Monte-Carlo Methods and the Step-Back Kalman Filter for Orbital State Estimation

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Abstract

This research assesses the performance of filtering schemes for tracking uncooperative satellites through space-based optical measurements, and identifies a simple and numerically stable methodology that ameliorates the poor performance of standard filtering schemes at a substantially reduced cost in comparison to nonlinear particle filter-based remedies. Traditional filtering schemes, such as the extended Kalman filter (EKF) and unscented Kalman filter (UKF), both diverge when tracking a resident space object (RSO) in geosynchronous orbit (GEO) when there is a long time duration between measurements. This divergence is identified as a consequence of nonlinearity in the dynamics and nonlinearity in the optical measurements, both of which cause the underlying density of the state to deviate from a Gaussian distribution. A Gaussian sum filter based on using a Gaussian mixture model (GMM) for the probability density function can be implemented in order to avoid this divergence, but this comes at a high computational cost and has numerical sensitivity problems under reasonable orbital conditions. An alternative filter algorithm has been developed, referred to as the extended step-back Kalman filter (ESBKF), which is shown to effectively track the RSO in GEO while avoiding the computational burden and numerical sensitivity of the GMM filter. This filter applies the measurement updates to statistics at a time in the past when the distribution was approximately Gaussian, and then propagates the updated statistics forward to the present. In this manuscript the mathematical structure and properties of the ESBKF are presented, and its utility is demonstrated on tracking an RSO in a GEO orbit with right-ascension and declination angle measurements from an observer satellite.

1 Introduction

The increasing abundance of orbital debris in GEO represents a significant challenge for the future of space travel and satellite operation. Avoiding collision events with high probability in real time will require orbit estimation algorithms that can utilize sparse observations while maintaining computational expediency [2]. For example, limited resources require tracking of objects in GEO with long time increments between observations. This long time increment allows for nonlinear dynamics to significantly deviate an initially Gaussian distribution, which estimates the debris location in state space, far from the Gaussian class.

The predominant filtering algorithm used for tracking at low computational cost is the Extended Kalman filter (EKF). The EKF propagates forward-in-time between measurements and approximates the mean and covariance as having been generated through linear dynamics [3]. If the duration between measurements is short, the updated statistics when a measurement is received tracks well with how the Kalman filter (KF) was developed. The Kalman filter was created based on continuous linear dynamics and discrete linear measurements. If the initial distribution of the state is Gaussian, and the measurement distribution is also Gaussian, then the Kalman filter yields the optimal estimation of the state when a measurement

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is processed by combining the state density and the measurement density into a new Gaussian distribution and the resultant mean and covariance from this distribution are what are produced by the Kalman update equations. A problem arises when one of the distributions is not well-approximated as a Gaussian due to nonlinearity. The primary distribution of concern is that of the state vector, since most dynamical systems encountered in real-world situations contain intrinsic nonlinearity. Nonlinear measurement effects do have an impact but in simulations conducted, the state vector distribution is the primary culprit of failure of the EKF.

Several alternative filtering algorithms have been developed over the last several decades to attempt to resolve nonlinear effects. A class of filters called particle filters use nodes or particles generated from an initial Gaussian distribution and propagate the particles through the nonlinear dynamics to obtain posterior mean and covariance via sample statistics. A subset of particle filters are sigma-point filters, of which one of them is called the Unscented Kalman filter (UKF). The UKF creates sigma points which are particles obtained deterministically based on the matrix square root of the covariance matrix. Associated weights are assigned to each sigma point particle and propagate through the dynamical model. Deterministically obtaining the particles via sigma points rather than Monte-Carlo sampling methods allows for yielding at least second-order accurate statistics without the need for orders of magnitude more particles [4]. The UKF also breaks down, however, when the posterior distribution becomes severely skewed. It does, however, yield a larger post measurement covariance therefore compensating for bias in the mean. Another common algorithm, a Gaussian Sum filter or Gaussian Mixture model (GMM), is a particle filter which attempts to approximate the state vector distribution by a sum of weighted Gaussian distributions [1]. The idea is that each particle will propagate the Gaussian statistics forward in time linearly and following the EKF update procedure. It obtains a better posterior estimate of the mean and variance since the geometry of the distribution will more closely resemble the true distribution since the geometry, which has skewness, is provided by the location of the various Gaussian distributions. This algorithm is computationally expensive and also requires updating the weights of each particle via evaluation of various Gaussian distributions which has been found to be numerically sensitive.

