254. Chain type system with wave excitation

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Abstract. Various systems based on waves and vibrations are used for displacing, grouping, classification of multi-dimensional media and bodies. This paper deals with such wave operation based systems augmented by self-stopping elements which ensure one-directional motion of the driven sub-systems. Such enhancements can improve some dynamical characteristics of the analysed systems. The objective of this paper is to develop models of systems and methods of analysis which would help to reveal nonlinear dynamical properties and phenomena of those systems. The conditions of solutions' existence and stability, basin boundaries are determined. Simpler cases are analysed analytically. The obtained relationships provide insight into nonlinear dynamics of complex systems which are analysed numerically.

Keywords: standing and propagating waves, self-stopping element, nonlinear dynamics

1. Introduction

Non-controlled or controlled self-stopping mechanisms play important role in different systems and help to improve dynamical characteristics of those systems and to simplify their structure. Chain type systems with self-stopping elements generalise some simplified models of fluids and granular type materials. This paper is the further investigation of vibrational and wave transportation [1, 2]. Despite of important scientific achievements [3, 4] it is important to analyse such new type of transportation systems and determine attractors, their basin boundaries, develop motion control strategies [5].

2. Model of the system

The analyzed system (Fig. 1) consists from the input member with the working profile 0 and the output member – the chain consisting from blocks *i*=1,…,*n*. The *i*-th member is in the contact with the working profile 0 at the contact point A_i .

The working profile of the input member is defined according to *x*- and *y*-axes by:

$$\eta = \eta(u,t)\,;\ \xi = \xi(u,t)\,. \qquad (1)$$

The *i*-th member with the mass m_i can move with respect to the member i_0 (with mass m_{i0}) according to *x*-axis. The member i_0 can move in the directional guides along *x*-axis. The self-stopping device is attached to the member i_0. It lets the i_0-th element to move only in one (positive) direction. The self stopping devices can be attached also to the *i*-th member and can limit the direction of motion with respect to the working profile 0. The separate members of the output system are connected by the elastic-dissipative elements.

The force of friction between the *i*-th member and the working profile 0 at the contact point A_i is acting in the tangential direction:

$$F_{fi} = N_i f_0 \operatorname{sign}\dot{s}_i + f_1 \dot{s}_i\,, \qquad (2)$$

where N_i is the normal reaction force acting to the *i*-th member at point A_i; f_0 and f_1 are the coefficients of dry and viscous friction respectively; $\dot{s}_i$ is the slippage velocity of the *i*-th member with respect to the working profile 0 at point A_i; top dot stands for d/dt.

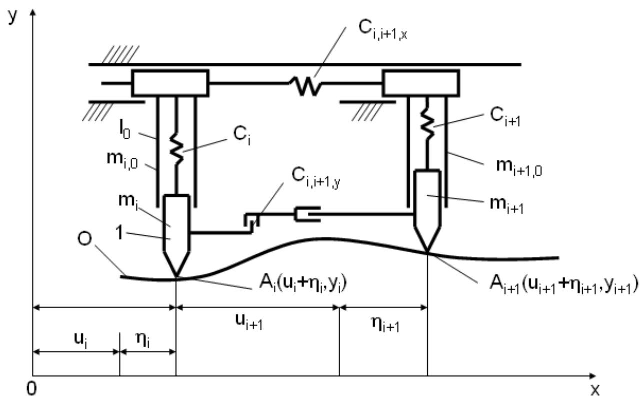

Fig. 1. The schematic diagram of the system

According to Fig. 1:

$$x_i = u_i + \eta_i;\ y_i = \xi_i, \tag{3}$$

From where, by taking into account eq. (1):

$$\dot\eta_i = \eta'_{iu_i}\dot u_i + \eta'_{it};$$
$$\ddot\eta_i = \eta'_{iu_i}\ddot u_i + \eta''_{iu_1-u_i}\dot u_i^{\,2} + 2\eta''_{iu_it}\dot u_i + \eta''_{itt}. \tag{4}$$

Similar relationships are valid for $\dot\xi_i$, $\ddot\xi_i$.

The angle α_i between the *x*-axis and the tangent at the point A_i is determined by the following equation:

$$\tan\alpha_i = \frac{\xi'_{iu_i}}{1+\eta'_{iu_i}}. \tag{5}$$

The velocity of slippage of the *i*-th member with respect to the working profile at the point A_i is:

$$\dot s_i = \left(\dot x_i - \eta'_{it}\right)\cos\alpha_i + \left(\dot y_i - \xi'_{it}\right)\sin\alpha_i,$$

or

$$\dot s_i = \dot u_i\sqrt{\left(1+\eta'_{iu_i}\right)^2 + \xi'^{2}_{iu_i}},$$
$$\dot s_i\big|_{\xi=0} = \left(1+\eta'_{iu_i}\right)\dot u_i,$$
$$\dot s_i\big|_{\eta=0} = \xi'_{iu_i}\dot u_i. \tag{6}$$

