PERTURBATION RESULTS AND THEIR APPLICATION TO THE STABILITY OF EVOLUTION EQUATIONS

by

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A Thesis

for the degree of Doctor of Philosophy submitted to the University of Warwick incorporating research conducted in the School of Engineering Science.

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I should like to express my warmest thanks to Dr A J Pritchard for suggesting this work and for the many stimulating discussions that I have had with him during its evolution. I should also like to thank Mrs Hilary Rodwell who produced an excellent typed copy of a difficult manuscript.

DECLARATION

The work presented in this thesis is original, with the exceptions stated below, and has not been submitted for another degree of this or any other University. The exceptions are:

1. The work contained in chapters 2 and 3 which provide some basic definitions and results in functional analysis and the theory of abstract evolution equations.
2. The perturbation theorems described in Chapter 4 along with the contents of sections 2 and 3 of Chapter 5 were described in the Control Theory Centre Report No. 45 of the University of Warwick. Some of the contents of this report had appeared earlier as a joint paper by A J Pritchard and W T F Blakeley which was presented at the IUTAM Symposium held at Marseilles in 1975. The paper was subsequently published in 'Lecture notes in Mathematics' Vol 503 by Springer-Verlag. A second joint paper has been accepted for presentation at the 2nd IFAC Symposium to be held at the University of Warwick in June 1977. This paper is based on the Control Centre Report and will be published as part of the Symposium proceedings.
3. The contents of Chapter 6 are based essentially on some results of Pritchard and Ichikawa (to be published).
4. The Liapunov theory for linear operators in Chapter 7 is not new. The results for non-linear operators are due to A J Pritchard and have not been published previously.
5. The examples used as illustrations of the theorems and results of the main body of the thesis are not new. References are given in each case of the source of each example where applicable.

4.6.77.

Summary

In this thesis a stability analysis of the abstract evolution equation is presented along with a discussion of the related questions of the existence and uniqueness of a global solution of such equations. The results are presented in a functional analytic framework using the theory of operators in normed linear spaces. The approach employed is that of perturbation theory, the evolution equation is taken to be of the form

(i) $$\dot{z}(t) + A(z,t) + B(z,t) = 0\,, \quad z(0) = z_o$$

with the operator $A(z,t)$ defined so that the solution of

(ii) $$\dot{z}(t) + A(z,t) = 0\,, \quad z(0) = z_o$$

is stable in the sense of Liapunov in the neighbourhood of the equilibrium state z_E.

For a strict solution of (i) there are already a number of perturbation theorems based on the theory of m-accretive operators which enable the size of allowable perturbations $B(z,t)$ to be determined which conserve the stability properties of the unperturbed system. The range of validity of these results has been extended first by showing that the conditions on $B(z,t)$ can be relaxed so increasing the size of allowable perturbations and then by deriving new theorems where the conditions are on $A+B$ rather than on B. The theorems are more relevant in this form since de-stabilizing perturbations can now be included in the applications. The operator A of (ii) may be non-linear which enables the results to be used to determine a class of allowable errors in the modelling of a physical system.

This thesis also contains proofs of the existence and uniqueness of a mild solution of (ii) using the concept of evolution operators. The underlying operator A is assumed to be linear but the perturbing operator can belong to one of several classes including those of unbounded linear operators and of non-linear operators. Estimates are made of the solution of the perturbed system to enable the stability properties of the solution to be deduced. The relation between these results and those obtained by the application of the methods of Liapunov is discussed. The results are applied to a number of problems in science and engineering.

§1 Introduction

The study of differential equations was initiated by Newton when he postulated his famous Laws of Motion, since that time the subject has developed into one of major importance because of the large number of physical laws that involve rates of change.

The classical methods of solution of differential equations were developed by Euler, Lagrange, Laplace and others, but the number of equations for which it is possible to find an explicit solution is relatively small. However, whether or not an explicit solution can be found, there are two fundamental questions that must be asked of any differential equation.

(i) Has the equation a solution? (The existence question)

(ii) Assuming an answer yes to (i), how many solutions are there? (The uniqueness question.)

In this thesis we are concerned with some answers to these questions when applied to partial differential equations that arise in the study of dynamical systems. We note here that many such systems can be modelled by a set of equations of the form

$$\frac{\partial z_i}{\partial t} + f_i = 0 \quad (i = 1, 2, \ldots k) \tag{1.1}$$

with a set of prescribed initial values of z_i, $z_i(0)$. The functions f_i may depend upon the time t, the dependent variables z_i $(i = 1, 2, \ldots k)$ and their derivatives with respect to the space variables x_j $(j = 1, 2, \ldots n)$. On introducing the vectors $z = (z_1, z_2, z_3, \ldots z_k)^T$ and $N(z,t) = (f_1, f_2, \ldots f_k)^T$ where the dependence upon z,t in N is in its most general sense as defined for the components f_i, we have one equation with a single initial condition

(1.2) $\dot{z} + N(z,t) = 0, \quad t \geq 0, \quad z(0) = z_o$

which we shall refer to as the general abstract evolution equation.

With evolution equations there is a third important property to be studied once the existence and uniqueness of the solution has been established and that is the behaviour of $z(t)$ as $t \to \infty$. In describing the latter we shall use the ideas and definitions of Liapunov, which were developed around 1890, and compare the solution $z(t)$ with that of the equilibrium state z_E which is the solution of the equation

$$N(z_E,t) = 0, \quad t \geq 0.$$

In particular if $\|z(t) - z_E\|$ is finite $\forall t$ and $\to 0$ as $t \to \infty$ for all initial states z_o in a neighbourhood of z_E then we shall say that the equilibrium point z_E is asymptotically stable and that the neighbourhood is a region of asymptotic stability. It is possible to discuss all three aspects of evolution equations without necessarily having a knowledge of explicit solution of the equation. This is particularly important when $N(z,t)$ is non-linear in z because it is usually impossible to solve (1.2) except by numerical techniques.

In this thesis we suppose that $N(z,t)$ is of the form

$$A(z,t) + B(z,t)$$

and require that most of the properties of $A(z,t)$ are known. In many of the examples $A(z,t)$ will be the operator obtained from $N(z,t)$ by the linearization method and be such that

(1.3) $\dot{z}(t) + A(z,t) = 0 \quad z(0) = z_o.$

is stable in the sense of Liapunov. The operator $B(z,t)$, which we shall refer to as the perturbation of the system (1.3), will be assumed to satisfy the condition $B(z_E,t) = 0$ for all $t \geq 0$. In addition to

(1.2) $\dot{z} + N(z,t) = 0, \quad t \geq 0, \quad z(0) = z_o$

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is stable in the sense of Liapunov. The operator $B(z,t)$, which we shall refer to as the perturbation of the system (1.3), will be assumed to satisfy the condition $B(z_E,t) = 0$ for all $t \geq 0$. In addition to analysing the uniqueness and existence of solutions of (1.3) we have attempted:

(iii) to find the conditions required of B(z,t) so that the perturbed system (1.2) is stable,

(iv) to find estimates of the solution of the perturbed system.

The results appertaining to (iii) are presented in the form of perturbation theorems and allow for A(z,t) to be a non-linear operator, but for (iv) most of the problems we examine will assume A(z,t) to be a linear operator.

The results obtained here under (iii) and (iv) have a wider applicability than just to the mathematical model (1.2). It is well known that in constructing a model of a complex physical system some effects or processes may be approximated or ignored and so the model is usually not capable of describing all of the dynamical characteristics of the actual system. The unidentified terms of the system may be regarded as a perturbation of the model (1.2) therefore once the stability and other properties have been established for the latter, we can use the perturbation theorems again to find the conditions that must be imposed on the unidentified part of the system so that the actual system has similar properties to that of the model. Similarly an unspecified forcing term can be included amongst the unknown perturbations of a system, our results for linear perturbations of a linear model include estimates of the effect of such terms.

The results obtained depend upon the definition of the term solution. In order to allow the greatest flexibility in this respect the material in this thesis is presented in a functional analytic framework, using in particular those ideas and results relating to the theory of operators in normed linear spaces. The definitions and basic facts of functional analysis required for the thesis are presented in Chapter §2 along with those from the Liapunov theory of stability. In §3 we present a survey of some of the more important known results concerning the existence and uniqueness of solutions of (1.2) in order to provide the background to the later chapters. In §4 we present the perturbation theorems which give existence and uniqueness results for stable solutions of (1.2) in the strictest sense of the term solution. These theorems include the case of A and B being non-linear operators as well as linear. In the next two chapters §5, §6 the idea of solution is relaxed slightly and we consider the existence and uniqueness of so-called 'mild' solutions of (1.2) along with methods of estimating the norm of the solution. These results are derived from the investigation of certain integral equations and are presented in two chapters. In §5 B(z,t) is taken to be linear whilst in §6, B(z,t) is assumed to be non-linear but in both cases A is a linear operator.

Many papers have been written applying the idea of Liapunov functionals to a study of the stability of partial differential equations. In chapter §7 we present some general results concerning these functionals and their use with (1.2) to illustrate an alternative approach to the methods of §5 to the problem of estimating the norm of the solution, it is shown that certain of the results for the linear equation can be obtained by either method.

In chapter §8 the results of §4-6 are applied to a wide variety of problems drawn from engineering and science. These include various formulations of the beam problem and two problems from chemical engineering where the basic system is described by a coupled pair of partial differential equations. Some conclusions and remarks are made in §9.

§2 Topics in Functional Analysis

2.1. Introduction

In this chapter we introduce those ideas and results of functional analysis which we require for the rest of the work, placing particular emphasis on the theory of mappings of a Banach space onto a Banach space. The proofs and further details can be found in the standard texts of Kato [1], Yosida [2] and Dunford and Schwartz [3].

2.2. Normed Linear Spaces

We commence with the definitions of the terms semi-norm, norm and normed linear space.

Definition 2.1. Let X be a linear space over a field of scalars K (real or complex). If for every $x \in X$ there is associated a real number $\|x\|$ such that for every $x,y \in X$

(2.1.)
(i) $\|x+y\| \leq \|x\|+\|y\|$
(ii) $\|\alpha x\| = |\alpha|.\|x\|$ for all $\alpha \in K$

then $\|.\|$ is called a semi-norm of X.

Definition 2.2. If $\|.\|$ is a semi-norm on a linear space X such that

(2.2) $\|x\| = 0$ iff $x = 0$

then $\|.\|$ is called a norm on X.

Definition 2.3. A linear space X with a norm $\|.\|$ defined on it is called a normed linear space.

The product XxY of two linear spaces X,Y over the same field of scalars K consists of all ordered pairs $\{x,y\}$ of elements $x \in X, y \in Y$. The set XxY is a vector space if the addition operation in XxY is defined by

$$\alpha_1\{x_1,y_1\} + \alpha_2\{x_2,y_2\} = \{\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 y_1 + \alpha_2 y_2\}$$

$\forall \alpha_1,\alpha_2 \in K$; $x_1,x_2 \in X$; $y_1,y_2 \in Y$. If, in addition X and Y are normed linear

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$\forall \alpha_1,\alpha_2 \in K$; $x_1,x_2 \in X$; $y_1,y_2 \in Y$. If, in addition X and Y are normed linear spaces with norms $\|.\|_X$, $\|.\|_Y$ respectively then XxY can be made a normed linear space. The norm in XxY may be chosen in many ways, the following are particularly useful

(2.3)
(i) $\|\{x,y\}\| = \|x\|_X + \|y\|_Y$, $\forall\ x\in X,\ y\in Y$
(ii) $\|\{x,y\}\| = (\|x\|_X^p + \|y\|_Y^p)^{1/p}$, $\forall\ x\in X,\ y\in Y,\ 1 \leq p \leq \infty$.

In any kind of analysis the idea of convergence is of fundamental importance but this requires a meaning for the distance d(x,y) between two elements x,y of a linear space satisfying the following axioms.

(2.4)
(i) $d(x,y) \geq 0$ and $d(x,y) = 0$ iff $x = y$
(ii) $d(x,y) \leq d(x,z) + d(z,y)$ (triangle inequality)
(iii) $d(x,y) = d(y,x)$

for all $x, y, z \in X$.

In a normed linear space this can be achieved by defining the distance function d(x,y) by

(2.5) $d(x,y) = \|x-y\|$.

Then we have the following definitions of convergence and of a Cauchy sequence.

<u>Definition 2.4.</u> A sequence $\{x_n\} \subseteq X$ converges strongly to x if $\lim_{n\to\infty} \|x_n-x\| = 0$. This is denoted by $x_n \to x$.

In a normed linear space X if $x_n \to x$, $y_n \to y$ and $\lim_{n\to\infty} \alpha_n = \alpha$, α_n scalars, then

(2.6)
(i) $\lim_{n\to\infty} \|x_n\| = \|x\|$
(ii) $\alpha_n x_n \to \alpha x$
(iii) $(x_n+y_n) \to x+y$

<u>Definition 2.5.</u> A sequence $\{x_n\} \subseteq X$ is a Cauchy sequence if given any $\varepsilon > 0$ there exists an integer $N = N(\varepsilon) > 0$ such that $\|x_m-x_n\| < \varepsilon$ for any $m,n > N$.

Every convergent sequence in X is a Cauchy sequence.

However not every Cauchy sequence converges to a limit $x \epsilon X$, This leads to the idea of completeness.

Definition 2.6. If every Cauchy sequence $\{x_n\} \epsilon X$ converges strongly to a limit $x \epsilon X$, then X is said to be complete. A complete normed space is called a Banach space (B-space).

One particular type of B-space of special interest later is a uniformly convex B-space.

Definition 2.7. A B-space is uniformly convex if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $\|x\| < 1$, $\|y\| < 1$ and $\|x-y\| \geqslant \varepsilon$ implies $\|x+y\| \leqslant 2(1-\delta)$.

We conclude this section by remarking that a given B-space X may have two (or more) norms defined on it. Equivalent norms are defined as follows.

Definition 2.8. If $\|.\|_1$, $\|.\|_2$ denote two different norms defined on a Banach space X then they are equivalent if $\exists \alpha, \beta$, $0 < \alpha < \beta < \infty$ such that

$$\alpha\|x\|_2 \leqslant \|x\|_1 \leqslant \beta\|x\|_2 \quad \text{for all } x \epsilon X.$$

2.3. Quasi-Norms and Fréchet Spaces

If the requirement (2.1) (ii) in the definition of a semi-norm is replaced by the weaker condition

$$(2.7) \qquad \|-x\| = \|x\|, \ \lim_{\alpha_n \to 0} \|\alpha_n x\| = 0 \text{ and } \lim_{\|x_n\| \to 0} \|\alpha x_n\| = 0$$

then $\|x\|$, now satisfying (2.1) (i), (2.2) and (2.7) is called a quasi-norm. The linear space X with quasi-norm $\|.\|$ defined on it is called a quasi-normed space. As with the norm of a normed space the concept of quasi-norm can be associated with the axioms of distance and used to establish the concept of strong convergence $\lim_{n\to\infty} d(x_n, x) = 0$, with the properties (2.4).

A complete quasi-normed space is called a Fréchet space, a B-space is a Fréchet space but not vice-versa.

2.4. Hilbert Spaces

In many applications of linear spaces the concept of norm on its own does not provide sufficient structure, in particular it is impossible to define angle in terms of norms. This difficulty is overcome by introducing the idea of inner product.

Definition 2.9. If to every pair of elements x,y belonging to a linear space H there is associated a complex number (x,y) satisfying

(2.8)

(i) $(\alpha x,y) = \alpha(x,y) \qquad \alpha\epsilon K$

(ii) $(x,y) = \overline{(y,x)}$ ($\overline{}$ denotes complex conjugate)

(iii) $(x+y,z) = (x,z) + (y,z)$

(iv) $(x,x) \geqslant 0$ and $(x,x) = 0$ iff $x = 0$

then (x,y) is called an inner (or scalar) product on H.

An inner product space H can be made into a normed linear space by defining an 'induced' norm $\|.\|$ on H with

(2.9) $\|x\| = (x,x)^{\frac{1}{2}}$.

A complete inner product space is called a Hilbert space. The following theorem provides a very useful test that can be applied to a Banach space to see if it is also a Hilbert space.

Theorem 2.1 If X is a Banach space and its norm satisfies the parallelogram law

(2.10) $\|x+y\|^2 + \|x-y\|^2 = 2\{\|x\|^2 + \|y\|^2\}$ for all $x,y\epsilon X$ then X is a Hilbert space.

It is easy to see from this formula that a Hilbert space is uniformly convex.

A Hilbert space may have two inner products defined on it, in which case we can define the concept of equivalent inner products.

Definition 2.10 If $(.,.)_1$, $(.,.)_2$ denote two distinct inner products on H then they are said to be equivalent if their corresponding norms are equivalent (c.f. Definition 2.8).

2.5 Subsets of a Banach space

Let A be a set contained in a Banach space X. A point $x \in X$ is a limit point of the set A iff there exists a sequence of distinct elements $\{x_n\} \subseteq A$ such that $\{x_n\}$ converges strongly to x. The set consisting of A and all its limit points is called the closure of A, denoted by $\overline{A}$. A set A is closed iff $A = \overline{A}$ and is said to be dense in X if $\overline{A} \equiv X$. If A is closed and dense then $A \equiv X$. The space X is separable if it has a countable dense subset.

Suppose $a \in A$ and $x \in X$ then $\|a-x\|$ is bounded from below for any $a \in A$ and therefore $\inf_{a \in A} \|a-x\|$ exists.

Definition 2.11 The quantity $\inf_{a \in A} \|a-x\|$ is called the distance of the point x from the set A and is denoted by $\rho(x,A)$. If A is closed and convex then there exists a point $b \in A$ such that

$$\rho(x,A) = \|x-b\|$$

We can now define the terms r - neighbourhood, bounded set and compact set.

Definition 2.12 The r - neighbourhood of a set $A \subset X$ is the set of all points $x \in X$ which have the property that

$$0 < \rho(x,A) < r$$

where $r > 0$ is a given number. This neighbourhood will be denoted by $S(A,r)$.

Definition 2.13 The set $A \subset X$ is called bounded if $\exists$ $M(<\infty)$ such that $\|x-y\| \leq M$ for all $x,y \in A$.

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Definition 2.14 The set $A \subset X$ is said to be compact in X if every infinite subset $B \subset A$ contains an infinite convergent sequence with limit in A.

2.6 Operators in Banach Spaces

Let A and B be two sets. The symbol $f:A \to B$ denotes a mapping or function that assigns to every element in A at least one element of B. Unless stated otherwise we shall assume that any function used in the following is single-valued, that is to each $a \in A$ there is assigned just one element $b \in B$. If in place of the two sets A and B we have two linear spaces U and V over the same field of scalars K then any mapping $T:u \to v = T(u) = Tu \in Y$ which maps all (or part of) U into V is called an operator or, in the special case when $V \equiv K$, a functional. The domain of T, D(T) is the set of all $u \in U$ such that there is a $v \in V$ for which $Tu = v$, the range of T,R(T) is the set $\{Tu:u \in D(T)\}$ and the kernel or null space of T is $N(T) = \{x:Tu = 0\}$. If D(T) is dense in X then T is said to be densely defined.

Definition 2.15 T is a linear operator (or linear functional if $V \equiv K$) if the domain D(T) is a linear subspace of U and if

$$T(\alpha u_1 + \beta u_2) = \alpha T(u_1) + \beta T(u_2) \quad \text{for all } \alpha, \beta \in K.$$

If T is linear and is a (1 - 1) map of D(T) onto R(T) there is an inverse operator T^{-1} which is a linear operator mapping R(T) onto D(T) such that

$$T^{-1} Tu = u \text{ for } u \in D(T) \text{ and } TT^{-1}v = v \text{ for } v \in R(T).$$

If T_1 and T_2 are two linear operators both on X into Y then $T_1 = T_2$ iff $D(T_1) = D(T_2)$ and $T_1x = T_2x$ for $x \in D(T_1) = D(T_2)$. If $D(T_1) \subseteq D(T_2)$ and $T_1x = T_2x$ for all $x \in D(T_1)$ then T_2 is called an extension of T_1 and

T_1 a restriction of T_2.

T is a closed linear operator if its graph G(T), where

$$G(T) = \{\{x,Tx\};\ x \in D(T),\ \{x,y\} \in X \times Y\}$$

is a closed linear subspace of $X \times Y$.

If T is a closed linear operator on X into X then $T-\xi I$ is also closed for all ξ in K.

In our work we shall be dealing with operators defined on normed linear spaces. The presence of the concept of norm allows the ideas of continuity and boundedness to be applied to such operators.

Definition 2.16 Let X,Y be normed linear spaces and let T be an operator on $D(T) \subseteq X$ into Y. T is continuous at $x_0 \in D(T)$ if given $\varepsilon > 0$ there exists $\delta > 0$ such that $\|Tx - Tx\|_Y < \varepsilon$ for all $x \in D(T)$ such that $\|x - x_0\|_X < \delta$. T is continuous if it is continuous at every point of D(T).

Definition 2.17 T is a bounded operator if there exists a constant M such that

(2.11) $\|Tx\| \leq M\|x\|$ for all $x \in D(T)$.

If T is bounded with constant M as in (2.11) then the result can be extended to all $x \in X$ if $\overline{D(T)} = X$, with the same M.

For a linear operator, continuity and boundedness are equivalent concepts.

The important concept of a contraction mapping is a special case of Definition 2.17.

Definition 2.18 The mapping F of a normed linear space into itself is a contraction mapping if there exists a k, $0 \leq k < 1$ such that

(2.12) $\|Fx - Fy\|_X < k\|x - y\|_X$ for all $x,y \in X$.

Theorem 2.2 Contraction mapping theorem

Let X be a Banach space and $F: X \to X$ a contraction mapping. Then there exists a unique point $x_0 \in X$ such that $Fx_0 = x_0$, x_0 is called the fixed point of F.

If T is a bounded operator the bound of T, denoted by $\|T\|$ is defined as

(2.13) $\|T\| = \sup(\|Tx\| : \|x\| \leq 1, \ x \in D(T))$.

The set of all bounded linear operators with domain X and range in Y and with norm defined as the bound of T constitutes a new normed linear space which is denoted by $\mathcal{L}(X,Y)$. If, further, Y is a Banach space then so is $\mathcal{L}(X,Y)$. When X = Y, $\mathcal{L}(X,X)$ is abbreviated to $\mathcal{L}(X)$. The topology induced on $\mathcal{L}(X,Y)$ by this norm is known as the uniform topology. There are alternative topologies produced by different norms. In particular there is the strong topology associated with the norm on the space Y, in this topology a sequence of operators $\{T_n\} \in \mathcal{L}(X,Y)$ converges to T iff $\{T_n x\}$ converges to Tx for every $x \in X$.

We consider now the special case of $\mathcal{L}(X,Y)$ where Y is the field K so that $\mathcal{L}(X,Y)$ is the space of all continuous linear functionals on X, called the dual (or conjugate) space of X and denoted by X^*. The topology on X^* in the sense of uniform topology on $\mathcal{L}(X,Y)$ described above is called the strong topology of X^* whilst the topology on X^* in the sense of the strong topology on $\mathcal{L}(X,Y)$ is called the weak* topology of X^*. Then X^* with the strong topology is denoted by X^*_s, the strong dual, whilst X^* with the weak* topology is denoted by $X^*_{W^*}$, the weak* dual.

The introduction of the dual space allows the introduction of the concept of weak convergence in $\mathcal{L}(X,Y)$. Denoting the value of the functional $x^* \in X^*$ at $x \in X$ by $\langle x, x^* \rangle$, a sequence $\{T_n\}$ is said to converge weakly if $\langle T_n x, f \rangle$ converges for each $x \in X$ and $f \in X^*$. For $\mathcal{L}(X,Y)$ uniform convergence implies strong convergence implies weak convergence.

We mention here a particular map F of X into X^* defined by

$$x^* \in Fx \quad \text{iff} \quad \langle x, x^* \rangle = \|x\|_X^2 = \|x^*\|_{X^*}^2$$

This map is called the duality map of X into X^*. F may be a multi-valued mapping although if X^* is uniformly convex then F(.)

is single-valued and uniformly continuous in every bounded set of X.

Since the dual space X^* is itself a normed linear space it is possible to form a dual space $(X^*)^*$ of X^*. This space is called the second dual space ofX and denoted by X^{**} By construction the elements of X^{**} are continuous linear functionals which map the linear functionals of X^* into K so if $F \in X^{**}$, $f \in X^*$

$$F(f) = \langle f, F \rangle = C \in K$$

$$= f(x)$$

for some $x \in X$. Thus to each $x \in X$ there corresponds an $F \in X^{**}$ the F corresponding x will be denoted by F_x

It can be shown that the uniform norm of F, $\|Fx\|$, is equal to $\|x\|$ so that the mapping $x \to F_x$ preserves norms. Also $x \to Fx$ is a linear map and is thus an isometric isomorphism so that X can be regarded as part of X^{**} without altering its structure as a normed linear space and we write $X \subseteq X^{**}$ i.e. X is contained in X^{**} algebraically and topographically.

If $X = X^{**}$ under this embedding then X is said to be reflexive. A uniformly convex Banach space is reflexive, consequently a Hilbert space is always reflexive.

We now return to the general theory of operators T on $D(T) \subset X$ into Y. For each linear operator T with $\overline{D(T)} = X$ there is an operator T^*, the dual (or conjugate) of T with $D(T^*) \in Y_S^*$ and $R(T^*) \in X_S^*$ such that

(2.14) $\langle Tx, y' \rangle = \langle x, T^* y' \rangle$ for all $x \in D(T)$ and all $y' \in D(T^*)$.

If $T \in \mathcal{L}(X,Y)$ then $T^* \in \mathcal{L}(Y^*,X^*)$ and $\|T\| = \|T^*\|$.

For X,Y Hilbert spaces the idea of the dual operator can be is single-valued and uniformly continuous in every bounded set of X.

Since the dual space X^* is itself a normed linear space it is possible to form a dual space $(X^*)^*$ of X^*. This space is called the second dual space ofX and denoted by X^{**} By construction the elements of X^{**} are continuous linear functionals which map the linear functionals of X^* into K so if $F \in X^{**}$, $f \in X^*$

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(2.14) $\langle Tx, y' \rangle = \langle x, T^* y' \rangle$ for all $x \in D(T)$ and all $y' \in D(T^*)$.

If $T \in \mathcal{L}(X,Y)$ then $T^* \in \mathcal{L}(Y^*,X^*)$ and $\|T\| = \|T^*\|$.

For X,Y Hilbert spaces the idea of the dual operator can be extended to that of the adjoint of T, adjT. We first note that for X a Hilbert space there is a one-one correspondence J_X between the elements $y \in X$ and $f \in X_S^*$ through the relation

$$f(x) = (x,y) \quad \text{for all } x \in X \text{ and } \|f\| = \|y\|$$

This correspondence is conjugate linear since

$$(\alpha_1 f_1 + \alpha_2 f_2) \leftrightarrow (\bar{\alpha}_1 f_1 + \bar{\alpha}_2 f_2) \qquad (\alpha_1, \alpha_2 \in K)$$

then if $\overline{D(T)} = X$ $\quad \langle Tx, y' \rangle = y'(Tx) = (Tx, J_Y y')_Y$

and $\quad \langle x, T^* y' \rangle = (T^* y')(x) = (x, J_X T^* y')_X$.

Hence $\quad (Tx, J_Y y')_Y = (x, J_X T^* y')_X$,

that is $\quad (Tx, y)_Y = (x, J_X T^* J_Y^{-1} y)_X$

In the special case when Y = X we can write

(2.15) $\quad \text{adj } T = J_X T^* J_X^{-1}$

Definition 2.19 . T is self-adjoint if adj T = T.

We now consider operator-valued functions $t \to T(t) \in \mathcal{L}(X,Y)$ of a real or complex variable t. As in operator theory there are three important kinds of convergence. T(t) is continuous in norm (uniform convergence) if $\| T(t+h) - T(t) \| \to 0$ as $h \to 0$, T(t) is strongly continuous if T(t) x is continuous with respect to the norm topology for each $x \in X$. T(t) is weakly continuous if $\langle T(t)x, f \rangle$ is continuous for each $x \in X$ and $f \in X^*$. In a similar way three kinds of differentiability can be introduced, in particular the strong derivative $T'(t) = \frac{dT(t)}{dt}$ is defined by

(2.16)
$$T'(t)x = \lim_{h \to 0} \frac{T(t+h)x - T(t)x}{h}$$

and similarly for the weak derivative.

There are also different forms of the integral $\int T(t)dt$. If $T(t)$ is continuous in norm then the integral can be defined as for numerically-valued functions. For a strongly continuous function $T(t)$ the integral $\int T(t)dt$ is the 'strong' integral with the properties that for each $x \in X$

$$(2.17)\quad \begin{aligned} &\left(\int T(t)dt\right) x = \int T(t)x\, dt, \quad \left\| \int T(t)dt \right\| \leqslant \int \| T(t) \| dt \\ &\frac{d}{dt} \int^{t} T(s)\, ds = T(t) \quad \text{(strong derivative).} \end{aligned}$$

2.7. Resolvent Set of an Operator

An operator T in a Banach space X can have eigenvalues and eigenvectors. An eigenvalue of T is a complex number $\lambda \in K$ such that there exists a non-zero $x \in D(T) \subset X$ with

$$(2.18)\qquad T x = \lambda x.$$

Thus λ is an eigenvalue if the null space of $(T - \lambda I)$ is not 0. Since T does not necessarily have any eigenvalues it is more useful to introduce the idea of a resolvent set.

Definition 2.20 Let T be a closed operator in X. Then $\xi \in K$ is said to belong to the resolvent set $P(T)$ of T if $T - \xi I$ is invertible with

$$(2.19)\qquad R(\xi) = R(\xi, T) = (T - \xi I)^{-1} \in \mathcal{L}(X).$$

The operator-valued function $R(\xi)$ is the resolvent of T, $R(\xi)$ has domain X and range $D(T)$ for any $\xi \in P(T)$.

2.8. Some Important Spaces

We now present some details concerning particular Banach and Hilbert space of functions defined on an open set Ω of R^n.

C(Ω) is the set of all continuous functions $u(x) = u(x_1, x_2, \ldots x_n)$ defined on a compact set Ω in R^n. It is a Banach space with norm

(2.20) $$\|u\| = \|u\|_{C(\Omega)} = \|u\|_\infty = \max_{x\in\Omega} |u(x)|.$$

$L^p(\Omega)$ $(1 \leqslant p \leqslant \infty)$ is the set of all Lebesgue-measurable functions u(x) on Ω, a measurable subset of R^n, such that $\int_\Omega |u(x)|^p dx$ is finite. It is a Banach space with norm

(2.21) $$\|u\| = \|u\|_{L^p} = \left(\int_\Omega |u(x)|^p dx\right)^{\frac{1}{p}} \qquad (p \geqslant 1)$$

In $L^p(\Omega)$ two functions u and v are identified if $u(x) = v(x)$ a.e. in Ω. $L^p(\Omega)$ is a separable space.

$L^2(\Omega)$ is a Hilbert space with inner product

(2.22) $$(u,v) = \int_\Omega u(x)\overline{v(x)}dx.$$

For bounded Ω, $L^p(\Omega) \subset L^q(\Omega)$ for $p \geqslant q$ algebraically and topologically i.e. $\|.\|_q \leqslant K\|.\|_p$, K a constant, since if $f \in L^p, g \in L^q$ then $fg \in L^s$ for $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$ with $\|fg\|_s \leqslant \|f\|_p \|g\|_q$ $(s,p,q \geqslant 1)$.

The above spaces are spaces of functions defined in the ordinary or classical sense. For the purpose of describing the solutions of partial differential equations they are too restrictive, to obtain a wider class of space the idea of a function is generalized using the theory of distributions due to Schwartz. We require first the following definition.

Definition 2.21 The support of a function f is the smallest closed set of Ω outside which f vanishes identically.

We note the following sets which are linear spaces but not Banach spaces.

$C^k(\Omega)$ $(0 \leqslant k < \infty)$ is the set of all complex-valued functions defined in Ω which have continuous partial derivatives up to and including k (of order $k < \infty$ if $k = \infty$).

$C_0^k(\Omega)$ is the set of functions $\in C^k(\Omega)$ with compact support.

Let K be any compact subset of Ω then we denote by $D_K(\Omega)$ the space of all $f \in C_0^\infty(\Omega)$ whose support is in K. A family of semi-norms can be defined on $D_K(\Omega)$ by

$$P_{K,m}(f) = \sup_{|s|<m,\, x\in K} |D^s f(x)| \qquad (m < \infty) \tag{2.23}$$

where $D^s f(x) = \dfrac{\partial^{s_1+s_2+\ldots+s_n}}{\partial x_1^{s_1}\partial x_2^{s_2}\ldots\partial x_n^{s_n}} f(x_1, x_2, \ldots x_n)$

$$|s| = |(s_1, s_2, \ldots s_n)| = \sum_{j=1}^{n} s_j$$

so that $D_K(\Omega)$ is a linear topological space. The inductive limit of the $D_K(\Omega)$'s as K ranges over all compact subsets of Ω is a linear topological space denoted by $D(\Omega)$.

<u>Definition 2.22</u> A linear functional f defined and continuous in $D(\Omega)$ is called a generalized function or distribution in $D(\Omega)$ and the value $f(\phi)$ is called the value of the generalized function f at the testing function $\phi \in D(\Omega)$.

If f is a generalized function in Ω then

$$g(\phi) = -f\left(\frac{\partial\phi}{\partial x_1}\right), \quad \phi \in D(\Omega)$$

defines another generalized function g in Ω called the generalized derivative or distributional derivative of f with respect to x_1. We note that any generalized function is infinitely differentiable in the sense of distributions as a consequence of the definition of $D(\Omega)$.

We now define the following Banach spaces which are based on the concept of distributions.

<u>$W^{k,p}(\Omega)$</u> (k a positive integer, $1 \leq p < \infty$) is the set of all complex valued functions f(x) defined in Ω such that f and its distributional derivatives $D^s f$ of order $|s| = \sum_{j=1}^{n} |s_j| \leq k \in L^p(\Omega)$. $W^{k,p}(\Omega)$ is a Banach space with norm

$$(2.24)\qquad \|f\|_{k,p} = \Big(\sum_{|s|\leq k}\int_\Omega |D^s f(x)|^p dx\Big)^{\frac{1}{p}}, \quad dx = dx_1 dx_2 \ldots dx_n$$

These spaces are usually referred to as Sobolev spaces.

