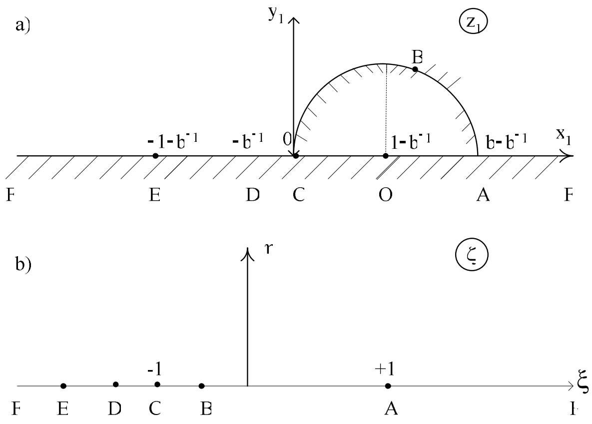

Fig. 3. Domains of the auxiliary variables z_1 and ζ

The image ABC in Fig. 3a is symmetric and tangents of the curve at A, B are parallel to the axis y_1. This can be proved when differentials $dx_1 = 0$, $dy_1 = \pm a e^a d\theta$ are derived for $\theta = 0$ and $\theta = \pi$. The line ABC in Fig. 3a can be assumed as an arc of a circle and then mapped to the upper half-plane [6] of the variable

$$\varsigma = \frac{k_1}{1 - \left(1 - \frac{e^a - e^{-a}}{z_1}\right)^2} + k_2 .$$

When $k_1 = 2, k_2 = -1$ are assumed and expression (1) is used the analytic function

$$\varsigma = \frac{\left(e^{az/r_o} - e^{-a}\right)^2 + \left(e^{az/r_o} - e^{a}\right)^2}{\left(e^{az/r_o} - e^{-a}\right)^2 - \left(e^{az/r_o} - e^{a}\right)^2} \tag{2}$$

or

$$2\varsigma = \frac{e^{az/r_o} - \cosh a}{\sinh a} + \frac{\sinh a}{e^{az/r_o} - \cosh a}. \tag{3}$$

These relatively simple formulae have the inverse expression

$$\frac{az}{r_o} = ln\left(\frac{2\,sinh\,a}{1-\sqrt{\frac{\varsigma-1}{\varsigma+1}}} + e^{-a}\right) \qquad (4)$$

or

$$\frac{az}{r_o} = ln\left[\left(1+\varsigma+\sqrt{\varsigma-1}\right)sinh\,a + e^{-a}\right]. \qquad (5)$$

Image of the conformal mapping (1) is not an exact circle, therefore image of the fluid flow domain, determined by Eq. (2) or Eq. (3), is not an exact upper half-plane $Im\varsigma \geq 0$. On the other hand image of the line $\eta = Im\varsigma = 0$ in the z plane is not exact circle. Calculations, made by the Eq. (5), shows that image of the line $-1 \leq \xi \leq +1, \eta = 0$ (Fig. 3b) is in close proximity to an ellipse. The big semi-axis of the ellipse coincides with the diameter AOC of the circle in z plane. The length of the small semi-axis depends on the a. If ratio h/r_o is 15, 10 and 5 the ratio of the small semi-axis to the big semi-axis is 0.993, 0.984 and 0.940 correspondingly. It may be considered that the circular wire is deformed by 0.72%, 1.6% and 6.0%. When the ellipses in the z plane are assumed, deviations of the image not exceed 0.0008%, 0.004% and 0.06%. So the images of the line $-1 \leq \xi \leq +1$ in the z plane can be assumd as an ellipses that differ little from the circle. The point $\varsigma = 0$ is not the highest point in the z plane. When $\varsigma = 0$ is inserted in Eq. (5) and identity

$$(1+i)\sinh a + e^{-a} = \cosh a + i\sinh a = \\ = \sqrt{1+2\sinh^2 a}\; e^{i\arctan\tanh a}$$

is substituted, expression, if a<<1,

$$\frac{z}{r_o} = \frac{\ln\left(1+2\sinh^2 a\right)}{2a} + i\,\frac{\arctan\tanh a}{a} \approx \\ \approx \left(a+\frac{a^3}{3}\right) + i\left(1-\frac{2a^2}{3}\right) \approx a+i \quad ,$$

can be proved. The term $2a^2/3$ is displacement down of the point $\varsigma = 0$, but this displacement is greater then deformation of the circle because image of $\varsigma = 0$ does not coincide with the highest point of the image curve. The equal intervals [–1;0], [0;+1] of the ς plane are mapped to unequal arcs.