On the basis of the equations of equilibrium with respect to the *i*-th member the differential equations of motion take the following form:

$$\dot F_{x_i} + f_1\dot s_i\cos\alpha_i + \left(F_{y_i} + f_1\dot s_i\sin\alpha_i\right)\cdot \frac{\xi'_{iu_i} + \left(1+\eta'_{iu_i}\right)f_0\operatorname{sign}\dot s_i}{1+\eta'_{iu_i} - \xi'_{iu_i}f_0\operatorname{sign}\dot s_i} = 0, \tag{7}$$

$$\left(1+\eta'_{iu_i}\right)F_{x_i} + \xi'_{iu_i}F_{y_i} + \left(-\xi'_{iu_i}F_{x_i} + \left(1+\eta'_{iu_i}\right)F_{y_i}\right)f_0\operatorname{sign}\dot s_i + \left(\left(1+\eta'_{iu_i}\right)^2 + \xi'^{\,2}_{iu_i}\right)f_1\dot u_i = 0, \tag{8}$$

where

$$F_{x_i} = \left(m_i + m_{i0}\right)\ddot x_i + F_{i,i-1,x} + F_{i,i+1,x} + H_{x_i}\dot x_i + Q_{x_i},$$
$$F_{y_i} = m_i\ddot\xi_i + F_{i,i-1,y} + F_{i,i+1,y} + F_{0y_i} + Q_{y_i},$$
$$F_{0y_i} = H_i\dot\xi_i + C_i\left(\xi_i - l_i\right),$$
$$F_{i,i-1,x} = H_{i,i-1,x}\left(\dot x_i - \dot x_{i-1}\right) + C_{i,i-1,x}\left(x_i - x_{i-1} - l_{i,i-1,x}\right),$$
$$F_{i,i-1,y} = H_{i,i-1,y}\left(\dot\xi_i - \dot\xi_{i-1}\right) + C_{i,i-1,y}\left(\xi_i - \xi_{i-1} - l_{i,i-1,y}\right). \tag{9}$$

Equations (8) are valid when there are no self-stopping devices. In case when self-stopping elements are attached between the member i_0 and the motionless basis, equations (8) are valid when

$$\dot x_i = \dot u_i + \dot\eta_i > 0, \tag{10}$$

and do not hold true when

$$\dot x_i = \dot u_i + \dot\eta_i \le 0. \tag{11}$$

In case when self-stopping elements are attached between members *i* and 0, equations (8) will be valid when

$$\dot s_i > 0, \tag{12}$$

and will not hold true when

$$\dot s_i \le 0. \tag{13}$$

Equations (8) can be simplified when parameters η and ξ oscillate with high frequency and small amplitudes, and assuming that:

$$1+\eta'_{iu_i} \approx 1;\ \left(1+\eta'_{iu_i}\right)^2 + \xi'^{2}_{iu_i} \approx 1; \tag{14}$$

and at the same time:

$$\dot x_i \approx \dot u_i + \eta'_{it};\ \ddot x_i \approx \ddot u_i + \eta''_{itt}. \tag{15}$$

Then, equations (8) are simplified to the following form:

$$F_{x_i} + \xi'_{iu_i} F_{y_i} + \left(-\xi'_{iu_i} F_{x_i} + F_{y_i}\right) f_0 \operatorname{sign}\dot{s}_i + f_1 \dot{u}_i = 0 . \quad (16)$$

Case when n = 2.

In this case $i = 1$, and according to equations (9):

$F_{x_1} = (m_1 + m_{10}) f_{x_1}$,
where
$f_{x_1} = \ddot{x}_1 + f_{12x} + h_{x_1}\dot{x}_1 + q_{x_1}$;
$F_{x_2} = (m_2 + m_{20}) f_{x_2}$,
where
$f_{x_2} = \ddot{x}_2 + \mu_x f_{12x} + h_{x_2}\dot{x}_2 + q_{x_2}$;
$F_{y_1} = m_1 f_{y_1}$,
where
$f_{y_1} = \ddot{\xi}_1 + f_{12y} + p_1^2(\xi_1 - l_1) + q_{y_1}$;
$F_{y_2} = m_2 f_{y_2}$,
where
$f_{y_2} = \ddot{\xi}_2 - \mu_y f_{12y} + p_2^2(\xi_2 - l_2) + q_{y_2}$;
$f_{12x} = h_{12x}(\dot{x}_1 - \dot{x}_2) + n_x^2(x_1 - x_2 - l_{12x})$;
$f_{12y} = h_{12y}(\dot{\xi}_1 - \dot{\xi}_2) + n_y^2(\xi_1 - \xi_2 - l_{12y})$;
(17)

$$h_{x_i} = \frac{H_{x_i}}{m_i + m_{i0}}; \quad q_{x_i} = \frac{Q_{x_i}}{m_i + m_{i0}}; \quad \mu_x = \frac{m_1 + m_{10}}{m_2 + m_{20}};$$
$$p_i = \sqrt{\frac{C_i}{m_i}}; \; q_{y_i} = \frac{Q_{y_i}}{m_i}; \; \mu_y = \frac{m_1}{m_2};$$
$$h_{12x} = \frac{H_{12x}}{m_1 + m_{10}}; \quad h_{12y} = \frac{H_{12y}}{m_1}; \quad n_x = \sqrt{\frac{C_{12x}}{m_1 + m_{10}}};$$
$$n_y = \sqrt{\frac{C_{12y}}{m_1}} . \quad (18)$$

Equation (16) is transformed to:

$$f_{x_i} + \mu_{yx_i}\xi'_{iui} f_{y_i} + \left(-\xi'_{iui} f_{x_i} + \mu_{y_i} f_{y_i}\right) f_0 sign\dot{u}_i + f_i^* \dot{u}_i = 0, \quad (19)$$

where

$$\mu_{yx_i} = \frac{m_i}{m_i + m_{i0}}; \; f_{1i}^* = \frac{f_1}{m_i + m_{i0}} . \quad (20)$$

Case when n=1.