$\underline{W^k(\Omega)}$ = $W^{k,p}(\Omega)$ with p = 2 is a Hilbert space.

$\underline{H^k_o(\Omega)}$. We have previously referred to the linear space $C^k_o(\Omega)$. It can be normed by

$$(2.25)\qquad \|f\|_k = \Big(\sum_{|s|\leq k}\int_\Omega |D^s f(x)|^2 dx\Big)^{\frac{1}{2}}$$

By the theorem of completion, Yosida [2], it is possible to complete this space. The completion will be denoted by $H^k_o(\Omega)$ which is in fact a Hilbert space and also a proper subspace of $W^k(\Omega)$ if $\Omega \subset R^n$. If $\Omega = R^n$, $H^k_o(R^n) = H^k(R^n) = W^k(R^n)$.

The above definition of $W^{k,p}(\Omega)$ is valid for k a positive integer. By use of the theory of the Fourier transform we can extend the range of definition of $W^k(\Omega)$ for $\Omega = R^n$ to all $k \in R$ as follows.

We first define the Fréchet space $\aleph(R^n)$ and the Fourier transform of an element of $\aleph(R^n)$.

$\underline{\aleph(R^n)}$ is the set of functions $f \in C^\infty(R^n)$ such that

$$\sup_{x\in R^n}\left|x^\beta D^\alpha f(x)\right| < \infty, \qquad \Big(x^\beta = \prod_{j=1}^n x_j^{\beta_j}\Big)$$

for every $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots \beta_n)$ with non-negative integers α_j and β_k. The topology on $\aleph(R^n)$ is defined by a family of semi-norms of the form

$$p(f) = \sup_{x\in R^n}|P(x)D^\alpha f(x)| \text{ where } P(x) \text{ is a polynomial.}$$

<u>Definition 2.23</u> Every $f \in \aleph(R^n)$ has a Fourier transform $\hat{f}$ defined by

$$(2.26)\qquad \hat{f}(\xi) = (2\pi)^{-n/2}\int_{R^n} e^{-i\langle\xi,x\rangle} f(x)dx$$

where $\xi = (\xi_1, \xi_2, \ldots \xi_n)$, $x = (x_1, x_2, \ldots x_n)$, $\langle\xi,x\rangle = \sum_{j=1}^n \xi_j x_j$ and $dx = dx_1 dx_2 \ldots dx_n$.

Theorem 2.3 (Plancherel's Theorem)

If $f \in L^2(R^n)$ then $\hat{f} \in L^2(R^n)$ and

$$\|f\|_{L^2} = \|\hat{f}\|_{L^2} .$$

In analogy with the idea of distribution defined above we have the following.

Definition 2.24 A linear functional T defined and continuous on $\mathcal{S}(R^n)$ is called a tempered distribution in R^n.

It can be proved that any function f in $L^p(R^n)$ $(p \geq 1)$ defines a tempered distribution. Thus in particular $f \in H^k(R^n)$ defines a tempered distribution T_f. Now the Fourier transform of T_f, $\hat{T}_f$, can be defined through

$$\hat{T}_f(f) = T_f(\hat{f}), \qquad f \in \mathcal{S}(R^n)$$

with the property that

(2.27) $$\widehat{D^\alpha T_f} = (i)^{|\alpha|} \prod_{j=1}^{n} x_j^{\alpha_j} \hat{T}_f$$

If $f \in H^k(R^n)$ then $D^\alpha T_f \in L^2(R^n)$ for $|\alpha| \leq k$ by definition and hence $\widehat{D_j^k T_f} \in L^2(R^n)$ $(j = 1,2,\ldots n)$. Now

$$(1+|x|^2)^k < (n+1)^{k-1}(1+x_1^{2k}+x_2^{2k}+ \ldots + x_n^{2k})$$

by Holder's inequality and so using (2.24)

$$\int_{R^n} (1+|x|^2)^k \hat{T}_f^2 \, dx < \infty$$

$$\text{i.e. } (1+|x|^2)^{k/2} \hat{T}_f \in L^2(R^n).$$

By Plancherel's theorem it can be shown that the norm $\|f\|_k$ of f in H^k given by (2.25) is equivalent to the norm

(2.28) $$\|f\|_k' = \|(1+|x|^2)^{k/2} \hat{T}_f\|_{L^2}.$$

Hence the space $H^k(R^n)$ can be renormed by $\|f\|_k'$ and thus defined as the totality of $f \in L^2(R^n)$ such that $\|f\|_k'$ is finite. With this definition k can be any real number, not just a natural number.

In order to define $H^k(\Omega)$ for any real number k it is necessary to restrict Ω to a smooth subset of R^n.

Definition 2.25 Let P be a point of the subset $A \subset R^n$. A is smooth near P if there exists a neighbourhood U of P and a mapping ϕ of U onto a spherical neighbourhood V of the origin such that

(i) ϕ and ϕ^{-1} are (1,1) and infinitely differentiable

(ii) $\phi(A \cap U) = V \cap \{x \in R^n | x_1 = 0\}$.

If A is smooth near each of its points it is a smooth subset of R^n.

Following Aubin [4] we have the following.

Definition 2.26 The space $H^s(\Omega)$ where Ω is a smooth bounded open subset of R^n is the space of the restrictions to Ω of functions f of $H^s(R^n)$.

The Sobolev spaces have the following embedding properties.

Theorem 2.4 Let $\Omega \subset R^n$ be a smooth bounded open set then

(i) $W_p^m(\Omega) \subset L^q(\Omega)$ algebraically and topologically for

$$\frac{1}{q} > \frac{1}{p} - \frac{m}{n}$$

(ii) $W_p^m(\Omega) \subset C^j(\bar{\Omega})$ for $j < m - \frac{n}{p}$

(iii) $H^s(\Omega) \subset C(\bar{\Omega})$ for $s > \frac{n}{2}$

where p,q,m,j are integers.

2.9. Liapunov Stability Theory

We have described in the Introduction §1 how a partial differential equation or a set of partial differential equations may be regarded as a single equation

(2.29) $$\dot{z}(t) + N(z,t) = 0.$$

We now regard z as a function on the finite real interval $I = [0,T]$ to a Banach space Z and N(z,t) as a given function or operator from IxZ to Z and associate with (2.29) an initial condition of the form

$$z(t_0) = z_0.$$

Following Zubov [5] we assume that for any element $z_o \in Z$ there exists a local solution of (2.29) which can be extended to a global solution $z(z_o,t,t_o)$ valid for all $t \geq t_o$ with the properties:-

(i) for any z_o, $z(z_o,t,t_o)$ is defined for all $t \geq t_o$ and $z(z_o,t,t_o) \in Z$ for all $t \geq t_o \geq 0$

(ii) $z(z_o,t,t_o) = z_o$ when $t = t_o$

Suppose that (2.29) has an equilibrium point z_E then

(2.30) $$z(z_E,t,t_o) = z_E \quad \text{for} \quad t \geq t_o$$

We have the following definitions, Zubov [5].

Definition 2.27 The equilibrium point z_E is stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|z_o - z_E\|_Z < \delta$ then $\|z(z_o,t,t_o) - z_E\| < \varepsilon$ for $0 \leq t_o \leq t$.

Definition 2.28 If z_E is stable and if

$$\|z(z_o,t,t_o) - z_E\| \to 0 \quad \text{as } t \to \infty$$

then z_E is said to be asymptotically stable.

Definition 2.29 The set of all $z_o \in Z$ such that z_E is asymptotically stable is called the region of asymptotic stability of the equilibrium solution z_E.

§3 The Abstract Evolution Equation

3.1. Introduction

In the previous chapter we have presented the essential features of functional analysis that we require for a discussion of the abstract evolution equation

(3.1) $$\dot{z}(t) + N(t,z) = 0, \qquad z(0) = z_o$$

where z is a function on the real interval I = [0,T] to a Banach space Z and N is a given function or operator from I x Z to Z. In studying the questions of existence and uniqueness of the solutions of (3.1) it is convenient to consider the equation in the following four different forms.

(i) The simplest type of equation is

(3.2) $$\dot{z}(t) + Az(t) = 0, \qquad z(0) = z_o$$

where A is a linear operator in Z which may be bounded or unbounded.

(ii) As (i) but with A dependent upon t, i.e.

(3.3) $$\dot{z}(t) + A(t)z(t) = 0, \qquad z(0) = z_o$$

for each t or almost all t > 0.

(iii) The semi-linear form

(3.4) $$\dot{z}(t) + A(t)z(t) = g(t,z), \qquad z(0) = z_o$$

where A(t) is as in (3.3) and g(t,z) is a non-linear function from I x Z to Z.

(iv) The general form (3.1) where N(t,z) = A(t)z(t) and A(t) is a non-linear operator in Z.

(3.5) $$\dot{z}(t) + A(t)z(t) = 0, \qquad z(0) = z_o$$

Each of the types (i) - (iv) is a homogeneous equation, we shall also refer to the corresponding in-homogeneous forms where there is a forcing term f(t) on the right-hand side of the equation.

3.2. The Linear Equation with A independent of t

We consider now in detail (3.2) viz

(3.6) $$\dot{z}(t) + Az(t) = 0 \qquad z(0) = z_o$$

where A is a linear operator independent of t.

If $A \in \mathcal{L}(Z)$ the operator

$$U(t) = U_t = e^{-tA} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^n A^n$$

is well-defined as the series is absolutely convergent for all real t. $e^{-tA} \in \mathcal{L}(Z)$ and satisfies the group property

(3.7) $$e^{-(s+t)A} = e^{-sA} e^{-tA}$$

Also

(3.8) $$\frac{d}{dt} e^{-tA} = -Ae^{-tA} = -e^{-tA}A$$

where the derivative is taken to be defined by norms

i.e. $\left\| \frac{U(t+h) - U(t)}{h} - \frac{dU}{dt} \right\| \to 0$ as $h \to 0$.

Thus $z(t) = e^{-tA} z_o$ is a solution of (3.6) for any $z_o \in Z$, we call it a strict solution because it satisfies the following definition.

<u>Definition 3.1</u> z(t) is called a strict solution of (3.2) if z(t) is continuous for $t \geq 0$, the strong derivative $\frac{dz}{dt}$ exists for $t > 0$, $z(t) \in D(A)$ for $t > 0$ and (3.2) holds true.

Suppose now A is an unbounded operator. The operator $U_t = e^{-tA}$ is no longer defined by a Taylor's series but by the limit

(3.9) $$e^{-tA} = \lim_{n \to \infty} (I + \frac{t}{n}A)^{-n}.$$

Kato [1] has shown that the limit exists if

(3.10)
(i) A is a closed operator in Z and D(A) is dense in Z

(ii) the negative real axis belongs to the resolvent set of A and the resolvent $(A+\xi I)^{-1}$ satisfies the inequality $\|(A + \xi I)^{-1}\| \leq \frac{1}{\xi}$, $\xi > 0$

and furthermore the limit, which we shall henceforth denote by U_t rather than e^{-tA}, has the properties of an exponential function, that is where A is a linear operator independent of t.

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(i) A is a closed operator in Z and D(A) is dense in Z

(ii) the negative real axis belongs to the resolvent set of A and the resolvent $(A+\xi I)^{-1}$ satisfies the inequality

$$\| (A + \xi I)^{-1} \| < \frac{1}{\xi}, \quad \xi > 0$$

and furthermore the limit, which we shall henceforth denote by U_t rather than e^{-tA}, has the properties of an exponential function, that is

(3.11) $$U_{t+s} = U_t U_s \qquad s,t \geq 0$$

The set $\{U_t\}_{t\geq 0}$ is said to form a one parameter semi-group of operators with the operator -A as the infinitesimal generator of the semi-group.

With A satisfying (3.10) $\|U(t)\| \leq 1$ and the semi-group is called a contraction semi-group. This result constitutes the Hille-Yosida theorem, the conditions (3.10) provide a necessary and sufficient set of conditions for the generation of a contraction semi-group. Also U_t is differentiable in t if $z \in D(A)$ with

(3.12) $$\frac{d}{dt} U_t z = -U_t Az = -AU_t z$$

so that U_t is a particular type of evolution operator. The unique strong solution of (3.6) is given by $z(t) = U_t z_o$ if z_o D(A).

The conditions of (3.10) on A are sufficient but not necessary for A to generate a semi-group. Kato [1] gives three alternative sets of conditions, two of which are less restrictive than (3.10). On the first of the less restrictive ones (3.10) (ii) is replaced by

(3.13) $$\|(A + \xi I)^{-k}\| \leq \frac{M}{\xi^k} \qquad \xi > 0, \quad k = 1, 2, 3, \ldots..$$

where M is a constant independent of ξ and k. The corresponding operator U_t is still strongly continuous for $t \geq 0$ and indeed all the previous results are valid except that now

$$\|U_t\| \leq M, \qquad U_o = 1.$$

In this case U_t is referred to as a bounded semi-group. (3.13) can in turn be replaced by the condition that the semi-infinite interval $\xi > \beta$ belongs to the resolvent set of -A and

(3.14) $$\|(A + \xi I)^{-k}\| \leq \frac{M}{(\xi-\beta)^k} \qquad \xi > \beta, \quad k = 1, 2, 3 \ldots$$

Then the operator $A + \beta I$ generates a bounded semi-group U_{1t} and the operator $U_t = e^{\beta t} U_{1t}$ has all the properties referred to above except that we have

$$\|U_t\| \leq Me^{\beta t}$$

in place of $\|U_t\| \leq M$. $\|U_t\|$ is not necessarily bounded as $t \to \infty$. In this case U_t is referred to as a quasi-bounded semi-group.

We consider next the following set of conditions on A where slightly more is assumed concerning the resolvent set of -A.

(i) A is closed and densely defined in Z

(ii) The resolvent set of -A contains the sector

(3.15) $$|\arg \zeta| \leq \frac{\pi}{2} + \omega \quad (\omega > 0)$$

(iii) For any $\varepsilon > 0$

$$\|(A + \zeta I)^{-1}\| \leq \frac{M_\varepsilon}{\zeta} \quad \text{for } |\arg \zeta| \leq \frac{\pi}{2} + \omega - \varepsilon,$$

with M_ε independent of ζ.

Any operator satisfying (3.15) generates a semi-group of operators U_t which are holomorphic for $|\arg t| < \omega$, uniformly bounded for $|\arg t| \leq \omega - \varepsilon$ and strongly continuous within the sector $|\arg t| \leq \omega - \varepsilon$ at $t = 0$ with $U_0 = I$. Such a semi-group is called a bounded holomorphic semi-group and has the property that $z(t) = U_t z_0$ satisfies $\frac{dz}{dt}(t) = -Az$ for $t > 0$ and any $z_0 \epsilon Z$ rather than any $z_0 \in D(A)$.

All the proofs for the above are given in Kato [1]. We note at this point that as far as the stability properties of the solution are concerned if the equilibrium point $z = 0$ of (3.8) is to be asymptotically stable in the sense of Liapunov then the semi-group must be such that $\|U_t\| \leq Me^{-\omega t}$ $(\omega > 0)$.

We consider now an alternative characterization of the generators of contraction semi-groups in terms of m-accretive operators.

Definition 3.2. The operator A (linear or non-linear) on a general Banach space Z is accretive if

(3.16) $$\|x - y + \lambda(Ax-Ay)\| \geq \|x - y\| \quad \text{for each } x,y \epsilon D(A) \text{ and } \lambda > 0.$$

If the range of $I + \lambda A$ is the whole of Z for some $\lambda > 0$ then A is m-accretive.

Theorem 3.1. (Lumer-Phillips)
$-A$ is the infinitesimal generator of a linear contraction semi-group on an arbitrary Banach space if and only if A is m-accretive and densely defined.

It is shown in [1] that an m-accretive operator A is necessarily densely defined on Z provided that Z is a reflexive Banach space.

We note here the following alternative condition for m-accretiveness.

Theorem 3.2. If A is a densely defined closed linear operator such that D(A) and R(A) are both in a Banach space Z and if A and its dual A^* are accretive then $-A$ generates a contraction semi-group.

In the special case when Z is a Hilbert space, $Z = H$ is reflexive and the condition (3.16) for accretiveness is equivalent to

$$\text{(3.17)} \qquad \operatorname{Re}\langle Az, z\rangle \geq 0 \qquad \text{for all } z \in D(A)$$

where $\langle , \rangle$ denotes the inner product on H.

We consider now the in-homogeneous form of (3.2)

$$\text{(3.18)} \qquad \dot{z}(t) + Az(t) = f(t), \quad z(0) = z_0.$$

where f(t) is a given function with values in Z. If z(t) is a solution of (3.18) then

$$\begin{aligned} \frac{d}{ds} U(t-s)z(s) &= -U'(t-s)z(s) + U(t-s)z'(s) \\ &= U(t-s)Az(s) + U(t-s)[f(s) - Az(s)] \quad \text{by (3.12)} \\ &= U(t-s)f(s). \end{aligned}$$

Integrating this on (0,t) gives

$$\text{(3.19)} \qquad z(t) = U(t)z_0 + \int_0^t U(t-s)f(s)ds$$

provided that U(t-s)f(s) is integrable.

We note that if $U(t)$ is a quasi-bounded semi-group and $f(t)$ is continuously differentiable then (3.19) is a strict solution of (3.18) for any $z_0 \epsilon D(A)$ but (3.19) does not provide a strict solution of (3.18) for every $f(t)$ and z_0. However we can regard (3.19) as a form of generalized solution of (3.18) and make the following definition.

Definition 3.3. For any $z_0 \epsilon Z$ and integrable $f(t)$

$$z(t) = U(t)z_0 + \int_0^t U(t-s)f(s)ds$$

is called a mild solution of (3.18).

3.3. The Linear Equation with A dependent upon t.

We now consider the linear equation (3.3)

$$\dot{z}(t) + A(t)z(t) = 0 \qquad z(0) = z_0 \tag{3.20}$$

and give some basic results concerning the existence and uniqueness of solutions. A fuller account can be found in Carroll [6].

The important difference between the time dependent and time independent cases is that the semi-group operator of the latter becomes an operator in two variables $G(t,\tau)$ called an evolution operator. The following is a basic result.

Theorem 3.3. If $t \rightarrow A(t) \in C^0(\mathcal{L}(Z))$ on the interval $I = [0,T]$, (3.20) has a unique strict solution given by $z(t) = G(t,0)z_0$ where $t \rightarrow G(t,\tau) \in C^1(\mathcal{L}(Z))$ is a solution of

$$\dot{G} + A(t)G = 0 \quad \text{with} \quad G(\tau,\tau) = I, \ t \geqslant \tau. \tag{3.21}$$

The strict solution referred to here is essentially as in Definition 3.1 but with $z(t) \epsilon D(A(t))$ for $t > 0$.

In addition $G(t,\tau)$ has the following properties.

(3.22)

(i) $G(t,\tau) = G(t,s)G(s,\tau), \quad \tau \leqslant s \leqslant t$

(ii) $G(\tau,t) = G(t,\tau)^{-1}$

(iii) the maps $z_0 \rightarrow G(\cdot,0)z_0 : Z \rightarrow C^0(Z)$ and $(t,s) \rightarrow G(t,s) : I \times I \rightarrow \mathcal{L}(Z)$ are continuous.

We now make a formal definition of a mild evolution operator.

Definition 3.4. A family of bounded linear operators $G(t,s)$ on Z to itself defined for $0 \leq s \leq t \leq T$, strongly continuous in the two variables jointly and satisfying

$$G(t,s) = G(t,r)G(r,s), \qquad G(r,r) = 1 \quad s \leq r \leq t$$

is called a family of mild evolution operators.

We introduce two further types of evolution operators by the following.

Definition 3.5. If a family of mild evolution operators satisfies

$$\frac{\partial G(t,s)z}{\partial t} = -A(t)G(t,s)z \qquad (s \leq t) \tag{3.23}$$

and

$$\frac{\partial G(t,s)z}{\partial s} = G(t,s)A(s)z \qquad (s \leq t) \tag{3.24}$$

for all $z \in D(A(t))$ then $\{G(t,s)\}$ is called a family of strict evolution operators. If however (3.24) is satisfied but (3.23) is not then the operators are called quasi-evolution operators.

Consider now the in-homogeneous equation with $A(t)$ an unbounded operator

$$\dot{z}(t) + A(t)z(t) = f(t) \qquad z(\tau) = z_0, \quad \tau \leq t \leq T. \tag{3.25}$$

As in §3.2 we can construct the solution

$$z(t) = G(t,\tau)z_0 + \int_\tau^t G(t,s)f(s)ds \tag{3.26}$$

provided $G(t,s)f(s)$ is integrable.

We quote the following theorem due to Kato [7] which gives conditions for (3.26) to be a strict solution of (3.25) when $A(t)$ is an unbounded operator and $-A(t)$ generates a strongly continuous contraction semi-group for each $t \in [0,T]$.

Theorem 3.4. Let $-A(t)$ be the generator of a strongly continuous contraction semi-group for $t \in [0,T]$ with $\|A(t)^{-1}\| \leq M$ (thus $\|\lambda I + A(t)^{-1}\| \leq 1/\lambda$

for $\lambda > 0$.) If $D(A(t)) = D$ with $t \to A(t)A^{-1}(0) \in C^1(\mathcal{L}_s(Z))$ then there exists a unique strong evolution operator $G(t,s)$ and if $f \in C^1(Z)$ and $z_0 \in D$ then (3.26) gives the unique strong solution of (3.25) on $[\tau,T]$.

There is a similar theorem due to Yosida [8] when $-A(t)$ generates a strongly continuous semi-group.

$C^0(\mathcal{L}_s(Z))$ denotes the space of bounded linear operators mapping Z into Z which are strongly continuous in t for $t \in [0,T]$.

It is clearly a possibility that (3.26) might be a mild solution of (3.25) under less restrictive conditions than are given in this theorem. This idea is explored in Chapter 5 which contains some new results in this direction.

3.4. The semi-linear form

We consider next the semi-linear form (3.4) viz

(3.27) $$\dot{z}(t) + A(t)z(t) = f(t,z) \qquad z(\tau) = z_0.$$

The mild solution can be constructed as in the previous cases giving

(3.28) $$z(t) = G(t,\tau)z(\tau) + \int_\tau^t G(t,s)f(s,z)ds$$

If $f(.,.)$ is continuous on $I \times B(b,z_0)$ where $B(b,z_0) = B = \{z \in Z;\ \|z-z_0\| \leq b\}$ then (3.28) is well-defined and is also the strict solution of (3.27) as the right-hand side can be differentiated with respect to t.

Other results where $f(t,z)$ is a bounded operator are given in Kato [9].

The case where $f(t,z)$ is not bounded is, not surprisingly, more complicated. If f is bounded relative to A there is a theorem due to Segal [10] for the case where $A(t) = A$ is independent of t and the semi-group generated by $-A$ is strongly continuous. In this result the domain $D(A)$ of A is regarded as a Banach space with the graph norm $\|z\| + \|Az\|$ and as such is denoted by $[D(A)]$.

for $\lambda > 0$.) If $D(A(t)) = D$ with $t \to A(t)A^{-1}(0) \in C^1(\mathcal{L}_s(Z))$ then there exists a unique strong evolution operator $G(t,s)$ and if $f \in C^1(Z)$ and $z_o \in D$ then (3.26) gives the unique strong solution of (3.25) on $[\tau,T]$.

There is a similar theorem due to Yosida [8] when $-A(t)$ generates a strongly continuous semi-group.

$C^o(\mathcal{L}_s(Z))$ denotes the space of bounded linear operators mapping Z into Z which are strongly continuous in t for $t \in [0,T]$.

It is clearly a possibility that (3.26) might be a mild solution of (3.25) under less restrictive conditions than are given in this theorem. This idea is explored in Chapter 5 which contains some new results in this direction.

3.4. The semi-linear form

We consider next the semi-linear form (3.4) viz

(3.27) $\dot{z}(t) + A(t)z(t) = f(t,z) \qquad z(\tau) = z_o.$

The mild solution can be constructed as in the previous cases giving

(3.28) $z(t) = G(t,\tau)z(\tau) + \int_\tau^t G(t,s)f(s,z)ds$

If $f(.,.)$ is continuous on $I \times B(b,z_o)$ where $B(b,z_o) = B = \{z \in Z;\ \|z-z_o\| \leq b\}$ then (3.28) is well-defined and is also the strict solution of (3.27) as the right-hand side can be differentiated with respect to t.

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Theorem 3.5. Let $f(t,z)$ be defined on $I \times [D(A)]$ to Z and differentiable in t and z. Let $f_t(t,z)$ be continuous on $I \times [D(A)]$ to Z and Lipschitz continuous in z. Let $f_z(t,z)$ have an extension as a bounded linear operator on Z to Z and let the map $t,z \to f_z(t,z)$ be continuous from $I \times [D(A)]$ to $\mathcal{L}(Z)$ and Lipschitz continuous in z. Then

$$\dot{z}(t) + Az(t) = f(t,z), \qquad z(\tau) = z_0$$

(where A is as in the previous paragraph) has a unique local strong solution $z \in C(I_0;[D(A)]) \cap C^1(I_0,Z)$ for any $z_0 \in D(A)$ where I_0 is some interval smaller than I i.e. $I_0 = [\tau,T_0]$ $(T_0 < T)$.

The local solution can be extended to a global solution in certain cases such as when the Lipschitz condition is satisfied uniformly for all $z \in Z$ or if the initial data and $f(t,z)$ are sufficiently small.

There are some more recent results concerning the global solution of (3.27) when $f(t,z) = -B(z)$ where B is a continuous, everywhere defined, non-linear operator from Z to itself. Webb [11] has shown that if $-A$ generates a contraction semi-group T_t, $t \geq 0$ and B is accretive then the solution of (3.27)

$$U(t)z = T_t z - \int_0^t T_{t-s} BU(s) z ds \tag{3.29}$$

exists and is unique for all $z \in Z$. Also the operator $U(t)$, $t \geq 0$ of (3.29) is a strongly continuous semi-group of non-linear contractions on Z i.e. $U(t)$ is a function from $[0,\infty) \times Z$ to Z such that

(i) $U(t)U(s)z = U(t+s)z$ for all $t,s \geq 0$, $z \in Z$

(ii) $\|U(t)y - U(t)z\| \leq \|y-z\|$ for $t \geq 0$ and $y,z \in Z$

(iii) $U(0)z = z$ for all $z \in Z$.

The operator A+B is m-accretive on Z and $-(A+B)$ is the infinitesimal generator of $U(t)$. The validity of this result was extended by Maruo and Yamada [12] to the case where A and B are both dependent on t (with T_t

becoming an evolution operator $U(t,s)$ with norm ≤ 1) whilst Maruo [13] has shown that the condition that $B(t)$ is everywhere defined can be relaxed provided $-A$ is the infinitesimal generator of an analytic semi-group.

3.5. The Nonlinear Evolution Equation

Finally we consider the abstract evolution equation (3.1) in its most general operator form (3.5) viz:-

(3.30) $\dot{z}(t) + A(t)z(t) = 0 \qquad z(0) = z_0$

where $A(t)$ is a non-linear operator with domain and range in Z. As in the linear case the existence of a strong solution of (3.30) is linked with the property of m-accretiveness of the operator $A(t)$ (or an equivalent condition).

Kato [14] has proved an important result concerning the strict solution of (3.30) when $A(t)$ satisfies the three conditions

(i) the domain D of $A(t)$ is independent of t

(ii) there is a constant L such that for all $y \in D$ and $s,t \in [0,T]$

(3.31) $\|A(t)y - A(s)y\| \leq L(t-s)(1+\|y\|+\|A(s)y\|)$

(iii) for each t, $A(t)$ is m-accretive, (i.e. $A(t)$ satisfies (3.16)).

(Thus $A(t)$ is assumed to be uniformly Lipschitz continuous in t). The proof of the theorem uses the concept of the duality map F from Z to Z^* (see §2). The duality map provides an alternative condition for accretiveness to (3.16) namely the following:

(3.32) there is an element $f \in F(x-y)$ such that $\mathrm{Re}\,(Ax-Ay,f) \geq 0$

for each x,y $D(A)$.

Kato's theorem is:

Theorem 3.6. Assume Z^* is uniformly convex and let $A(t)$ satisfy (3.31). For each $z_0 \epsilon D$ there exists a unique function $z(t) \epsilon Z$ on $[0,T]$ satisfying (3.30) such that

(a) $z(t)$ is uniformly Lipschitz continuous on $[0,T]$ with $z(o) = z_o$,

(b) $z(t) \epsilon D$ for each $t \epsilon [0,T]$ and $A(t)z(t)$ is weakly continuous on $[0,T]$

(c) the weak derivative of $z(t)$ exists for all $t \epsilon [0,T]$ and equals $-A(t)z(t)$.

(d) $z(t)$ is an indefinite integral of $-A(t)z(t)$, which is Bochner integrable, so that the strong derivative of $z(t)$ exists almost everywhere and equals $-A(t)z(t)$.

If, further, Z is uniformly convex then the strong derivative $\frac{dz}{dt}(t) = -A(t)z(t)$ exists and is strongly continuous except at a countable number of values of t.

The conditions in the above theorem are not necessary conditions, in a further paper Kato [15] introduced an alternative set of sufficient conditions more general than the first. The condition is given in the original paper in terms of the canonical restriction of an m-accretive (multiple-valued) operator depending upon t smoothly.

Definition 3.6. A is the canonical restriction of a multiple-valued operator B if Az is the set of all $y \epsilon Bz$ such that $\|y\| = \inf_{u \epsilon Bz} \|u\|$.

Theorem 3.7. Let Z be a Banach space such that Z and Z^* are uniformly convex. Suppose A is m-accretive and $A(t)z = Az + \gamma z - b(t)$, $0 \leq t < \infty$ with $D(A(t)) = D(A)$ where γ is a real constant and $b \in W_1^1(I,Z)$ for each interval $I \subset [0,\infty)$. Then for each $z_o \in D(A)$ the equation

$$\dot{z}(t) \in -A(t)z(t) \qquad z(0) = z_o$$

has a unique strong solution $z(t)$ on $[0,\infty)$ with $z(0) = z_o$ having the properties (i) $z(t) \in D(A)$ for all $t \geq 0$

(ii) $\|A(t)z(t)\|$ is of bounded variation on any finite subinterval of $[0,\infty)$ with no positive jumps,

(iii) $\dot{z}(t) \in -A(t)^{o}z(t)$ for almost all $t \geq 0$

where $A^{o}(t)$ is the canonical restriction of $A(t)$.

$W_1^1(I,Z)$ is the set of all Z-valued functions u on I such that u is the indefinite integral of a strongly integrable function v on I.

When $A(t) = A$ is independent of t and m-accretive these results can be used to introduce a semi-group of non-linear operators as in Webb's results described in the previous section. By setting $z(t) = U_t z_o$ a family of single-valued operators $\{U_t\}$, $0 \leq t < \infty$, is defined with domain D(A) and ranges in D(A). $\{U_t\}$ is a semi-group since $U_t U_s = U_{t+s}$ and contractive since

$$\|U_t a - U_t b\| \leq \|a-b\|$$

$U_t a$ is strongly continuous in t and by continuity the U_t can be extended to be operators with domain $\overline{D(A)}$ and range in $\overline{D(A)}$.

The general problem of the generation of semi-groups of non-linear transformations on general Banach spaces has been discussed in depth by Crandall and Liggett [16], and Miyadera [17], and on Hilbert spaces by Crandall and Pazy [18]. In order to describe their results which are essentially extensions of the Hille-Yosida theorem to non-linear operators let the semi-groups U(t) on C, a subset of Z, be characterized as follows.

A semi-group on $C \subset Z$ is a function U(t) on $[0,\infty)$ such that U(t) maps C into C for each $t \geq 0$ and satisfies

(i) $U(t+s) = U(t)\,U(s)$ for $t,s \geq 0$ and

(ii) $\lim_{t \downarrow 0} U(t)z = U(0)z = z$ for $z \in C$

Further $U(t) \in Q_\omega(C)$ if there exists a real number ω such that

$$\|U(t)x - U(t)y\| \leq e^{\omega t} \|x-y\|$$

for $t \geq 0$ and $x,y \in C$.

The non-linear operators A are viewed as subsets of $Z \times Z$ and then

(i) $Ax = \{y : [x,y] \in A\}$

(ii) $D(A) = \{x : Ax \neq \emptyset\}$

(iii) $R(A) = \cup\{Ax : x \in D(A)\}$

For Hilbert spaces Crandall and Pazy [18] have obtained a complete extension of the Hille-Yosida theorem to cover all of $Q_\omega(C)$ when C is a closed convex set. In this case, $Q_\omega(C)$ can be put into 1 - 1 correspondence with the set of subsets A of $Z \times Z$ for which $A+\omega I$ is accretive, $R(I+\lambda A) = Z$ for all $\lambda > 0$ and $\omega\lambda < 1$, and $\overline{D(A)} = C$. In the case of a general Banach space, it has been remarked earlier after Theorem 3.7. that if Z* is uniformly convex then an m-accretive operator A can be associated with a non-linear semi-group. In the general case when Z* is not uniformly convex there is the following result due to Crandall and Liggett [16].

Theorem 3.8. Let $A \subset Z \times Z$ and let ω be a real number such that $A+\omega I$ is accretive. If $R(I+\lambda A) \supset \overline{D(A)}$ for all sufficiently small $\lambda (> 0)$ the

(3.33) $$\lim_{n\to\infty} (I + \frac{t}{n}A)^{-n}z.$$

exists for $z \in \overline{D(A)}$ and $t > 0$. If $U(t)z$ is defined as the limit (3.33) then $U(t) \in Q_\omega(\overline{D(A)})$. Furthermore if A is a closed subset of $Z \times Z$ and $z \in D(A)$ and $0 < T \leq \infty$ then the conditions (i) and (ii) below are equivalent:

(i) $z(t) = \operatorname{limit}_{n\to\infty} (I + \frac{t}{n}A)^{-n}z$ for $t \in [0,T)$ and $z(t)$ strongly differentiable almost everywhere,

(ii) $z(t)$ is a strong solution of

$$0 \in \dot{z}(t) + Az \quad , \qquad z(0) = z_o$$

on $[0,T)$. This is the non-linear version of the result for linear contraction semi-groups.