3. Potential flow in the channel

The complex potential $w = w(z)$ of the ideal incompressible fluid can be found if analytic function $z = z(\varsigma)$ is determined. If cylinder moves with the velocity U_c in the direction of the x axis (Fig. 1), the normal components of the cylinder and fluid velocity coincide: $v_n = U_c\,cos\,\theta = U_c\,cos\frac{s}{r_o}$, where s is the length of the arc from A. On the other hand $v_n = \frac{\partial\varphi}{\partial n} = \frac{\partial\psi}{\partial s}$, where φ is velocity potential, ψ is stream function : $w = \varphi + i\psi$. Therefore, the stream function $\psi = \int v_n ds = U_c r_o \sin\frac{s}{r_o}$ is determined on the surface of the circular cylinder. When imaginary part ψ of the analytic function w is deduced the whole function w can be expressed [7]

$$w(\varsigma) = \frac{U_c r_o}{\pi}\int_{-1}^{+1}\frac{\sin\theta}{\xi-\varsigma}d\xi - C_o. \qquad (6)$$

Parameter ς in this integral is a point inside the upper half-plane domain $\eta > 0$ (Fig. 3b). When ς approaches the border point $\varsigma = \xi_o$, $-1 < \xi_o < +1$, the Plemelj formula have to be applied. The real part of the potential $\varphi(\xi_o) = \frac{U_c r_o}{\pi}\int_{-1}^{+1}\frac{sin\,\theta}{\xi-\xi_o}d\xi$ is a singular integral. The angle θ (Fig. 2) have to be determined as a function of ξ and is given in Eq. (5). If $\left(1+\xi+i\sqrt{1-\xi^2}\right)sinh\,a + e^{-a} = \rho e^{i\vartheta}$ then $z = \frac{r_o}{a}\left(ln\,\rho + i\vartheta\right)$ and $Im\,z = \frac{r_o}{a}\vartheta = r_o\,sin\,\theta$. It follows that $a\,sin\,\theta = arctan\frac{\sqrt{1-\xi^2}}{1+\xi+\beta_s}$, $\beta_s = \frac{e^{-a}}{sinh\,a}$. Potential of the fluid velocity is expressed

$$\varphi(\xi_o) = \frac{U_c h}{\pi}\int_{-1}^{+1}\frac{arctan\frac{\sqrt{1-\xi^2}}{1+\xi+\beta_s}}{\xi-\xi_o}d\xi. \qquad (7)$$

When $h \to \infty$ the parameter $a \to 0$ and $\beta_s \to a^{-1}$. The function

$$arctan\frac{\sqrt{1-\xi^2}}{1+\xi+\beta_s} \to a\sqrt{1-\xi^2}\;.$$

The Tricomi integrals [7]

$$\int_{-1}^{+0}\frac{T_k(\xi)}{\xi-\xi_o}\frac{d\xi}{\sqrt{1-\xi^2}} = \pi U_{k-1}(\xi_o), k = 1,2,...$$

now can be used in Eq. (7). $T_k(\xi), U_k(\xi)$ are orthogonal Chebyshev polynomials of the first and the second type $T_1(\xi)=\xi$, $T_2(\xi)=2\xi^2-1$, $U_1(\xi)=2\xi$, … The velocity potential from Eq.(7) in the case $a\to 0$ is $\varphi(\xi)=-U_c r_o \xi$. If $h\to\infty$ and circle is not restricted, the conformal mapping of the fluid flow domain is presented by simple function

$$z=r_o\left(\varsigma+\sqrt{\varsigma^2-1}\right). \quad (8)$$

On the circle $z=r_o(\cos\theta+i\sin\theta)$, thus $\xi=\cos\theta$ and it follows from Eq. (7) $\varphi(\xi)=-U_c r_o \cos\frac{s}{r_o}$. The tangent component of the velocity

$$v_\tau=\frac{\partial\varphi}{\partial s}=U_c\sin\theta. \quad (9)$$

This value coincides with the well-known classical solution.