Another algorithm has been developed in this research effort, the Extended Step-Back Kalman filter (ESBKF), which reduces the concern of approximating a non-Gaussian distribution of the state vector. The strategy for this algorithm is to apply the measurement update to the last point in time when the distribution is either exactly Gaussian or well-approximated as a Gaussian immediately following a measurement update, and then propagating the new distribution forward to the present. The ESBKF avoids the need to approximate a non-Gaussian distribution by maintaining a more precise and accurate Gaussian approximation for the state vector, and it does this at a computational cost on the same order of magnitude as the EKF. We will: (1) explain the development of the ESBKF; (2) demonstrate that, with linear dynamics and linear measurements, it coincides exactly with the standard Kalman Filter; (3) illustrate its utility in surmounting the shortcomings of other filtering strategies for this problem.

2 Kalman Filter Background

At each step, the Kalman filter [3] begins by propagating the mean of the state vector $\hat{X}$ and its covariance matrix P forward in time by,

$$\hat{X}(t^-) = \Phi(t, t_0)\hat{X}(t_0^-), \tag{1}$$

$$\begin{aligned} P(t^-) &= \Phi(t, t_0)P(t_0^-)\Phi^T(t, t_0) \\ &\quad + \int_{t_0}^{t} \Phi(\tau, t_0)GQG^T\Phi^T(\tau, t_0)\,\mathrm{d}\tau, \end{aligned} \tag{2}$$

where Φ is the state transition matrix of the dynamical system.

Once a measurement $\tilde{Z}$ is available, the statistics are updated by

$$\begin{aligned} K(t) &= P(t^-)H^T(t)[H(t)P(t^-)H^T(t) + R]^{-1}, \\ \hat{X}(t^+) &= \hat{X}(t^-) + K(t)(\tilde{Z} - \hat{Z}), \\ P(t^+) &= [I - K(t)H(t)]P(t^-)[I - K(t)H(t)]^T \\ &\quad + K(t)RK^T(t), \end{aligned} \tag{3}$$

where H is the measurement geometry matrix, and R is the covariance matrix of the measurement noise.

Derivation of the Kalman update equations (3) was based on both the state vector and measurement probability density functions being Gaussian. The dynamics were also assumed linear, so that if the initial distribution is Gaussian, it will remain Gaussian as it propagates forward-in-time. Below we will find that the step-back Kalman filter that we derive in this work will match the Kalman filter definition when both the dynamics and measurement models are linear.

2.1 Extended Kalman filter

In the situation where nonlinearities are present in the dynamics and measurement model, the Kalman filter can be reformulated based on first order approximations. This reformulation is called the Extended Kalman filter [3]. Here we review its derivation for the purpose of exposing the structure of our modified approach. The first order approximation of the propagation of the statistics through nonlinear dynamics is as follows. The nonlinear dynamics are defined by the integral equation

$$X(t^-) = X(t_0^-) + \int_{t_0}^{t} \dot{X}(X(\tau), \tau)\, \mathrm{d}\tau. \tag{4}$$

The State Transition Matrix (STM) Φ is also the Jacobian of the dynamics, and is given by,

$$\Phi(t, t_0^-) = \left.\frac{\partial X(t)}{\partial X(t_0)}\right|_{X(t_0)=\hat{X}(t_0^-)}. \tag{5}$$

It is calculated by integrating through time based on a formulation from the Leibniz's Integral Rule. The Leibniz's Integral Rule is given by

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^b f(x,t)\,\mathrm{d}t\right) = \int_a^b \frac{\partial}{\partial x} f(x,t)\,\mathrm{d}t.$$