Although the proofs of Crandall and Pazy are not valid when the assumption on A is of the form

$$\| (I+\lambda A)^{-n} \| \leq M(1-\lambda\omega)^{-n}, \qquad \text{Re } \lambda > \omega,$$

as in the Hille-Yosida theorem, they can be extended to the case of A dependent on t to provide a theorem which gives the existence of a non-linear evolution operator. This theorem, which extends Theorem 3.7. has like Kato's result the condition that $D(A(t))$ is independent of t. A similar result to these two but with the condition on $D(A(t))$ relaxed and Z a Hilbert space is reported by Watanabe [19].

§4. Perturbation Theorems

4.1. Introduction

In the previous chapter we have reviewed the known results concerning the existence of the various forms of solution of the abstract evolution equation in its most general form

(4.1) $\dot{z}(t) + N(t)z(t) = 0, \quad z(\tau) = z_o$

We now suppose that (4.1) can be written in the form

(4.2) $\dot{z}(t) + A(t)z(t) + B(z,t) = f(t), \; z(\tau) = z_o$

where $A(t)$ is a linear operator but $B(z,t)$ may be linear or non-linear. Equation (4.2) provides a particularly useful form of (4.1) if $A(t)$ is not the null operator for then we can regard $B(z,t)$ as a perturbation of the operator $A(t)$ i.e. $B(z,t)$ is a perturbation of the linear system

(4.3) $\dot{z}(t) + A(t)z(t) = 0, \qquad z(\tau) = z_o$

$f(t)$ is simply a forcing term in the system (4.2).
In this thesis we are particularly interested in the problems of finding the conditions on $B(z,t)$ that ensure that there exists a solution of the perturbed system

(4.4) $\dot{z}(t) + A(t)\, z(t) + B(z,t) = 0, \; z(\tau) = z_o$

which is stable in the sense of Liapunov given that there exists a stable solution of the basic unperturbed linear system (4.3).

In view of the theorem of Lumer and Phillips the important part of the analysis of the perturbed system is to obtain the conditions on $B(z,t)$ under which $A(t) + B(z,t)$ is m-accretive given that $A(t)$ is m-accretive. In this chapter we present some new theorems which provide results of this form. They are essentially extensions of the theorems of Kato [1,15] Nelson and Gustafson [20]. and Okazawa [21,22] for linear and non-linear m-accretive time-independent operators.

4.2. Linear Operators

Kato [1] has shown that if $-A$, $-B$ both generate linear contraction semi-groups on a Banach space Z with $D(B) \supset D(A)$ and if B is relatively bounded with respect to A with A bound less than $\frac{1}{2}$, that is

$$\|Bz\| \leq a\|z\| + b\|Az\|, \quad a \geq 0, \quad 0 < b < \tfrac{1}{2}, \quad z \in D(A)$$

then $-(A+B)$ generates a contraction semi-group on Z. Nelson and Gustafson [20] modified this result to show that for the same conditions on A but with B accretive and $b < 1$, then $-(A+B)$ generates a contraction semi-group. It has been noted earlier in §3 that $-A$ generates a contraction semi-group if and only if A is m-accretive and densely defined. If Z is reflexive and A m-accretive then A is automatically densely defined and for such spaces Okazawa [21] has extended the Nelson and Gustafson result to $b = 1$.

For Hilbert spaces Okazawa [22] has shown that if A is m-accretive, B accretive, $D(B) \supset D(A)$ and there are non-negative constants a and $b \leq 1$ such that

$$0 \leq \mathrm{Re}\langle Az, Bz\rangle + a\|z\|^2 + b\|Az\|^2, \qquad z \in D(A)$$

then $A+B$ is m-accretive if $b < 1$ and the closure of $A+B$ is m-accretive if $b = 1$.

We extend these results in the following theorems.

Theorem 4.1 Let Z be a reflexive Banach space and A be m-accretive, B accretive with $D(B) \supset D(A)$. If for any integer $n \geq 0$ there exists an $a \geq 0$ such that

$$(4.5) \qquad \|Bz\| \leq a\|z\| + \left\| \left[2 - \frac{1}{2^{(2^n-1)}}\right]Az + \left[1 - \frac{1}{2^{(2^n-1)}}\right]Bz \right\| \quad z \in D(A)$$

then $A+B$ is m-accretive. In the limit as $n \to \infty$ this condition becomes

$$(4.6) \qquad \|Bz\| \leq a\|z\| + \|(2A+B)z\|, \qquad z \in D(A), \qquad a \geq 0.$$

Proof:- Split the operator by writing

$$A + B = A + \beta B + (1-\beta)B \quad \text{where } 0 < \beta < 1.$$

Since B is accretive and $D(B)\supset D(A)$ then βB is accretive and $D(\beta B)\supset D(A)$ and so we may apply Okazawa's result [21] to conclude that $A+\beta B$ is m-accretive if there exists an $a\geqslant 0$ such that

(4.7) $\quad \|\beta Bz\| \leqslant a\|z\| + \|Az\|, \qquad z\in D(A).$

If (4.7) is valid then since $D((1-\beta)B)\supset D(A+\beta B)$ we may again apply Okazawa's theorem to the perturbation $(1-\beta)B$ of $A+\beta B$ to conclude that $A+B$ is m-accretive if there exists $a'\geqslant 0$ such that

(4.8) $\quad \|(1-\beta)Bz\| \leqslant a'\|z\| + \|(A+\beta B)z\|, \qquad z\in D(A)$

We now choose β so that (4.8) implies (4.7). Clearly if (4.8) is valid then

$$(1-\beta)\|Bz\| \leqslant a'\|z\| + \|Az\| + \beta\|Bz\|$$

or $\quad (1-2\beta)\|Bz\| \leqslant a'\|z\| + \|Az\|$

and this implies (4.7) if $a' = a$ and $1-2\beta = \beta$ i.e. $\beta = \frac{1}{3}$ We have thus shown that $A+B$ is m-accretive if there exists $a\geqslant 0$ such that

(4.9) $\quad \|Bz\| \leqslant \frac{3}{2}\|z\| + \|(\frac{3}{2}A + \frac{1}{2}B)z\|, \ z\in D(A).$

We note that if B satisfies

(4.10) $\quad \|Bz\| \leqslant a\|z\| + \|Az\|$

then

$$\|Bz\| \leqslant \tfrac{3}{2}a\|z\| + \tfrac{3}{2}\|Az\| - \tfrac{1}{2}\|Bz\|$$

hence

$$\|Bz\| \leqslant \tfrac{3}{2}a\|z\| + \|(\tfrac{3}{2}A + \tfrac{1}{2}B)z\|$$

so that any B which satisfies the Okazawa hypthesis (4.10) also satisfies (4.9). However the converse is not true because $B = \rho A$, $1 < \rho \leqslant 3$ satisfies (4.9) but not (4.10).

Since B is accretive and $D(B)\supset D(A)$ then βB is accretive and $D(\beta B)\supset D(A)$ and so we may apply Okazawa's result [21] to conclude that $A+\beta B$ is m-accretive if there exists an $a\geqslant 0$ such that

(4.7) $\quad \|\beta Bz\| \leqslant a\|z\| + \|Az\|, \quad z\in D(A).$

If (4.7) is valid then since $D((1-\beta)B)\supset D(A+\beta B)$ we may again apply Okazawa's theorem to the perturbation $(1-\beta)B$ of $A+\beta B$ to conclude that $A+B$ is m-accretive if there exists $a'\geqslant 0$ such that

(4.8) $\quad \|(1-\beta)Bz\| \leqslant a'\|z\| + \|(A+\beta B)z\|, \quad z\in D(A)$

We now choose β so that (4.8) implies (4.7). Clearly if (4.8) is valid then

$$(1-\beta)\|Bz\| \leqslant a'\|z\| + \|Az\| + \beta\|Bz\|$$

or

$$(1-2\beta)\|Bz\| \leqslant a'\|z\| + \|Az\|$$

and this implies (4.7) if $a' = a$ and $1-2\beta = \beta$ i.e. $\beta = \frac{1}{3}$ We have thus shown that $A+B$ is m-accretive if there exists $a \geqslant 0$ such that

(4.9) $\quad \|Bz\| \leqslant \frac{3}{2}\|z\| + \|(\frac{3}{2}A + \frac{1}{2}B)z\|, \quad z\in D(A).$

We note that if B satisfies

(4.10) $\quad \|Bz\| \leqslant a\|z\| + \|Az\|$

then

$$\|Bz\| \leqslant \tfrac{3}{2}a\|z\| + \tfrac{3}{2}\|Az\| - \tfrac{1}{2}\|Bz\|$$

hence

$$\|Bz\| \leqslant \tfrac{3}{2}a\|z\| + \|(\tfrac{3}{2}A + \tfrac{1}{2}B)z\|$$

so that any B which satisfies the Okazawa hypthesis (4.10) also satisfies (4.9). However the converse is not true because $B = \rho A$, $1 < \rho \leqslant 3$ satisfies (4.9) but not (4.10).

With n = 1, (4.5) is identical to (4.9). To prove (4.5) with n = 2 we let $A+B = A+\gamma B+(1-\gamma)B$ where $0 < \gamma < 1$ and derive two conditions similar to (4.7) and (4.8) but using (4.9) instead of Okazawa's condition (4.10). These conditions are

$$(4.11) \qquad \|\gamma Bz\| \leq \tfrac{3}{2}\|z\| + \|(\tfrac{3}{2}A+\tfrac{1}{2}\gamma B)z\| \qquad z \in D(A)$$

and

$$(4.12) \qquad \|(1-\gamma)Bz\| \leq \tfrac{3}{2}\|z\| + \|[\tfrac{3}{2}(A+\gamma B) + \tfrac{1}{2}(1-\gamma)B]z\|, \quad z \in D(A)$$

If (4.12) is valid then

$$\|(1-\gamma)Bz\| \leq \tfrac{3}{2}\|z\| + \|(\tfrac{3}{2}A+\tfrac{1}{2}\gamma B)z\| + \|\gamma Bz\| + \|\tfrac{1}{2}(1-\gamma)Bz\|, \; z \in D(A)$$

which is identical to (4.11) if $\frac{1-3\gamma}{2} = \gamma$ i.e. $\gamma = \frac{1}{5}$. With this value of γ (4.12) becomes

$$\|Bz\| \leq a\|z\| + \|(\tfrac{15}{8}A+\tfrac{7}{8}B)z\| \qquad z \in D(A)$$

which is (4.5) with n = 2.

The general result can be obtained by an induction argument.

In applications of the above theorem the assumption that B is accretive is very restrictive. The following theorem relaxes this condition.

<u>Theorem 4.2.</u> Let A be m-accretive and A+B be accretive on a reflexive Banach space Z with $D(B) \supset D(A)$. If there exists an $a \geq 0$ and an α, $0 \leq \alpha < 1$ such that

$$\|(A+B)z\| \leq \|\left(\frac{2-\alpha}{\alpha}A+B\right)z\| + a\|z\| \qquad z \in D(A)$$

then $A+\alpha B$ is m-accretive.

<u>Proof:-</u> By setting $A+\alpha B = (1-\alpha)A + \alpha(A+B)$ for $0 \leq \alpha < 1$ and applying Theorem 4.1. we find that $A+\alpha B$ is m-accretive if there exists an $a \geq 0$ s.t.

$$\|\alpha(A+B)z\| \leq a\|z\| + \|2(1-\alpha)Az + \alpha(A+B)z\| \qquad z \in D(A)$$

which is the desired condition.

For Hilbert spaces we have a similar result to Okazawa's theorem [22].

<u>Theorem 4.3.</u> Let A be m-accretive and A+B accretive on a Hilbert space H with $D(B) \supset D(A)$. If there exists $a' \geq 0$, $b > 1$ such that

$$0 \leq \mathrm{Re} \langle Az, Bz \rangle + a'\|z\|^2 + b\|Az\|^2 \qquad z \in D(A)$$

then $A + \frac{1}{b}B$ is m-accretive.

<u>Proof:-</u> It is easy to show that

$$\left\|\left(\frac{2-\alpha}{\alpha}A+B\right)z\right\|^2 - \|(A+B)z\|^2 = \frac{4(1-\alpha)}{\alpha}\left[\frac{1}{\alpha}\|Az\|^2 + \mathrm{Re}\langle Az, Bz\rangle\right] \tag{4.13}$$

Put $\alpha = \frac{1}{b}$ then using the condition

$$0 \leq \mathrm{Re} \langle Az, Bz \rangle + a'\|z\| + b\|Az\|^2$$

we have $\quad \|(A+B)z\|^2 \leq a\|z\|^2 + \left\|\left(\frac{2-\alpha}{\alpha}A+B\right)z\right\|^2$

where $a = \frac{4a'}{\alpha}(1-\alpha)$. Hence by Theorem 4.2. $A + \frac{1}{b}B$ is m-accretive.

<u>Corollary</u> Let A be m-accretive and $\frac{1}{b}A+\tilde{B}$ $(b > 1)$ be accretive on a Hilbert space H with $D(\tilde{B}) \supset D(A)$. If there exists a constant $a \geq 0$ such that

$$0 \leq \mathrm{Re}\langle Az, \tilde{B}z\rangle + a\|z\|^2 + \|Az\|^2 \qquad z \in D(A)$$

then $A+\tilde{B}$ is m-accretive.

<u>Proof:-</u> Set $B = b\tilde{B}$ in Theorem 4.3. If $b > 1$ A+B is accretive if $\frac{1}{b}(A+B)$ is accretive and the result follows directly.

<u>Theorem 4.4.</u> Let A be m-accretive and A+B accretive on a Hilbert space H with $D(B) \supset D(A)$. If there exist real constants $a \geq 0$ and $b > 1$ such that

$$\|Bz\| \leq a'\|z\| + b\|Az\| \tag{4.14}$$

then $A + \frac{1}{b}B$ is m-accretive.

<u>Proof:-</u> Set $b = \frac{2-\alpha}{\alpha} - \frac{\alpha}{4}\gamma$ where $\gamma = \left(\frac{2-\alpha}{\alpha}\right)^2 - 1$ then $b = \frac{1}{\alpha}$. With $a' = \frac{a}{2}$ (4.14) gives

$$0 \leq \|Az\|^2 - \alpha\|Az\|\left(\frac{a}{2}\|z\| + b\|Az\|\right) + \frac{a^2}{\gamma}\|z\|^2 + \frac{2a}{\gamma}\|z\|.\left\|\frac{2-\alpha}{\alpha}.\ Az\right\| - \frac{2a}{\gamma}\|z\|.\|Bz\|$$

and so

$$0 \leq \|Az\|^2 + \alpha \mathrm{Re}\langle Az, Bz\rangle + \frac{a^2}{\gamma}\|z\|^2 + \frac{2a}{\gamma}\|z\|.\|\frac{2-\alpha}{\alpha}Az\| - \frac{2a}{\gamma}\|z\|.\|Bz\|$$

from which it can be deduced that

$$\|(A+B)z\| \leq a\|z\| + \left\|\frac{2-\alpha}{\alpha}Az + Bz\right\|.$$

Since $b > 1$ we have $\alpha < 1$ so the conditions of Theorem 4.2. are satisfied hence $A + \frac{1}{b}B$ is m-accretive.

<u>Corollarv:-</u> Let A be m-accretive and $A+b\tilde{B}$ be accretive, if there exists $a > 0$ such that

$$\|\tilde{B}z\| \leq a\|z\| + \|Az\|$$

then $A+\tilde{B}$ is m-accretive.

<u>Proof:-</u> Set $B = b\tilde{B}$ $(b > 1)$ in Theorem (4.4)

We can obtain similar results for the theorems of Kato [1] and Nelson and Gustafson [20].

4.3. Non-linear Operators

We now proceed to consider non-linear operators. We will generalize the following two perturbation theorems of Kato [15] by essentially the same methods as we have used for the linear case. Kato's results are as follows.

Let A and B be m-accretive operators, possibly non-linear and multiple-valued.

(i) Let B be locally A-bounded so that $D(B) \supset D(A)$ and for each $z \in Z$ there are a neighbourhood U of z and constants a and b such that

(4.15) $\quad |||Bz||| \leq a + b|||Az||| \quad$ for $z \in D(A) \cap U$

where $|||Az||| = \inf_{s \in Az}\|s\|$. If $b < 1$ then A+B is m-accretive.

(ii) Let A be m-accretive and B single-valued and accretive. Let B satisfy the conditions of (i) above. Furthermore assume that for each and so

$$0 \leq \|Az\|^2 + \alpha \mathrm{Re}\langle Az, Bz\rangle + \frac{a^2}{\gamma}\|z\|^2 + \frac{2a}{\gamma}\|z\| . \|\frac{2-\alpha}{\alpha}Az\| - \frac{2a}{\gamma}\|z\| . \|Bz\|$$

from which it can be deduced that

$$\|(A+B)z\| \leq a\|z\| + \left\|\frac{2-\alpha}{\alpha}Az + Bz\right\| .$$

Since $b > 1$ we have $\alpha < 1$ so the conditions of Theorem 4.2. are satisfied hence $A + \frac{1}{b}B$ is m-accretive.

Corollary:- Let A be m-accretive and $A+b\tilde{B}$ be accretive, if there exists $a > 0$ such that

$$\|\tilde{B}z\| \leq a\|z\| + \|Az\|$$

then $A+\tilde{B}$ is m-accretive.

Proof:- Set $B = b\tilde{B}$ $(b > 1)$ in Theorem (4.4)

We can obtain similar results for the theorems of Kato [1] and Nelson and Gustafson [20].

4.3. Non-linear Operators

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where $|||Az||| = \inf_{s \in Az} \|s\|$. If $b < 1$ then A+B is m-accretive.

(ii) Let A be m-accretive and B single-valued and accretive. Let B satisfy the conditions of (i) above. Furthermore assume that for each

$z_o \in D(A)$ there are a neighbourhood U of z_o and constants a' and b' such that

$$\|By-Bz\| \leq a'\|y-z\| + b'|||Ay-Az||| \quad \text{for } y,z \in D(A) \cap U \tag{4.16}$$

Then A+B is m-accretive if $b < 1$ and $b' < 1$.

We derive first the following extension of (i).

Theorem 4.5. With the same conditions on the operators A and B as in (i) above but with (4.15) replaced by

$$|||Bz||| \leq a\|z\| + b\left|\left|\left|\left(2 - \frac{1}{2^{(2^n-1)}}\right)Az + \left(1 - \frac{1}{2^{(2^n-1)}}\right)Bz\right|\right|\right|, z \in D(A) \cap U \tag{4.17}$$

for any integer $n \geq 0$, $a > 0$, $b < 1$ then A+B is m-accretive. In the limit as $n \to \infty$ this condition becomes

$$|||Bz||| \leq a\|z\| + b|||(2A+B)z||| \qquad z \in D(A) \cap U. \tag{4.18}$$

Proof:- Let $|||Az||| = \|z_p\|$ where $z_p \in \{Az\}$, $|||\beta Bz||| = \|z_q\|$ where $z_q \in \{\beta Bz\}$ then

$$|||(A+\beta B)z||| = \inf \|(A+\beta B)z\| \leq \|z_p + z_q\| \leq \|z_p\| + \|z_q\| \leq |||Az||| + |||\beta Bz|||$$

hence the proof of this theorem is similar to the linear case. The essential difference will be illustrated in the proof of the next theorem.

Theorem 4.6. With the same conditions on the operators A and B as in (ii) above but with (4.15) replaced by (4.17) and (4.16) replaced by

$$\|Bz-By\| \leq a'\|z-y\| + b'\left|\left|\left|\left(2 - \frac{1}{2^{(2^n-1)}}\right)(Az-Ay) + \left(1 - \frac{1}{2^{(2^n-1)}}\right)(Bz-By)\right|\right|\right| \tag{4.19}$$

then A+B is m-accretive if $b < 1$, $b' < 1$.

In the limit as $n \to \infty$ this condition becomes

$$\|Bz-By\| \leq a'\|z-y\| + b'|||2(Az-Ay) + Bz-By||| \quad \text{for } z,y \in D(A) \cap U.$$

<u>Proof:-</u> Split the operator A+B by writing it as $A+\beta B+(1-\beta)B$, $0 < \beta < 1$, as in the proof of Theorem 4.1. Applying Kato's result (ii) regarding B as a perturbation of A then $(1-\beta)B$ as a perturbation of $A+\beta B$ we find that A+B is m-accretive if there exist non-negative constants a, a_1, a_2, a_3, b, b_1, b_2, b_3 with each $b < 1$ such that for $y,z \in D(A) \cap U$

(4.20) $\quad \|\beta Bz\| \leq a + b\|Az\|$

(4.21) $\quad \|\beta(Bz-By\| \leq a_1\|z-y\| + b_1\|Az-Ay\|$

(4.22) $\quad \|(1-\beta)Bz\| \leq a_2 + b_2\|(A+\beta B)z\|$

(4.23) $\quad \|(1-\beta)(Bz-By)\| \leq a_3\|z-y\| + b_3\|(A+\beta B)z - (A+\beta B)y\|$

If (4.22) holds then we have

$$(1-\beta-b_2\beta)\|Bz\| \leq a_2+b_2\|Az\|$$

which gives (4.20) if we set $a = \frac{\beta a_2}{1-\beta(1+b_2)}$ and $b = \frac{\beta b_2}{1-\beta(1+b_2)}$. We require $b < 1$ i.e. $\beta b_2 < 1-\beta(1+b_2)$ which gives $\beta < \frac{1}{1+2b_2}$. Since $b_2 < 1$ we can choose $\beta = \frac{1}{3}$. In a similar way we can show that (4.23) implies (4.21) if $\beta = \frac{1}{3}$ so that equations (4.20) - (4.23) can be replaced by the two conditions

(4.24) $\quad \|Bz\| \leq \frac{3}{2}a_2 + b_2\|(\frac{3}{2}A+\frac{1}{2}B)z\| \qquad z \in D(A) \cap U$

and

(4.25) $\quad \|Bz-By\| \leq \frac{3}{2}a_3\|z-y\| + b_3\|\frac{3}{2}(Az-Ay) + \frac{1}{2}(Bz-By)\|; z,y \in D(A) \cap U$

The rest of the proof follows as in the linear case.

The accretiveness condition on B can be relaxed as in Theorem 4.2. to obtain the following results.

<u>Theorem 4.7.</u> Let A be single-valued and m-accretive and A+B single-valued and accretive with $D(B) \supset D(A)$. If there exist non-negative constants a, a_1, b, b_1 with $b < 1$, $b_1 < 1$ and an α, $0 \leq \alpha < 1$ such that

(4.26) $$\|(A+B)z\| \leqslant a\|z\| + b\|\frac{2-\alpha}{\alpha}Az+Bz\| \qquad z \in D(A)\cap U$$

(4.27) $$\|Az-Ay+Bz-By\| \leqslant a_1\|z-y\| + b_1\|\frac{2-\alpha}{\alpha}(Az-Ay) + Bz-By\|; \; z,y \in D(A)\cap U$$

then $A+\alpha B$ is m-accretive.

<u>Proof</u>:- See proof of Theorem 4.2.

<u>Theorem 4.8.</u> Let A be single valued and m-accretive, A+B be single-valued and accretive on a Hilbert space H with $D(B) \supset D(A) \supset 0$. If $A0 = B0 = 0$ and there exists a, b > 1 such that

(4.28) $$\|Bz-By\|^2 \leqslant a^2\|z-y\|^2 + b^2\|Az-Ay\|^2 \quad z,y \in D(A)\cap U$$

then $A + \frac{2}{1+b^2}B$ is m-accretive.

<u>Proof</u>:- If

(4.29) $$(1-b_1^2)\|Bz-By\|^2 \leqslant a_1^2\|z-y\|^2 + \left(\frac{1}{b_1^2} - 1\right)\|Az-Ay\|^2$$

and $\quad b_1^2(2-\alpha)/\alpha = 1 \quad$ then

$$\begin{aligned}\|Az-Ay+Bz-By\|^2 &= \|Az-Ay\|^2 + \langle Az-Ay, Bz-By\rangle + \langle Bz-By, Az-Ay\rangle \\ &\quad + (1-b_1^2)\|Bz-By\|^2 + b_1^2\|Bz-By\|^2 \\ &\leqslant \|Az-Ay\|^2 + \langle Az-Ay, Bz-By\rangle + \langle Bz-By, Az-Ay\rangle \\ &\quad + a_1^2\|z-y\|^2 + \left(\frac{1}{b_1^2} - 1\right)\|Az-Ay\|^2 + b_1^2\|Bz-By\|^2 \\ &= a_1^2\|z-y\|^2 + \|\frac{1}{b_1}(Az-Ay) + b_1(Bz-By)\|^2 \\ &= a_1^2\|z-y\|^2 + b_1^2\|\frac{2-\alpha}{\alpha}(Az-Ay) + (Bz-By)\|^2\end{aligned}$$

Now (4.29) is valid if $b^2 = 1/b_1^2$, $a^2 = a_1^2/(1-b_1^2)$, using (4.28), and so we have (4.27) of Theorem 4.7. Moreover since $A0 = B0 = 0$ the above is true if $y = 0$ and we obtain

$$\|Az+Bz\| \leqslant a_1\|z\| + b\|\frac{2-\alpha}{\alpha}Az+Bz\| \qquad z \in D(A)\cap U$$

which is (4.26) of Theorem 4.7. Since $b > 1$ then $b_1 < 1$ so all the conditions of Theorem 4.7. are satisfied and we have $A+\alpha B = A + \frac{2}{1+b^2}B$ is m-accretive.

<u>Corollary</u> With the same conditions as in Theorem 4.8. but with $(1-\varepsilon)A+B$ accretive instead of A+B accretive and (4.28) replaced by

$$\|\tilde{B}z-\tilde{B}y\|^2 \leq a^2\|z-y\|^2 + (1-\varepsilon^2)\|Az-Ay\|^2 \quad z,y\in D(A)\cap U$$

then A+$\tilde{B}$ is m-accretive.

Proof:- Set $B = \left(\frac{1+b^2}{2}\right)\tilde{B}$ and $\frac{2}{1+b^2} = 1-\varepsilon$. If $(1-\varepsilon)A+\tilde{B}$ is accretive then so is $(1-\varepsilon)(A+B)$ and also A+B.

If $\|\tilde{B}z-\tilde{B}y\|^2 \leq a^2\|z-y\|^2 + (1-\varepsilon^2)\|Az-Ay\|^2$ then

$$\|Bz-By\|^2 \leq a^2\left(\frac{1+b^2}{2}\right)^2\|z-y\|^2 + \frac{(1-\varepsilon^2)}{(1-\varepsilon)^2}\|Az-Ay\|^2$$
$$= a'^2\|z-y\|^2 + b^2\|Az-Ay\|^2$$

which is (4.28). Hence by Theorem 4.8. $A+(1-\varepsilon)B$ i.e. A+$\tilde{B}$ is m-accretive.

Theorem 4.9. Let A be single-valued and m-accretive, A+B be single-valued and accretive on a real Hilbert space H, with $D(B)\supset D(A)\supset 0$. If $A0 = B0 = 0$ and there exists non-negative constants a', $b' > 1$ such that

(4.30) $$\|Bz-By\| \leq a'\|z-y\| + b'\|Az-Ay\|$$

then for any $\delta > 0$, $A + \frac{1}{(1+\delta)b'-\delta}B$ is m-accretive.

Proof:- Choose α s.t. $0 \leq \alpha < \frac{1}{b'}$, and let $b = \dfrac{b'-1}{\frac{2-\alpha}{\alpha} - b'}$ then $b < 1$ and $\alpha < 1$ since $b' > 1$. Set $a = a'(1+b)$ then squaring (4.30) gives

$$0 \leq \left[b^2\left(\frac{2-\alpha}{\alpha}\right)^2 - 1\right]\|Az-Ay\|^2 + 2ab\|z-y\|.\|\frac{2-\alpha}{\alpha}(Az-Ay)\| + (b^2-1)\|Bz-By\|^2 - 2ab\|z-y\|.$$
$$\left[a'\|z-y\| + b'\|Az-Ay\|\right] + a^2\|z-y\|^2 - 2\left[b^2\frac{2-\alpha}{\alpha} - 1\right]\|Az-Ay\|\left[a'\|z-y\| + b'\|Az-Ay\|\right]$$

which implies

$$\|Az-Ay\|^2 + 2\langle Az-Ay, Bz-By\rangle + \|Bz-By\|^2$$
$$\leq a^2\|z-y\|^2 + b^2\|\frac{2-\alpha}{\alpha}(Az-Ay) + Bz-By\|^2 + 2ab\|z-y\|.\|\frac{2-\alpha}{\alpha}(Az-Ay) + Bz-By\|$$

i.e. $$\|Az-Ay + Bz-By\| \leq a\|z-y\| + b\|\frac{2-\alpha}{\alpha}(Az-Ay) + Bz-By\|$$

Moreover if $A0 = B0 = 0$ we can see that (4.30) implies

$$\|(A+B)z\| \leq a\|z\| + b\|\frac{2-\alpha}{\alpha}Az-Bz\|$$

so by Theorem 4.7. $A+\alpha B$ is m-accretive.

Since $b' = \frac{1}{1+b}\left(\frac{2-\alpha}{\alpha}b+1\right)$ we have $\alpha = \frac{1}{(1+\delta)b'-\delta}$ with $\delta = \frac{1-b}{2b}$. Since $b < 1$ it follows that $\delta > 0$.

<u>Corollary</u> With the same conditions as in Theorem 4.9. but with $(1-\varepsilon)A+\tilde{B}$ accretive instead of A+B accretive and with (4.30) replaced by

(4.31) $$\|\tilde{B}z-\tilde{B}y\| \leq a\|z-y\| + b'(1-\varepsilon)\|Az-Ay\|$$

then $A+\tilde{B}$ is m-accretive.

<u>Proof:-</u> Set $\tilde{B} = \frac{1}{1+\delta(b'-1)} B = (1-\varepsilon)B$ in Theorem 4.9.

In Chapter 8 we shall illustrate these results by applying the following version of the above corollary. We conjecture, in the light of the above results, that it is true.

<u>Corollary</u>. Let A be single-valued and m-accretive, $(1-\varepsilon)A+\tilde{B}$ be single-valued and accretive on a neighbourhood Uof the origin of a real Hilbert space H with $D(\tilde{B})\supset D(A)\supset U\ni 0$.

If $A0=\tilde{B}0 = 0$ and there exists non-negative constants $a', b' > 1$ such

that $$\|\tilde{B}z - \tilde{B}y\| \leq a'\|z-y\| + b'(1-\varepsilon)\|Az-Ay\|$$

then $A + \tilde{B}$ is m-accretive in U.

<u>Note</u>

In the above results we have not proved that the hypotheses of theorems 4.1 and 4.5 can be weakened to

$$\|Bz\| \leq a\|z\| + \|2Az + Bz\|, \qquad z\in D(A)$$

but the subsequent results can be obtained without actually taking the limit as $n\to\infty$.

§5 Perturbations of Linear Semi-Groups

5.1. Introduction

We recall that in the survey of existence and uniqueness theorems in Chapter 3 we introduced the concept of a mild solution of a differential equation. For the linear homogeneous equation

(5.1) $$\dot{z}(t) + A(t)z(t) + B(t)z(t) = 0 \quad z(0) = z_o$$

the mild solution is given by

(5.2) $$z(t) = T(t,0)z_o - \int_o^t T(t,\rho)B(\rho)z(\rho)d\rho$$

where $T(t,s)$ is the evolution operator associated with $A(t)$. Writing $z(t) = U(t,s)z(s)$ we call $U(t,s)$ the evolution operator associated with the operator $A(\cdot) + B(\cdot)$ on $\Delta(T) = \{(s,t),\ 0 \leq s < t \leq T\}$ and define it by the integral equation

(5.3) $$U(t,s)z = T(t,s)z - \int_s^t T(t,\rho)B(\rho)U(\rho,s)z\, d\rho$$

which can be derived from (5.2) using the properties of $T(t,s)$.

In this chapter we discuss the properties of $U(t,s)$ in the following situations:-

(i) $T(t,s) = T_{t-s}$ the strongly-continuous semi-group on Z which has $-A$ as its infinitesimal generator and is such that

(5.4) $$\|T_t\| \leq Me^{-\omega t} \qquad (\omega > 0)$$

so that

$$z(t) = T_t z_o$$

is the asymptotically stable strict solution of

$$\dot{z}(t) + Az(t) = 0 \quad z(0) = z_o \in D(A).$$

For $B(\cdot)$ we consider the two cases of $B(\cdot)$ bounded and $B(\cdot)$ unbounded.

(ii) $T(t,s)$ is the evolution operator generated by $-A(t)$ and is also a quasi-evolution operator in the sense that

$$\frac{\partial}{\partial s}T(t,s)z = T(t,s)A(s)z \qquad z \in D(A) \quad \text{a.e.}$$

with $B(\cdot)$ the class of (possibly) unbounded operators such that

$$\|T(t,s)B(s)z\| \leq g(t-s)\ \|z\| \qquad z \in Z$$

where g is a locally integrable positive function.

In each case we show that $U(t,s)$ is a quasi-evolution operator with an exponential bound of the form

(5.5) $$\|U(t,s)\| \leq Me^{-(\omega-\omega_1)(t-s)}.$$

From these properties we can derive conditions under which the perturbed system (5.1) is asymptotically stable in the sense of Liapunov and can also show that it is possible to estimate the solution of the in-homogeneous equation

(5.6) $$\dot{z}(t) + A(t)z(t) + B(t)z(t) = f(t).$$

We make frequent use of the following inequality. (For a proof see Carroll [4]).

Gronwall's Inequality

Let $a \in \mathcal{L}^1(\tau,T)$, $a(t) \geq 0$, $z \in \mathcal{L}^\infty(\tau,T)$ and assume b is absolutely continuous on $[\tau,T]$. If

$$z(t) \leq b(t) + \int_\tau^t a(\mu)z(\mu)\,d\mu$$

then

$$z(t) \leq b(\tau)\exp\int_\tau^t a(\mu)d\mu + \int_\tau^t b'(s)\exp\left(\int_s^t a(\mu)d\mu\right)ds.$$

5.2. Bounded Perturbations of a Semi-Group Operator

Let T_t be a strongly continuous semi-group of operators with $-A$ as infinitesimal generator such that (5.4) is valid. Let $B \in B_\infty([0,T],\mathcal{L}(Z))$, the space of $\mathcal{L}(Z)$ valued operators strongly measurable and essentially bounded in $[0,T]$ so that there exists K such that

(5.7) $$\|B(t)\|_{\mathcal{L}(Z)} \leq K \quad \text{a.e. on } [0,T]$$

We derive all the properties of $U(t,s)$ in the following theorem. The results can easily be generalized to the case where $\|B(t)\|_{\mathcal{L}(Z)} \leq K(t)$, [23].