In this case the equation (8) or (16) holds true. When n = 1 and according to eq. (9) and taking into account $u_1 = u$; $x_1 = x$; $\eta_1 = \eta$; $\xi_1 = \xi$; …; the following relationships are obtained:

$F_x = (m + m_0) f_x$, where $f_x = \ddot{x} + h_x \dot{x} + q_x$;
$F_y = m f_y$, where $f_y = \ddot{\xi} + h_y \dot{\xi} + p^2(\xi - l) + q_y$;
(21)

where h_x, h_y, q_x, q_y are determined analogously to eq. (18). Eq. (16) is transformed to:

$$f_x + \mu_{yx}\xi'_u f_y + f_1^* \dot{u} + \left(-\xi'_u f_x + \mu_{yx} f_y\right) f_0 \operatorname{sign}\dot{u} = 0 . \quad (22)$$

3. Excitation by travelling waves.

According to eq. (1)

$$\eta = \eta(\omega t - ku); \; \xi = \xi(\omega t - ku), \quad (23)$$

where η and ξ are periodic functions of their arguments. In this case the derivatives in eq. (3-6) take the following form:

$$\eta_i = \eta(\omega t - ku_i); \; \eta'_{iui} = -k\eta'_i; \; \eta''_{iu_iu_i} = k^2\eta''_i; \; \eta'_{it} = \omega\eta'_i;$$
$$\eta''_{itt} = \omega^2\eta''_i; \; \eta''_{iu_it} = -\omega k\eta''_i; \quad (24)$$

where

$$\eta'_i = \frac{\partial\eta_i}{\partial(\omega t - ku_i)}; \; \eta''_i = \frac{\partial^2\eta_i}{\partial(\omega t - ku_i)^2} . \quad (25)$$

Eq. (4) simplifies to:

$$\dot{\eta}_i = (\omega - k\dot{u}_i)\eta'_i; \; \ddot{\eta}_i = -k\eta'_i\ddot{u}_i + (\omega - k\dot{u}_i)^2\eta''_i . \quad (26)$$

Analogous relationships can be obtained for ξ.

In that case eq. (5, 6) are transformed to:

$$\tan\alpha_i = \frac{-k\xi'_i}{1 - k\eta'_i}; \quad \dot{s}_i = \dot{u}_i\sqrt{(1 - k\eta'_i)^2 + k^2\xi_i'^2};$$
$$\dot{s}_i\Big|_{\xi=0} = (1 - k\eta'_i)\dot{u}_i; \; \dot{s}_i\Big|_{\eta_i=0} = -k\xi'_i\dot{u}_i . \quad (27)$$

Next we assume harmonic waves

$$\eta_i = A\cos(\omega t - ku_i); \; \xi_i = B\sin(\omega t - ku_i). \quad (28)$$

For example, eq. (8) at n=1 becomes:

$$(1 - k\eta')f_x - \mu_{yx}k\xi' f_y + \left((1 - k\eta')^2 + k^2\xi'\right)f_1^* + \left(k\xi' f_x + (1 - k\eta')f_y\right)f_0 \operatorname{sign}\dot{s} = 0; \quad (29)$$

where f_x, f_y, μ_{yx} are determined by eq. (20, 21).

By taking into account that the amplitudes of excitation are small and the frequency is high; also $f_0 = l = q_y = 0$, eq. (29), taking into account (14) and (15) becomes:

$$\ddot{u} + \left(h_x + f_1^*\right)\dot{u} = -\eta''_{tt} - h_x\eta'_t - q_x + \mu_{yx}k\xi'\left(\xi''_{tt} + h_y\xi'_t + p^2\xi\right), \quad (30)$$

where derivatives of η and ξ are calculated from eq. (28):

$$\eta'_t = -\omega A\sin(\omega t - ku), \ \ldots \quad (31)$$

Eq. (30) can be expressed in the following form:

$$\ddot{u} + h\dot{u} = a + f(\omega t - ku). \quad (32)$$

Case 1. The output system moves with the velocity of the propagating wave.

In that case

$$u_i = \frac{\omega}{k}t + \overline{u_i}, \quad (33)$$

where $\overline{u_i} = const$. In that case $\dot{\eta}_i = \ddot{\eta}_i = \dot{\xi}_i = \ddot{\xi}_i = 0$ and

$$\eta_i = \overline{\eta_i} = \eta\left(-k\overline{u_i}\right); \qquad \eta'_i = \overline{\eta'_i} = \eta'\left(-k\overline{u_i}\right);$$
$$\eta''_i = \overline{\eta''_i} = \eta''\left(-k\overline{u_i}\right). \quad (34)$$

In case of harmonic waves and taking into account (28, 33, 34):

$$\overline{\eta_i} = A\cos k\overline{u_i}\,;\ \overline{\eta'_i} = A\sin k\overline{u_i}\,;\ \overline{\eta''_i} = -A\cos k\overline{u_i}\,;$$
$$\overline{\xi_i} = -B\sin k\overline{u_i}\,;\ \overline{\xi'_i} = B\cos k\overline{u_i}\,;\ \overline{\xi''_i} = B\sin k\overline{u_i}\,. \quad (35)$$