Applying the chain rule to obtain the derivative with respect to the initial condition(s),

$$\frac{\mathrm{d}}{\mathrm{d}x_0}\left(\int_a^b \mathbb{E} f(x,t)\,\mathrm{d}t\right) = \int_a^b \left(\frac{\partial}{\partial x} f(x,t)\right)\left(\frac{\partial x}{\partial x_0}\right)\mathrm{d}t. \tag{6}$$

Therefore, the STM is calculated by combining (4), (5), and (6), to give

$$\Phi(t, t_0^-) = I + \int_{t_0}^{t} \frac{\partial \dot{X}}{\partial X} \Phi(\tau, t_0^-)\,\mathrm{d}\tau.$$

Next recall that the statistics of the state vector at t_0 are defined by

$$\begin{aligned}
&\hat{X}(t_0^-) = \mathbb{E}[X(t_0^-)],\\
&P(t_0^-) = \mathbb{E}\Big[(X(t_0^-) - \mathbb{E}[X(t_0^-)])(X(t_0^-) - \mathbb{E}[X(t_0^-)])^T\Big].
\end{aligned}$$

Applying a first order Taylor expansion to the dynamics centered around $\hat{X}(t_0^-)$, we find

$$X(t^-) = \hat{X}(t^-) + \Phi(t, t_0^-)(X(t_0^-) - \hat{X}(t_0^-)).$$

The propagated mean value and covariance matrix will then be calculated by using the expectation operator. First, the mean value will be determined,

$$\begin{aligned}
\mathbb{E}[X(t^-)] &= \mathbb{E}[\hat{X}(t^-) + \Phi(t, t_0^-)(X(t_0^-) - \hat{X}(t_0^-))]\\
&= \mathbb{E}[\hat{X}(t^-)] + \Phi(t, t_0^-)\mathbb{E}[X(t_0^-)] - \Phi(t, t_0^-)\mathbb{E}[\hat{X}(t_0^-))]\\
&= \hat{X}(t^-).
\end{aligned}$$

Then the covariance matrix before incorporating process noise is obtained as follows,

$$\begin{aligned}
P(t^-) &= \mathbb{E}[(X(t^-) - \mathbb{E}[X(t^-)])(X(t^-) - \mathbb{E}[X(t^-)])^T]\\
&= \mathbb{E}[\Phi(t, t_0^-)(X(t_0^-) - E[X(t_0^-)])\\
&\quad \cdot (X(t_0^-) - \mathbb{E}[X(t_0^-)])^T \Phi^T(t, t_0^-)]\\
&= \Phi(t, t_0^-)\mathbb{E}(X(t_0^-) - \mathbb{E}[X(t_0^-)])\\
&\quad \cdot (X(t_0^-) - \mathbb{E}[X(t_0^-)])^T]\Phi^T(t, t_0^-)\\
&= \Phi(t, t_0^-)P(t_0^-)\Phi^T(t, t_0^-)
\end{aligned}$$

Adding the process noise to the covariance matrix will give the same result as (2).

$$\begin{aligned}
P(t^-) =& \Phi(t, t_0^-)P(t_0^-)\Phi^T(t, t_0^-)\\
&+ \int_{t_0}^{t} \Phi(\tau, t_0^-) G Q G^T \Phi^T(\tau, t_0^-)\,\mathrm{d}\tau.
\end{aligned}$$

When a measurement is available, the mean and covariance will be updated following the same linearization methodology. The nonlinear measurement model and measurement Jacobian are the following, respectively,

$$
\begin{aligned}
Z =& g(X(t)), \\
H(t) =& \frac{\partial Z}{\partial X(t)}\bigg|_{X(t)=\hat{X}(t)}.
\end{aligned}
$$