<u>Theorem 5.1</u>: There exists a unique solution of (5.3) with the following properties,

(i) $U(t,.)$ is strongly continuous on $[0,T]$ and $U(.,s)$ is strongly continuous on $[s,T]$.

(ii) $U(t,r)U(r,s) = U(t,s)$, $U(t,t) = I$ $\quad 0 \leq s \leq r \leq t \leq T$

(iii) $\|U(t,s)\| \leq Me^{-(\omega-MK)(t-s)}$

(iv) $U(t,s)$ is a quasi-evolution operator [23] in the sense

$$\frac{\partial}{\partial s}U(t,s)z = U(t,s)(A+B(s))z \;, \; z \in D(A) \quad \text{a.e.}$$

<u>Proof</u>:- The proof of parts (i), (ii) and (iv) of this theorem can be found in [23], consequently it is not necessary to give all the details here. It is sufficient to observe the following remarks concerning the existence of a solution of (5.3).

We construct $U(t,s)z$ by the method of successive approximations so that $U(t,s)z = \sum_{n=0}^{\infty} U_n(t,s)z$ with

(5.8) $$U_0(t,s)z = T_{t-s}z$$

(5.9) $$U_n(t,s)z = -\int_s^t T_{t-\rho}B(\rho)U_{n-1}(\rho,s)z \, d\rho$$

The integral in (5.9) is a well-defined Bochner integral since $B(.) \in B_\infty([0,T],(Z))$. Taking norms we obtain

$$\|U_n(t,s)z\| \leq \int_s^t \|T_{t-\rho}B(\rho)\| \; \|U_{n-1}(\rho,s)z\| d\rho.$$

By writing $\|U_n(t,s)z\| = g_n(t,s)$ $(n \geq 0)$ and $\|T_{t-\rho}B(\rho)\| = K(t,\rho)$ we have

(5.10) $$g_n(t,s) \leq \int_s^t K(t,\rho)g_{n-1}(\rho,s)d\rho.$$

For a given kernel $K(t,\rho)$ we require minimally that $g_n(t,s) \to 0$ as $n \to \infty$. For our bounded operator B we have $K(t,\rho) = MKe^{-\omega(t-\rho)}$ and it is easy to show by induction that

$$g_n(t,s) \le M(MK)^n \|z\| \frac{(t-s)^n}{n!} e^{-\omega(t-s)},$$

hence $\|\sum_0^N U_n(t,s)z\| \le \sum_0^N g_n(t,s) \le M \exp[(MK-\omega)(t-s)]$ for all N and so $\sum_{n=0}^{\infty} U_n(t,s)$ is convergent on $\Delta(\tau)$ in the uniform topology. We can easily show that $U(t,s) = \sum_{n=0}^{\infty} U_n(t,s)$ is a solution of (5.3) and so we have proved the existence of a solution with

$$\|U(t,s)\| \le Me^{-(\omega-MK)(t-s)} \tag{5.11}$$

We can now estimate the effect of the perturbation operator B(.) and the forcing term f(.). We first define the 'mild' solution of (5.6) to be

$$z(t) = U(t,0)z_o + \int_o^t U(t,\rho)f(\rho)d\rho \tag{5.12}$$

so that if for example $f \in L_2[0,T,Z]$ we have $z(.) \in C[[0,T],Z]$. Note however that in general we are not able to differentiate (5.12) and obtain (5.6). To do so requires further assumptions on f and that U(t,s) is a strict evolution operator in the sense that it satisfies

$$\frac{\partial}{\partial t}U(t,s)z = -(A+B(t))U(t,s)z \quad z \in D(A).$$

as well as the properties defined in our Theorem 5.1.

It is easy to show that (5.12) is equivalent to

$$z(t) = T_t z_o - \int_o^t T_{t-\rho}B(\rho)z(\rho)d\rho + \int_o^t T_{t-\rho}f(\rho)d\rho \tag{5.13}$$

Using (5.11) and (5.12) together gives

$$\|z(t)\| \le Me^{-(\omega-MK)t}\|z_o\| + \int_o^t Me^{-(\omega-MK)(t-\rho)}\|f(\rho)\|d\rho \tag{5.14}$$

from which we can estimate the effect of the perturbation operator B(.) and the forcing term f(.).

5.3. Unbounded Perturbations of a Semi-Group Operator

The second class of perturbation operator B(.) that we wish to consider is that of unbounded perturbations with the properties

(5.15) $\overline{D(B(t))} = Z$ for all t,

(5.16) $\|T_{t-s}B(s)z\| \leq \frac{N}{(t-s)^{\alpha}} \|z\|$, $t > s$, $z \in D(B(s))$, $0 \leq \alpha < 1$.

We require conditions such that A+B(.) generates an evolution operator U(t,s) with

(5.17) $\|U(t,s)\| \leq M' e^{-\omega_1 (t-s)}$, $\omega_1 > 0$.

We note first that (5.16) implies that for any $t > s$ the operator $T_{t-s}B(s)$ has extension $\overline{T_{t-s}B(s)}$ to all of Z with

(5.18) $\|\overline{T_{t-s}B}(s)\| \leq \frac{N}{(t-s)^{\alpha}}$ $t > s$ and

(5.19) $T_{\rho}\overline{T_{t-s}B}(s) = \overline{T_{t+\rho-s}B}(s)$ $t > s$, $\rho \geq 0$.

Equation (5.3) is now taken as

(5.20) $U(t,s)z = T_{t-s}z - \int_s^t \overline{T_{t-\rho}B(\rho)}U(\rho,s)z \, d\rho$ $0 \leq s < t \leq T$.

We prove the following theorem which is essentially the same as the first two parts of Theorem 5.1.

<u>Theorem 5.2.</u> With $\|\overline{T_{t-s}B(s)}\|$ given by (5.18) and $\|T_t\| \leq M$ there exists a unique solution of (5.20) with the following properties:-

(i) U(t,.) is strongly continuous on [0,T] and U(.,s) is strongly continuous on [s,T].

(ii) $U(t,r)U(r,s) = U(t,s)$, $U(t,t) = I$, $0 \leq s \leq r \leq t \leq T$.

<u>Proof</u>:- We construct U(t,s) by successive iterations as in the proof of Theorem 5.1. We have (5.10) with $K(t,\rho) = \frac{N}{(t-\rho)^{\alpha}}$

(5.21) i.e. $g_n(t,s) \leq \int_s^t \frac{N}{(t-\rho)^{\alpha}} g_{n-1}(\rho,s) \, d\rho$, $0 \leq \alpha < 1$.

Although this kernel $K(t,\rho)$ has a weak singularity at $\rho = t$ we can, following Micklin [24], show that $g_n(t,s)$ is bounded by $h_n(t,s)$ where

$$(5.22)\qquad h_n(t,s) = \frac{M[N\Gamma(1-\alpha)]^n}{\Gamma(n-n\alpha)} \int_s^t \frac{1}{(t-\rho)^{\alpha-(n-1)(1-\alpha)}}\, d\rho$$
$$= \frac{M[N\Gamma(1-\alpha)]^n}{\Gamma[n+1-n\alpha]} (t-s)^{n(1-\alpha)}, \quad (\Gamma \equiv \text{the gamma function}).$$

$\sum_{n=0}^{\infty} h_n(t,s)$ is a power series in $(t-s)^{1-\alpha}$, uniformly convergent for all $t < \infty$ hence $\sum_0^{\infty} g_n(t,s)$ converges uniformly on $[0,T]$. As in the case of a bounded operator we can then deduce that $\sum_{n=0}^{\infty} U_n(t,s)$ is convergent on $\Delta(\tau)$ in the uniform topology and that it is a solution of (5.20). We also have $U(t,t) = I$.

To show that the solution is unique we suppose that there is another solution $U_1(t,s)$ and let

$$R(t,s) = U(t,s) - U_1(t,s) .$$

Then
$$R(t,s)z = \int_s^t \overline{T_{t-\rho} B(\rho)} R(\rho,s)z \, d\rho$$

and
$$\|R(t,s)z\| \leq \int_s^t \frac{N}{(t-\rho)^{\alpha}} \|R(\rho,s)z\| \, d\rho.$$

which is the same form as (5.21). If we iterate this inequality a sufficient number of times we obtain

$$\|R(t,s)z\| \leq C \int_s^t \|R(\rho,s)z\| \, d\rho \qquad (C > 0)$$

so $R(t,s)z = 0$ for all $z \in Z$ by Gronwall's inequality.

The semi-group property can be proved in a similar way.

$$U(t,r)U(r,s)z = T_{t-r}T_{r-s}z - T_{t-r}\int_s^r \overline{T_{r-\rho}B(\rho)} U(\rho,s)z \, d\rho$$
$$- \int_r^t \overline{T_{t-\rho'}B(\rho')} U(\rho',r)U(r,s)z \, d\rho'.$$

Using the semi-group property for T_t along with (5.19) and setting $U(t,r)U(r,s) - U(t,s) = R(t,r,s)$ we have

$$R(t,r,s)z = -\int_r^t \overline{T_{t-\rho'}B(\rho')}\, R(\rho',r,s)z\, d\rho'$$

and so by Gronwall's inequality we have $R(t,r,s)z = 0\ \forall z\in Z$ and $s \leq r \leq t$. Hence $U(t,r)U(r,s) - U(t,s) = 0$.

For the proof of continuity we write

$$U(t,s)z = T_{t-s}z - \int_s^t \overline{T_{t-\rho}B(\rho)}U(\rho,s)zd\rho = T_{t-s}z + \phi(t,s)z$$

and note that T_{t-s} is strongly continuous in s and t. Therefore we need consider only $\phi(t,s)z$. Take $h > 0$, $t_1\in[s,T)$, $t_2\in(s,T]$ then we have

$$\phi(t_1+h,s)z - \phi(t_1,s)z = \int_s^{t_1} (\overline{T_{t_1+h-\rho}B(\rho)} - \overline{T_{t_1-\rho}B(\rho)})U(\rho,s)zd\rho + \int_{t_1}^{t_1+h} \overline{T_{t_1+h-\rho}B(\rho)}U(\rho,s)zd\rho$$

and

$$\phi(t_2,s)z - \phi(t_2-h,s)z = \int_s^{t_2-h} (\overline{T_{t_2-\rho}B(\rho)} - \overline{T_{t_2-h-\rho}B(\rho)}U(\rho,s)zd\rho + \int_{t_2-h}^{t_2}\overline{T_{t_2-\rho}B(\rho)}U(\rho,s)zd\rho$$

$$\therefore \|\phi(t_1+h,s)z - \phi(t_2,s)z\| \leq \|(T_h-I)\int_s^{t_1}\overline{T_{t_1-\rho}B(\rho)}U(\rho,s)zd\rho\| + \int_{t}^{t_1+h} \frac{N}{(t_1+h-\rho)^\alpha}\|U(\rho,s)z\|d\rho$$

using (5.19). Then $\|\phi(t_1+h,s)z - \phi(t_1,s)z\| \to 0$ as $h \to 0$ by the strong continuity of T_t and the fact that $0 \leq \alpha < 1$ and $\|U(\rho,s)z\|$ is bounded. $\phi(t_2,s)z - \phi(t_2-h,s)z \to 0$ as $h \to 0$ for similar reasons.

Now take $h > 0$, $s_1\in[0,T)$ and $s_2\in(0,T]$ then

$$\|\phi(t,s_1+h)z - \phi(t,s_1)z\| \leq \int_{s_1+h}^t \|\overline{T_{t-\rho}B(\rho)}\|\ \|U(\rho,s_1+h)z - U(\rho,s_1)z\|d\rho + \int_{s_1}^{s_1+h} \|\overline{T_{t-\rho}B(\rho)}\|\ \|U(\rho,s_1)z\|d\rho$$

and

$$\|\phi(t,s_2-h)z - \phi(t,s_2)z\| \leq \int_{s_2}^t \|\overline{T_{t-\rho}B(\rho)}\|\ \|U(r,s_2-h)z - U(r,s_2)z\|d\rho + \int_{s_2-h}^{s_2} \|\overline{T_{t-\rho}B(\rho)}\|\ \|U(r,s_2-h)z\|d\rho$$

Since $U(\rho,s_1) = U(\rho,s_1+h)U(s_1+h,s_1)$ as $h > 0$ we have

$$\|U(\rho,s_1+h)z - U(\rho,s_1)z\| \leq \|U(\rho,s_1+h)\| \; \|I - U(s_1+h,s_1)z\|$$

hence $\|\phi(t,s_1+h)z - \phi(t,s_1)z\| \to 0$ as $h \to 0$ by the strong continuity of $U(.,s)$ on $[s,T]$, the boundedness of $\|U(t,s)\|$ and the property of $\|T_{t-\rho}B(\rho)\|$. $\|\phi(t,s_2-h)z - \phi(t,s_2)z\| \to 0$ as $h \to 0$ for similar reasons.

We now have to consider the problem of obtaining an estimate for $\|U(t,s)\|$ when $\|T_t\| \leq Me^{-\omega t}$ $(\omega > 0)$ similar to that given by (iii) of Theorem 5.1. for the case of a bounded operator. For simplicity in the following we assume that B is independent of t so that the operator $U(t,s)$ is in fact a semi-group $U(t,s) = U_{t-s}$ where

$$U_t z = T_t z - \int_o^t \overline{T_{t-\rho}B}\, U_\rho z d\rho$$

An estimate of U_t could be found by summing the series

$$\sum_0^\infty \frac{M[N\Gamma(1-\alpha)]^r}{\Gamma[r+1-r\alpha]} t^{r(1-\alpha)}$$

but in applications this could be difficult. An alternative approach is to iterate the inequality (5.21) $\overline{N}$ times only where $\overline{N}$ is determined by

$$(5.23) \qquad \begin{cases} n(1-\alpha) - \alpha \geq 0 & \text{for} \quad n \geq \overline{N} \\ n(1-\alpha) - \alpha < 0 & \text{for} \quad n < \overline{N} \end{cases}$$

Then we have

$$(5.24) \qquad g(t) \leq \sum_0^{\overline{N}} g_n(t) + T^{\overline{N}(1-\alpha)-\alpha} \frac{[N\Gamma(1-\alpha)]^{\overline{N}+1}}{\Gamma((\overline{N}+1)(1-\alpha))} \int_o^t g(\rho)d\rho$$

with $g_o(t) = Me^{-\omega t}$, and we may use Gronwall's lemma to obtain

$$(5.25) \qquad g(t) \leq Me^{Ct} + e^{Ct}\int_o^t b'(\rho)e^{-C\rho}d\rho$$

where $b(t) = \sum_0^{\overline{N}} g_n(t)$ and $C = T^{\overline{N}(1-\alpha)-\alpha} \dfrac{[N\Gamma(1-\alpha)]^{\overline{N}+1}}{\Gamma((\overline{N}+1)(1-\alpha))}$.

Since $C > 0$ this result has no use for our purposes.

We can find a better estimate if we use the semi-group property of T_t to modify the estimate of $\|T_t B\|$ given by (5.18) and used above. From (5.19) with $s = \gamma t$ and using (5.4) we find that

$$\|T_{(1+\gamma)t}B\| \leq Me^{-\omega\gamma t}\frac{N}{t^\alpha}, \quad t > 0$$

or

$$\|T_t B\| \leq \frac{MN}{t^\alpha}(1+\gamma)^\alpha e^{-\omega\gamma t/(1+\gamma)}, \quad t > 0$$

(5.26) i.e. $\|T_t B\| \leq \frac{N_1}{t^\alpha} e^{-\beta t}, \quad t > 0$ where

(5.27) $N_1 = MN(1+\gamma)^\alpha, \qquad \beta = \omega\gamma/(1+\gamma).$

Theorem 5.2. is still valid as the additional term in the kernel $K(t,\rho)$ is $e^{-\beta(t-\rho)}$ which is bounded by unity. The corresponding $g_n(t)$ are given by

$$g_n(t) = Me^{-\beta t}\frac{[N_1\Gamma(1-\alpha)]^n}{\Gamma(n-n\alpha)}\int_0^t \frac{e^{-(\omega-\beta)\rho}}{(t-\rho)^{\alpha-(n-1)(1-\alpha)}}d\rho, \quad n \geq 1$$

and (5.24) is replaced by

$$g(t) \leq Me^{-\omega t} + \sum_1^{\bar{N}} g_n(t) + T^{\bar{N}(1-\alpha)-\alpha}\frac{[N_1\Gamma(1-\alpha)]^{\bar{N}+1}}{\Gamma[(\bar{N}+1)(1-\alpha)]}\int_0^t e^{-\beta(t-\rho)}g(\rho)d\rho$$

We now use Gronwall's lemma to estimate $e^{\beta t}g(t)$ and have

$$g(t)e^{\beta t} \leq Me^{Ct} + b(t)e^{Ct}$$

where $b(t) = \sum_1^{\bar{N}} g_n(t)e^{\beta t}$ and $C = T^{\bar{N}(1-\alpha)-\alpha}\frac{[N_1\Gamma(1-\alpha)]^{\bar{N}+1}}{\Gamma[(\bar{N}+1)(1-\alpha)]}$

Thus in place of (5.25) we have

(5.28) $g(t) \leq [M+b(t)]e^{-(\beta-C)t}.$

Clearly this estimate will depend upon T unless $\alpha = 0, \frac{1}{2}, \frac{2}{3}, \ldots \frac{\bar{N}}{\bar{N}+1} \ldots$ and so will be of no use as such when we consider $T \to \infty$. However since we have

$$\|U_t\| \leq g(t) \leq M_1 e^{-\omega_1 t} \qquad 0 \leq t \leq T$$

using the semi-group property of U_t we can show that

$$\|U_{nT}\| < M_1^n e^{-\omega_1 nT}$$

If $\gamma = \log M_1$ then

$$\|U_{nT}\| < e^{-(\omega_1 - \frac{\gamma}{T})\, nT}$$

hence $$\|U_{nT+t}\| < M_1 e^{-(\omega_1 - \frac{\gamma}{T})(nT+1)} \qquad \text{if } \gamma > 0$$

so we have

(5.29) $$\|U_t\| < M_1 e^{-(\omega_1 - \frac{\gamma}{T})t} \qquad (t > 0).$$

Since $\omega_1 = \beta - C$, ω_1 depends on T so that in determining conditions on the operator B, through the parameter N_1, for the origin of the perturbed system to be asymptotically stable we must optimise on T.

If N_1 is small enough then we can find T such that

$$\beta - C > \frac{\log M_1}{T} = \frac{\log[M+b(t)]}{T}$$

since C and b(T) increase continuously with N_1 and T and are zero when $N_1 = 0$.

In contrast to the above analysis we will now find a stability criterion by estimating directly the solution of the integral equation

(5.30) $$h(t) < Me^{-\omega' t} + N_1 \int_0^t \frac{h(\rho)}{(t-\rho)^\alpha}\, d\rho$$

(5.31) where $h(t) = e^{\beta t}\|U_t\|$ and $\omega' = \omega - \beta$.

Now suppose we can find a function H(t) which satisfies

(5.32) $$H(t) > Me^{-\omega' t} + N_1 \int_0^t \frac{H(\rho)}{(t-\rho)^\alpha}\, d\rho \qquad \forall t > 0$$

then $$H(t) > Me^{-\omega' t} + MN_1 \int_0^t \frac{e^{-\omega'\rho}}{(t-\rho)^\alpha}\, d\rho + \ldots\ldots$$

$$= h_o(t)$$

where $h_o(t)$ is the solution of

$$h_o(t) = Me^{-\omega' t} + N_1 \int_0^t \frac{h_o(\rho)}{(t-\rho)^\alpha}\, d\rho$$

Moreover $h_o(t)$ is greater than any h(t) which satisfies (5.30) hence $H(t) > h(t)$ for all $t > 0$.

We look for H(t) in the form of an exponential bound $M_1 e^{\Omega t}$ and note that Ω must be positive since if $H(t) = M_1 e^{-\sigma t}$ $(\sigma > 0)$, when we substitute into (5.32) we require

$$M_1 e^{-\sigma t} > Me^{-\omega' t} + N_1 M_1 \int_0^t \frac{e^{-\sigma\rho}}{(t-\rho)^\alpha} d\rho \quad \forall t > 0$$

i.e. $$M_1 > e^{-(\sigma-\omega)t} + N_1 M_1 \int_0^t \frac{e^{\sigma y}}{y^\alpha} dy$$

for which there is no solution if $\sigma > 0$ since the integral diverges as $t \to \infty$. With $H(t) = M_1 e^{\Omega t}$ $(\Omega > 0)$ we solve

(5.33) $$M_1 e^{\Omega t} > Me^{-\omega' t} + N_1 M_1 \int_0^t \frac{e^{\Omega\rho}}{(t-\rho)^\alpha} d\rho \quad \forall t > 0$$

Since $$\int_0^t \frac{e^{\Omega\rho}}{(t-\rho)^\alpha} d\rho = \frac{e^{\Omega t}}{1-\alpha} \int_0^{t^{1-\alpha}} e^{-\Omega y^{\frac{1}{1-\alpha}}} dy < \frac{e^{\Omega t}}{1-\alpha} \int_0^\infty e^{-\Omega y^{\frac{1}{1-\alpha}}} dy = \frac{e^{\Omega t}}{\Omega^{1-\alpha}} \Gamma(1-\alpha).$$

we can see that (5.33) is satisfied if Ω is chosen so that

(5.34) $$M_1 e^{\Omega t} > Me^{-\omega' t} + \frac{N_1 M_1}{\Omega^{1-\alpha}} e^{\Omega t} \Gamma(1-\alpha).$$

If there exists $\delta > 0$ such that

(5.35) $$1 = \frac{N_1 \Gamma(1-\alpha)}{\Omega^{1-\alpha}} + \delta$$

then (5.34) is satisfied for all $t > T$ such that

(5.36) $$\frac{M}{M_1} e^{-(\omega'+\Omega)T} = \delta$$

and if $M_1\delta > M$ then (5.34) is valid for all $t > 0$. Hence we can find $M_1, \Omega(>0)$ such that

$$h(t) < M_1 e^{\Omega t} \qquad \forall t > 0,$$

an estimate for this value of Ω being given by

(5.37) $$1 > \frac{N_1}{\Omega^{1-\alpha}} \Gamma(1-\alpha).$$

Using (5.31) and (5.37) we now have an estimate of $\|U_t\|$ in the form

(5.38) $$\|U_t\| = g(t) = e^{-\beta t} h(t) < M_1 e^{(\Omega-\beta)t}.$$

where $\Omega^{1-\alpha} > N_1 \Gamma(1-\alpha)$.

For stability we require $\Omega < \beta$ so on using (5.27) we need

$$\Omega < \frac{\omega\gamma}{1+\gamma}$$

(5.39) i.e. $$N_1\Gamma(1-\alpha) < \left(\frac{\omega\gamma}{1+\gamma}\right)^{1-\alpha}.$$

But $N_1 = MN(1+\gamma)^{\alpha}$ so (5.39) becomes

$$MN\Gamma(1-\alpha) < \frac{(\omega\gamma)^{1-\alpha}}{1+\gamma}.$$

The parameter γ is still at our disposal. The maximum value of $\frac{\gamma^{1-\alpha}}{1-\gamma}$ occurs when $\gamma = \frac{1-\alpha}{\alpha}$ so that the best estimate is

(5.40) $$MN\Gamma(1-\alpha) < \omega^{1-\alpha}\alpha^{\alpha}(1-\alpha)^{1-\alpha}.$$

In chapter §8 Applications we are concerned specifically with the case $\alpha = \frac{1}{2}$. We now obtain a comparison of the estimates obtained by the direct method (5.28) and by (5.40).

With $\alpha = \frac{1}{2}$, $\overline{N}$ is equal to 1 so that (5.28) gives

$$g(t) \leq Me^{-\omega t} + Me^{-\beta t}N_1\int_0^t \frac{e^{-(\omega-\beta)\rho}}{(t-\rho)^{\frac{1}{2}}}d\rho + N_1^2\pi\int_0^t e^{-\beta(t-\rho)}g(\rho)d\rho$$

The first integral on the right-hand side can be estimated from the general result that

(5.41) $$\int_0^t \frac{e^{-\sigma\rho}}{(t-\rho)^{\alpha}}d\rho \leq \frac{1}{1-\alpha}\left(\frac{\alpha}{\sigma}\right)^{1-\alpha} \qquad t > 0, \quad \tfrac{1}{2} \leq \alpha < 1$$

so that

$$g(t) \leq Me^{-\omega t} + MN_1\sqrt{\frac{2}{\omega-\beta}}\,e^{-\beta t} + N_1^2\pi\int_0^t e^{-\beta(t-\rho)}g(\rho)d\sigma.$$

Multiplying by $e^{\beta t}$ and using Gronwall's lemma as before gives

$$g(t) \leq e^{(N_1^2\pi-\beta)t}\left[M+MN_1\sqrt{\frac{2}{\omega-\beta}} + \frac{M(\beta-\omega)}{\omega-\beta+N_1^2\pi}\left[1-e^{-(\omega-\beta+N_1^2\pi)t}\right]\right]$$

Now $\omega > \beta$ so for stability we require $\beta > N_1^2\pi$. Using (5.27) this becomes the condition that

$$\frac{\omega\gamma}{1+\gamma} > M^2N^2(1+\gamma)\pi$$

i.e. $$\left(\frac{\omega\gamma}{1+\gamma}\right)^{\frac{1}{2}} > MN\sqrt{\pi}$$

which is identical to (5.40) with $\alpha = \frac{1}{2}$.

We observe that (5.38) provides for our class of unbounded operators the analogous result to (iii) of Theorem 5.1. for bounded operators. We can thus proceed to obtain estimates of the effect of a forcing term as was done for bounded operators. Following (5.12) through to (5.14) we find

$$\|z(t)\| \leq M_1 e^{-(\beta-\omega)t}\|z_o\| + \int_o^t M_1 e^{-\beta t} e^{\Omega(t-\rho)}\|f(\rho)\|d\rho. \tag{5.42}$$

Finally we prove that the operator $U(t,s)$ defined by (5.20) with $\|\overline{T_{t-\rho}B(s)}\| \leq \frac{N}{(t-\rho)^{\alpha}}$ is a quasi-evolution operator in the sense that if $D(B(S)) \supset D(A)$ for almost all s,

$$\frac{\partial}{\partial s} U(t,s)z = (U(t,s)A + \overline{U(t,s)B(s)})z\ , \quad z \in D(A) \tag{5.43}$$

where $\overline{U(t,s)B(s)}$ is the extension of $U(t,s)B(s)$ to all of Z.

We first obtain an estimate on $\|\overline{U(t,s)B(s)}z\|$. Since

$$U(t,s)z = T_{t-s}z - \int_s^t \overline{T_{t-\rho}B}(\rho)U(\rho,s)zd\rho \tag{5.44}$$

then
$$\overline{U(t,s)B}(s)z = \overline{T_{t-s}B(s)}z - \int_s^t \overline{T_{t-\rho}B}\ \overline{U(\rho,s)B}(s)z\ d\rho$$

hence
$$\|\overline{U(t,s)B(s)}z\| \leq \frac{N}{(t-s)^{\alpha}}\|z\| + \int_s^t \frac{N}{(t-\rho)^{\alpha}}\|\overline{U(\rho,s)B}(s)z\|d\rho$$

By comparison with the proof of Theorem 3.2. we can see that

$$\|U(t,s)B(s)z\| \leq \frac{NH(t,s)}{(t-s)^{\alpha}}\|z\| \tag{5.45}$$ where $H(t,s)$ is a bounded continuous function for $0 \leq s \leq t \leq T$.

<u>Theorem 5.3.</u> $U(t,s)$ defined by (5.44) is the unique solution of

$$U(t,s)z = T_{t-s}z - \int_s^t \overline{U(t,\rho)B}(\rho)T_{\rho-s}zd\rho \tag{5.46}$$

<u>Proof</u>:- Let $U_o'(t,s)z = T_{t-s}z$

$$U_n'(t,s)z = -\int_s^t \overline{U_{n-1}'(t,\rho)B}(\rho)T_{\rho-s}zd\rho$$

then $U'(t,s)z = \sum_0^{\infty} U_n'(t,s)z$ is the unique solution of (5.46).

To show $U_n'(t,s)z = U_n(t,s)z$ for all n, where $U(t,s)z = \sum_o^{\infty} U_n(t,s)z$ is the solution of (5.44), we use a proof by induction.

Suppose the assertion is true when n = k-1 and k-2 then

$$U'_k(t,s)z = -\int_s^t \overline{U'_{k-1}(t,\rho)B}(\rho)T_{\rho-s}zd\rho$$
$$= -\int_s^t \overline{U_{k-1}(t,\rho)B}(\rho)T_{\rho-s}zd\rho$$
$$= \int_s^t \int_\rho^t \overline{T_{t-r}B(r)}\ \overline{U_{k-2}(r,\rho)B(\rho)}T_{\rho-s}zdr\, d\rho$$

Changing the order of integration which is valid because of (5.45) and using the assumption that $U_{k-2}(t,s)z = U'_{k-2}(t,s)z$ we have

$$U'_k(t,s)z = \int_s^t \int_s^r \overline{T_{t-r}B(r)}\ \overline{U'_{k-2}(r,\rho)B(\rho)}T_{\rho-s}zd\rho\, dr$$
$$= -\int_s^t \overline{T_{t-r}B(r)}U'_{k-1}(r,s)zdr$$
$$= U_k(t,s)z$$

Since $U'_o(t,s) = U_o(t,s) = T_{t-s}$ and $U'_1(t,s)z = -\int_s^t \overline{T_{t-\rho}B}(\rho)T_{\rho-s}zd\rho = U_1(t,s)z$ we have the result.

Finally, using (5.46) we have for $z\in D(A)$:

$$\int_s^t U(t,\rho)Azd\rho = \int_s^t T_{t-\rho}Azd\rho - \int_s^t \int_\rho^t \overline{U(t,r)B}(r)T_{t-\rho}Azdrd\rho.$$

Changing the order of integration (again valid because of (5.45)) we have

$$\int_s^t U(t,\rho)Azd\rho = \int_s^t T_{t-\rho}Azd\rho - \int_s^t \overline{U(t,r)B}(r)\int_s^r T_{r-\rho}Azd\rho dr.$$

Now $T_t z = z - \int_o^t T_\tau Azd\tau$ and $\int_s^r T_{r-\rho}Azd\rho = \int_o^{r-s} T_\tau Azd\tau$ hence

$$\int_s^t U(t,\rho)Azd\rho = z - T_{t-s}z - \int_s^t \overline{U(t,s)B}(r)\,[z-T_{r-s}z]\,dr$$
$$= z - U(t,s)z - \int_s^t \overline{U(t,r)B}(r)zdr$$

hence $U(t,s)z - z = -\int_s^t (U(t,\rho)A + \overline{U(t,\rho)B}(\rho))zd\rho$

so that U(t,s) is a quasi-evolution operator.

5.4. Perturbation of an Evolution Operator

We now wish to show that an analysis similar to the above can be carried out in the more general case where A = A(t) so that the evolution operator $T(t,s)$ associated with $-A(t)$ is not a semi-group. We assume that (5.16) is replaced by the condition

(5.47) $$\|T(t,s)B(s)z\| \leq g(t-s)\|z\|$$

where $g(t)$ is a locally integrable positive function and $T(t,s)$ is a quasi-evolution operator. (5.47) implies that for any $t > s$ the operator $T(t,s)B(s)$ has an extension $\overline{T(t,s)B(s)}$ to all of Z with

(5.48) $$\|\overline{T(t,s)B(s)}\| \leq g(t-s) \qquad t > s \quad \text{and}$$

(5.49) $$T(t,\rho)\overline{T(\rho,s)B(s)} = \overline{T(t,s)B(s)} \qquad s < \rho < t$$

We prove first a lemma and then obtain a generalized version of Gronwall's lemma. The proof of the lemma requires the following theorem, c.f. [3].

<u>Theorem 5.4.</u> Let $f \in L^p$ and $g \in L^r$ where $\frac{1}{p} + \frac{1}{r} \geq 1$, $p \geq 1$ and $r \geq 1$. The convolution integral

$$h(x) = \int_0^x f(x-y)g(y)\, dy$$

exists for almost all x and defines a function in L^s where $s^{-1} = r^{-1} + p^{-1} - 1$ and $\|h\|_s \leq \|f\|_p \|g\|_r$.

<u>Lemma 5.1.</u> Let

(5.50) $$f(t) = h(t) + \int_0^t g(t-s)f(s)ds$$

where $h \in L^p[0,T]$, $g \in L^1[0,T]$ and h,g are positive functions. Then there exists a unique solution of the integral equation $f \in L^p[0,T]$.

<u>Proof:-</u> For $g \in L^1[0,T]$ we can set

(5.51) $$\int_0^T e^{-\omega t} g(t)dt = M_\omega \quad \text{for } \omega > 0$$

and note that for ω sufficiently large $M_\omega < 1$.