Eq. (8) by taking into account (33, 34) becomes:

$$\overline{\phi_i} = \left(1 - k\overline{\eta_i}\right) + k\overline{\xi'_i}f_0\overline{F_{x_i}} + \left(-k\overline{\xi'_i} + \left(1 - k\overline{\eta'_i}\right)f_0\right)\overline{F_{y_i}} +$$
$$\left(\left(1 - k\overline{\eta'_i}\right)^2 + k^2\overline{\xi'_i}^2\right)f_1\frac{\omega}{k} = 0, \quad (36)$$

where

$$\overline{F_{x_i}} = \overline{F_{i,i-1,x}} + \overline{F_{i,i+1,x}} + H_{x_i}\frac{\omega}{k} + Q_{x_i}\,;$$
$$\overline{F_{y_i}} = \overline{F_{i,i-1,y}} + \overline{F_{i,i+1,y}} + \overline{F_{0y_i}} + Q_{y_i}\,;$$
$$\overline{F_{0y_i}} = C_i\left(\overline{\xi_i} - l_i\right); \qquad \overline{F_{i,i-1,x}} = C_{i,i-1,x}\left(\overline{x_i} - \overline{x_{i-1}} - l_{i,i-1,x}\right);$$
$$\overline{F_{i,i+1,y}} = C_{i,i+1,y}\left(\overline{\xi_i} - \overline{\xi_{i-1}} - l_{i,i-1,y}\right). \quad (37)$$

It is possible to find $\overline{u_i}$ from eq. (36). The existence of their real values is the necessary condition of existence of motions. The stability of those motions can be determined from linear variational differential equations constructed for eq. (8 and 36):

$$u_i = \overline{u_i} + \delta u_i\,, \quad (38)$$

where δu_i is the variation.

In case 1.1, when n = 1, it is assumed that

$$x_1 = x\,;\ y_i = y\,;\ u_1 = u\,; \quad (39)$$

and from eq. (36), by taking into account (35), it is obtained:

$$\overline{\phi} = \left(1 - kA\sin k\overline{u}\right) + f_0kB\cos k\overline{u}\,, \quad (40)$$

where

$$\overline{f_x} = h_x\frac{\omega}{k} + q_x\,; \qquad \overline{f_y} = -p^2\left(B\sin k\overline{u} + l\right) + q_y\,;$$

$$h_x = \frac{H_x}{m + m_0}\,;\ q_x = \frac{Q_x}{m + m_0}\,;$$

$$\mu_{yx} = \frac{m}{m + m_0}\,; \quad p = \sqrt{\frac{C}{m}}\,; \quad q_y = \frac{Q_y}{m}\,; \quad f_1^* = \frac{f_1}{m + m_0}\,. \quad (41)$$

Eq. (40) is transformed to:

$$\overline{\phi} = a_2\cos 2k\overline{u} + b_2\sin 2k\overline{u} + a_1\cos k\overline{u} + b_1\sin k\overline{u} + a_0 = 0, \quad (42)$$

where

$$a_2 = -0.5f_0\mu_{yx}p^2kAB + 0.5f_1^*\omega k\left(-A^2 + B^2\right);$$
$$b_2 = 0.5\mu_{yx}p^2kB^2\,;$$
$$a_1 = \left(f_0\overline{f_x} - \mu_{yx}\left(q_y - p^2l\right)\right)kB\,;$$
$$b_1 = -f_0\mu_{yx}\left(p^2B + \left(q_y - p^2l\right)kA\right) - 2f_1^*\omega A - \overline{f_x}kA\,;$$
$$a_0 = \overline{f_x} + f_0\mu_{yx}\left(q_y - p^2l + 0.5p^2kAB\right) +$$
$$+ f_1^*\frac{\omega}{k}\left(1 + 0.5k^2\left(A^2 + B^2\right)\right). \quad (43)$$

The condition of stability is:

$$\frac{\partial\overline{\phi}}{\partial\overline{u}} = -2a_2\sin 2k\overline{u} + 2b_2\cos 2k\overline{u} - a_1\sin k\overline{u} + b_1\cos k\overline{u} > 0\,. \quad (44)$$

In case when $f_0 = B = 0$, the conditions of existence and stability are:

$$\left| \frac{\overline{f_x} + f_1^* \frac{\omega}{k}\left(1 + 0.5k^2 A^2\right)}{\left(k\overline{f_x} + 2f_1^*\omega\right)A} \right| < 1; \quad (45)$$

$$\cos k\bar{u} > 0 . \quad (46)$$

It follows from (45) that the existing stable motions are in the interval

$$k\bar{u} \in \left(0, \frac{\pi}{2}\right),$$

and unstable –

$$k\bar{u} \in \left(\frac{\pi}{2}, \pi\right). \quad (47)$$

In case when

$$A = f_0 = f_1 = q_y = 0 , \quad (48)$$

the conditions of existence and stability are:

$$\left| \frac{\overline{f_x}}{0.5\mu_{yx} p^2 kB^2} \right| < 1; \; -\cos 2k\bar{u} > 0 . \quad (49)$$

Stable motions exist in intervals $k\bar{u} \in \left(\frac{\pi}{2}, \frac{3}{4}\pi\right)$ and $k\bar{u} \in \left(\frac{3}{2}\pi, \frac{7}{4}\pi\right)$; and unstable ones in

$$k\bar{u} \in \left(\frac{3}{4}\pi, \pi\right) \text{ and } k\bar{u} \in \left(\frac{7}{4}\pi, 2\pi\right). \quad (50)$$

In case of longitudinal travelling waves determined by (45 – 47) two types of motions can coexist, and in case of transverse travelling waves determined by (48 – 50) four types of motions can coexist.

Those are the limiting cases, in general case 2 or 4 types of motions can coexist, from which half are stable, and half are unstable.

Case 1.2, when n = 2.