In the same manner as the original Kalman filter, the update equations are the following,

$$
\begin{aligned}
K(t) &= P(t^-)H^T(t)[H(t)P(t^-)H^T(t) + R]^{-1}, \\
\hat{X}(t^+) &= \hat{X}(t^-) + K(t)(\tilde{Z} - \hat{Z}), \\
P(t^+) &= [I - K(t)H(t)]P(t^-)[I - K(t)H(t)]^T \\
&\quad + K(t)RK^T(t).
\end{aligned}
$$

2.2 Divergence of Extended Kalman filter

When tracking a RSO in GEO through space-based optical measurements, when the duration between measurements is small, the linear approximation behind the EKF is accurate, and it yields a covariance matrix such that when implemented into calculating the Kalman Gain matrix, the mean and covariance are correctly updated after a measurement. However, when the duration between measurements is too long, equation (3) loses fidelity, and the propagated covariance matrix is not sufficiently accurate. When this happens, the updated mean value can be moved farther from the truth and the covariance matrix can become overly confident about the wrong mean value. As seen in Figure 1, this leads to skewed filter statistics after the measurement update, and subsequent filter divergence.

The reason why the updated state distribution is not reasonably accurate when using the EKF update equations is that the intersection of the measurement distribution and the state vector Gaussian approximation places the updated distribution in the incorrect location. The correct location is where the measurement distribution intersects the true state vector distribution, which has skewness and is non-Gaussian. Figure 2 illustrates the density functions utilized in the EKF and how it compares with the true distribution.

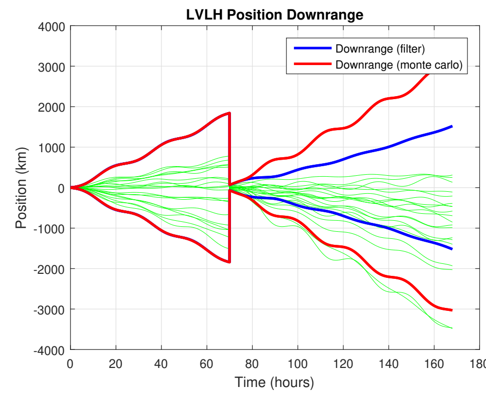

Figure 1: Extended Kalman filter divergence

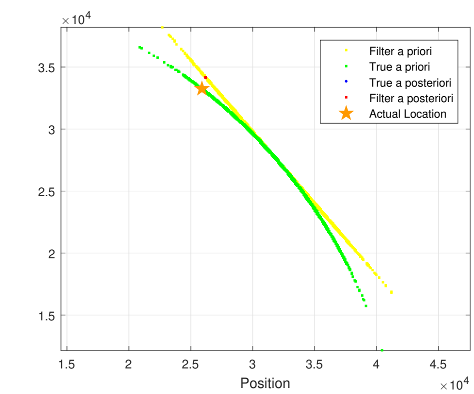

Figure 2: Extended Kalman filter density functions

3 Unscented Kalman filter

As a widely used alternative filtering algorithm for handling nonlinear dynamics, the UKF can resolve the divergence problem if skewness in the state vector distribution is not too severe. This is shown in Figure 3. However, the UKF diverges when updates are conducted at a time when the state vector distribution is severely skewed. The covariance matrix propagated by the sigma points is larger that what the EKF yields when skewness in the true distribution is inherited. This is illustrated in Figure 4, which also demonstrates the true distribution having skewness and the Gaussian distribution updated by UKF algorithm.