Now $g * h = \int_0^t g(t-s)h(s)ds \leq e^{\omega t}\int_0^t e^{-\omega(t-s)} g(t-s)h(s)ds$ thus

$$\|g * h\|_{L^p[0,T]} \le e^{\omega T} \| e^{-\omega t}g(t) * h(t)\|_{L^p[0,T]}$$
$$\le e^{\omega T} \| e^{-\omega t}g(t)\|_{L^1[0,T]} \|h(t)\|_{L^p[0,T]}$$

by Theorem 5.4, and so by (5.51) we have

(5.52) $$\|g * h\|_{L^p[0,T]} \le e^{\omega T}M_\omega \|h(t)\|_{L^p[0,T]}.$$

Thus the Volterra operator G is well-defined where

$$(Gh)(t) = \int_0^t g(t-s)h(s)ds.$$

Equation (5.50) can now be solved by the method of successive approximations giving

(5.53) $$f(t) = h(t) + \sum_{n=1}^{\infty} (G^n h)(t)$$

where $$(G^n h)(t) = \int_0^t g_n(t-s)h(s)ds \qquad (n \ge 1)$$

with $$g_1(t) = g(t)$$
$$g_n(t) = \int_0^t g(t-s)g_{n-1}(s)ds. \qquad (n \ge 2).$$

To prove that the series part of (5.53) is convergent we prove by induction that $\int_0^T e^{-\omega t}g_n(t)dt \le M_\omega{}^n$ as follows. The result is true for n = 1 by (5.51). If the result is valid for n-1 we have

$$\int_0^T e^{-\omega t}g_n(t)dt = \int_0^T e^{-\omega t}\int_0^t g(t-s)g_{n-1}(s)ds\, dt$$
$$= \int_0^T \int_s^T e^{-\omega t}g(t-s)g_{n-1}(s)dt\, ds$$

Setting t = ρ+s we obtain

$$\int_0^T e^{-\omega t}g_n(t)dt \le \int_0^T \int_0^T e^{-\omega(\rho+s)}g(\rho)g_{n-1}(s)d\rho ds$$
$$= M_\omega{}^n.$$

Thus $\|(G^n h)(t)\|_{L^p} \le e^{\omega T}M_\omega^n \|h\|_{L^p}$ by (5.52). By choosing ω large enough so that $M_\omega < 1$ the series part of (5.53) is convergent in $L^p[0,T]$. Hence (5.53) is the unique solution of (5.50).

We can obtain a generalized form of Gronwall's Inequality as a Corollary to the above Lemma.

$$\|g * h\|_{L^p[0,T]} \leq e^{\omega T} \| e^{-\omega t} g(t) * h(t)\|_{L^p[0,T]}$$
$$\leq e^{\omega T} \| e^{-\omega t} g(t)\|_{L^1[0,T]} \| h(t)\|_{L^p[0,T]}$$

by Theorem 5.4, and so by (5.51) we have

(5.52) $$\|g * h\|_{L^p[0,T]} \leq e^{\omega T} M_\omega \| h(t)\|_{L^p[0,T]}.$$

Thus the Volterra operator G is well-defined where

$$(Gh)(t) = \int_0^t g(t-s)h(s)ds.$$

Equation (5.50) can now be solved by the method of successive approximations giving

(5.53) $$f(t) = h(t) + \sum_{n=1}^{\infty} (G^n h)(t)$$

where $$(G^n h)(t) = \int_0^t g_n(t-s)h(s)ds \qquad (n \geq 1)$$

with $$g_1(t) = g(t)$$

$$g_n(t) = \int_0^t g(t-s)g_{n-1}(s)ds. \qquad (n \geq 2).$$

To prove that the series part of (5.53) is convergent we prove by induction that $\int_0^T e^{-\omega t} g_n(t)dt \leq M_\omega{}^n$ as follows. The result is true for n = 1 by (5.51). If the result is valid for n-1 we have

$$\int_0^T e^{-\omega t} g_n(t)dt = \int_0^T e^{-\omega t} \int_0^t g(t-s)g_{n-1}(s)ds\, dt$$
$$= \int_0^T \int_s^T e^{-\omega t} g(t-s)g_{n-1}(s)dt\, ds$$

Setting $t = \rho+s$ we obtain

$$\int_0^T e^{-\omega t} g_n(t)dt \leq \int_0^T \int_0^T e^{-\omega(\rho+s)} g(\rho)g_{n-1}(s)d\rho ds$$
$$= M_\omega{}^n.$$

Thus $\| (G^n h)(t)\|_{L^p} \leq e^{\omega T} M_\omega^n \|h\|_{L^p}$ by (5.52). By choosing ω large enough so that $M_\omega < 1$ the series part of (5.53) is convergent in $L^p[0,T]$. Hence (5.53) is the unique solution of (5.50).

We can obtain a generalized form of Gronwall's Inequality as a Corollary to the above Lemma.

Corollary: If $f(t) \leq h(t) + \int_0^t g(t-s)f(s)ds$ with $h \in L^p[0,T]$, $g \in L^1[0,T]$ and h,g positive then

$$f(t) \leq h(t) + \sum_{n=1}^{\infty} (G^n h)(t),$$

and in particular if h = 0 then f = 0.

We can now proceed to the main theorem.

Theorem 5.5. If T(t,s) is a quasi-evolution operator satisfying (5.47) then there exists a unique solution of the equation

$$U(t,s)z = T(t,s)z - \int_s^t \overline{T(t,\rho)B(\rho)}\ U(\rho,s)z d\rho \tag{5.54}$$

with the properties

(i) U(t,.) is strongly continuous on [0,T] and U(.,s) is strongly continuous on [s,T].

(ii) $U(t,r)\ U(r,s) = U(t,s)$, $U(t,t) = I$, $0 \leq s \leq T \leq t \leq T$

(iii) U(t,s) is a quasi-evolution operator

$$\frac{\partial}{\partial s} U(t,s)z = U(t,s)(A(s) + B(s))z,\ z \in D(A(t)) \text{ a.e.}$$

Proof: The evolution-type operator U(t,s) of (5.54) is constructed by means of the iterative scheme

$$U_o(t,s) = T(t,s)$$

$$U_n(t,s)z = (-1)^n \int_0^t \overline{T(t,\rho)B(\rho)}\ U_{n-1}(\rho,s)z\, d\rho$$

so that $U(t,s) = \sum_0^\infty U_n(t,s)$. If $\|T(t,s)z\| \leq \bar{h}(t-s)\|z\|$ and $\bar{h} \in L^p[0,T]$ then by our lemma $\sum_0^\infty U_n(t,s)$ is majorized by the series $\bar{h}(t-s) + \sum_1^\infty (G^n\bar{h})(t-s)$. Hence $\sum_0^\infty U_n(t,s)$ converges and the limit U(t,s) must satisfy (5.54).

The proofs of the uniqueness of the solution and the semi-group properties are similar to the earlier proofs, the generalized Gronwall inequality being used as necessary. Further the continuity proof is identical to the earlier proof of continuity for unbounded operators since the quasi-evolution operator T(t,s) is strongly continuous in t and s, g(t) is locally integrable and $\|U_n(\rho,s)z\|$ is bounded. The proof that U(t,s) is a quasi-evolution operator also follows the lines of the corresponding proof for unbounded operators but we offer the following proof as an alternative. From (5.54) we have

$$\overline{U(t,s)B(s)}z = \overline{T(t,s)B(s)}z - \int_s^t \overline{T(t,\rho)B(\rho)}\ \overline{U(\rho,s)B(s)}z\,d\rho \tag{5.55}$$

Also since T(t,s) is the quasi-evolution operator associated with -A(t) we have

$$T(t,s)z - z = -\int_s^t T(t,\alpha)A(\alpha)z\,d\alpha$$

so on replacing z by $A(\alpha)z$ in (5.54) and integrating from s to t we have

$$\begin{aligned}\int_s^t U(t,\alpha)A(\alpha)z d\alpha &= -T(t,s)z + z - \int_s^t \int_\alpha^t \overline{T(t,\rho)B(\rho)}\ U(\rho,\alpha)A(\alpha)z\,d\rho d\alpha \\ &= -T(t,s)z + z - \int_s^t \int_s^\rho \overline{T(t,\rho)B(\rho)}\ U(\rho,\alpha)A(\alpha)z\,d\alpha d\rho. \end{aligned} \tag{5.56}$$

From (5.55) we have

$$\begin{aligned}\int_s^t \overline{U(t,\alpha)B(\alpha)}z d\alpha &= \int_s^t \overline{T(t,\alpha)B(\alpha)}z d\alpha - \int_s^t \int_\alpha^t \overline{T(t,\rho)B(\rho)}\ \overline{U(\rho,s)B(s)}z d\rho d\alpha \\ &= \int_s^t \overline{T(t,\alpha)B(\alpha)}z d\alpha - \int_s^t \int_s^\rho \overline{T(t,\rho)B(\rho)}\ \overline{U(\rho,s)B(s)}z d\alpha d\rho \end{aligned} \tag{5.57}$$

Then on setting $f(t,s)x = \int_s^t (U(t,\alpha)A(\alpha)z + \overline{U(t,\alpha)B(\alpha)}z)d\alpha$ and adding (5.56) to (5.57) we have

$$\begin{aligned} f(t,s)z &= -T(t,s)z + z + \int_s^t \overline{T(t,\alpha)B(\alpha)}z d\alpha - \int_s^t \overline{T(t,\rho)B(\rho)}\ f(\rho,s)z d\rho \\ &= -U(t,s)z - \int_s^t \overline{T(t,\rho)B(\rho)}U(\rho,s)z d\rho + z \\ &\quad + \int_s^t \overline{T(t,\alpha)B(\alpha)}z d\alpha - \int_s^t \overline{T(t,\rho)B(\rho)}\ f(\rho,s)z d\rho. \end{aligned}$$

If $R(t,s)z = f(t,s)z + U(t,s)z - z$ then we have

$$R(t,s)z = -\int_s^t \overline{T(t,\rho)B(\rho)}\ R(\rho,s)z\,d\rho$$

Taking norms and using the generalized Gronwall lemma gives

$$R(t,s)z = 0$$

and hence

$$U(t,s)z - z = -\int_s^t [U(t,\alpha)A(\alpha) + \overline{U(t,\alpha)B(\alpha)}]z\, d\alpha \tag{5.58}$$

We now proceed to obtain a stability criterion by the direct method used earlier. We first note that if

$$\|T(t,s)\| \le Me^{-\omega(t-s)}, \quad \omega > 0, \quad t > s \tag{5.59}$$

then using (5.49) and taking norms we have

$$\|\overline{T(t,s)B(s)}\| \le Me^{-\omega(t-\rho)}g(\rho-s), \quad s \le \rho \le t \tag{5.60}$$

so on setting $t-\rho = \alpha(t-s)$ we obtain

$$\|\overline{T(t,s)B(s)}\| \le Me^{-\omega\alpha(t-s)}g((1-\alpha)(t-s)), \ 0 \le \alpha \le 1. \tag{5.61}$$

We require an estimate of $h(t,s)$ where

$$h(t,s) \le \|T(t,s)x\| + \int_s^t \|\overline{T(t,\rho)B(\rho)}\|\ h(\rho,s)\, d\rho \tag{5.62}$$

Using (5.60) and (5.61) and setting $h(t,s)e^{\omega\alpha(t-s)} = k(t,s)$ (5.62) becomes

$$k(t,s) \le Me^{-\omega(1-\alpha)(t-s)} + M\int_s^t g((1-\alpha)(t-\rho))k(\rho,s)\, d\rho \tag{5.63}$$

Noting that $k(t,s)$ is a function of $t-s$ and following the argument given earlier in the derivation of the stability criteria for $-A$ generating a semi-group, we can show that

$$k(t) \le K(t) \tag{5.64}$$

where $K(t)$ is given by

$$K(t) \ge Me^{-\omega(1-\alpha)t} + M\int_o^t g((1-\alpha)(t-\rho))K(\rho)\, d\rho$$

We estimate $K(t)$ by setting

$$K(t) = Pe^{\Omega t} \tag{5.65}$$

then P,Ω must satisfy

$$Pe^{\Omega t} \ge Me^{-\omega t(1-\alpha)} + \int_o^t g((1-\alpha)(t-\rho)\ Pe^{\Omega\rho}\, d\rho \tag{5.66}$$

Now $\int_0^t Pe^{\Omega\rho} g((1-\alpha)(t-\rho))d\rho$

$$= \frac{Pe^{\Omega t}}{1-\alpha} \int_0^{(1-\alpha)t} e^{-\frac{\Omega\tau}{1-\alpha}} g(\tau)d\tau$$

(5.67) $$= \frac{Pe^{\Omega t}}{1-\alpha} \; N \quad \text{(say)}$$

Thus P,Ω satisfy (5.66) if

(5.68) $$Pe^{\Omega t} \geq Me^{-\omega t(1-\alpha)} + \frac{Pe^{\Omega t}}{1-\alpha} N \; .$$

If P is large enough and Ω is estimated from the condition

(5.69) $$\frac{N}{1-\alpha} < 1$$

then (5.68) is valid for all $t \geq s$.

Using (5.64) and (5.65) we now have an estimate of $\|U(t,s)\|$ in the form

$$\|U(t,s)\| = h(t,s) \leq e^{-\omega\alpha(t-s)}k(t,s) \leq Pe^{(\Omega-\omega\alpha)(t-s)}$$

For stability we require $\omega\alpha > \Omega$. α is a parameter and is chosen to maximise the allowable $g(\tau)$ in the estimate (5.69) i.e.

(5.70) $$\int_0^{(1-\alpha)t} \exp\left(-\frac{\Omega\tau}{1-\alpha}\right).g(\tau)d\tau < 1-\alpha.$$

subject to the condition $\Omega < \omega\alpha$.

§6 Non-Linear Semi-Groups

6.1. Introduction

In §5 we have considered methods for estimating $\|U(t,s)z\|$ when $U(t,s)$ is the semi-group generated by $-(A+B)$ where A and B are linear operators. We now study semi-groups generated by $-(A+B)$ where $-A$ is a linear operator generating a semi-group T_t and B is a non-linear operator. The mild solution of

$$(6.1) \qquad \dot{z}(t) + Az(t) + Bz(t) = 0 \qquad z(0) = z_o$$

is given by the solution of the integral equation

$$(6.2) \qquad z(t) = T_t z_o - \int_o^t \overline{T_{t-s}} Bz(s)ds$$

the kernel of which is now a non-linear function of $z(s)$.

We have reported in §3 that Webb [11] has shown that (6.2) has a unique solution of the form $z(t) = U(t)z_o$ if T_t is the semi-group of operators which has as its infinitesimal generator $-A$, where A is a linear m-accretive operator, and if B is a continuous everywhere defined non-linear accretive operator from Z to itself. The operator $U(t)$, $t \geq 0$ is a strongly continuous semi-group of non-linear contractions on Z with $-(A+B)$ as the infinitesimal generator. These conditions are quite severe. Consider for example the operator $Bz = z^3$, $z \in L^2[0,1]$ then $\int_0^1 z^2 dx$ is finite, however it does not follow that $\int_0^1 z^6 dx$ is finite so that $Bz \notin L^2[0,1]$ and hence the operator B does not map Z to itself, it maps Z into a larger space Z_1 such that $z \in Z_1$. Therefore Webb's result is not applicable to this problem. In this chapter we show that it is possible for (6.2) to have a unique solution even if $B: Z \to Z_1$, $Z \subset Z_1$ by assuming smoothness properties for the semi-group T_t, that is T_t maps the larger space Z_1 back to Z for $t > 0$.

Following Pritchard and Ichikawa [25] we first develop local existence, uniqueness and regularity results and then consider the extension of the results to global solutions in order to discuss the stability of the solution. We can show that if T_t satisfies an estimate of the form

$$\|T_t\| \leq Me^{-\omega t} \qquad \omega > 0$$

we are able to construct a ball of initial states centred at the origin for which the non-linear system is asymptotically stable. However the results have a wider application than this as they are applicable to cases where the linear system is unstable but

$$\|z(t)\| \leq \|z_o\|$$

for all z_o outside a region centred at the origin. We can show that the solution is global and can obtain stability results. The results are shown to be applicable in particular to non-linearities which are polynomials in z.

6.2. Local Existence, Uniqueness and Regularity Theory

The following theorem gives a set of conditions which ensure that the mild solution (6.2) is unique and belongs to the space $L^r[0,T,V]$, $r \geq 1$ where V is a Banach space such that $V \subset Z$. In a corollary to this theorem we show that $z \in C[0,T:Z]$. For convenience we abbreviate $L^p[0,T]$ to L^p and $L^p[0,T:Z]$ to $L^p[Z]$.

Theorem 6.1. Let V, Z_1, Z_2 be Banach spaces with $V \subset Z_1$, $V \subset Z_2$ and p_1, p_2, q, r, s, a, b be positive real constants such that $p_1 \geq r \geq 1$, $p_2 \geq q \geq 1$, $s \geq 1$ and $r^{-1} = q^{-1} + s^{-1} - 1$.

Assume that

(i) $T_t \in \mathcal{L}(Z_1, V) \cap \mathcal{L}(Z_2, V)$ for $t > 0$ with $\|T_t z\|_V \leq g_1(t)\|z\|_{Z_1}$ $t > 0$, $\forall z \in Z_1$ and $\|T_t z\|_V \leq g_2(t)\|z\|_{Z_2}$, $t > 0$, $\forall z \in Z_2$ where $g_1(t) \in L^{p_1}$ and $g_2(t) \in L^{p_2}$.

(ii) $B : V \to Z_2$ such that if $z \in L^r[V]$ with $\|z\|_{L^r[V]} \leqslant a$ then $Bz \in L^s[Z_2]$ and there exists b depending upon a with $\|Bz\|_{L^s[Z_2]} \leqslant b$.

(iii) $\|Bz_1 - Bz_2\|_{L^s[Z_2]} \leqslant k(\|z_1\|_{L^r[V]}, \|z_2\|_{L^r[V]})\|z_1 - z_2\|_{L^r[V]}$ where $k : R^+ \times R^+ \to R^+$.

(iv) for z_1, z_2 with $\|z_1\|_{L^r[V]} \leqslant a$, $\|z_2\|_{L^r[V]} \leqslant a$, then

$$\|g_2\|_{L^q} \; k(\|z_1\|_{L^r[V]}, \; \|z_2\|_{L^r[V]}) < 1,$$

(v) for $z_o \in Z_1$

$$\|g_1\|_{L^r} \; \|z_o\|_{Z_1} + \|g_2\|_{L^q} \; b \leqslant a$$

then there exists a unique solution of (6.2) in $L^r[V]$.

<u>Proof</u>:- Let $\Omega : V \to V$ be the map defined by

$$(\Omega z)(t) = T_t z_o - \int_o^t T_{t-s} Bz(s) ds.$$

By assumptions (i) and (ii)

(6.3) $$\|(\Omega z)(t)\|_V \leqslant g_1(t)\|z_o\|_{Z_1} + \int_o^t g_2(t-s)\|Bz(s)\|_{Z_2} ds$$

The integral on the right-hand side is a convolution so by Theorem 5.4 and the assumption that $r^{-1} = q^{-1} + s^{-1} - 1$ we have

(6.4) $$\|(\Omega z)(t)\|_{L^r[V]} \leqslant \|g_1(t)\|_{L^r}\|z_o\|_{Z_1} + \|g_2\|_{L^q} \cdot \|Bz\|_{L^s[Z_2]}$$

Thus if $\|z\|_{L^r[V]} \leqslant a$ we have

(6.5) $$\|(\Omega z)\|_{L^r[V]} \leqslant a$$

by assumptions (ii) and (v), hence Ω maps the closed ball of radius a in $L^r[V]$ into itself.

Now $(\Omega z_1)(t) - (\Omega z_2)(t) = \int_o^t T_{t-s}(Bz_1(s) - Bz_2(s)) ds$

so again by Theorem 5.4 and the assumption $r^{-1} = q^{-1} + s^{-1} - 1$ we have

(6.6) $$\|\Omega z_1 - \Omega z_2\|_{L^r[V]} \leqslant \|g_2\|_{L^q[0,T]} \|Bz_1 - Bz_2\|_{L^s[Z_2]}$$

By assumption (iii), (iv) we have

(6.7) $$\|\Omega z_1 - \Omega z_2\|_{L^r[V]} \leqslant K\|z_1 - z_2\|_{L^r[V]}$$

with $0 < K < 1$, therefore $\Omega:V \to V$ is a contraction mapping and hence there is a unique fixed point $z(t) = \Omega z(t)$ in the closed ball of radius a in $L^r[V]$.

Corollary 6.1. If we assume $T_t \in \mathcal{L}(Z_2,Z_1)$ for $t > 0$ with

$$(6.8) \qquad \|T_t z_2\|_{Z_1} \leq g_3(t)\|z_2\|_{Z_2}$$

and $g_3(t) \in L^p[0,T]$ where $p^{-1} = 1-s^{-1}$, then the unique solution of (6.2) in the closed ball in $L^r[V]$ also lies in $C[0,T;Z_1]$.

Proof:- The term $T_t z_o$ in (6.2) clearly belongs to the space $C[0,T;Z_1]$ by the strong continuity of T_t. Let $u(t) = \int_o^t T_{t-s} z_2(s)ds$ with $z_2 \in L^s[Z_2]$

We have
$$\|u(t)\|_{Z_1} \leq \int_o^t g_3(t-s)\|z_2(s)\|_{Z_2} ds$$

$$(6.9) \qquad \leq \|g_3\|_{L^p} \|z_2\|_{L^s[Z_2]}$$

by Holder's inequality since $p^{-1}+s^{-1} = 1$.

Further if $h > 0$

$$\|u(t+h) - u(t)\|_{Z_1} \leq \|(T_h-I)u(t)\|_{Z_1} + \|\int_t^{t+h} T_{t+h-s} z_2(s)ds\|_{Z_1}$$

$$(6.10) \qquad \leq \|(T_h-I)u(t)\|_{Z_1} + \|g_3\|_{L^p[0,h]} \|z_2\|_{L^s[t,t+h,Z_2]}$$

By the strong continuity of T_t, used on the right-hand side of (6.10) we can conclude that

$$\underset{h \to 0}{\text{limit}} \|u(t+h) - u(t)\|_{Z_1} = 0$$

i.e. $u(t)$ is continuous from the right.

For $t > \varepsilon > h > 0$

$$u(t) - u(t-h) = (T_t - T_{t-h})u(t-\varepsilon) + \int_{t-\varepsilon}^{t-h} T_{t-h-s} z_2(s)ds + \int_{t-\varepsilon}^{t} T_{t-s} z_2(s)ds$$

hence
$$\|u(t) - u(t-h)\|_{Z_1} \leq \|(T_t - T_{t-h})u(t-\varepsilon)\|_{Z_1} + \|g_3\|_{L^p[0,\varepsilon-h]} \|z_2\|_{L^s[t-\varepsilon,t-h;Z_2]}$$
$$+ \|g_3\|_{L^p[0,\varepsilon]} \|z_2\|_{L^s[t-\varepsilon,t:Z_2]}$$

from which

$$\underset{\varepsilon,h \to 0}{\text{limit}} \|u(t) - u(t-h)\|_{Z_1} = 0.$$

i.e. $u(t)$ is continuous from the left. Hence $u(t)$ and therefore $z(t) \in C[0,T;Z_1]$.

In most examples it is possible to choose $Z_1 = Z_2$ but this is not necessary and in many cases may not be desirable as the form of the non-linearity will often suggest the choice of Z_2 in a natural way. There is no unique choice for V and Z_1 either, in the applications it will be shown that there are many different pairs of V and Z_1 for a particular problem and that for each pair an optimal value of r can be obtained.

In the next section we will show that for certain initial states it is possible to extend the solution for all time. The application of Liapunov theory to this problem requires that the solution is more regular than in the above theorem. We have the following corollary.

Corollary 6.2.

Let $V_1 \subset V$ be a Banach space such that

(i) $T_t \in \mathcal{L}(Z_1, V_1)$, $t > 0$, with

$\|T_t z\|_{V_1} \leq g_1(t) \|z\|_{Z_1}$, $t > 0$, for all $z \in Z_1$ and $g_1 \in L^{p_1}[\varepsilon,T]$ for any $\varepsilon > 0$, $p_1 \geq 1$.

(ii) $T_t \in \mathcal{L}(Z_2, V_1)$, $t > 0$, with

$\|T_t z\|_{V_1} \leq g_2(t) \|z\|_{Z_2}$, $t > 0$ for all $z \in Z_2$ and $g_2 \in L^{p_2}[0,T]$, $p_2 \geq 1$.

Then for any s allowed in the assumption (ii) of Theorem 6.1 and for m which satisfies $m^{-1} = w^{-1} + s^{-1} - 1$ with $m \leq p_1$, $w \leq p_2$, the solution $z(.)$ of (6.2) lies in $L^m([\varepsilon,T];V_1)$.

Proof:- We have

$$\|z(t)\|_{V_1} \leq g_1(t) \|z_o\| + \int_o^t g_2(t-s) \|Bz(s)\|_{Z_2} ds$$

Thus

$$\|z\|_{L^m[\varepsilon,T;V_1]} \leq \|g_1\|_{L^m(\varepsilon,T)} \|z_o\|_{Z_1} + \|g_2\|_{L^w[0,T]} \|Bz\|_{L^s[Z_2]}$$

by Theorem 5.4. The result follows directly.

6.3. Extension of a Local Solution to a Global Solution

Theorem 6.1 in the previous section gives conditions for the existence and uniqueness of a local solution of (6.2) in $[0,T]$. We now wish to derive conditions under which the local solution can be extended to a global solution. We note first that the form of the assumptions (iv) and (v) of Theorem 6.1 is such that $\|z_o\|$ can be chosen as large as we like by decreasing T. Alternatively if $\|z_o\|$ is chosen sufficiently small with

(6.11)
$$\text{(i)} \quad b \sim a^{1+\alpha}, \quad \alpha > 0 \quad \text{as} \quad a \to 0$$
$$\text{(ii)} \quad k(x,y) \sim a^{\beta}, \quad \beta > 0 \quad x \leq a, \quad y \leq a \quad \text{as } a \to 0$$

then for any finite T there is a local solution of (6.2) at least in a ball of finite radius a in $L^r[V]$.

The first objective is to determine the maximal time interval for any given initial state. We consider various cases.

(i) Let $g_1 \in L^r[0,\infty]$, $g_2 \in L^q[0,\infty]$, $g_3 \in L^p[0,\infty]$, $p^{-1}+r^{-1} = q^{-1}$ and set

(6.12)
$$\|g_1\|_{L^r[0,\infty]} = \gamma_1, \quad \|g_2\|_{L^q[0,\infty]} = \gamma_2, \quad \|g_3\|_{L^p[0,\infty]} = \gamma_3$$

If there exists an a such that

(6.13)
$$\gamma_2 k(x,y) < 1 \qquad \text{for all } x,y \leq a$$
$$\gamma_1 \|z_o\| + \gamma_2 b \leq a$$

then the solution with initial state z_o exists for all time and by optimizing the above inequalities with respect to a we can find the maximum ball (centred at the origin) of initial states for which this is true.

(ii) Let there exist M,ω such that

(6.14)
$$\|T_t\|_{\mathcal{L}(Z_1)} \leq Me^{-\omega t} \qquad \omega > 0$$

then by Corollary 6.1

(6.15)
$$\|z(t)\|_{Z_1} \leq Me^{-\omega t}\|z_o\| + \gamma_3 b$$

and since $\|T_t\|_{\mathcal{L}(Z_1,V)} = \|T_{(1-\lambda)t+\lambda t}\|_{\mathcal{L}(Z_1,V)}$ for any $0 \leq \lambda \leq 1$ we have

$$\|T_t\|_{\mathcal{L}(Z_1,V)} \leq \|T_{(1-\lambda)t}\|_{\mathcal{L}(Z_1,V)} \|T_{\lambda t}\|_{\mathcal{L}(Z_1)}$$

(6.16) $$\leq Me^{-\omega t} g((1-\lambda)t)$$

and similar expressions for $\|T_t\|_{\mathcal{L}(Z_2,V)}$, $\|T_t\|_{\mathcal{L}(Z_2,Z_1)}$. With these results, since $\omega > 0$, it is possible to show that (6.12) holds if we only assume g_1, g_2, g_3 are locally r, q, p - integrable respectively.

We now examine the behaviour of $z(t)$ as $t \to \infty$. for various cases. For (i) we can prove under the additional assumption (6.14) that

(6.17) $$z(t) \to 0 \quad \text{as} \quad t \to \infty$$

if the injection of V into Z_1 is continuous.

Since $z \in L^r[0,\infty;V]$ there exists for any given $\varepsilon > 0$ a time T* such that

$$\|z(T^*)\|_{Z_1} \leq \varepsilon \|z_0\|$$

Then (6.13) ensures that the solution with initial state $z(T^*)$ exists and on replacing b by εb in (6.15) we have the estimate

$$\|z(t)\| \leq \varepsilon M e^{-\omega(t-T^*)} \|Z_0\| + \varepsilon b \gamma_2 \quad \text{for} \quad t \geq T^*$$

and the result follows. Case (ii) above is similar.

We now consider the situation where g_1, g_2, g_3 are locally r, q, p - integrable and only the existence of a local solution is assured. Consider a sequence $\{T_i\}$ $i = 0, 1, 2 \ldots$ such that $T_i > T_{i-1} > 0$ and assume there exist constants $\gamma_{1,i}$; $\gamma_{2,i}$; $\gamma_{3,i}$ such that

$$\|g\|_{L^r[0,T_i-T_{i-1}]} = \gamma_{1,i} \qquad i > 0$$

(6.18) $$\|g\|_{L^q[0,T_i-T_{i-1}]} = \gamma_{2,i} \qquad i > 0$$

$$\|g\|_{L^p[0,T_i-T_{i-1}]} = \gamma_{3,i} \qquad i > 0$$

Then for a fixed initial state z_0 we can choose a value of a, a_1 (say) and since $\|T_t\|_{\mathcal{L}(Z_1,V)} = \|T_{(1-\lambda)t+\lambda t}\|_{\mathcal{L}(Z_1,V)}$ for any $0 \leq \lambda \leq 1$ we have

$$\|T_t\|_{\mathcal{L}(Z_1,V)} \leq \|T_{(1-\lambda)t}\|_{\mathcal{L}(Z_1,V)}\|T_{\lambda t}\|_{\mathcal{L}(Z_1)}$$

(6.16) $$\leq Me^{-\omega t}g(1-\lambda)t)$$

and similar expressions for $\|T_t\|_{\mathcal{L}(Z_2,V)}$, $\|T_t\|_{\mathcal{L}(Z_2,Z_1)}$. With these results, since $\omega > 0$, it is possible to show that (6.12) holds if we only assume g_1, g_2, g_3 are locally r, q, p - integrable respectively.

We now examine the behaviour of $z(t)$ as $t \to \infty$. for various cases. For (i) we can prove under the additional assumption (6.14) that

(6.17) $$z(t) \to 0 \quad \text{as} \quad t \to \infty$$

if the injection of V into Z_1 is continuous.

Since $z \in L^r[0,\infty;V]$ there exists for any given $\varepsilon > 0$ a time T* such that

$$\|z(T^*)\|_{Z_1} \leq \varepsilon\|z_o\|$$

Then (6.13) ensures that the solution with initial state $z(T^*)$ exists and on replacing b by εb in (6.15) we have the estimate

$$\|z(t)\| \leq \varepsilon Me^{-\omega(t-T^*)}\|Z_o\| + \varepsilon b\gamma_2 \quad \text{for} \quad t \geq T^*$$

and the result follows. Case (ii) above is similar.

We now consider the situation where g_1, g_2, g_3 are locally r, q, p - integrable and only the existence of a local solution is assured. Consider a sequence $\{T_i\}$ i = 0, 1, 2 ... such that $T_i > T_{i-1} > 0$ and assume there exist constants $\gamma_{1,i}$; $\gamma_{2,i}$; $\gamma_{3,i}$ such that

$$\|g\|_{L^r[0,T_i-T_{i-1}]} = \gamma_{1,i} \qquad i > 0$$

(6.18) $$\|g\|_{L^q[0,T_i-T_{i-1}]} = \gamma_{2,i} \qquad i > 0$$

$$\|g\|_{L^p[0,T_i-T_{i-1}]} = \gamma_{3,i} \qquad i > 0$$

Then for a fixed initial state z_o we can choose a value of a, a_1 (say) and a value of T, T_1 (say) such that the solution exists in [0,T) provided that

$$(6.19)\qquad \begin{array}{ll} \gamma_{2,1}\, k(x,y) < 1 & \text{for all } x,y \leq a_1 \\ \gamma_{1,1}\, \|z_o\|_{Z_1} + \gamma_{2,1}\, b_1 \leq a_1 & \end{array}$$

An estimate of $\|z(T_1)\|$ is then obtained and the process repeated giving a new value of a (a_2 say) and a value of T (T_2 say). This requires that

$$(6.20)\qquad \begin{array}{ll} \gamma_{2,2}\, k(x,y) < 1 & \text{for all } x,y \leq a_2 \\ \gamma_{1,2}\, \|z(T_1)\| + \gamma_{2,2}\, b_2 \leq a_2 & \end{array}$$

This process can be repeated indefinitely so that we have an infinite sequence $\{T_i\}$ depending upon the sequence $\{a_i\}$. Then the $\lim_{i \to \infty} \{T_i\}$ can be maximised for the choice of $\{a_i\}$. If $T_i \to \infty$ with i the solution can be extended for all time. Furthermore the value of the $\lim_{i \to \infty} a_i$ will give some information on the behaviour of z(t) as $t \to \infty$.

The crucial step in the above is the estimation of $\|z(T_i)\|_{Z_1}$. In the general case we can use corollary 6.1 and write

$$\|z(T_{i+1})\|_{Z_1} \leq \|T_{T_{i+1}} z(T_i)\| + \gamma_{3,i+1} b_i$$

but in case of Hilbert spaces we can use Liapunov theory to provide a second method. This alternative will be discussed in the next chapter.

Some examples illustrating the results are given in §8.

6.4. Nonlinearities of Polynomial Form

Theorem 6.1 enables us to handle perturbations of the polynomial type

$$(6.21)\qquad Bz = \sum_{i=2}^{n} a_i z^i .$$

The following lemma is useful in showing that Bz given by (6.21) satisfies assumption (iii) of that theorem.

Lemma 6.1 Let u_1, u_2 be positive real numbers then

$$u_1^r + u_1^{r-1}u_2 + u_1^{r-2}u_2^2 + \dots + u_2^r \leq \frac{r+1}{2}(u_1^r + u_2^r) \text{ if } r \geq 1.$$

Proof:- The result is trivial for r = 1. For r = 2 we have

$$u_1^2 + u_1u_2 + u_2^2 \leq \tfrac{3}{2}(u_1^2+u_2^2)$$

since $u_1^2 - 2u_1u_2 + u_2^2 \geq 0$.