In this case, from eq. (8, 17, 18) it is obtained:

$$\left(1 - k\overline{\eta_1'} + k\overline{\xi_1'} f_0\right)\overline{f_{x_1}} + \left(-k\overline{\xi_1'} + \left(1 - k\overline{\eta_1'}\right) f_0\right)\mu_{yx_1}\overline{f_{y_1}} + \\ + \left(\left(1 - k\overline{\eta_1'}\right)^2 + k^2\overline{\xi_1'^2}\right) f_1^* \frac{\omega}{k} = 0;$$

$$\left(1 - k\overline{\eta_2'} + k\overline{\xi_2'} f_0\right)\overline{f_{x_2}} + \left(-k\overline{\xi_2'} + \left(1 - k\overline{\eta_2'}\right) f_0\right)\mu_{yx_2}\overline{f_{y_2}} + \\ + \left(\left(1 - k\overline{\eta_2'}\right)^2 + k^2\overline{\xi_1'^2}\right) f_1^* \frac{\omega}{k}\mu_{12} = 0; \quad (51)$$

where

$$f_{x_1} = n_x^2\left(\overline{u_1} - \overline{u_2} + \overline{\eta_1} - \overline{\eta_2} - l_{12x}\right) + h_{x_1}\frac{\omega}{k} + q_{x_1};$$

$$f_{x_2} = -\mu_{12} n_x^2\left(\overline{u_1} - \overline{u_2} + \overline{\eta_1} - \overline{\eta_2} - l_{12x}\right) + h_{x_2}\frac{\omega}{k} + q_{x_2};$$

$$\overline{f_{y_1}} = n_y^2\left(\overline{\xi_1} - \overline{\xi_2} - l_{12}\right) + p_1^2\left(\overline{\xi_1} - l_1\right) + q_{y_1};$$

$$\overline{f_{y_2}} = -\mu_{y12} n_y^2\left(\overline{\xi_1} - \overline{\xi_2} - l_{12}\right) + p_2^2\left(\overline{\xi_2} - l_2\right) + q_{y_2};$$

$$\mu_{yx_1} = \frac{m_i}{m_i + m_{i0}}, \quad (i = 1,2); \quad \mu_{12} = \frac{m_1 + m_{10}}{m_2 + m_{20}};$$

$$f_1^* = \frac{f_1}{m_1 + m_{10}}; \; n_x = \sqrt{\frac{c_{12x}}{m_1 + m_{10}}};$$

$$h_{x_i} = \frac{H_{x_i}}{m_i + m_{i0}}; \quad q_{x_i} = \frac{Q_{x_i}}{m_i + m_{i0}}; \quad n_y = \sqrt{\frac{c_{12y}}{m_1 + m_{10}}};$$

$$p_i = \sqrt{\frac{C_i}{m_i}}; \; q_{y_i} = \frac{Q_{y_i}}{m_i}. \quad (52)$$

Conditions of existence and stability are determined from eq. (51).

Case 2: The output system moves with a small velocity with respect to the velocity of the propagating wave:

$$u_i = vt + \tilde{u}_i; \quad (53)$$

where

$$v << \frac{\omega}{k}; \; \overline{\tilde{u}_i} = \frac{1}{T}\int_t^{t+T} \tilde{u}_i dt = 0; \quad (54)$$

T is period of $\tilde{u}_i$. In a separate case it is shown that at n = 1 and $f_0 = 0$, the differential equations of motion becomes eq. (32). For the determination of steady state modes of motion this equation, by taking into account (53 and 54), is rearranged into the following form:

$$\ddot{\tilde{u}} + h\dot{\tilde{u}} = f(\delta t) + \varepsilon\left(f\left(\delta t - k\tilde{u}\right) - f(\delta t) - hv\right); \quad (55)$$

where ε is a small parameter; $\delta = \omega - kv$. The steady state motion according to (55) is sought with the help of the power series:

$$\tilde{u} = \tilde{u}_0 + \varepsilon\tilde{u}_1 + \varepsilon^2 \ldots, \quad (56)$$

and the equations for determination of $\tilde{u}_j$; j = 0,1,… become:

$$\ddot{\tilde{u}}_0 + h\dot{\tilde{u}}_0 = f(\delta t);$$
$$\ddot{\tilde{u}}_1 + h\dot{\tilde{u}}_1 = f(\delta t - k\tilde{u}_0) - f(\delta t) - hv;\ldots \quad (57)$$

The condition of periodicity of $\tilde{u}_1$, by taking into account only the linear part of $f(\delta t - k\tilde{u}_0)$ by power series in terms of $k\tilde{u}_0$, is obtained in the following form:

$$\overline{-k\tilde{u}_0 \frac{\partial f(\delta t)}{\partial \delta t}} - hv = 0; \quad (58)$$

from where v is found.

In case of harmonic waves and $f_0 = h_y = 0$ eq. (30) takes the following form:

$$\ddot{u} + (h_x + f_1^*)\dot{u} = \omega^2 A\cos(\omega t - ku) + \omega h_x A\sin(\omega t - ku) - \\ - q_x + 0.5\mu_{yx}(p^2 - \omega^2)kB^2 \sin 2(\omega t - ku). \quad (59)$$

Analogously:

$$\ddot{\tilde{u}}_0 + (h_x + f_1^*)\dot{\tilde{u}}_0 = \omega^2 A\cos(\omega t - ku) + \omega h_x A\sin(\omega t - ku) - \\ - q_x + 0.5\mu_{yx}(p^2 - \omega^2)kB^2 \sin 2(\omega t - ku),$$

$$\ddot{\tilde{u}}_1 + (h_x + f_1^*)\dot{\tilde{u}}_1 = -(h_x + f_1^* v) - q_x + \\ + k\tilde{u}_0(\omega^2 A\sin\delta t - \omega h_x A\cos\delta t - \mu_{yx}(p^2 - \omega^2)kB^2 \cos 2\delta t),$$

$$\ldots; \quad (60)$$

from where:

$$u_0 = \frac{\omega A}{\delta(\delta^2 + (h_x + f_1^*)^2)}\begin{pmatrix} -(\delta\omega + h_x(h_x + f_1^*))\cos\delta t + \\ + (-\delta h_x + \omega(h_x + f_1^*))\sin\delta t \end{pmatrix} +$$

$$+ \frac{\mu_{yx}(p^2 - \omega^2)kB^2}{4\omega^2(4\delta^2 + (h_x + f_1^*)^2)}(-2\omega^2 \cos 2\delta t + \delta(h_x + f_1^*)\sin 2\delta t). \quad (61)$$