4. Velocity on the cylinder surface

If circular cylinder is in a channel, the function (2), instead of (8), should be applied. The constant a is assumed less then 1. Since $z=r_o e^{i\theta}=r_o(\cos\theta+i\sin\theta)$ the approximation

$$\varsigma=\cos\theta-a\sin^2\theta-a^2\sin^2\theta\frac{5\cos\theta-i\sin\theta}{6}+\ldots \quad (10)$$

follows from Eq. (2). When $h/r_o=15;10;7$ the error of the real part approximation is correspondingly 0.3%; 1.0%; 2.9%. If approximations

$$\beta_s^{-1}=e^a\sinh a\approx a+a^2\frac{2}{3}a^3,\ \arctan x\approx x\left(1-\frac{x^2}{3}\right)$$

are used, then

$$\arctan\frac{\sqrt{1-\xi^2}}{1+\xi+\beta_s}\approx a\left(1+a+\frac{2}{3}a^2\right)\sqrt{1-\xi^2}B_1(\xi),$$

where

$$B_1(\xi)=1-a(1+a)(1+\xi)+a^2(1+\xi)^2-a^2\frac{1-\xi^2}{3}.$$

Thus, from Eq. (7) and Tricomi integrals,

$$\varphi(\xi)=U_c r_o\left(1+a+\frac{2}{3}a^2\right)B_2(\xi),$$

where

$$B_2(\xi)=-a\frac{1-a}{2}-\left(1-a-a^2\right)\xi+a(1-a)\xi^2-\frac{4}{3}a^2\xi^3.$$

If the real part of Eq.(10) is applied and expression of the arc length $s=r_o\theta$ is used the relative component of the velocity

$$v_{\tau rel}=\frac{\partial\varphi}{\partial s}=U_c\left[\left(1+\frac{a^2}{3}\right)\sin\theta-\frac{a^2}{2}\sin^3\theta\right] \quad (11)$$

can be deduced. This is velocity of the fluid when the cylinder moves in a channel with fluid stacionary in infinity. The tangent component of the fluid velocity relative to the cylinder can be obtained if the component (11) is added to the translation velocity $v_{\tau tr}=U_c\sin\theta$. This velocity component presents projection of the velocity $-U_c$ to the tangent τ (Fig. 1), i. e. velocity of the fluid at the infinity with respect to the cylinder. Therefore,

$$v_\tau=U_c\left[\left(2+\frac{a^2}{3}\right)\sin\theta-\frac{a^2}{2}\sin^3\theta\right] \quad (12)$$

is velocity component of the fluid relative to the cylinder when terms with a^3 and less are ignored.

Conclusions

Conformal mapping of the fluid flow domain to the half-plane can be realized by relatively simple analytic function (2) or (3) , and the inverse function (4) or (5) can be solved. These functions present an approximate mapping of a domain with circular cylinder in the middle of the channel.

The boundary value problem is solved to obtaine fluid flow velocity in the fluid domain and on the cylinder surface. The explicit solutions (11), (12) of the fluid velocity on the circular cylinder are deduced for the case when width of the channel $2h$ is much more then the diameter of the cylinder $2r_o$.

If h is very large and $a=\pi r_o/h$ can be neglected in comparison with 1, velocity solution (11) coincides with well-known solution (9) for an unbounded domain. Nevertheless, the same solution is valid when not a but $a^2<<1$ and can be neglected. So boundaries of the channel have to be sufficiently close to the cylinder to influence the fluid flow velocity on the cylinder surface.

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