Assume the result is true for r = n-2, i.e.

$$u_1^{n-2} + u_1^{n-3}u_2 + \ldots + u_2^{n-2} \leq \frac{n-1}{2}(u_1^{n-2}+u_2^{n-2})$$

We first note that from Holder's inequality

$$x^\alpha y^\beta + x^\beta y^\alpha \leq x+y \qquad (\alpha+\beta = 1, \quad x,y \geq 0)$$

we can deduce that

$$a^{r-s}b^s + a^s b^{r-s} \leq a^r + b^r \qquad (a,b > 0, \quad 0 \leq s \leq r)$$

Now

$$2(u_1^n+u_1^{n-1}u_2+\ldots+u_2^n) = (u_1^{n-2}+u_1^{n-3}u_2+\ldots+u_n^{n-2})(u_1^2+u_2^2)+u_1^n+u_1^{n-1}u_2+\ldots+u_2^n$$

$$\therefore 2(u_1^n+u_1^{n-1}u_2+\ldots+u_2^n) \leq \frac{n-1}{2}(u_1^{n-2}+u_2^{n-2})(u_1^2+u_2^2) + 2(u_1^n+u_2^n)$$

$$= \frac{n+3}{2}(u_1^n+u_2^n) + \frac{n-1}{2}(u_1^{n-2}u_2^2+u_1^2u_2^{n-2})$$

$$\leq (\frac{n+3}{2}+\frac{n-1}{2})(u_1^n+u_2^n)$$

$$\therefore u_1^n + u_1^{n-1}u_2 + \ldots + u_2^n \leq \frac{n+1}{2}(u_1^n+u_2^n)$$

Q.E.D.

We will now investigate the application of Theorem 6.1 to nonlinearities of the form (6.21). We assume that (i) is satisfied for $Z_2 = L^\alpha[0,1]$ $(\alpha \geq 1)$ $V = L^{n\alpha}[0,1]$ and some Banach space Z_1. Then

(6.22) $$\|z^i\|_{Z_2} = (\int_0^1 z^{i\alpha}dx)^{\frac{1}{\alpha}} \leq (\int_0^1 z^{i\alpha k}dx)^{\frac{1}{\alpha k}} \leq (\int_0^1 z^{n\alpha}dx)^{\frac{i}{\alpha n}} = \|z\|_V^i$$

if $i < n$, and so $B:V \to Z_2$ since

(6.23) $$\|Bz\|_{Z_2} \leq \sum_{i=2}^{n} |a_i| \; \|z\|_V^i$$

Furthermore by Minkowski's inequality

(6.24) $$\|Bz\|_{L^s[Z_2]} \leq \sum_{i=2}^{n} |a_i| \; \|z\|^i_{L^{is}[V]} \quad \text{if} \quad s > 1$$

Therefore if ns = r there exists b depending upon a such that if $\|z\|_{L^r[V]} \leq a$ then $\|Bz\|_{L^s[Z_2]} \leq b$.

For assumption (iii) we require an estimate of $\|Bz_1 - Bz_2\|_{Z_2}$.
Since $\alpha \geq 1$,

$$\|Bz_1-Bz_2\|_{Z_2} \leq \sum_{i=2}^{n} |a_i| \left(\int_0^1 (z_1^i - z_2^i)^\alpha dx \right)^{\frac{1}{\alpha}}$$

$$= \sum_{i=2}^{n} |a_i| \left(\int_0^1 (z_1-z_2)^\alpha (z_1^{i-1}+\ldots+z_2^{i-2})^\alpha dx \right)^{\frac{1}{\alpha}}$$

$$\leq \sum_{i=2}^{n} |a_i| \left\{ \int_0^1 (z_1-z_2)^{n\alpha} dx \right\}^{\frac{1}{n\alpha}} \left\{ \int_0^1 (z_1^{i-1}+\ldots+z_2^{i-1})^{\frac{n\alpha}{n-1}} dx \right\}^{\frac{n-1}{n\alpha}}$$

which by Lemma 6.1 gives

$$(6.25)\quad \|Bz_1-Bz_2\|_{Z_2} \leq \sum_{i=2}^{n} \frac{i}{2} |a_i| \; \|z_1-z_2\|_V \left\{ \int_0^1 (|z_1|^{i-1}+|z_2|^{i-1})^{\frac{n\alpha}{n-1}} dx \right\}^{\frac{n-1}{n\alpha}}$$

Since $n\alpha > n-1$ and $\int_0^1 |z_1|^{(i-1)\frac{n\alpha}{n-1}} dx \leq \left(\int_0^1 |z_1|^{n\alpha} dx \right)^{\frac{i-1}{n-1}}$ $\quad (i \leq n)$

we have

$$\|Bz_1-Bz_2\|_{Z_2} \leq \sum_{i=2}^{n} \frac{i}{2} |a_i| \; \|z_1-z_2\|_V \; \{ \|z_1\|_V^{i-1} + \|z_2\|_V^{i-1} \}$$

Consider now the map

$$\Omega z(t) = T_t z_o - \int_0^t T_{t-s} Bz(s) ds$$

Let W be the space $L^r[V]$ then

$$(6.26)\quad \left\| \int_0^t T_{t-s} Bz(s) dx \right\|_W = \left(\int_0^T \left\| \int_0^t T_{t-s} Bz ds \right\|_V^r dt \right)^{\frac{1}{r}}$$

$$\leq \left(\int_0^T \left(\int_0^t g_2(t-s) \sum_{i=2}^{n} |a_i| \; \|z(s)\|_V^i ds \right)^r dt \right)^{\frac{1}{r}}$$

$$\leq \sum_{i=2}^{n} |a_i| \; \|g_2\|_{L^{p_i}} \; \| \, \|z(s)\|_V^i \, \|_{L^{q_i}}$$

by Theorem 5.4, where

$$(6.27)\quad p_i^{-1} + q_i^{-1} \geq 1, \quad p_i, q_i \geq 1 \quad \text{and} \quad p_i^{-1} + q_i^{-1} - 1 = r^{-1}.$$

Also $\quad \|T_t z_o\|_W = \left(\int_0^T \|T_t z_o\|_V^r dt \right)^{\frac{1}{r}}$

$$(6.28)\quad \leq \left(\int_0^T g_1^r(t) \|z_o\|_{Z_1}^r dt \right)^{\frac{1}{r}} \quad \text{by assumption (i)}$$

$$= \|g_1\|_{L^r} \|z_o\|_{Z_1}$$

Thus Ω is a map of $W \to W$ if $g_1 \in L^r$, $g_2 \in L^{p_i}$ and $\|z(s)\|_V^i \in L^{q_i}$

Therefore we require that

(6.29) $$p_i \leq p_2, \quad p_1 \geq r \quad \forall i \leq n .$$

Furthermore if we choose

(6.30) $$iq_i = r$$

then $\| \, \|z(s)\|_V^i \|_{L^{q_i}} = (\int_0^T \|z(s)\|_V^{iq_i} dt)^{\frac{1}{q_i}} = (\int_0^T \|z(s)\|_V^r dt)^{\frac{1}{r}} = \|z\|_W^i$

so that we have

(6.31) $$\|\Omega z(t)\|_W \leq \|g_1\|_{L^r} \|z_0\|_{Z_1} + \sum_{i=2}^n |a_i| \, \|g_2\|_{L^{p_i}} \|z\|_W^i .$$

We now seek conditions under which Ω is a contraction map. We have

(6.32)
$$\begin{aligned}\|\Omega z_1(t)-\Omega z_2(t)\|_W &= (\int_0^T (\int_0^t g_2(t-s)\|Bz_1-Bz_2\|_{Z_2} ds)^r dt)^{\frac{1}{r}} \\ &\leq \sum_{i=2}^n |a_i| \, \|g_2\|_{p_i'} \, \| \, \|z_1^i - z_2^i\|_{Z_2} \|_{L^{q_i}[V]}\end{aligned}$$

by Theorem 5.4 and Lemma 6.1 where

(6.33) $$p_i'^{-1} + q_i^{-1} - 1 = r^{-1}, \quad p_i'^{-1} + q_i^{-1} \geq 1, \quad p_i', q_i \geq 1 .$$

Applying (6.25) equation (6.32) becomes

(6.34)
$$\begin{aligned}\|\Omega z_1(t)-\Omega z_2(t)\|_V &\leq \sum_{i=2}^n |a_i| \tfrac{i}{2} \|g_2\|_{p_i'} \| \, \|z_1-z_2\|_V (\|z_1\|_V^{i-1} + \|z_2\|_V^{i-1}) \|_{L^{q_i}[V]} \\ &\leq \sum_{i=2}^n |a_i| \tfrac{i}{2} \|g_2\|_{p_i'} (\int_0^T (\|z_1\|_V^{i-1} + \|z_2\|_V^{i-1})^{\gamma_i} dt)^{\frac{1}{\gamma_i}} . \\ &\quad (\int_0^T \|z_1-z_2\|^r)^{\frac{1}{r}}\end{aligned}$$

where

(6.35) $$q_i^{-1} = r^{-1} + \gamma_i^{-1}$$

If we choose $iq_i = r$ as in (6.30) then

(6.36) $$(i-1)\gamma_i = r$$

and we have

$$(\int_0^T \|z\|_V^{(i-1)\gamma_i} dt)^{\frac{1}{\gamma_i}} = \|z_1\|_W^{i-1}$$

hence (6.34) gives

(6.37) $$\|\Omega z_1-\Omega z_2\|_W \leq \sum_{i=2}^{n} \frac{i}{2}|a_i| \; \|g_2\|_{p_i'} (\|z_1\|_W^{i-1}+\|z_2\|_W^{i-1}) \cdot \|z_1-z_2\|_W$$

Therefore $\|\Omega z_1-\Omega z_2\|_W$ is bounded if

(6.38) $$p_i' \leq p_2 \qquad \forall i \leq n.$$

We require that the conditions on p_1, p_2, p_i, p_i', q_i, r, n, γ_i are consistent. Condition (6.33) implies (6.27) if we take $p_i' = p_i$ and (6.38) is then consistent with (6.29). Eliminating q_i between (6.27) and (6.30) gives

$$p_i^{-1} + \frac{i}{r} - 1 = \frac{1}{r}$$

(6.39) $$\text{i.e. } p_i = \frac{r}{r-i+1}$$

so that to satisfy $p_i \leq p_2$ $\forall i \leq n$, we require

(6.40) $$p_2 \geq \frac{r}{r+1-n}$$

in addition to $p_1 \geq r$.

In the applications in §8 we will demonstrate that the value of n is restricted by the form of the operator A.

§7. Use of Liapunov Functionals

7.1. Liapunov Theory for Linear Operators

In Chapter 5 we have shown how to obtain estimates of the semi-group of a perturbed operator where the perturbation operator is either bounded or belongs to a certain class of unbounded operators. These results were then used to estimate the effect of a forcing term in the differential equation. However in applications it may be easier to estimate these effects via a Liapunov functional as in Plaut and Infante [26], constructed from the unperturbed homogeneous system

(7.1) $$\dot{z}(t) + Az(t) = 0 \qquad z(0) = z_0,$$

further the functional can also be used to obtain the estimate (5.4.) i.e.

$$\|T_t\| \leq Me^{-\omega t} \qquad (\omega > 0)$$

for the semi-group T_t. For completeness we present here some results of Pritchard [27] which provide an important link between the theory of §5 and the use of Liapunov functionals because they provide a justification of the use of functionals.

Suppose that Z is a Hilbert space and that there exists an operator $P(t) \in \mathcal{L}(Z)$ for $t \in [0,T]$ such that

(7.2) $$\frac{d}{dt}\langle y,P(t)z\rangle - \langle y,(PA+A^*P-W)z\rangle = 0,$$
$$P(T) = G$$

where $W > 0$, $G \geq 0$, $\langle y,P(t)z\rangle \in C^1[0,T]$ for $y,z \in D(A)$, $\langle\ ,\ \rangle$ is the inner product on Z and

(7.3) $$\inf \langle z,P(t)z\rangle \geq p\|z\|^2, \qquad p > 0, \quad t \in [0,T].$$

If we consider the Liapunov functional

$$V(t) = \langle z(t),P(t)z(t)\rangle$$

then formally for the unperturbed system (7.1.) we have

$$\dot{V}(t) = \frac{d}{dt}\langle z(t),P(t)z(t)\rangle$$

$$(7.4) \quad = \langle \dot{z}(t),P(t)z(t)\rangle + \langle z(t),\dot{P}(t)z(t)\rangle + \langle z(t),P\dot{z}(t)\rangle$$

$$= -\langle Az(t),P(t)z(t)\rangle + \langle z(t),\dot{P}(t)z(t)\rangle - \langle z(t),PAz(t)\rangle$$

$$= -\langle z(t),Wz(t)\rangle$$

and for the perturbed in-homogeneous system (5.6.) we have

$$(7.5) \quad \dot{V}(t) = -\langle z(t),Wz(t)\rangle - \langle B(t)z(t),P(t)z(t)\rangle$$
$$- \langle z(t),P(t)B(t)z(t)\rangle + \langle f,P(t)z(t)\rangle + \langle z(t),P(t)f\rangle$$

Before we use these results it is necessary to justify the formal differentiation. Since the strict solution of $\dot{z} = -Az$ is $z(t) = T_t z_o$ for $z_o \in D(A)$ the proof of (7.4) is straightforward. However if we have only a mild solution (5.12) of (5.6) the derivation of (7.5) is not so immediate, we need the following theorem.

Theorem 7.1. If $P(t)$ satisfies (7.2) then

$$(7.6) \quad P(t)z = T^*_{T-t}GT_{T-t}z + \int_t^T T^*_{s-t}WT_{s-t}z\,ds$$

and if $z(t)$ is given by (5.13) then

$$(7.7) \quad \langle z(t),P(t)z(t)\rangle = \langle z(t),Gz(T)\rangle + \int_t^T [\langle z(s),P(s)B(s)z(s)\rangle$$
$$- \langle z(s),P(s)f(s)\rangle + \langle P(s)B(s)z(s),z(s)\rangle - \langle P(s)f(s),z(s)\rangle$$
$$+ \langle z(s),Wz(s)\rangle]\,ds$$

Proof:- Proof of (7.7) is obtained by substituting (5.13) directly into (7.6). (For more details see [27]).

To prove (7.6) we note that T^*_t is the dual semi-group of T_t and since Z is a Hilbert space T^*_t is strongly continuous. $P(t)$ given by (7.6) satisfies (7.2) so the only problem is uniqueness.

Let $Q(t)$ be another solution then $R(t) = P(t)-Q(t)$ must satisfy

$$\frac{d}{dt}\langle y,R(t)z\rangle - \langle Ay,R(t)z\rangle - \langle R(t)y,Az\rangle = 0, \quad R(T) = 0.$$

If we let $S(t) = T^*_{t-s}R(t)T_{t-s}$ for any $s \in [0,t)$ we obtain

$$\frac{d}{dt}\langle y,S(t)z\rangle = 0 \qquad y,z \in D(A).$$

Since $R(T) = 0$ we have $S(T) = 0$ hence $\langle y, S(t)z \rangle = 0$, $y, z \in D(A)$ so that $\langle T_{t-s}y, R(t)T_{t-s}z \rangle = 0$. But $D(A)$ is dense in Z so that on letting $s \to t$ we obtain $P(t) = Q(t)$ and so the solution is unique. Differentiating (7.7) we obtain (7.5) for almost all $t \in [0,T]$.

From the conditions on $P(t)$ and W there exist positive constants $\bar{p}, w$ such that $W > wI$, $\bar{p} > \sup_{t[0,T]} \|P(t)\|$

Hence from (7.3) and (7.4)

$$p\|z(t)\|^2 \leq V \leq \bar{p}\|z(t)\|^2$$

and
$$\dot{V} \leq -w\|z(t)\|^2 \leq -\frac{w}{\bar{p}} V$$

so that
$$\|z(t)\| \leq \sqrt{\frac{\bar{p}}{p}}\, e^{-wt/2\bar{p}} \|z_0\|$$

and
$$\|T_t\| \leq \sqrt{\frac{\bar{p}}{p}}\, e^{-wt/2\bar{p}}.$$

Using (7.5), if $w > 2\bar{p}K$ where $\bar{p}K\|z\|^2 \geq |\langle Pz, Bz \rangle|$ we obtain

$$\dot{V} \leq -\left(\frac{w-2\bar{p}K}{\bar{p}}\right) V(t) + \frac{2V^{\frac{1}{2}}}{p^{\frac{1}{2}}} \bar{p}\, \|f(t)\| \quad \text{a.e.}$$

Setting $V(t) = U^2(t)$ we find

$$U(t) \leq e^{-\lambda t} U(0) + \frac{\bar{p}}{p^{\frac{1}{2}}} \int_0^t e^{-\lambda(t-s)} \|f(s)\| ds$$

where $\lambda = \frac{w-2\bar{p}K}{2\bar{p}}$.

Hence $\|z(t)\| \leq \left(\frac{\bar{p}}{p}\right)^{\frac{1}{2}} e^{-\lambda t}\|z_0\| + \frac{\bar{p}}{p^{\frac{1}{2}}} \int_0^t e^{-\lambda(t-s)} \|f(s)\| ds$

which is a similar result to (5.14) or (5.42).

For many applications it is convenient to take the operator P to be independent of t so that

$$Pz = \int_0^\infty T_t^* W T_T z\, dt$$

which is well-defined if $\|T_t\| \leq Me^{-\omega t}$ $(\omega > 0)$. In this case the analysis can be simplified by the introduction of a new Hilbert space Z_0 with inner product $\langle y, z \rangle_{Z_0} = \langle y, Pz \rangle_Z$

(7.8) Then $V(t) = \| z(t)\|^2_{Z_o}$ and $p = \bar{p} = 1$.

From (7.4) $\dot{V}(t) = -\langle z(t),Wz(t)\rangle_Z = \dfrac{\langle z(t),Wz(t)\rangle_Z}{\langle z(t),Pz(t)\rangle_Z} V$

(7.9) $\leq -\| W^{-\frac{1}{2}}PW^{-\frac{1}{2}}\|^{-1}_{\mathcal{L}(Z)} V = -2\mu V$ (say)

since $\dfrac{\langle z,Wz\rangle}{\langle z,Pz\rangle} = \dfrac{\langle W^{\frac{1}{2}}z,W^{\frac{1}{2}}z\rangle}{\langle z,Pz\rangle} = \dfrac{\langle y,y\rangle}{\langle W^{-\frac{1}{2}}y,PW^{-\frac{1}{2}}y\rangle} \geq \dfrac{\| y\|^2}{\| W^{-\frac{1}{2}}PW^{-\frac{1}{2}}\| \, \| y\|^2}$

From (7.9) $\| z(t)\|^2_{Z_o} \leq e^{-2\mu t}\| z(0)\|^2_{Z_o}$

and $\| T_t\|_{\mathcal{L}(Z_o)} \leq e^{-\mu t}$.

The estimate equivalent to (5.14) will now have the form

(7.10) $\| z(t)\|_{Z_o} \leq e^{-(\mu-K)t}\| z(0)\|_{Z_o} + \int_o^t e^{-(\mu-K)(t-\rho)}\| f(\rho)\|_{Z_o} d\rho$

where $\operatorname{ess\,sup}_{t\in[0,T]} \| B(t)\|_{\mathcal{L}(Z_o)} \leq K$.

Since the space Z_o and the constants μ,K are essentially determined by the operator W a variety of estimates of the form (7.10) can be obtained. This could lead to an optimisation problem if suitable criteria could be formulated. We note that the dependence on K in the estimate (7.10) is such that it cannot be improved by splitting the operator B into $\beta B+(1-\beta)B$ as in §4.

7.2 Liapunov Theory for Non-linear Semi-groups

In this section we develop a theory of Liapunov functionals based on the local existence theory of §6.2 which will enable us to extend the local solution for all time.

We assume that all the spaces are real Hilbert spaces and consider Liapunov functionals of the form

(7.11) $V(t) = \langle z(t),z(t)\rangle_{Z_1}$

in order to obtain estimates of $\| z(t)\|_{Z_1}$. In most cases it is not possible to develop a Liapunov theory on $[0,T]$ but by using the regularity results of Corollary 6.2. a theory can be given for the interval $[\varepsilon,T]$.

We assume further that

(7.12) $$f(v) = -\langle Av,v\rangle_{Z_1} - \langle v,Av\rangle_{Z_1} = -\|Dv\|^2_{Z_1} + \langle Ev,v\rangle_Z$$

for $v\in D(A)$, $E\in\mathcal{L}(Z_1)$, and the space V_1 of Corollary 6.2. can be chosen so that $D\in\mathcal{L}(V_1,Z_1)$. Furthermore let

(7.13) $$D(A)\subset V_1\subset Z_2^*\subset Z_1\subset Z_2\subset V_1^*$$

where each space is dense in the larger one, all injections are continuous and $Z_1^* = Z_1$.

Under these assumptions we have the following theorem.

Theorem 7.2. If the conditions of Corollaries 6.1. and 6.2. hold with $m = 2$, so that $z\in L^2[\varepsilon,T;V_1]$ and if $z\in L^{s'}[\varepsilon,T;Z_2^*]$ with $s^{-1} + (s')^{-1} = 1$ and s as defined in Theorem 6.1. then

(7.14) $$\|z(T)\|^2_{Z_1} - \|z(\varepsilon)\|^2_{Z_1} \leq \int_\varepsilon^T[-\|D\ z(s)\|^2_{Z_1} + \langle Ez(s),z(s)\rangle_{Z_1} + 2\langle z(s),Bz(s)\rangle_{Z_2^*Z_2}]ds$$

where z is the unique solution of (6.1) as given in Theorem 6.1.

Proof Consider the equation

(7.15) $$z(t) = T_tz_o + \int_o^t T_{t-s}\bar{z}(s)ds$$

where $z_o\in D(A)$ and $\bar{z}\in C^1[0,T;Z_1]$. Then by (3.19) z is the solution of

(7.16) $$\dot{z} + Az = \bar{z}, \quad z(0) = z_o.$$

For V defined by (7.11) and z defined by (7.15) we have

(7.17) $$\dot{V}(t) = -\langle z(t),Az(t)\rangle_{Z_1} - \langle Az(t),z(t)\rangle_{Z_1} + 2\langle z(t),\bar{z}(t)\rangle_{Z_1}$$

but $z(t)\in D(A)$ so by use of (7.12) we obtain

$$\dot{V}(t) = \|Dv(t)\|^2_{Z_1} + \langle Ev(t),v(t)\rangle_{Z_1} + 2\langle z(t),\bar{z}(t)\rangle_{Z_1}$$

Hence

$$\|z(T)\|^2_{Z_1} - \|z(\varepsilon)\|^2_{Z_1} \leq \int_\varepsilon^T[-\|Dv(t)\|^2_{Z_1} + \langle Ez(s),z(s)\rangle_{Z_1} + 2\langle z(s),\bar{z}(s)\rangle_{Z_1}]ds$$

Now $D(A)$ is dense in Z_1 and $C^1[0,T;Z_1]$ is dense in $L^2[0,T;Z_2]$ so there exist sequences $\{z_o^{(n)}\}$, $\{\bar{z}^{(n)}\}$ which converge to arbitrary $z_o\in Z_1$ and $\bar{z}\in L^s[0,T;Z_2]$ respectively. We set

$$z^{(n)}(t) = T_t z_o^{(n)} + \int_o^t T_{t-s} \bar{z}^{(n)}(s)ds$$

then using the same estimates as in Theorem 6.1 it is easy to show that $z^{(n)}(t) \to z(t)$ in $L^r[0,T;V]$ where $z(t)$ satisfies

$$z(t) = T_t z_o + \int_o^t T_{t-s} \bar{z}(s)ds .$$

Therefore for $\bar{z} = -Bz$ where z is the unique solution of (6.1) we find $z^{(n)}$ converges in $L^r[0,T;V]$ to this solution and under the conditions of Corollaries 6.1 and 6.2 $z^{(n)}(t) \to z(t)$ in Z_1 for each $t \in [0,T]$ and $z^{(n)} \to z$ in $L^2[\varepsilon,T;V_1]$. Further since $z^{(n)} \in L^2[\varepsilon,T,V]$ we have

$$\|z^{(n)}(T)\|^2_{Z_1} - \|z^{(n)}(\varepsilon)\|^2_{Z_1} \leq \int_\varepsilon^T [-\|Dv(t)\|^2_{Z_1} + \langle Ez^{(n)}(s), z^{(n)}(s)\rangle_{Z_1} + 2\langle z^{(n)}(s), \bar{z}^{(n)}(s)\rangle_{Z_2^* Z_2}]ds.$$

The continuity of the quadratic terms in the above can be easily proved so that the limit as $n \to \infty$ can be taken. This completes the proof.

In the applications of this theorem, we will use inequalities to estimate the right-hand side of (7.14), then if $z \in C[0,T;Z_1]$ so that $\|z(\varepsilon) - z_o\|_{Z_1}$ is small if ε is small, we can estimate $\|z(T)\|_{Z_1}$ in terms of $\|z_o\|_{Z_1}$.

§8 Applications

8.1. Introduction

In the previous chapters we have developed three particular sets of results relevant to the theory of abstract evolution equations, namely perturbation theorems for m-accretive operators and perturbation results for linear semi-groups and non-linear semi-groups. We now look at various problems drawn from engineering and science and demonstrate how our results can be applied to them. The objective is not necessarily to obtain new information for particular problems but rather to demonstrate that our methods are of a general nature with a wide range of applicability.

8.2. Application of Perturbation Theorems

We commence by applying our perturbation theorems to a set of related problems based on a beam subject to transverse vibrations. The basic model for such a system can be described by the non-dimensional equation

$$(8.1) \qquad y_{tt} + y_{xxxx} = 0 \qquad 0 \leq x \leq 1, \quad t \geq 0$$

with appropriate initial conditions and boundary conditions to describe the method of support or fixing of the ends of the beam. In our work here we will confine our attention to the problem of a simply-supported beam so that

$$(8.2) \qquad y = y_{xx} = 0 \quad \text{at} \quad x = 0,1$$

The other important end conditions can usually be accommodated by the simple procedure of altering the values of the constants in the fundamental inequalities that we use. See Freund and Plaut [28] for full details.

The model (8.1) can be refined in many ways c.f. Ball [29], Sharma and Dasgupta [30], in particular by including a term to describe the damping of the system due to a resistive medium. The resulting equation can then be written as

(8.3) $$y_{tt} + 2\xi y_t + y_{xxxx} = 0 \quad 0 < x < 1, \quad t > 0, \quad \xi > 0$$

with boundary conditions as in (8.2). We take this model as our underlying linear system because the solutions are asymptotically stable in the sense of Liapunov whereas the solutions of (8.1) are only stable. Equation (8.3) can be written in the abstract form

(8.4) $$\dot{w} + Aw = 0$$

by setting $w = [y,v]^T$ with $v = y_t$, and introducing the real Hilbert space H with inner product

(8.5) $$\langle w_1,w_2\rangle = \int_0^1 [y_{1_{xx}} y_{2_{xx}} + v_1 v_2 + \xi(v_1 y_2 + v_2 y_1) + 2\xi^2 y_1 y_2]\,dx.$$

The operator A is formally defined by

(8.6) $$Aw = [-v, 2\xi v + y_{xxxx}]^T$$

and $D(A) = \{w \in H : Aw \in H,\ v = y = v_{xx} = y_{xx} = 0 \text{ at } x = 0,1\}$ Then A is m-accretive because

$$\langle Aw,w\rangle = \xi \int_0^1 (v^2 + y_{xx}^2)\,dx \geq 0 \quad \text{for all } w \in D(A)$$

and the range of $(I+\lambda A)$ is the whole of H if $\lambda \geq 0$.

We now consider some further refinements of (8.3). In [26] the motion of a shallow arch is assumed to be

(8.7) $$y_{tt} + 2\xi y_t + (y-y_o)_{xxxx} + 2y_{xx}\int_0^1 [(y_o)_x^2 - y_x^2]\,dx = 0,$$
$$0 < x < 1, \qquad t > 0$$

where $y_o(x)$ denotes the equilibrium position of the arch. The non-linear term is the axial constraint effect caused by the ends of the beam being held a fixed distance apart. Taking $y_o(x) = 0$ for simplicity, the non-linear term can be regarded as a perturbation of (8.3), so that (8.7) takes the abstract form

(8.8) $$\dot{w} + Aw + Bw = 0,$$

with Aw defined by (8.6) and

(8.9) $$Bw = [\,0,\ -2y_{xx}\int_0^1 y_x^2dx]^T.$$

B is not accretive because

$$<Bw,w> = \int_0^1 y_x^2dx(2\xi\int_0^1 y_x^2dx - \int_0^1 2vy_{xx}dx) \not\geq 0.$$

We now apply our non-linear theory of Chapter 4, in particular the Corollary to Theorem 4.9 to determine a neighbourhood of the equilibrium point within which A+B is m-accretive.

We note first that if $\|w\| \leq k$ then

(8.10) $$k^2 \geq \int_0^1 (y_{xx}^2+\xi^2y^2)dx \geq \left(\frac{\xi^2+\pi^4}{\pi^2}\right)\int_0^1 y_x^2dx,$$

and, on writing $c_i = \int_0^1 y_{i_x}^2dx$ $(i = 1,2)$ we have

$$\tfrac{1}{2}\int_0^1 (c_1y_{1_{xx}}-c_2y_{2_{xx}})^2dx \leq \int_0^1 (c_1-c_2)^2y_{1_{xx}}^2dx + c_2^2\int_0^1(y_{1_{xx}}-y_{2_{xx}})^2dx$$

since $(a+b)^2 \leq 2(a^2+b^2)$ if a,b are real. Also

$$\begin{aligned}(c_1-c_2) &= \int_0^1 (y_{1_x}^2-y_{2_x}^2)\,dx\\ &\leq \left[\int_0^1 (y_{1_x}-y_{2_x})^2dx\right]^{\frac{1}{2}}\left[\int_0^1 (y_{1_x}+y_{2_x})^2dx\right]^{\frac{1}{2}}\\ &\leq \left[\frac{4k^2}{\xi^2+\pi^4}\int_0^1 (y_{1_{xx}}-y_{2_{xx}})^2dx\right]^{\frac{1}{2}}\end{aligned}$$

since $$\pi^2\int_0^1 y_x^2dx \leq \int_0^1 y_{xx}^2dx$$

if $y = y_{xx} = 0$ at $x = 0,1$.

Combining these results we have

(8.11) $$\begin{aligned}\int_0^1 (c_1y_{1_{xx}}-c_2y_{2_{xx}})^2dx &\leq \left(\frac{8k^4}{\xi^2+\pi^4} + \frac{2\pi^2k^2}{(\xi^2+\pi^4)^2}\right)\int_0^1 (y_{1_{xx}}-y_{2_{xx}})^2dx\\ &= \frac{(8\xi^2+10\pi^4)k^4}{(\xi^2+\pi^4)^2}\int_0^1 (y_{1_{xx}}-y_{2_{xx}})^2dx.\end{aligned}$$

Now $\|Bw_1-Bw_2\|^2 = 4\int_0^1 (c_1y_{1_{xx}}-c_2y_{2_{xx}})^2dx$ so we can conclude from (8.11) that $\exists a > 0$ such that

$$\|Bw_1-Bw_2\|^2 \leq a^2\int_0^1 (y_{1_{xx}}-y_{2_{xx}})^2dx \leq a^2\|w_1-w_2\|^2$$

with Aw defined by (8.6) and

(8.9) $$Bw = [0, -2y_{xx}\int_0^1 y_x^2 dx]^T.$$

B is not accretive because

$$\langle Bw, w\rangle = \int_0^1 y_x^2 dx\left(2\xi\int_0^1 y_x^2 dx - \int_0^1 2vy_{xx} dx\right) \not\geq 0.$$

We now apply our non-linear theory of Chapter 4, in particular the Corollary to Theorem 4.9 to determine a neighbourhood of the equilibrium point within which A+B is m-accretive.

We note first that if $\|w\| \leq k$ then

(8.10) $$k^2 \geq \int_0^1 (y_{xx}^2+\xi^2y^2)dx \geq \left(\frac{\xi^2+\pi^4}{\pi^2}\right)\int_0^1 y_x^2 dx,$$

and, on writing $c_i = \int_0^1 y_{i_x}^2 dx$ $(i = 1,2)$ we have

$$\tfrac{1}{2}\int_0^1 (c_1y_{1_{xx}}-c_2y_{2_{xx}})^2 dx \leq \int_0^1 (c_1-c_2)^2 y_{1_{xx}}^2 dx + c_2^2\int_0^1 (y_{1_{xx}}-y_{2_{xx}})^2 dx$$

since $(a+b)^2 \leq 2(a^2+b^2)$ if a,b are real. Also

$$\begin{aligned}(c_1-c_2) &= \int_0^1 (y_{1_x}^2-y_{2_x}^2)\, dx \\ &\leq \left[\int_0^1 (y_{1_x}-y_{2_x})^2 dx\right]^{\frac{1}{2}}\left[\int_0^1 (y_{1_x}+y_{2_x})^2 dx\right]^{\frac{1}{2}} \\ &\leq \left[\frac{4k^2}{\xi^2+\pi^4}\int_0^1 (y_{1_{xx}}-y_{2_{xx}})^2 dx\right]^{\frac{1}{2}}\end{aligned}$$

since $$\pi^2\int_0^1 y_x^2 dx \leq \int_0^1 y_{xx}^2 dx$$

if $y = y_{xx} = 0$ at $x = 0,1$.