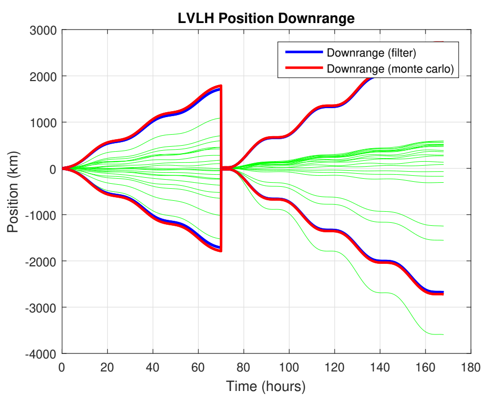

Figure 3: Unscented Kalman filter convergence

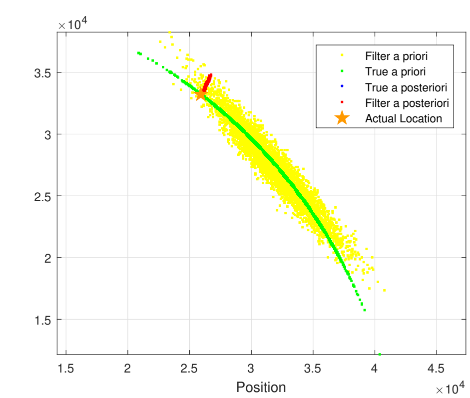

Figure 4: Unscented Kalman filter density functions

The updated statistics shown in Figure 4 lead to divergence when propagated forward in the future. This divergence behavior is illustrated in Figure 5. The break-

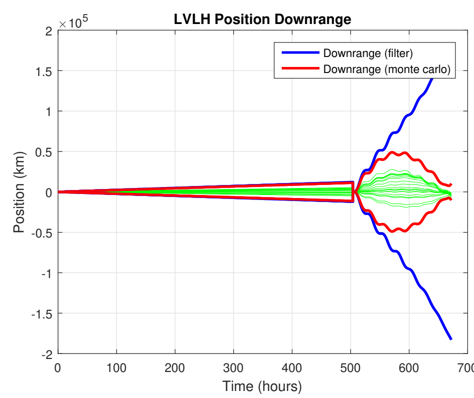

Figure 5: Unscented Kalman filter divergence

down of traditional Kalman filtering methods for tracking an RSO in GEO provides our motivation to obtain some alternative means to acquire more accurate statistics.

4 Step-Back Kalman filter Equivalence for a Linear System

The premise for the SBKF is to apply the measurement update at a time in the past where the probability distribution of the state is Gaussian, or at least well-approximated by a Gaussian. The distribution at the start of all the previously described filter algorithms is modeled as a Gaussian. After a measurement update is performed, the resulting mean and covariance are then modeled as describing a new Gaussian distribution. The SBKF begins exactly the same as the Kalman filter by propagating forward in time and obtaining a measurement prediction, $\hat{Z}$, and measurement Jacobian, $H(t)$. Using the chain rule, the measurement Jacobian with respect to the state vector at time t_0 can be calculated by

$$H(t_0) = H(t)\Phi(t, t_0). \tag{7}$$

Also, the contribution of process noise added to the covariance matrix during propagation from t_0 to t is given as,

$$P_q = \int_{t_0}^{t} \Phi(\tau, t_0) G Q G^T \Phi^T(\tau, t_0) \, \mathrm{d}\tau.$$

Applying $H(t_0)$ and P_q to find the Kalman update at time t_0 and corresponding statistics at time t is given by

$$\begin{aligned}
P_{q0} &= \Phi^{-1}(t, t_0) P_q \Phi^{-T}(t, t_0), \\
K(t_0) &= [P(t_0^-) + P_{q0}] H^T(t_0) \\
&\quad \cdot [H(t_0)[P(t_0^-) + P_{q0}] H^T(t_0) + R]^{-1}, \\
\hat{X}(t_0^+) &= \hat{X}(t_0^-) + K(t_0)(\tilde{Z} - \hat{Z}), \\
\hat{X}(t^+) &= \Phi(t, t_0)\hat{X}(t_0^+), \\
P(t_0^+) &= [I - K(t_0)H(t_0)][P(t_0^-) + P_{q0}] \\
&\quad \cdot [I - K(t_0)H(t_0)]^T + K(t_0) R K^T(t_0), \\
P(t^+) &= \Phi(t, t_0) P(t_0^+) \Phi^T(t, t_0).
\end{aligned} \tag{8}$$

In the case where the both the dynamics and measurement models are linear, the SBKF can be shown to yield identical results as the Kalman filter for $\hat{X}(t^+)$ and $P(t^+)$. Equation set (8) will match equation set (3) in a linear system as demonstrated by the following arguments.