The condition of periodicity of $\tilde{u}_1$ from (58, 60, 61) yields:

$$v = \frac{1}{h_x + f_1^*}\begin{pmatrix} -q_x + \dfrac{\omega^2(\omega^2 + h_x^2)(h_x + f_1^*)}{2\delta(\delta^2 + (h_x + f_1^*)^2)}kA^2 + \\ + \dfrac{\mu_{yx}{}^2(p^2 - \omega^2)^2}{4(4\delta^2 + (h_x + f_1^*)^2)}k^3B^4 \end{pmatrix}. \quad (62)$$

The coefficient of non-uniformity:

$$\delta\tilde{u}\Big|_{B=q_x=0} = \frac{\tilde{u}_{\max}}{v} = \frac{2\delta}{\omega kA}\sqrt{\frac{\delta^2 + (h_x + f_1^*)^2}{\omega^2 + h_x^2}};$$

$$\delta\tilde{u}\Big|_{A=q_x=0} = \frac{\delta(h_x + f_1^*)\sqrt{4\omega^4 + \delta^2(h_x + f_1^*)^2}}{\mu_{yx}\omega^2(p^2 - \omega^2)k^2B^2}. \quad (63)$$

4. Excitation by standing waves.

In this case eq. (1) have the following form:

$$\eta_i = \eta_*(ku_i)\eta(\omega t); \; \xi_i = \xi_*(ku_i)\xi(\omega t), \quad (64)$$

where functions η, ξ, η_*, ξ_* are periodic functions. In case of harmonic waves:

$$\eta_i = A\cos ku_i \cos\omega t; \; \xi_i = B\cos ku_i \cos\omega t. \quad (65)$$

Case 1: n = 1.

In this case the investigation of the system dynamics is concentrated on a such type of equation:

$$\ddot{u} + h\dot{u} = a + f(ku, \omega t), \quad (66)$$

where f is a periodic function. The motion of the system is divided into the slow and fast motions:

$$u = \bar{u} + \tilde{u}, \quad (67)$$

and eq. (66) is rearranged in the following way:

$$\ddot{u} + h\dot{u} = f(k\bar{u}, \omega t) + \varepsilon(a + f(ku, \omega t) - f(k\bar{u}, \omega t)). \quad (68)$$

The solution of eq. (63) is sought in analogy with (55-57):

$$\tilde{u} = \tilde{u}_0 + \varepsilon u + \varepsilon^2 \ldots, \quad (69)$$

what produces

$$\ddot{\tilde{u}}_0 + h\dot{\tilde{u}}_0 = f(k\bar{u}, \omega t);$$
$$\ddot{\tilde{u}}_1 + h\dot{\tilde{u}}_1 = a + f(k(\bar{u} + \tilde{u}_0), \omega t) - f(k\bar{u}, \omega t); \ldots \quad (70)$$

Taking into account the condition of periodicity of u_1 and linear part of the expansion of $f(k(\bar{u} + \tilde{u}_0), \omega t)$ in respect of $k\tilde{u}_0$ yield the following relationship:

$$\overline{a + k\tilde{u}_0 \left(\frac{\partial f\left(k\left(\overline{u} + \tilde{u}_0\right), \omega t\right)}{\partial \tilde{u}_0} \right)_{\tilde{u}_0 = 0}} = 0 . \qquad (71)$$

The existence of real values of $\overline{u}$ is the necessary condition of existence of the analyzed mode of motion. Its stability can be determined from

$$\frac{\overline{\tilde{u}_0 \partial \phi\left(k\left(\overline{u} + \tilde{u}_0\right), \omega t\right)}}{\partial \overline{u}} < 0 . \qquad (72)$$

In case of harmonic waves according to eq. (19) and (18, 20) at $i = 1$; $n = 1$; $x_1 = x$; $y_1 = y$; $u_1 = u$; and by taking into account (14, 15, 28, 30) and $f_0 = h_y = l = q_y = 0$ the equation of motion yields:

$$\ddot{u} + \left(h_x + f_1^*\right)\dot{u} = \omega^2 A \cos ku \cos \omega t + h_x \omega A \cos ku \sin \omega t - $$
$$- q_x + 0.25 \mu_{yx}\left(p^2 - \omega^2\right) k B^2 \sin 2ku \left(1 + \cos 2\omega t\right). \qquad (73)$$

In case when $B = 0$:

$$\ddot{u} + \left(h_x + f_1^*\right)\dot{u} = \omega A \cos k\overline{u}\left(\omega \cos \omega t + h_x \sin \omega t\right) +$$
$$+ \varepsilon\left(-q_x + \omega A\left(\cos ku - \cos k\overline{u}\right)\left(\omega \cos \omega t + h_x \sin \omega t\right)\right). \qquad (74)$$