Combining these results we have

(8.11) $$\int_0^1 (c_1y_{1_{xx}}-c_2y_{2_{xx}})^2 dx \leq \left(\frac{8k^4}{\xi^2+\pi^4} + \frac{2\pi^2k^2}{(\xi^2+\pi^4)^2}\right)\int_0^1 (y_{1_{xx}}-y_{2_{xx}})^2 dx$$

$$= \frac{(8\xi^2+10\pi^4)k^4}{(\xi^2+\pi^4)^2}\int_0^1 (y_{1_{xx}}-y_{2_{xx}})^2 dx.$$

Now $\|Bw_1-Bw_2\|^2 = 4\int_0^1 (c_1y_{1_{xx}}-c_2y_{2_{xx}})^2 dx$ so we can conclude from (8.11) that $\exists a > 0$ such that

$$\|Bw_1-Bw_2\|^2 \leq a^2\int_0^1 (y_{1_{xx}}-y_{2_{xx}})^2 dx \leq a^2\|w_1-w_2\|^2$$

therefore $\|Bw_1 - Bw_2\| \leqslant |a| \ \|w_1 - w_2\|$

Also $\langle (1-\varepsilon)A(w_1-w_2), w_1-w_2 \rangle + \langle Bw_1-Bw_2, w_1-w_2 \rangle$

$$= (1-\varepsilon)\xi \int_0^1 [(v_1-v_2)^2 + (y_{1_{xx}}-y_{2_{xx}})^2]dx$$

$$- 2\int_0^1 (c_1 y_{1_{xx}} - c_2 y_{2_{xx}})(v_1-v_2)dx + \int_0^1 2\xi(y_1-y_2)(c_2 y_{2_{xx}} - c_1 y_{1_{xx}})dx$$

$$\geqslant (1-\varepsilon)\xi \int_0^1 [v_1 - v_2 - \frac{1}{(1-\varepsilon)\xi}(c_1 y_{1_{xx}} - c_2 y_{2_{xx}})]^2 dx$$

$$+ \left[(1-\varepsilon)\xi - \frac{k^4}{(1-\varepsilon)\xi}\frac{(8\xi^2+10\pi^4)}{(\xi^2+\pi^4)^2}\right]\int_0^1 (y_{1_{xx}} - y_{2_{xx}})^2 dx$$

since

$$\int_0^1 (y_1-y_2)(c_2 y_{2_{xx}} - c_1 y_{1_{xx}})dx = c_2^2 + \int_0^1 (c_1 y_2 y_{1_{xx}} + c_2 y_1 y_{2_{xx}})dx + c_1^2$$

$$\geqslant c_2^2 - (c_1+c_2)\sqrt{c_1 c_2} + c_1^2 \geqslant 0$$

and by (8.11). Hence we can choose $\varepsilon > 0$ so that $(1-\varepsilon)A+B$ is accretive if

$$\xi^2 \geqslant \frac{k^4(8\xi^2+10\pi^4)}{(\xi^2+\pi^4)^2}$$

(8.12) i.e. $k^2 \leqslant \dfrac{\xi(\xi^2+\pi^4)}{(8\xi^2+10\pi^4)^{\frac{1}{2}}}$

Thus provided (8.12) is satisfied all the conditions of the Corollary to Theorem 4.9 are fulfilled so that A+B is m-accretive in a neighbourhood of $w = 0$ and hence $-(A+B)$ generates a non-linear contraction semi-group. We note that assuming ξ is given (8.12) effectively determines the size of the neighbourhood for which the system is stable through the value of k. Furthermore if k is such that the inequality holds in (8.12) then there is a certain allowable class of perturbations of the non-linear system which will ensure existence, uniqueness and stability of the perturbed system. This class of perturbations will be determined by application of the corollary to Theorem 4.9. again but with A replaced by the operator A+B of (8.8).

therefore $\quad \|Bw_1-Bw_2\| \leq |a| \; \|w_1-w_2\|$

Also $\quad \langle(1-\varepsilon)A(w_1-w_2),w_1-w_2\rangle + \langle Bw_1-Bw_2,w_1-w_2\rangle$

$$= (1-\varepsilon)\xi\int_0^1[(v_1-v_2)^2 + (y_{1_{xx}}-y_{2_{xx}})^2]dx$$

$$- 2\int_0^1(c_1y_{1_{xx}}-c_2y_{2_{xx}})(v_1-v_2)dx + \int_0^1 2\xi(y_1-y_2)(c_2y_{2_{xx}}-c_1y_{1_{xx}})dx$$

$$\geq (1-\varepsilon)\xi\int_0^1[v_1-v_2-\frac{1}{(1-\varepsilon)\xi}(c_1y_{1_{xx}}-c_2y_{2_{xx}})]^2dx$$

$$+ \left[(1-\varepsilon)\xi - \frac{k^4}{(1-\varepsilon)\xi}\frac{(8\xi^2+10\pi^4)}{(\xi^2+\pi^4)^2}\right]\int_0^1(y_{1_{xx}}-y_{2_{xx}})^2dx$$

since

$$\int_0^1(y_1-y_2)(c_2y_{2_{xx}}-c_1y_{1_{xx}})dx = c_2^2 + \int_0^1(c_1y_2y_{1_{xx}}+c_2y_1y_{2_{xx}})dx + c_1^2$$

$$\geq c_2^2 - (c_1+c_2)\sqrt{c_1c_2} + c_1^2 \geq 0$$

and by (8.11). Hence we can choose $\varepsilon > 0$ so that $(1-\varepsilon)A+B$ is accretive if

$$\xi^2 > \frac{k^4(8\xi^2+10\pi^4)}{(\xi^2+\pi^4)^2}$$

(8.12) i.e. $k^2 < \dfrac{\xi(\xi^2+\pi^4)}{(8\xi^2+10\pi^4)^{\frac{1}{2}}}$

Thus provided (8.12) is satisfied all the conditions of the Corollary to Theorem 4.9 are fulfilled so that A+B is m-accretive in a neighbourhood of $w = 0$ and hence $-(A+B)$ generates a non-linear contraction semi-group. We note that assuming ξ is given (8.12) effectively determines the size of the neighbourhood for which the system is stable through the value of k. Furthermore if k is such that the inequality holds in (8.12) then there is a certain allowable class of perturbations of the non-linear system which will ensure existence, uniqueness and stability of the perturbed system. This class of perturbations will be determined by application of the corollary to Theorem 4.9. again but with A replaced by the operator A+B of (8.8).

We can obtain an estimate of the solution directly by considering the Liapunov function $V = \|w\|^2$ so that

$$\dot{V} = -2\langle w,(A+B)w\rangle = -2\xi\int_0^1 (v^2+y_{xx}^2)dx - 4\xi\left(\int_0^1 y_x^2 dx\right)^2 + 4\int_0^1 vy_{xx}\left(\int_0^1 y^2 dx\right)dx$$

It is easy to show from this that if

$$\|w_o\| < \sqrt{\frac{\xi(\pi^4+\xi^2)(1-\alpha)}{\pi^2}}$$

then $\|w(t)\| \leq e^{-\alpha\xi[1-\xi/(\xi^2+\pi^4)^{\frac{1}{2}}]t}\|w_o\|$, $\quad w_o \in D(A)\cap U$. Hence the origin is asymptotically stable in the sense of Liapunov.

We now consider the bending problem of a beam on a non-linear foundation following Sharma and DasGupta [30], taking as the equation of motion

(8.13) $\qquad y_{tt} + 2\xi y_t + y_{xxxx} + \mu \sinh \alpha y = 0 \quad 0 \leq x \leq 1, \quad t \geq 0$

with $\alpha > 0$. If we use the same abstract formulation as before A is defined by (8.6) and the perturbing term is

$$Bw = [0, \mu \sinh \alpha y]$$

Then $\langle Bw,w\rangle = \int_0^1 (\mu v \sinh\alpha y + \xi y \sinh \alpha y)dx \not\geq 0$

although $y \sinh\alpha y \geq 0$ for all $\alpha \geq 0$, $y \in R$, hence B is not accretive. In order to show that the conditions for the Corollary to Theorem 4.9 apply in a neighbourhood of the equilibrium point $w = 0$ we observe first that if $y_1 \geq y_2$ then

$$\sinh\alpha y_1 - \sinh\alpha y_2 = \alpha(y_1-y_2)\left[1 + \frac{\alpha^2}{3!}(y_1^2+y_1y_2+y_2^2) + \frac{\alpha^4}{5!}(y_1^4+\ldots)+\ldots\right]$$
$$\leq \frac{\alpha}{2}(y_1-y_2)\left[2 + \frac{\alpha^2}{2!}(y_1^2+y_2^2) + \frac{\alpha^4}{4!}(y_1^4+y_2^4)+\ldots\right]$$

(by Lemma 6.1)

$$= \frac{\alpha}{2}(y_1-y_2)(\cosh\alpha y_1+\cosh\alpha y_2)$$

Further if $\|w\|^2 \leq k^2$ by using the methods of Freund & Plaut [26] we can establish the inequality (see Appendix I)

$$k^2 \geq \int_0^1 (y_{xx}^2 + \xi^2 y^2)dx \geq c^2\beta^2 \quad \text{where} \quad \beta \equiv \max_{0\leq x\leq 1} |y|$$

and $c^2 = 16\lambda^3 \dfrac{\sinh^2\frac{\lambda}{2} + \cos^2\frac{\lambda}{2}}{\sinh\lambda - \sin\lambda}$ where $2\lambda^2 = \xi$. Hence

(8.14) $$\sinh\alpha y_1 - \sinh\alpha y_2 \leqslant \alpha(y_1-y_2)\cosh\alpha\beta.$$

Consider now the operator $(1-\varepsilon)A+B$,

$$\begin{aligned}
&\langle w_1-w_2,(1-\varepsilon)A(w_1-w_2)\rangle + \langle w_1-w_2, Bw_1-Bw_2\rangle \\
&= (1-\varepsilon)\int_0^1 [\xi(v_1-v_2)^2 + \xi(y_{1_{xx}}-y_{2_{xx}})^2]dx + \int_0^1 \mu(v_1-v_2)(\sinh\alpha y_1-\sinh\alpha y_2)dx \\
&+ \int_0^1 \xi\mu(y_1-y_2)(\sinh\alpha y_1-\sinh\alpha y_2)dx \\
&= (1-\varepsilon)\xi\int_0^1 \{[v_1-v_2+\mu\frac{(\sinh\alpha y_1-\sinh\alpha y_2)}{2(1-\varepsilon)\xi}]^2 - \frac{\mu^2}{4(1-\varepsilon)\xi}(\sinh\alpha y_1-\sinh\alpha y_2)^2\}dx \\
&+ (1-\varepsilon)\xi\int_0^1 (y_{1_{xx}}-y_{2_{xx}})^2dx + \xi\mu\int_0^1 (y_1-y_2)(\sinh\alpha y_1-\sinh\alpha y_2)dx.
\end{aligned}$$

Hence, using (8.13) and the inequality

(8.15) $$\pi^4\int_0^1 y^2dx \leqslant \int_0^1 y_{xx}^{\,2}dx \qquad \text{(see [28])}$$

we can see that it is possible to choose $\varepsilon > 0$ so that $(1-\varepsilon)A+B$ is accretive if k is such that

(8.16) $$\begin{cases} \text{either} & (1-\varepsilon)\pi^4\xi + \xi\mu\alpha - \dfrac{\mu^2\alpha^2}{4\xi(1-\varepsilon)}\cosh^2\dfrac{\alpha k}{c} \geqslant 0 \quad (\mu > 0) \\ \text{or} & (1-\varepsilon)\pi^4\xi + \xi\mu\alpha\cosh\dfrac{\alpha k}{c} - \dfrac{\mu^2\alpha^2}{4\xi(1-\varepsilon)}\cosh^2\dfrac{\alpha k}{c} \geqslant 0 \quad (\mu < 0) \end{cases}$$

We also have

$$\begin{aligned}
\|Bw_1-Bw_2\|^2 &= \mu^2\int_0^1 (\sinh\alpha y_1-\sinh\alpha y_2)^2dx \\
&\leqslant \mu^2\alpha^2\cosh^2\alpha\beta\int_0^1 (y_1-y_2)^2dx \quad \text{by (8.14)} \\
&\leqslant \frac{\mu^2\alpha^2\cosh^2\alpha\beta}{\xi^2}\|w_1-w_2\|^2
\end{aligned}$$

hence $$\|Bw_1-Bw_2\| \leqslant \frac{|\mu\alpha|}{\xi}\cosh\frac{\alpha k}{c}\|w_1-w_2\| .$$

Thus provided (8.16) can be satisfied all the conditions of the Corollary to Theorem 4.9 are fulfilled so that A+B is m-accretive in a neighbourhood of $w = 0$ and hence $-(A+B)$ generates a non-linear contraction semi-group. We note that assuming ξ, μ and α are given then (8.16) effectively determines the size of the neighbourhood for which the system is stable through the value of $k = c\beta$. Furthermore as with the previous example if k is such that the inequality holds in (8.15) then there is a certain allowable class of perturbations of the non-linear system which will ensure existence, uniqueness and stability of the perturbed system. This class of perturbations will be determined by application of the Corollary to Theorem 4.9 with A replaced by the operator A+B derived from (8.13).

Finally for the beam problem we consider the stability of the non-planar, non-linear oscillations of a beam described by the coupled pair of equations Ho, Scott and Esley [31].

$$(8.17)\quad \begin{cases} y_{tt} + \gamma_1 y_{xxxx} + 2\xi y_t - \frac{1}{2} y_{xx} \int_0^1 (y_x^2 + z_x^2)\,dx = 0 \\ z_{tt} + \gamma_2 z_{xxxx} + 2\xi z_t - \frac{1}{2} z_{xx} \int_0^1 (y_x^2 + z_x^2)\,dx = 0 \end{cases}$$

where y,z are the components of the displacement of a point on the neutral axis with respect to the two axes of symmetry for the cross-section of the beam perpendicular to the neutral axis. The parameters γ_1, γ_2 are proportional to the two different values of inertia with respect to the two axes of symmetry.

We set $w = [y,u,z,v]^T$ where $y_t = u$ and $z_t = v$ and choose the inner product

$$\langle w_1,w_2\rangle = \int_0^1 [\gamma_1 y_{1_{xx}} y_{2_{xx}} + u_1u_2 + \xi(y_1u_2+y_2u_1) + 2\xi^2 y_1y_2 + \gamma_2\, z_{1_{xx}} z_{2_{xx}} + v_1v_2 + \xi(z_1v_2+z_2v_1) + 2\xi^2 z_1z_2]\,dx$$

then with the operator A describing the basic linear system of (8.17) defined by

$$Aw = [-u, 2\xi u + y_{xxxx}, -v, 2\xi v + z_{xxxx}]$$

we have

$$\langle Aw,w\rangle = \int_0^1 [\xi(u^2+v^2) + \xi\gamma_1 y_{xx}^2 + \xi\gamma_2 z_{xx}^2]\,dx \geq 0$$

if $\xi > 0$ and A is m-accretive so the linear system is asymptotically stable.

Taking the perturbing term Bw as the non-linear term in (8.17) we have

$$Bw = [\,0,\ -\tfrac{1}{2}y_{xx}\int_0^1 (y_x^2+z_x^2)dx,\ 0,\ -\tfrac{1}{2}z_{xx}\int_0^1 (y_x^2+z_x^2)dx]^T$$

so that $\langle Bw_1-Bw_2, w_1-w_2\rangle$

$$(8.18)\quad \begin{aligned} &= \tfrac{1}{2}\int_0^1 [\,(u_1-u_2)(c_2y_{2_{xx}}-c_1y_{1_{xx}}) + (y_1-y_2)(c_2y_{2_{xx}}-c_1y_{1_{xx}}) \\ &\quad + (v_1-v_2)(c_2z_{2_{xx}}-c_1z_{1_{xx}}) + (z_1-z_2)(c_2z_{2_{xx}}-c_1z_{1_{xx}})]dx. \end{aligned}$$

where $c_i = \int_o^1 (y_{i_x}^2+z_{i_x}^2)dx$.

Let $\|w\| \leq k$ then

$$(8.19)\quad k^2 \geq \int_0^1 (\gamma_1 y_{xx}^2 + \xi^2y^2 + \gamma_2 z_{xx}^2 + \xi^2z^2)dx \geq 2\int_0^1 (\sqrt{\gamma_1}\,y_x^2 + \sqrt{\gamma_2}\,z_x^2)dx$$

from which we can determine bounds on c_i. Also it is not difficult to show that

$$\int_0^1 [\,(y_1-y_2)(c_2y_{2_{xx}}-c_1y_{1_{xx}}) + (z_1-z_2)(c_2z_{2_{xx}}- c_1z_{1_{xx}})]dx \geq 0$$

so that the problem of showing that there exists $\varepsilon > 0$ such that

$$\langle (1-\varepsilon)A(w_1-w_2), w_1-w_2\rangle + \langle Bw_1-Bw_2, w_1-w_2\rangle \geq 0$$

effectively splits into two parts, each of which is similar to the one-dimensional problem discussed earlier.

Also, since the c_i are bounded, we can show $\exists$ an 'a' such that

$$\begin{aligned} \|Bw_1-Bw_2\|^2 &= \tfrac{1}{4}\int_0^1 [\,(c_1y_{1_{xx}}-c_2y_{2_{xx}})^2 + (c_1z_{1_{xx}}-c_2z_{2_{xx}})^2]dx \\ &\leq a\|w_1-w_2\|^2 \end{aligned}$$

as in the previous one-dimensional case. Hence the conditions of the Corollary to Theorem 4.9 can be applied and we can derive a stability condition under which the non-planar oscillations of the beam will be asymptotically stable in the sense of Liapunov. With this problem also it will be possible to use the corollary again to determine an allowable class of perturbations of the non-linear system which will ensure existence, uniqueness and stability of the perturbed system.

We now turn from problems concerning the beam to two further problems which are described by diffusion equations in the classical sense.

The stability of a chemical reaction in a porous catalyst in the presence of heat and mass has been investigated in a limited way by Wei [32] using Liapunov methods in the sense that they were applied to linearized versions of the partial differential equations governing the heat and mass transfer of the chemical reactions and not to the full non-linear versions. As has been pointed out by Pritchard [33] there is the possibility that even if the perturbations of the linearized system grow in time the actual perturbations may be bounded in time. To investigate whether this occurs or not, the non-linear terms must be considered in the analysis.

Following Wei we consider the pair of partial differential equations

$$(8.20) \begin{cases} \text{(i)} \quad \frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} - \phi^2 y \exp \frac{\beta\gamma(1-y)}{1+\beta(1-y)} \\ \text{(ii)} \quad \frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + \phi^2 \beta(1+\beta-z) \exp \left(\gamma \frac{z-1}{z}\right), \quad 0 < x < 1 \end{cases}$$

with $\frac{\partial y}{\partial x} = \frac{\partial z}{\partial x} = 0$ at $x = 0$, $y = z = 1$ at $x = 1$. y is the dimensionless concentration $\frac{c}{c_0}$ inside the catalyst, z the dimensionless temperature T/T_0 and the three parameters are:

ϕ^2 = Thiele modulus

$\beta = \Delta T_{max}/T_0$

$\gamma = E/RT_0$

Equations (8.20) describe the reaction when the Lewis number is equal to unity provided that the perturbations are restricted to those that do not change the value of $(\beta y+z)$ inside the particle. In the steady state $\beta y+z = 1+\beta$, under this condition equations (8.20) are equivalent to each other so we need consider only (8.20(i)). Also $0 \leq y \leq 1$ and $0 \leq z \leq 1$.

Let $y_E(x)$ be the equilibrium function for y i.e. the solution of

$$\frac{\partial^2 y}{\partial x^2} - \phi^2 y \exp \frac{\beta\gamma(1-y)}{1+\beta(1-y)} = 0$$

with y = 1 at x = 1 and $\frac{\partial y}{\partial x} = 0$ at x = 0.

Writing $u = y-y_E$ in (8.20(i)) and setting $f(y) = \exp \frac{\beta\gamma(1-y)}{1+\beta(1-y)}$ for convenience we have

(8.21) $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \phi^2 y_E f(y_E) - \phi^2 y f(y)$$

with u = 0 at x = 1 and $\frac{\partial u}{\partial x} = 0$ at x = 0.

This equation can be considered as

$$\dot{u} + Au + Bu = 0$$

where $Au = -\frac{\partial^2 u}{\partial x^2}$ and $Bu = \phi^2(yf(y) - y_E f(y_E)$.

Taking H as the real Hilbert space $L^2[0,1]$ with inner product

$$\langle u_1,u_2 \rangle = \int_0^1 u_1 u_2 dx$$

we have $\langle Au,u \rangle = -\int_0^1 \frac{\partial^2 u}{\partial x^2} u\, dx = \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx \geq 0$ and $A = A^*$ hence A is m-accretive. Now

$$\langle Bu,u \rangle = \phi^2 \int_0^1 (yf(y) - y_E f(y_E))(y-y_E)dx$$

$$= \phi^2 \int_0^1 [y^2 f(y) - yy_E(f(y)+f(y_E)) + y_E^2 f(y_E)]dx$$

which, since $f(y) > 0$, is ≥ 0 if and only if

$$[f(y_E) + f(y)]^2 \leq 4f(y)f(y_E)$$

This is impossible hence B is not accretive.

However, we have

$$|Bu_1-Bu_2| = \phi^2 |y_1 f(y_1) - y_2 f(y_2)|$$

$$\leq \phi^2 |y_1-y_2| \; f(y_2)$$

$$\leq \phi^2 |y_1-y_2| \; e^{\gamma\beta/1+\beta}$$

since $1 < f(y) < e^{\beta\gamma/1+\beta}$ and $f(y_1) < f(y_2)$ if $y_1 > y_2$.

Therefore there exists α such that

$$\|Bu_1-Bu_2\|^2 \leq \alpha \|u_1-u_2\|^2.$$

hence $$\|Bu_1-Bu_2\| \leq |\alpha|^{\frac{1}{2}} \|u_1-u_2\|$$

Furthermore

$$\langle (1-\varepsilon)A(u_1-u_2),u_1-u_2\rangle + \langle Bu_1-Bu_2,u_1-u_2\rangle$$
$$= (1-\varepsilon)\int_0^1 \left(\frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial x}\right)^2 dx + \phi^2 \int_0^1 (y_1 f(y_1) - y_2 f(y_2))(y_1-y_2)dx$$
$$\geq 0$$

(8.22) if $(1-\varepsilon)\frac{\pi^2}{4} + \phi^2(1 - \frac{\beta\gamma}{(1+\beta)^2}e^{\beta\gamma/1+\beta}) \geq 0$

since $\int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx \geq \frac{\pi^2}{4}\int_0^1 u^2 dx$ if $u(1) = 0$ and $\frac{\partial u}{\partial x} = 0$ at $x = 0$

and we can show that

$$[y_1 f(y_1) - y_2 f(y_2)](y_1-y_2) = \phi^2[(y_1-y_2)f(y_1) + y_2(f(y_1)-f(y_2))][y_1-y_2]$$
$$\geq \phi^2(y_1-y_2)^2 + \phi^2[f(y_1)-f(y_2)][y_1-y_2]$$
$$\geq \phi^2(y_1-y_2)^2 \left(1 - \frac{\beta\gamma}{(1+\beta)^2}\right)e^{\beta\gamma/1+\beta}$$

by using the fact that $f(y)$ has greatest negative slope at $y = 0$.

Hence, if $$\frac{\pi^2}{4} + \phi^2(1 - \frac{\beta\gamma}{(1+\beta)^2}e^{\beta\gamma/1+\beta}) > 0$$

there exists $\varepsilon > 0$ such that (8.22) is valid. Thus all the conditions for the Corollary to Theorem 4.9 are satisfied so that A+B is m-accretive and -(A+B) generates a non-linear contraction semi-group. The corollary can also be applied to the non-linear model to determine an allowable class of perturbations.

Pao[34] has investigated a more general model than Wei, described by the equations

$$\text{(8.23)} \quad \begin{cases} \text{(i)} & u_t = a_1 u_{xx} - u_x - cu + d_1 v \exp\left(\frac{u}{1+\gamma u}\right) + h_1 \\ \text{(ii)} & v_t = a_2 v_{xx} - v_x - d_2 v \exp\left(\frac{u}{1+\gamma u}\right) \end{cases}$$

for $t > 0$, $0 < x < 1$, with initial and boundary conditions in the form

$$(8.24)\quad \begin{cases} a_1u_x(t,0) - u(t,0) = 0, & u_x(t,1) = 0 \\ a_2v_x(t,0) - v(t,0) = 0, & v_x(t,1) = 0 \end{cases}$$

$$(8.25)\qquad u(0,x) = \phi_1(x), \quad v(0,x) = \phi_2(x)$$

and with the ranges of u,v as $0 \leq u < \infty$ and $0 \leq v \leq 1$.

If (u_E, v_E) is the equilibrium solution of (8.23) we have on writing $y = u-u_E$, $w = v-v_E$

$$(8.26)\quad \begin{cases} \text{(i)} & \frac{\partial y}{\partial t} = a_1y_{xx} - y_x - cy + d_1\left[v\exp\left(\frac{u}{1+\gamma u}\right) - v_E\exp\left(\frac{u_E}{1+\gamma u_E}\right)\right] \\ \text{(ii)} & \frac{\partial w}{\partial t} = a_2w_{xx} - w_x - d_2\left[v\exp\left(\frac{u}{1+\gamma u}\right) - v_E\exp\left(\frac{u_E}{1+\gamma u_E}\right)\right] \end{cases}$$

Let H be the real Hilbert space of ordered pairs $z = (y,w)^T$ with inner product

$$\langle z_1, z_2 \rangle = \int_0^1 (y_1y_2 + w_1w_2)dx$$

Then the coupled pair of equations (8.26) can be written as a single abstract evolution equation

$$\dot{z} + Az + Bz = 0$$

where $Az = (-a_1y_{xx} + y_x + cy, \; -a_2w_{xx} + w_x)^T$

and $\quad Bz = [\,v\exp\left(\frac{u}{1+\gamma u}\right) - v_E\exp\left(\frac{u_E}{1+\gamma u_E}\right)]\;(-d_1, d_2)^T.$

$$\text{Then}\quad \langle Az, z\rangle = \int_0^1 [\,(-a_1y_{xx} + y_x + cy)y + (-a_2w_{xx} + w_x)w]\,dx$$

$$= \int_0^1 [\,a_1y_x{}^2 + cy^2 + a_2w_x{}^2]\,dx$$

$$+ \tfrac{1}{2}[\,y^2(t,0) + y^2(t,1) + w^2(t,0) + w^2(t,1)]$$

using the boundary conditions (8.24).

Since the constants a_1, a_2, c are positive in the applications considered by Pao, A is an accretive operator in these cases. The range of $I+\lambda A$ is H for all $y \in D(A)$ therefore A is m-accretive.

For the non-linear operator B, we have, on writing $f(y) = \exp\left(\frac{y}{1+\gamma y}\right)$,

$$\langle Bz, z\rangle = \int_0^1 [\,vf(u) - v_Ef(u_E)][-d_1y + d_2w]\,dx$$

$\not\geq 0$ for all $z \in H$ so that B is not accretive.

For $0 \leq v \leq 1$, we have

(8.27) $$|v_1 f(u_1) - v_2 f(u_2)| \leq e^{\frac{1}{\gamma}}\{|u_1-u_2| + |v_1-v_2|\}$$

hence

$$\|Bz_1-Bz_2\|^2 = \int_0^1 (d_1^2+d_2^2)(v_1 f(u_1) - v_2 f(u_2))^2 dx$$
$$\leq (d_1^2+d_2^2)e^{2/\gamma}\int_0^1 \{|u_1-u_2| + |v_1-v_2|\}^2 dx$$

therefore there exists an $\alpha > 0$ such that

$$\|Bz_1-Bz_2\|^2 \leq \alpha\|z_1-z_2\|^2.$$

In order to show that there exists an $\varepsilon > 0$ such that $(1-\varepsilon)A+B$ is accretive we require the following inequality.

Consider $I = \int_0^1 (a_1 y_x^2 - k^2y^2)dx + \frac{1}{2}(y_0^2 + y_1^2)$ where $a_1 y_x - y = 0$ at $x = 0$ and $y_x = 0$ at $x = 1$. The integral part of I has an extreme value if y satisfies the Euler-Lagrange equation which for this integral is

(8.28) $$a_1 y_{xx} + k^2 y = 0.$$

This equation has a non-trivial solution fitting the specified boundary conditions if

(8.29) $$\tan \frac{k}{\sqrt{a_1}} = \frac{1}{k\sqrt{a_1}} ,$$

the solution being $y = A \cos \frac{k}{\sqrt{a_1}}(x-1)$.

The smallest positive root k_1 of (8.29) corresponds to a minimum value of the integral and gives the value of

$$\frac{A^2}{2}\left[1 - \cos^2\frac{k_1}{\sqrt{a_1}}\right]$$

for I. Hence we have the inequality

(8.30) $$\int_0^1 a_1 y_x^2 dx + \tfrac{1}{2}[y_0^2 + y_1^2] \geq k_1 \int_0^1 y^2 dx$$

We can now obtain conditions under which $(1-\varepsilon)A+B$ is accretive.

We have

$$< (1-\varepsilon)A(z_1-z_2), z_1-z_2 > + < Bz_1-Bz_2, z_1-z_2 >$$

$$= (1-\varepsilon)\{\int_0^1 a_1(y_{1x}-y_{2x})^2 + c(y_1-y_2)^2 + a_2(w_{1x}-w_{2x})^2dx$$

$$+ \tfrac{1}{2}[(y_1-y_2)_0^2 + (y_1-y_2)_1^2 + (w_1-w_2)_0^2 + (w_1-w_2)_1^2]\}$$

$$+ \int_0^1 \{d_1(y_1-y_2)(v_2f(u_2) - v_1f(u_1))$$

$$+ d_2(w_1-w_2)(v_1f(u_1) - v_2f(u_2))\}dx$$

$$\geq (1-\varepsilon)(c+k_1^2)\int_0^1 (y_1-y_2)^2dx + (1-\varepsilon)k_2^2\int_0^1 (w_1-w_2)^2dx$$

$$- d_1e^{\frac{1}{\gamma}}\int_0^1 (|u_1-u_2| + |v_1-v_2|)(y_1-y_2)dx$$

$$- d_2e^{\frac{1}{\gamma}}\int_0^1 (|u_1-u_2| + |v_1-v_2|)(w_1-w_2)dx$$

by (8.27) and (8.30), k_2 is the smallest positive root of the equation

$$\tan\frac{k}{\sqrt{a_2}} = \frac{1}{k\sqrt{a_2}} \cdot$$

Further

$$d_1\int_0^1 (|u_1-u_2| + |v_1-v_2|)(y_1-y_2)dx$$

$$+ d_2\int_0^1 (|u_1-u_2| + |v_1-v_2|)(w_1-w_2)dx$$

$$< \int_0^1 (d_1|y_1-y_2|^2 + (d_1+d_2)|y_1-y_2||v_1-v_2| + d_2|v_1-v_2|^2)dx$$

$$< \int_0^1 \{\left(\frac{3d_1+d_2}{2}\right)(y_1-y_2)^2 + \left(\frac{d_1+3d_2}{2}\right)|v_1-v_2|^2\}dx$$

so that $\exists\ \varepsilon > 0$ such that $(1-\varepsilon)A+B$ is accretive if the following conditions are satisfied simultaneously:

$$c + k_1^2 - e^{\frac{1}{\gamma}}\left(\frac{3d_1+d_2}{2}\right) > 0$$

(8.31)

$$k_2^2 - e^{\frac{1}{\gamma}}\left(\frac{d_1+3d_2}{2}\right) > 0$$

These conditions determine the allowable size of the perturbation Bz for which the perturbed system is asymptotically stable through the values of d_1 and d_2.

8.3. Application of the Linear Semi-group Results

We demonstrate the applicability of the results of §5 by considering three examples. The first is a one-dimensional diffusion process, where the unperturbed system is

$$z_t = z_{xx}, \qquad z(x,0) = z_o(x)$$

$$z(0,t) = z(1,t) = 0, \quad \text{and} \quad z = L^2[0,1].$$

Abstracting this equation it is easy to show that the solution is given in terms of a semigroup T_t, where

$$T_t z_o = \Sigma 2e^{-n^2\pi^2 t} \sin n\pi x \int_0^1 \sin n\pi y z_o(y)dy$$

and $\|T_t\|_{\mathcal{L}(z)} \leq e^{-\pi^2 t}$.

Hence the class of perturbation operators we can allow must satisfy (5.18) and if the perturbed semigroup is to be negatively exponentially bounded we require (5.40).

For example if $Bz = \alpha z_x$ then it is easy to show that

$$\|T_t B\| \leq \frac{|\alpha|}{\sqrt{2te}}$$

and so we require $|\alpha| < \sqrt{\frac{\pi e}{2}} \simeq 2.06$.

For our second example we consider the heat conduction problem in R^n, that is,

$$z_t = \nabla^2 z$$

with $Z = L^2(\Omega)$ where Ω is an open bounded set in R^n with boundary Γ. On Γ we assume that part of the boundary Γ_1 is insulated so that $\frac{\partial z}{\partial n} = 0$ on Γ_1, and the rest of the boundary Γ/Γ_1 is kept at ambient temperature which we take to be zero. Under certain smoothness conditions on Ω and Γ it is possible to show that the solution is given in terms of a semigroup T_t such that

$$\|T_t\| \leq Me^{-\omega t} \quad \text{for some } \omega > 0.$$

Now let us assume that the boundary part Γ_1 is not perfectly insultated and in fact

$$\left.\frac{\partial z}{\partial n}\right|_{\Gamma} = kz(\Gamma)$$

then we can regard this new problem as a perturbation of the original problem. Using the methods of Curtain and Pritchard [34] we can reformulate the problem so that the analysis of §5 can be applied. Now B will map $L^2(\Gamma)$ functions into a larger space than $L^2(\Omega)$. However the semigroup T_t is smoothing in that we are able to show that

$$\|T_t B\| \leq \frac{N}{t^{1/4}}$$

Therefore the results show the existence, uniqueness of the mild solution of the perturbed problem and stability will follow if

$$4MN\Gamma(\tfrac{3}{4}) < (3\omega)^{\frac{3}{4}}$$

To illustrate the estimates for non-homogeneous equations we consider the non-dimensional equation [29] for the displacement y(x,t). of a simply supported column subject to an axial load p(t) and a distributed transverse load q(x,t),

(8.32) $$y_{tt} + 2\xi y_t + y_{xxxx} + p(t)y_{xx} = q(x,t) \quad 0 \leq x \leq 1, \; t > 0$$

The boundary conditions are $y = y_{xx} = 0$ at $x = 0,1$ $\forall t \geq 0$.

If we take H as the same Hilbert space as in the earlier examples on the beam problem and assume that

$$\operatorname*{ess\,sup.}_{t\in[0,T]} |p(t)| < p \quad \text{and} \quad q \in L_2[0,T;L_2[0,1]]$$

then equation (5.4) may be written as

$$\dot{w} + Aw + Bw = Q$$

where Aw is as (8.6), $B(t)w = [0,p(t)y_{xx}]^T, Q = [0,q(x,t)]^T$.