Inserting the measurement Jacobian relationship (7) into the Kalman Gain at t_0,

$$\begin{aligned}
K(t_0) = [P(t_0^-) &+ P_{q0}] \Phi^T(t, t_0) H^T(t) \\
&\cdot [H(t)\Phi(t, t_0)[P(t_0^-) + P_{q0}] \\
&\cdot \Phi^T(t, t_0) H^T(t) + R]^{-1}.
\end{aligned}$$

With some simplifcation, we find

$$\begin{aligned}
\Phi(t, t_0)[P(t_0^-) + &P_{q0}]\Phi^T(t, t_0) \\
&= \Phi(t, t_0) P(t_0^-) \Phi^T(t, t_0) \\
&\quad + \Phi(t, t_0) P_{q0} \Phi^T(t, t_0) \\
&= P(t^-).
\end{aligned} \tag{9}$$

Therefore, if $K(t_0)$ is multiplied by $\Phi(t, t_0)$, then

$$\Phi(t, t_0) K(t_0) = K(t). \tag{10}$$

Applying (10) toward $\hat{X}(t^+)$ as given in (8),

$$\begin{aligned}
\hat{X}(t^+) =& \Phi(t, t_0)\hat{X}(t_0^+) \\
=& \Phi(t, t_0)[\hat{X}(t_0^-) + K(t_0)(\tilde{Z} - \hat{Z})] \\
=& \hat{X}(t^-) + K(t)(\tilde{Z} - \hat{Z}).
\end{aligned} \tag{11}$$

This matches $\hat{X}(t^+)$ as given in (3). Now, the covariance matrix $P(t^+)$ will be demonstrated to match by similar arguments. We modify expressions as follows

$$\begin{aligned}
\Phi(t, t_0)&[I - K(t_0)H(t_0)] \\
&= \Phi(t, t_0)[I - K(t_0)H(t)\Phi(t, t_0)] \\
&= \Phi(t, t_0)[I - K(t_0)H(t)\Phi(t, t_0)] \\
&= \Phi(t, t_0) - K(t)H(t)\Phi(t, t_0) \\
&= [I - K(t)H(t)]\Phi(t, t_0),
\end{aligned} \tag{12}$$

By expanding $P(t^+)$ represented in (8) and applying (9), (10), and (12),

$$\begin{aligned}
&P(t^+) \\
&\quad = \Phi(t, t_0)\Big[[I - K(t_0)H(t_0)][P(t_0^-) + P_{q0}] \\
&\qquad\qquad \cdot [I - K(t_0)H(t_0)]^T \\
&\qquad\qquad + K(t_0) R K^T(t_0)\Big]\Phi^T(t, t_0) \\
&\quad = [I - K(t)H(t)]\Phi(t, t_0)[P(t_0^-) + P_{q0}] \\
&\qquad \cdot \Phi^T(t, t_0)[I - K(t)H(t)]^T \\
&\qquad \cdot \Phi(t, t_0) K(t_0) R K^T(t_0) \Phi^T(t, t_0) \\
&\quad = [I - K(t)H(t)] P(t^-) [I - K(t)H(t)]^T \\
&\qquad + K(t) R K^T(t).
\end{aligned} \tag{13}$$

Therefore, $P(t^+)$ in (13) matches the same expression in (3). The equivalence between the Kalman filter and the Step-Back Kalman filter in a linear system has now been demonstrated, however, the primary purpose of development of the SBKF is the significant advantage over the KF when using the extended forms, EKF and ESBKF. The extended forms are used when nonlinear dynamics and/or nonlinear measurements are present.