Analogously to (69 – 72):

$$u_0 = \frac{A \cos k\overline{u}}{\omega^2 + \left(h_x + f_1^*\right)^2} \begin{pmatrix} \left(-\omega^2 + \left(h_x + f_1^*\right)h_x\right)\cos \omega t + \\ + \omega f_1^* \sin \omega t \end{pmatrix};$$

$$\frac{\omega^2\left(\omega^2 + h_x^2\right)}{4\left(\omega^2 + \left(h_x + f_1^*\right)^2\right)} kA \sin 2k\overline{u} - q_x = 0 .$$

The conditions of existence and stability yield:

$$\left| \frac{4 q_x \left(\omega^2 + \left(h_x + f_1^*\right)^2\right)}{\omega^2\left(\omega^2 + h_x^2\right) k A^2} \right| < 1; \ \cos 2k\overline{u} < 0 ; \qquad (75)$$

and stable regimes of motion are in the intervals $k\overline{u} \in \left(\frac{\pi}{4}; \frac{\pi}{2}\right)$ and $k\overline{u} \in \left(\frac{5\pi}{4}; \frac{3\pi}{2}\right)$. Unstable motions are at

$$k\overline{u} \in \left(0; \frac{\pi}{4}\right) \text{ and } k\overline{u} \in \left(\pi; \frac{5\pi}{4}\right). \qquad (76)$$

When $q_x = 0$ stable points are nodes and unstable – saddle points.

Case when $A = 0$

$$\ddot{u} + \left(h_x + f_1^*\right)\dot{u} = -q_x + 0.25 \mu_{yx}\left(p^2 - \omega^2\right) k B^2 \sin 2k\overline{u}\left(1 + \cos 2\omega t\right) +$$
$$+ \varepsilon\left(0.25 \mu_{yx}\left(p^2 - \omega^2\right) k B^2 \left(\sin 2ku - \sin 2k\overline{u}\right)\left(1 + \cos 2\omega t\right)\right). \qquad (77)$$

Zero approximation yields
$-q_x + 0.25 \mu_{yx}\left(p^2 - \omega^2\right) k B^2 \sin 2k\overline{u} = 0$.
The conditions of existence and stability take form:

$$\left| \frac{4 q_x}{\mu_{yx}\left(p^2 - \omega^2\right) k B^2} \right| < 0 ; \ \left(p^2 - \omega^2\right)\cos 2k\overline{u} < 0 . \qquad (78)$$

In case of longitudinal and transverse travelling waves the conditions of existence and stability (75, 78) are similar, but there is an essential difference. In case of transverse waves the stable and unstable conditions when $p^2 - \omega^2 > 0$ are quite similar to the case of longitudinal waves, but when

$$p^2 - \omega^2 < 0 , \qquad (79)$$

they change essentially, that is stable regimes are in the intervals $k\overline{u} \in \left(\frac{3\pi}{4}; \pi\right)$ and $k\overline{u} \in \left(\frac{7\pi}{8}; 2\pi\right)$; and unstable – in the intervals

$$k\overline{u} \in \left(\frac{\pi}{2}; \frac{3\pi}{4}\right) \text{ and } k\overline{u} \in \left(\frac{3\pi}{2}; \frac{7\pi}{6}\right). \qquad (80)$$

When $q_x = 0$ and $p^2 - \omega^2 > 0$ stable points are the nodes and unstable – saddles. When $p^2 - \omega^2 < 0$ stable points become saddles and unstable – nodal points.

Case 2: n = 2; transverse standing waves.

According to the simplified equations (16 – 20) in case of transverse standing waves $\eta = 0$ and differential equation of motion yields:

$$\ddot{u}_1 + h_{12x}\left(\dot{u}_1 - \dot{u}_2\right) + n_x^2\left(u_1 - u_2 - l_x\right) + h_{x_1}\dot{u}_1 + q_{x_1} +$$
$$+ \mu_{yx_1}\xi'_{1u_1}\left(\xi''_{1tt} + p_1^2\left(\xi_1 - l_1\right) + h_{y_1}\xi'_{1t} + q_{y_1}\right) + f_1^*\dot{u}_1 = 0;$$

$$\ddot{u}_2 - \mu_x\left(h_{12x}\left(\dot{u}_1 - \dot{u}_2\right) + n_x^2\left(u_1 - u_2 - l_x\right)\right) + h_{x_2}\dot{u}_2 + q_{x_2} +$$
$$+ \mu_{yx_2}\xi'_{2u_2}\left(\xi''_{2tt} + p_2^2\left(\xi_2 - l_2\right) + h_{y_2}\xi'_{2t} + q_{y_2}\right) + \mu_x f_1^*\dot{u}_2 = 0; \qquad (81)$$

where it is assumed that $f_0 = 0$; $l_{12x} = l_x$.

In case of harmonic standing transverse waves determined by eq. (65) when $l_i = h_{y_i} = q_{y_i} = 0$; i = 1,2; the steady state regime yields from eq. (81):

$$n_x^2(\bar{u}_1 - \bar{u}_2 - l_x) + q_{x_1} - 0.25\mu_{yx_1}(p_1^2 - \omega^2)kB^2 \sin 2k\bar{u}_1 = 0;$$
$$-\mu_x n_x^2(\bar{u}_1 - \bar{u}_2 - l_x) + q_{x_2} - 0.25\mu_{yx_2}(p_2^2 - \omega^2)kB^2 \sin 2k\bar{u}_2 = 0. \quad (82)$$

Eq. (82) leads to:

$$\mu_x q_{x_1} + q_{x_2} - 0.25kB^2 \begin{pmatrix} \mu_x \mu_{yx_1}(p_1^2 - \omega^2)\sin 2k\bar{u}_1 + \\ \mu_{yx_2}(p_2^2 - \omega^2)\sin 2k\bar{u}_2 \end{pmatrix} = 0. \quad (83)$$