(8.33) Then $$\langle Bw,w\rangle \leq \frac{p}{2}\frac{\pi^2}{\sqrt{\pi^4+\xi^4}}\|w\|^2 = \mu^1\|w\|^2 \quad \text{(say)}$$

and $$\|Q(\cdot,t)\|^2 = \int_0^1 q^2(x,t)\,dx.$$

The semi-group T_t for the unperturbed system can be estimated by considering the Liapunov functional $V = \|w\|^2$, we have

(8.34) $$\dot{V}(t) \leq -2\xi \int_0^1 (v^2+y_{xx}^2)dx \leq -2\xi\left[1 - \frac{\xi}{(\xi^2+\pi^4)^{\frac{1}{2}}}\right] V = -2\mu V \text{ (say)}$$

so that $\|T_t\| \leq e^{-\mu t}$.

Hence for the perturbed system (5.14) which is equivalent to (7.10) gives

(8.35) $$\|w(t)\| \leq e^{-(\mu-\mu^1)t}\|w_o\| + \int_0^t e^{-(\mu-\mu^1)(t-\rho)}\|Q(\cdot,\rho)\|d\rho.$$

Since it is easy to find a constant λ such that

(8.36) $$\left[\int_0^1 y_{xx}^2 dx\right]^{\frac{1}{2}} \geq \lambda \sup_{x\in[0,1]} |y(x,t)|$$

then using (8.33) and (8.35) we have the following bound on the maximum displacement $y_m(t)$.

(8.37) $$y_m(t) \leq \frac{1}{\lambda}\{e^{-(\mu-\mu^1)t}\|w_o\| + \int_0^t e^{-(\mu-\mu^1)(t-\rho)}\|Q(\cdot,\rho)\|d\rho\}$$

This result may be compared with

(8.38) $$y_m(t) \leq \frac{\pi^2}{\lambda(\pi^4-\xi^2)^{\frac{1}{2}}}\int_0^t e^{-(\xi-\pi^2\xi/2(\pi^4-\xi^2)^{\frac{1}{2}})(t-\rho)}\|Q(\cdot,\rho)\|d\rho$$

which is the result given in [26] for the case of small damping $(\xi < \frac{\pi}{\sqrt{2}})$ and $\|w_o\| = 0$.

We note that (8.38) can be obtained using our methods by taking the functional $V = \int_0^1 (v^2+2\xi vy+y_{xx}^2)dy$. Then in place of (8.33) and (8.34) we have

(8.39) $$|\langle Bw,w\rangle| \leq K\|w\|^2$$

where $2K = |p(t)|\frac{\pi^2}{\sqrt{\pi^4-\xi^2}}$ and

(8.40) $$\dot{V} = -2\xi V.$$

Also $$\int_0^1 y_{xx}^2 dx \leq \frac{\pi^4}{\pi^4-\xi^2} V$$

so that using (8.36) and (7.10) with $\|w_o\| = 0$ and $\mu = \xi$ we have

$$y_m(t) \leq \frac{1}{\lambda}\frac{\pi^2}{\sqrt{\pi^4-\xi^2}}\int_0^t e^{-(\xi-\pi^2 p/2(\pi^4-\xi^2)^{\frac{1}{2}})(t-\rho)}\|Q(\cdot,\rho)\|\,d\rho.$$

8.4. Applications of the Results on Non-Linear Semi-groups

We consider three non-linear problems to illustrate various aspects of the theory of §6 concerned with the estimation of non-linear semi-groups.

The first example is the heat equation in R^1 defined by the non-linear diffusion equation

(8.41) $$\dot{z}(t) = z_{xx} + z^3 \qquad 0 \leq x \leq 1$$

with $z(x,0) = z_0(x), z(o,t) = z(1,t) = 0$. We consider the application of Theorem 6.1 to this problem. The semi-group T_t is that generated on $L^2[0,1]$ by the operator $-A = \frac{d^2}{dx^2}$ $D(A) = H_0^2[0,1] \cap H_0^1[0,1]$. We take $V = L^{3\alpha}[0,1]$, $Z_2 = L^{\alpha}[0,1]$, $\alpha \geq 1$ and $Nz = -z^3$. To avoid difficulties in estimating the norms of semi-groups in L^p spaces we use embedding theorems (Theorem 2.4) and work with the Sobolev spaces $H^m[0,1]$.

We have $H^m[0,1] \subset L^{3\alpha}[0,1]$ if $m \geq \frac{1}{2} - \frac{1}{3\alpha}$ (see §2) so there exist positive constants c_1, c_2, c_3, c_4 such that the following inequalities are valid.

(i) $$\|T_t z\|_V \leq \|T_t z\|_{H^m} \leq \frac{c_1}{t^{m/2}}\|z\|_{Z_1} = \frac{c_1}{t^{\frac{1}{4}-\frac{1}{6\alpha}}}\|z\|_{Z_1}, \qquad t > 0$$

(see Appendix 2). Thus we require $r \leq p_1 < 1/\frac{1}{4} - \frac{1}{6\alpha}$.

(ii) $N: z \to z^3$ maps V to Z_2 and

$$\|Nz\|_{Z_2} = \|z\|_V^3, \qquad \|Nz\|_{L^s[Z_2]} = \|z\|^3_{L^{3s}[V]}$$

therefore we have to take $r = 3s$ and $b = a^3$.

(iii) For $1 \leq \alpha < 2$

$$\|T_t z_1\|_V \leq \|T_t z_1\|_{H^m} \leq \frac{c_2}{t^{\frac{m}{2}+\frac{1}{4}}}\|z_1\|_{L^1} \leq \frac{c_2}{t^{\frac{1}{2}-\frac{1}{6\alpha}}}\|z_1\|_{Z_2} = g_2(t)\|z_1\|_{Z_2}$$

or if $\alpha \geqslant 2$, since $L^{\alpha} \subset L^2$

$$\|T_t z_1\|_V \leqslant \|T_t z_1\|_{H^m} \leqslant \frac{c_1}{t^{m/2}} \|z_1\|_{L^2} \leqslant \frac{c_1}{t^{\frac{1}{4} - \frac{1}{6\alpha}}} \|z_1\|_{Z_2} = g_2(t)\|z_1\|_{Z_2}$$

Thus we require $p_2 < 1/\tfrac{1}{2} - \frac{1}{6\alpha}$ if $1 \leqslant \alpha < 2$ or $p_2 < 1/\tfrac{1}{4} - \frac{1}{6\alpha}$ if $\alpha \geqslant 2$.

(iv) Using Lemma 6.1 and the Holder and Minkowski inequalities we can show that

$$\|Nz_1 - Nz_2\|_{L^s[Z_2]} \leqslant \tfrac{3}{2}\|z_1 - z_2\|_{L^{3s}[V]} (\|z_1\|^2_{L^{3s}[V]} + \|z_2\|^2_{L^{3s}[V]})$$

(v) $\|T_t z_1\|_{Z_2} = \|T_t z_1\|_{H^0} \leqslant \frac{c_3}{t^{\frac{1}{4}}} \|z_1\|_{Z_2} = g_3(t)\|z_1\|_{Z_2}$ if $1 \leqslant \alpha < 2$

or $\|T_t z_1\|_{Z_2} \leqslant c_4\|z_1\|_{Z_2} = g_3(t)\|z_1\|_{Z_2}$ if $\alpha \geqslant 2$.

Let $r^{-1} = q^{-1} + s^{-1} - 1$ for $r = 3s \geqslant 3$. We require $p_1 \geqslant r \geqslant 1$ hence $r < 1/\tfrac{1}{4} - \frac{1}{6\alpha}$, also we need $p_2 \geqslant q \geqslant 1$ where p_2 is given by (8.42). Solving for q in terms of s we have

(8.43) $$q = \frac{3s}{3s-2}$$

and provided there is a choice of admissible s, q, r we can choose a T such that

(8.44) $$3a^2\|g_1\|_{L^q} < 1$$

and z_o such that

(8.45) $$\|g_1\|_{L^r}\|z_o\|_{Z_1} + \|g_2\|_{L^q} a^3 \leqslant a.$$

We require with (8.43)

$$1 \leqslant s < 1/3(\tfrac{1}{4} - \tfrac{1}{6\alpha})$$

and $$1 \leqslant q < \begin{cases} 1/(\frac{1}{2} - \frac{1}{6\alpha}), & 1 \leqslant \alpha < 2 \\ 1/(\frac{1}{4} - \frac{1}{6\alpha}), & \alpha \geqslant 2 \end{cases}$$

and amongst permissible values we have

$\alpha = 1$, $p_1 < 12$, $r = 12 - \varepsilon$, $q = \frac{6}{5}$

$\alpha = 2$, $p_1 < 6$, $r = 6 - \varepsilon$, $q = \frac{3}{2}$

$\alpha = \infty$, $p_1 < 4$, $r = 4 - \varepsilon$, $q = 2$

so that, for example, the solution lies in

$$L^{12-\varepsilon}[0,T;L^3[0,1]] \cap L^{6-\varepsilon}[0,T;L^6[0,1]] \cap L^{4-\varepsilon}[0,T;L^\infty[0,1]].$$

We consider next the application of corollary 6.1. We have

$$\|T_t z\|_{Z_1} \leq \begin{cases} \frac{c}{t^{\frac{1}{4}}}\|z\|_{Z_2}, & 1 \leq \alpha < 2 \\ c'\|z\|_{Z_2}, & \alpha \leq 2. \end{cases}$$

hence for $1 \leq \alpha < 2$ we require $p < 4$ whilst for $\alpha \geq 2$ we require $p \geq 1$. In each case there exist admissible pairs p, s so by Corollary 6.1 $z \in C[0,T;Z_1]$.

The above calculations establish the conditions for the existence and uniqueness of a local solution in $[0,T]$. To obtain a global solution we proceed as follows.

The semi-group T_t generated by $-A$ satisfies the inequality

$$\|T_t\|_{\mathcal{L}(z_1)} \leq e^{-\pi^2 t}$$

so we can replace g_1, g_2 by $e^{-\lambda\pi^2 t}g_1((1-\lambda)t)$ and $e^{-\lambda\pi^2 t}g_2((1-\lambda)t)$ respectively. Then (8.44) and (8.45) become

$$3\gamma_2 a^2 < 1$$

$$\gamma_1\|z_0\| + \gamma_2 a^3 \leq a$$

where γ_1, γ_2 are defined in (6.12). The largest value of $\|z_0\|$ will be obtained when $3a^2\gamma_2 = 1$ so if

$$\|z_0\| \leq \frac{2}{3\gamma_1\sqrt{3\gamma_2}} \tag{8.46}$$

the solution can be extended for all time and the region defined by (8.46) will be asymptotically stable. For the best result the quantity $2/3\gamma_1\sqrt{3\gamma_2}$ should be maximised on λ.

The multistage time process based on (6.18) could also be applied to this example as could the Liapunov theory of §7. We will consider briefly the latter method. The regularisation result Corollary 6.2 can be applied with $Z_1 = Z_2$ and we find $z \in L^2[\varepsilon,T;H^1]$. Applying the Liapunov theory of §7 with

$$\langle Az,z\rangle_{Z_1} + \langle z,Az\rangle_{Z_1} = -2\int_0^1 z_x^2(x,t)dx, \qquad z \in D(A)$$

we have

$$\|z(t)\|_{Z_1}^2 - \|z(\varepsilon)\|_{Z_1}^2 = -2\int_\varepsilon^t \int_0^1 [z_x^2(x,s) - z^4(x,s)]dx\,ds$$

For $z \in H^1[0,1]$, $z(0) = z(1) = 0$

$$\int_0^1 z_x^2 dx \geq 4\|z\|_{C[0,1]}^2$$

hence

$$\|z(T)\|_{Z_1}^2 - \|z(\varepsilon)\|_{Z_1}^2 \leq -2\int_\varepsilon^T \|z(s)\|_{C[0,1]}^2[4 - \|z(s)\|_{Z_1}^2]ds$$

is valid for all $\varepsilon > 0$ and so

$$\|z(t)\|_{Z_1} \leq \|z_o\|_{Z_1}$$

if $\|z_o\| < 2$. Therefore the solution can be extended for all time if $\|z_o\| < 2$, and this condition defines a region of asymptotic stability.

In §6.4 we investigated the application of the local existence theorem 6.1 to the abstract evolution equation with a perturbation of polynomial type.

$$Bz = -\sum_{i=2}^{n} a_i z^i.$$

Consider now the equation

$$z_t = z_{xx} + \sum_{i=2}^{n} a_i z^i \qquad 0 < x < 1 \tag{8.47}$$

with $z(x,o) = z_o(x)$, $z(o,t) = z(1,t) = 0$ so that the semi-group is the same as in the previous example. With $V = L^{n\alpha}[0,1]$ $Z_2 = L^{\alpha}[0,1]$ and following the calculations for that example we require for the local existence theorem that

$$(8.48)\quad \begin{array}{ll} \text{(i)} & m \geq \frac{1}{2} - \frac{1}{n\alpha} \\ \text{(ii)} & r \leq p_1 < 1/\frac{1}{4} - \frac{1}{2n\alpha} \\ \text{(iii)} & ns = r \\ \text{(iv)} & p_2 < \begin{cases} 1/\frac{1}{2} - \frac{1}{2n\alpha} & \text{if } 1 \leq \alpha < 2 \\ 1/\frac{1}{4} - \frac{1}{2n\alpha} & \text{if } \alpha \geq 2. \end{cases} \\ \text{and (v)} & p_2 \geq \frac{r}{r+1-n} \end{array}$$

These conditions imply that if $1 \leq \alpha \leq 2$ then

$$n\alpha < 3\alpha+2$$

$$\text{i.e.} \quad \alpha < \frac{2}{n-3}$$

but $\alpha \geq 1$ so $n < 5$. Alternatively if $\alpha \geq 2$ these conditions give $n\alpha < 4\alpha+2$ and again $n < 5$ so that the maximum value of n for this problem is 4 if Z_1 is chosen to be $L^2[0,1]$. A higher value than this could be obtained by choosing a smoother space for the initial data.

The Corollaries 6.1 and 6.2 can be applied as in the previous example. For Corollary 6.1 for example if $\alpha < 2$ then we require $p^{-1} = 1 - s^{-1}$ where

$$p < 4$$

hence by (8.48(iii)) we need (for $n = 4$) that $\frac{r}{r-4} < 4$ i.e. $r > \frac{16}{3}$. This is compatible with (8.48(ii)) which gives the condition $r < 8\alpha/2\alpha-1$. For the regularity result Corollary 6.2 with $Z_1 = Z_2 = L^2[0,1]$ and $V_1 = H^1$ we require

$$\frac{1}{2} = \frac{1}{w} + \frac{n}{r} - 1$$

if z is to belong to $L^2[\xi,T;V]$. Now w must satisfy

$$w < 2$$

so that we must have

$$(8.49)\qquad \frac{n}{r} < 1 \qquad \text{i.e. } r > 4.$$

Now $n\alpha = 8$ and (8.48 (iv) and (v)) require

$$\frac{16}{3} > r > \frac{16.3}{13}$$

which can be reconciled with (8.49).

In both of these examples we have found that the solution lies in the intersection of a number of different spaces.
The results can be stated in a particularly neat form by using the interpolation result of Adams [36] which states that

$$L^p[0,T;L^q(\Omega)] \cap L^\infty[0,T;L^2(\Omega)] \subset L^\rho[0,T;L^\sigma(\Omega)]$$

where $\quad \frac{1}{\rho} = \frac{1-\theta}{p}, \quad \frac{1}{\sigma} = \frac{1-\theta}{q} + \frac{\theta}{2}.$

For the first example the solution was in

$$L^{12-\varepsilon}[0,T;L^3[0,1]] \cap L^{6-\varepsilon}[0,T;L^6[0,1]] \cap C[0,T;L^2[0,1]]$$

and on using the interpolation result we find that the solution lies in $L^\rho[0,T;L^\sigma(\Omega)]$ where $\frac{4-\varepsilon}{\rho} = 1 - \frac{2}{\sigma}$.

For our third and final example we consider one of Burgers models of turbulence in hydrodynamics [37] namely the system described by

(8.50) $$z_t = \frac{1}{R}z_{xx} + z - Rz\|z\|^2_{L^2[0,1]} - 2zz_x, \quad 0 \leq x \leq 1$$

with $z(o,t) = z(1,t) = 0$. R is the Reynolds number and the model represents flow in a channel. The stability of the solution of this equation has been discussed previously by Pritchard [33]. The operator A is taken to be

$$Az = -\frac{1}{R}z_{xx} - z$$

with $D(A) = H^2[0,1] \cap H^1_o[0,1]$, so that $-A$ generates a strongly continuous semi-group T_t. There are two non-linearities in (8.50) so we set

$$Bz = B_1 z + B_2 z$$

where $B_1 z = Rz\|z\|^2_{L^2[0,1]}$ and $B_2 z = 2zz_x$. Then

$$\|B_1 z\|_{H^1} \leq R\|z\|_{H^1} \|z\|^2_{H^0}$$

$$\|B_1 z\|_{H^{\frac{1}{2}+}} \leq R\|z\|_{H^{\frac{1}{2}+\varepsilon}} \|z\|^2_{H^0}$$

and $$\|B_2 z\|^2_{H^0} \leq H \sup (z^2) \int_0^1 z_x^2 dx \leq C\|z\|^2_{H^{\frac{1}{2}+}} \|z\|^2_{H^1}$$

where $H^{\frac{1}{2}+}$ denotes $H^{\frac{1}{2}+\varepsilon}$ and C is some generic constant.

The direct approach of the first two examples cannot be used in this problem because of the presence of two non-linearities. We look for solutions in the Banach space W where

$$W = L^r[0,T;H^1] \cap L^u[0,T;H^{\frac{1}{2}+}]$$

and $\|z\|_W = \|z\|_{L^r[0,T;H^1]} + \|z\|_{L^u[0,T:H^{\frac{1}{2}+}]}$

Noting that for the semi-group T_t we have on the interval $[0,T]$,

$$\|T_t\|_{\mathcal{L}(H^{\frac{1}{2}+})} \leq C, \quad \|T_t\|_{\mathcal{L}(H^1)} \leq C, \quad \|T_t\|_{\mathcal{L}(H^0_, H^{\frac{1}{2}+})} \leq \frac{c}{t^{\frac{1}{4}+}},$$

$$\|T_t\|_{\mathcal{L}(H^0, H^1)} \leq \frac{c}{t^{\frac{1}{2}}}, \quad t > 0$$

so that

$$\left\| \int_0^t T_{t-s} B_1 z(s) ds \right\|_{H^1} \leq RC \int_0^t \|z(s)\|_{H^1} \|z(s)\|^2_{H^0} ds$$

and hence by the convolution theorem, Theorem 5.4, we have

$$\left\| \int_0^t T_{t-s} B_1 z(s) ds \right\|_{L^r[0,T;H^1]} \leq RC\|z\|_{L^r[0,T:H^1]} \|z\|^2_{L^u[0,T;H^{\frac{1}{2}+}]}$$

if $u > 2$. Similarly

$$\left\| \int_0^t T_{t-s} N_1 z(s) ds \right\|_{L^u[0,T;H^{\frac{1}{2}+}]} \leq RC\|z\|_{L^u[0,T;H^{\frac{1}{2}+}]} \|z\|^2_{L^u[0,T;H^{\frac{1}{2}+}]}$$

For the second non-linearity $B_2 z$ we have

$$\left\| \int_0^t T_{t-s} B_2 z(s) ds \right\|_{L^u[0,T;H^{\frac{1}{2}+}]} \leq C\|z\|_{L^u[0,T:H^{\frac{1}{2}+}]} \|z\|_{L^r[0,T;H^1]}$$

if $r > 4/3$ and

$$\left\| \int_0^t T_{t-s} B_2 z(s) ds \right\|_{L^r[0,T;H^{\frac{1}{2}+}]} \leq C\|z\|_{L^u[0,T:H^{\frac{1}{2}+}]} \|z\|_{L^r[0,T;H^1]}$$

if $u > 2$. Finally

$$\|T_t z_o\|_{L^r[0,T;H^1]} \leq C\|z_o\|_{H^0} \quad \text{if } r < 2$$

and $$\|T_t z_o\|_{L^u[0,T;H^{\frac{1}{2}+}]} \leq C\|z_o\|_{H^0} \quad \text{if } u < 4.$$

With these estimates we can deduce that if the mapping Ω is defined by

$$(\Omega z)(t) = T_t z_o - \int_o^t T_{t-s} B_1 z(s)ds - \int_o^t T_{t-s} B_2 z(s)ds$$

then $\Omega : W \to W$ for $z_o \in H^0$, and there is a closed ball or radius a in W such that Ω maps this ball into itself. If a is suitably small or the time interval is short enough Ω is a contraction on this ball and so there exists a local solution on W for $\frac{4}{3} < r < 2$ and $2 < u < 4$. The upper constraint on r is essentially imposed by the condition $\| T_t \|_{\mathcal{L}(H^0,H^1)} \leq \frac{c}{t^{\frac{1}{2}}}$ so that it is easy to show that $z \in L^2[\varepsilon,T;H^1]$. It is also straightforward to show that $z \in C[0,T:H^0]$.

The Liapunov theory of §7.2 can now be applied. We have

$$-\langle Az,z\rangle_{H^0} - \langle z,Az\rangle_{H^0} = + 2\int_0^1 (\tfrac{1}{R} z_{xx} + z)z \, dx$$

$$= 2\int_0^1 (z^2 - \tfrac{1}{R} z_x^2)dx$$

and $\quad -\langle Bz,z\rangle_H = -\int_0^1 (R\|z\|^2_{H^0} + 2zz_x)z \, dx$

$$= - R\|z\|^4_{H^0}, \quad z \in D(A).$$

Moreover $\quad \int_0^1 z_x^2 dx \geq \pi^2 \int_0^1 z^2 dx \quad , \quad z \in H_0^1[0,1]$.

hence by Theorem 7.2

$$\|z(T)\|^2_{H^0} - \|z(\varepsilon)\|^2_{H^0} \leq \int_\varepsilon^T [2(z^2 - \tfrac{1}{R} z_x^2) - 2R\|z\|^4_{H^0}]ds$$

$$\leq 2R\int_\varepsilon^T \|z(s)\|^2_{H^0}\left[\tfrac{1}{R}(1 - \tfrac{\pi^2}{R}) - \|z(s)\|^2_{H^0}\right]ds$$

Therefore if $R < \pi^2$ the solution exists for all time and $\|z(t)\| \to 0$ as $t \to \infty$. If $R > \pi^2$ and $\|z_0\|^2 < \frac{1}{R}(1 - \frac{\pi^2}{R})$ then the solution exists for all time and $\|z(t)\|^2 \leq \frac{1}{R}(1 - \frac{\pi^2}{R})$. However if $R > \pi^2$ and $\|z_o\|^2 \geq \frac{1}{R}(1 - \frac{\pi^2}{R})$ then again the solution exists for all time and $\|z(t)\|_{H^0} \to \left[\frac{1}{R}(1 - \frac{\pi^2}{R})\right]^{\frac{1}{2}}$ or $\|z(t)\| < \left[\frac{1}{R}(1 - \frac{\pi^2}{R})\right]^{\frac{1}{2}}$ as $t \to \infty$.

§9. Conclusions and Suggestions for Further Work

In this thesis we have developed two methods for analysing the stability of the solution of non-linear evolution equations. The results provide a useful addition to the known methods for such work, for example those of classical analysis using eigenfunction expansions and the direct method of Liapunov. As seen by the work of Chapter 7 the methods described in this thesis are related to and in some ways complementary to the latter approach.

In principle neither type of result is difficult to apply, however for some problems it will not be easy to find the appropriate inequalities required to verify the conditions of the perturbation theorem. (Theorem 4.9 and its Corollary). For the second method relating to the semi-group and mild solution of the equation the theorems depend upon estimates of $\|T_t B\|$ in the form

(9.1) $$\|T_t B\| \leq \frac{C}{t^\alpha} \qquad 0 < \alpha \leq 1$$

where T_t is the unperturbed semi-group and B is the perturbing operator. Whilst it may not be too difficult to establish the value of α we should note that in order to find the optimum result by establishing the largest region of asymptotic stability the value of C must be determined as accurately as possible. Very rough estimates may be found quite easily but values within 10% say of the optimum value will be difficult to obtain. As is usual in such cases the complexity of the calculation increases with the degree of precision that is required. For a given problem there could be much difficult computation to be done in this direction of necessity involving some computing and numerical analysis.

The Liapunov theory of §7 is applicable only when the underlying spaces are Hilbert spaces. The development of a similar theory for

Banach spaces would seem to be a desirable theoretical advance but will probably prove to be a very difficult task.

Despite the above comments the methods would seem to have a wide range of applicability to real problems in science and engineering particularly since most problems in these fields would be based on Hilbert spaces rather than Banach spaces. There would seem to be many possibilities concerning further applications of the results to problems other than those considered in §8 particularly to those posed on R^n with $n = 2$ or 3. It can be seen from the nature of the imbedding theorems that the value of n will play a vital role in the application of the results for non-linear semi-groups (§6). Problems where the perturbation occurs on the boundary of the system are also of particular interest. One of these has been considered briefly in §8 using a result from [35]. Further work in this direction would seem to be desirable.

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§11 APPENDICES

Appendix 1

We require an inequality of the form

(A.1) $$\int_0^1 (y_{xx}^2+\xi^2y^2)dx \geq c^2\beta^2 \quad \text{where} \quad \beta = \max_{0\leq x\leq 1} [y]$$

where y is an admissible deflection of a simply-supported uniform beam. We use the method of Freund and Plaut [28] as follows.

Consider all admissible deflections y(x) such that

(A.2) $$\int_0^1 (y_{xx}^2+\xi^2y^2)dx = A^2 \quad (A \text{ a constant}).$$

We maximise y(a) for any point $0 \leq a \leq 1$ subject to the condition (A.2) by application of Lagrange's method of undetermined multipliers. We thus determine the function y such that

$$F[y] = y(a) + \lambda[\int_0^1 (y_{xx}^2+\xi^2y^2)dx - A^2]$$

is stationary, where λ is the undetermined multiplier. Setting the first variation equal to zero gives

$$\eta(a) + 2\lambda\int_0^1 (y_{xx}\eta_{xx}+\xi^2y\eta)dx = 0$$

and on integrating by parts and using the boundary conditions $y = y_{xx} = 0$ at $x = 0,1$ we find that y must satisfy

(A.3) $$y_{xxxx} + \xi^2y = -\frac{1}{2\lambda}\delta(x-a),$$

where δ denotes the Dirac delta function.

The solution of (A.3) is

$$y(x) = \frac{AG(x,a)}{\sqrt{G(a,a)}}$$

where G(x,a) is the influence function for the beam and λ has been determined in terms of A using (A.2). By symmetry the maximum value of y occurs at the centre point of the beam i.e. $x = \frac{1}{2}$ and hence is given by

$$\beta = A\sqrt{G(\tfrac{1}{2},\tfrac{1}{2})}$$

The optimum value of c^2 is then given by

$$c^2 = \int_0^1 (y_{xx}^2+\xi^2 y^2)\,dx/\beta^2 = \frac{1}{G(\frac{1}{2},\frac{1}{2})} .$$

$G(\frac{1}{2},\frac{1}{2})$ is found in the usual way by finding the solution of the equation

$$u_{xxxx} + \xi^2 u = \delta(x-\tfrac{1}{2})$$

such that u, $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$ are continuous for $0 \leqslant x \leqslant 1$ whilst $\frac{d^3u}{dx^3}$ has a finite discontinuity of value 1 at $x = \frac{1}{2}$, subject to the boundary conditions $u = u_{xx} = 0$ at $x = 0, 1$.

Appendix 2

In this appendix we wish to illustrate the derivation of some of the estimates used in the examples. Consider the one-dimensional diffusion equation

(A.4) $$z_t = z_{xx}, \qquad z(0) = z_o$$

with boundary conditions $z(0,t) = z(1,t) = 0 \ \forall\ t > 0$. Let T_t be the semi-group generated by the operator $-\frac{\partial^2}{\partial x^2}$. The solution of (A.4) is well-known to be

(A.5) $$T_t z = z(x,t) = \sum_1^\infty b_n e^{-n^2\pi^2 t}\sin n\pi x, \qquad |x| \leqslant 1$$

where $b_n = 2\int_0^1 z_o \sin n\pi x\, dx$. The Fourier transform of $T_t z$ is given by

$$\hat{T_t z} = \frac{-i}{\sqrt{2\pi}} \sum_1^\infty b_n e^{-n^2\pi^2 t} \int_{-1}^1 \sin n\pi x \sin \zeta x dx$$

$$= \frac{-i}{\sqrt{2\pi}} \sum_1^\infty b_n e^{-n^2\pi^2 t}(-1)^n\left[\frac{1}{\zeta - n\pi} - \frac{1}{\zeta + n\pi}\right]\sin\zeta$$

By (2.28) $$\|T_t z\|^2_{H^k} = \int_{-\infty}^\infty (1+|\zeta|^2)^k\ |\hat{T_t z}|^2 d\zeta$$

and for $k \leqslant 1$ it follows from Parseval's identity that

$$\|T_t z\|^2_{H^k} \leqslant \sum_1^\infty 2\frac{b_n^2}{\pi} e^{-2n^2\pi^2 t} \int_{-\infty}^\infty (1+|\zeta|^2)^k \frac{\sin^2\zeta}{(\zeta - n\pi)^2} d\zeta.$$

Using the inequality

$$(1 + |x+y|^2)^s \leqslant 2^{|s|}(1 + |x|^2)^s(1 + |y|^2)^{|s|}$$

valid for $x \in R^n$, $y \in R^n$, $-\infty < s < \infty$ it is not difficult to show that

$$\int_{R^n} |h(x-y)|^2(1 + |x|^2)^s dx \leqslant 2^{|s|}(1 + |y|^2)^{|s|} \int_{R^n}|h|^2(1 + |x|^2)^s dx$$

Thus $\|T_t z\|^2_{H^k} \leqslant \sum_1^\infty \frac{2b_n^2}{\pi} e^{-2n^2\pi^2 t} 2^k(1 + (n\pi)^2)^k \int_{-\infty}^\infty (1+\zeta^2)^k \frac{\sin^2\zeta}{\zeta^2} d\zeta$

Now $e^{-2x^2 t}(1+x^2)^k$ has a maximum value when $1+x^2 = \frac{k}{2t}$ so that

$$\|T_t z\|^2_{H^k} \leqslant \frac{2k^k}{\pi t^k} \sum_1^\infty b_n^2 \cdot \int_{-\infty}^\infty (1+\zeta^2)^k \frac{\sin^2\zeta}{\zeta^2} d\zeta.$$

and provided the integral exists we have

$$\|T_t z\|_{H^k} < \frac{c}{t^{k/2}} \|z_o\|_{L^2}$$

Appendix 2

In this appendix we wish to illustrate the derivation of some of the estimates used in the examples. Consider the one-dimensional diffusion equation

(A.4) $$z_t = z_{xx}, \qquad z(0) = z_o$$

with boundary conditions $z(0,t) = z(1,t) = 0 \;\forall\; t > 0$. Let T_t be the semi-group generated by the operator $-\frac{\partial^2}{\partial x^2}$. The solution of (A.4) is well-known to be

(A.5) $$T_t z = z(x,t) = \sum_1^\infty b_n e^{-n^2\pi^2 t}\sin n\pi x, \qquad |x| \leq 1$$

where $b_n = 2\int_0^1 z_o \sin n\pi x\, dx$. The Fourier transform of $T_t z$ is given by

$$\hat{T_t z} = \frac{-i}{\sqrt{2n}} \sum_1^\infty b_n e^{-n^2\pi^2 t} \int_{-1}^{1} \sin n\pi x \sin \zeta x dx$$

$$= \frac{-i}{\sqrt{2n}} \sum_1^\infty b_n e^{-n^2\pi^2 t} (-1)^n \left[\frac{1}{\zeta - n\pi} - \frac{1}{\zeta + n\pi}\right] \sin\zeta$$

By (2.28) $$\| T_t z \|^2_{H^k} = \int_{-\infty}^{\infty} (1+|\zeta|^2)^k \; |\hat{T_t z}|^2 d\zeta$$

and for $k \leq 1$ it follows from Parseval's identity that

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Using the inequality

$$(1 + |x+y|^2)^s \leq 2^{|s|}(1 + |x|^2)^s (1 + |y|^2)^{|s|}$$

valid for $x \in R^n$, $y \in R^n$, $-\infty < s < \infty$ it is not difficult to show that

$$\int_{R^n} |h(x-y)|^2 (1 + |x|^2)^s dx \leq 2^{|s|}(1 + |y|^2)^{|s|} \int_{R^n} |h|^2 (1 + |x|^2)^s dx$$

Thus $$\| T_t z \|^2_{H^k} \leq \sum_1^\infty \frac{2b_n^2}{\pi} e^{-2n^2\pi^2 t} 2^k (1 + (n\pi)^2)^k \int_{-\infty}^{\infty} (1+\zeta^2)^k \frac{\sin^2\zeta}{\zeta^2} d\zeta$$

Now $e^{-2x^2 t}(1+x^2)^k$ has a maximum value when $1+x^2 = \frac{k}{2t}$ so that

$$\| T_t z \|^2_{H^k} \leq \frac{2k^k}{\pi t^k} \sum_1^\infty b_n^2 \cdot \int_{-\infty}^{\infty} (1+\zeta^2)^k \frac{\sin^2\zeta}{\zeta^2} d\zeta .$$

and provided the integral exists we have

$$\| T_t z \|_{H^k} < \frac{c}{t^{k/2}} \| z_o \|_{L^2}$$

Alternatively since $|b_n| < \|z_o\|_{L^1}$, we have

$$\|T_t z\|^2_{H^k} \leq C\|z_o\|^2_{L^1} \sum_1^\infty (1+n^2\pi^2)^k \, e^{-2n^2\pi^2 t}.$$

Now $(1+n^2\pi^2)^k e^{-n^2\pi^2 t} \leq \left(\frac{2k}{t}\right)^k$ and $\sum_1^\infty e^{-n^2\pi^2 t} \leq \int_0^\infty e^{-x^2 t} dt = \frac{\sqrt{\pi}}{2\sqrt{t}}$

so that there exists a constant c' such that

$$\|T_t z\|^2_{H^k} \leq \frac{c'}{t^{k+\frac{1}{2}}} \|z_o\|^2_{L^1}. \qquad (t > 0)$$