5 Extended Step-Back Kalman filter

Nonlinearities present in either or both the dynamics and measurement model, an extended form of the SBKF can be formulated and implemented. The ESBKF follows the same linearization approach as the EKF.

$$H(t_0) = H(t)\Phi(t, t_0^-).$$

Also, the contribution of process noise added to the covariance matrix during propagation from t_0 to t is given as,

$$P_q = \int_{t_0}^{t} \Phi(\tau, t_0^-) G Q G^T \Phi^T(\tau, t_0^-)\, \mathrm{d}\tau,$$

Following the update procedure in (8), the ESBKF methodology for updating statistics at time t is given as follows:

$$\begin{aligned}
H(t_0) =& H(t)\Phi(t, t_0^-),\\
P_{q0} =& \Phi^{-1}(t, t_0^-) P_q \Phi^{-T}(t, t_0^-),\\
K(t_0) =& [P(t_0^-) + P_{q0}] H^T(t_0)\\
& [H(t_0)[P(t_0^-) + P_{q0}] H^T(t_0) + R]^{-1},\\
\hat{X}(t_0^+) =& \hat{X}(t_0^-) + K(t_0)(\tilde{Z} - \hat{Z}),\\
\hat{X}(t^+) =& \hat{X}(t_0^+) + \int_{t_0}^{t} \dot{X}(X(\tau), \tau)\, \mathrm{d}\tau,\\
\Phi(t, t_0^+) =& I + \int_{t_0}^{t} \frac{\partial \dot{X}}{\partial X} \Phi(\tau, t_0^+)\, \mathrm{d}\tau.\\
P(t^+) =& \Phi(t, t_0^+) P(t_0^+) \Phi^T(t, t_0^+).
\end{aligned}$$

5.1 ESBKF Results

When nonlinear effects are present in a dynamical model, an initially Gaussian distribution may evolve into a distribution with non-zero skewness. Approximating this distribution as a Gaussian is essentially what the EKF and UKF attempt to do but if the actual location of the state vector is in a region where a Gaussian distribution is a poor approximation, and the Kalman update equations will not yield an accurate updated location. Avoidance of having to deal with a non-Gaussian distribution for the state vector would be ideal which is the primary motive behind the development of the ESBKF.

To illustrate ESBKF resolves this issue of divergence of the EKF, the same orbital mechanics model was considered involving an observer satellite tracking an RSO in GEO by using optical angle measurements. Sample statistics were generated and used to compare the probability distributions generated by the EKF, ESBKF, and the skewed distribution from the propagation through nonlinear orbital dynamics. Initial statistics of the GEO object and the time when the measurement is taken were selected to give a clear demonstration of divergence and improvement by the ESBKF. The density functions generated are depicted in Figure 6.

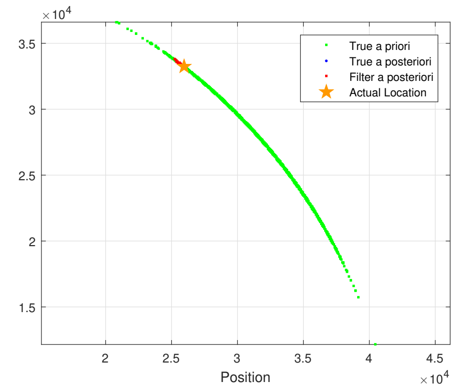

Figure 6: Step-Back Kalman filter density functions

The two optical angle measurements taken by the observer satellite essentially yield a line of sight from the observer to the GEO object. This line of sight does not however give any direct measurement for distance along the line of sight vector. Therefore, when the EKF generates an update it does so by placing a higher weight at the intersection of the line of sight and the assumed Gaussian distribution of the state. To expand further, the Kalman update equations were derived based on the recognition that the resultant probability distribution from the product of two Gaussian distributions, in this case for the state vector and the measurement model, is a new unique Gaussian distribution. In the orbital dynamics simulation, the measurement model has a Gaussian distribution in the angle measurement domain which translates to a non-Gaussian distribution in the state domain but is located in the neighborhood of the line of sight from the observer to the GEO object. The intersection of the measurement distribution and the assumed Gaussian distribution for the state generates a Gaussian distribution whose location deviates from the true location because the assumption that the state is Gaussian distributed is not valid at the time when the measurement is received (at three weeks). In contrast, the updated distribution in Figure 6 lies within the true non-Gaussian state density function prior to the update. Therefore, the filter statistics forward from this point in the future will yield convergent results, as seen in Figure

7.