In case when

$$u_2 = u_1 + \frac{\lambda}{4}, \quad (84)$$

where $\lambda = \frac{2\pi}{k}$ is the wavelength, eq. (83) yields:

$$\sin 2k\bar{u}_1 = \frac{\mu_x q_{x_1} + q_{x_2}}{0.25kB^2\left(\mu_x \mu_{yx_1}(p_1^2 - \omega^2) - \mu_{yx_2}(p_2^2 - \omega^2)\right)},$$

or by taking into account (18, 20):

$$\sin 2k\bar{u}_1 = \frac{Q_{x_1} + Q_{x_2}}{0.25kB^2\left((m_1 - m_2)\omega^2 - (c_1 - c_2)\right)}. \quad (85)$$

The condition of existence of such regimes is:

$$\left|\frac{Q_{x_1} + Q_{x_2}}{0.25kB^2\left((m_1 - m_2)\omega^2 - (c_1 - c_2)\right)}\right| < 1, \quad (86)$$

and their condition of stability on the basis of eq. (81, 83-85) is:

$$\left((m_1 - m_2)\omega^2 - (c_1 - c_2)\right)\cos 2k\bar{u}_1 < 0. \quad (87)$$

In case when

$$\left((m_1 - m_2)\omega^2 - (c_1 - c_2)\right) > 0, \quad (88)$$

stable regimes are located at $k\bar{u}_1 \in \left(\frac{\pi}{4}; \frac{\pi}{2}\right)$ and $k\bar{u}_1 \in \left(\frac{5\pi}{4}; \frac{3\pi}{2}\right)$; and unstable – at

$$k\bar{u}_1 \in \left(0; \frac{\pi}{4}\right) \text{ and } k\bar{u}_1 \in \left(\pi; \frac{5\pi}{4}\right). \quad (89)$$

In case when

$$\left((m_1 - m_2)\omega^2 - (c_1 - c_2)\right) < 0, \quad (90)$$

stable regimes are located at $k\bar{u}_1 \in \left(\frac{3\pi}{4}; \pi\right)$ and $k\bar{u}_1 \in \left(\frac{7\pi}{4}; \pi\right)$; and unstable – at

$$k\bar{u}_1 \in \left(\frac{\pi}{2}; \frac{3\pi}{4}\right) \text{ and } k\bar{u}_1 \in \left(\frac{3\pi}{2}; \frac{7\pi}{4}\right). \quad (91)$$

Case 3: n= 2; longitudinal standing waves.

According to the simplified equations (16 – 20) and at $\xi = 0$ the differential equations of motion yield:

$$\ddot{u}_1 + \left(h_{12x} + h_{x_1} + f_1^*\right)\dot{u}_1 - h_{12x}\dot{u}_2 + \eta''_{1tt} + \left(h_{12x} + h_{x_1} + f_1^*\right)\eta'_{1t} -$$
$$- h_{12x}\eta'_{2t} + q_{x_1} + n_x^2(u_1 + \eta_1 - u_2 - \eta_2 - l_x) = 0,$$

$$\ddot{u}_2 + \left(\mu_x h_{12x} + h_{x_2} + \mu_x f_1^*\right)\dot{u}_2 - \mu_x h_{12x}\dot{u}_1 + \eta''_{2tt} +$$
$$+ \left(\mu_x h_{12x} + h_{x_2} + \mu_x f_1^*\right)\eta'_{2t} -$$
$$- \mu_x h_{12x}\eta'_{1t} + q_{x_2} - \mu_x n_x^2(u_1 + \eta_1 - u_2 - \eta_2 - l_x) = 0. \quad (92)$$

In case when the excitation is in the form of longitudinal standing harmonic wave, then according to (65) and ignoring dissipative elements produces:

$$n_x^2(\bar{u}_1 - \bar{u}_2 - l_{x_1}) + q_{x_1} - 0.25kA^2(\omega^2 - 2n_x^2)\sin 2k\bar{u}_1 = 0;$$
$$-\mu_x n_x^2(\bar{u}_1 - \bar{u}_2 - l_{x_1}) + q_{x_2} + 0.25kA^2(\omega^2 - 2\mu_x n_x^2)\sin 2k\bar{u}_1 = 0. \quad (93)$$

From (93) it is obtained

$$0.25kA^2(1 - \mu_x)\omega^2 \sin 2k\bar{u}_1 + \mu_x q_{x_1} + q_{x_2} = 0,$$

and the following conditions of existence and stability are obtained:

$$\left|\frac{Q_{x_1} + Q_{x_2}}{0.25kB^2(m_1 - m_2)\omega^2}\right| < 1,$$

$$(m_1 - m_2)\cos 2k\bar{u}_1 > 0. \quad (94)$$

Stable points are the nodes in which the larger mass is located.

Conclusions

A sequential chain comprised from masses linked with elastic elements is analysed. The masses of the chain are excited by propagating wave profile. Several particular cases are analysed in detail. When the propagating wave is periodic the steady state motion mode is analysed at the condition that the average velocity of the chain elements is equal to the velocity of wave propagation. Conditions of existence and stability of such motions are derived.

When the average velocity of the chain elements is small compared to the velocity of the propagating wave the steady state motion modes are analysed exploiting approximate analytical techniques. Average velocity, oscillations around the average velocity and the properties of solutions are analysed.

When the excitation wave is a standing wave the motions are analysed in the nodal and maximum velocity nodes. Motion characteristics, conditions of existence and stability are investigated.

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