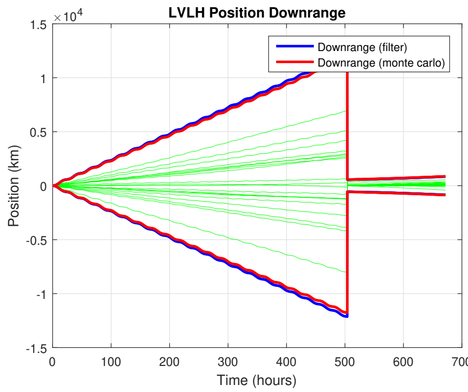

Figure 7: Extended Step-Back Kalman filter convergence

6 Unscented Step-Back filter

An unscented form for the Step-Back Kalman filter has also been developed in this research effort and has allowed for handling not only the nonlinear dynamics better, which the ESBKF does effectively, but the nonlinear measurements as well. Essentially the algorithm propagates deterministically selected particles forward in time and when a measurement is received, the measurement estimates for each particle are applied to each particle at the previous point in time in the past when the state distribution was well-approximated as a Gaussian distribution. The updated particles are then propagated forward to the current time and from that point the filter sets the current time as the point when the next measurement will be applied.

7 Monte-Carlo Particle filter

Monte-Carlo particle filters generate a large number of state vectors $\{x_i\}$ sampled from initial conditions, referred to as particles. These particles are propagated forward in time and the filter statistics are the sample statistics computed from these particles, namely the sample mean

$$\hat{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

and sample covariance

$$P = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \hat{x})(x_i - \hat{x})^T. \tag{14}$$

An example is a Gaussian Sum filter based on a GMM, for which deterministic particles or nodes are selected based on the initial statistics and propagated forward in time [1]. The measurement update relies on the assumption that each particle has a unique underlying Guassian distribution for the state and therefore a weighted update for the statistics is carried out. This methodology allows for a more accurate representation of the true distribution prior to an update because the collection of weighted Gaussian distributions fills out more of the geometry of the actual distribution. A major drawback for this algorithm is that, if the particle number is too small, it is numerically sensitive when the assigned weights for each particle are updated after a measurement is received. It was found that a GEO orbit model only allowed for numerically stable weight updates with reasonable particle numbers to be obtained at points in time when even the EKF was functioning effectively. The premise behind the GMM is sound but due to numerical limitations and computational burdens of handling large numbers of particles, the ESBKF was found to resolve the convergence issue without the numerical sensitivity or computational cost constraints.

8 Conclusion

Computationally efficient tracking and surveillance of RSOs in orbit around Earth, particularly in GEO where communication satellites are most present, is critical for the future of space exploration. The traditionally used low cost filtering algorithm for nonlinear dynamical behavior is the EKF but it has limitations based on the underlying assumptions of linearized dynamics and measurements and presumed Gaussianity of distributions. For short durations between measurements, the EKF handles the filtering process well, but once nonlinear effects become significant, the algorithm will place an updated state estimate in the incorrect location. Two particle filter algorithms, the UKF and GMM, both attempt to avoid the divergence problem by more accurately representing the state distribution prior to a measurement update by increasing the covariance of the Gaussian distribution or modeling with multiple weight Gaussian distributions, respectively. Each of these have limitations and will succumb to nonlinear effects of gravitational nonlinear dynamics. A new technique, the ESBKF, was developed in this research to mitigate the effect of the nonlinear dynamics. The primary idea behind this algorithm is that when a measurement is available to process for an update of the state vector statistics, the update is applied to the last point in time when the state distribution is known to be well-approximated as a Gaussian (for example, the last time an update was performed), and the filter is then propagated forward. This results in a distribution post-update that resides near the actual location of the RSO being tracked, which was validated for an RSO in GEO by Monte-Carlo simulation. Furthermore, the computational burden of the ESBKF is on the same order of magnitude as the EKF.

References

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