DYNAMICAL MODELS IN DISEQUILIBRIUM THEORY

by

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CONTENTS.

Preface.

This thesis contains the results of my studies during the period October 1976 to March 1979 at the University of Warwick, with Professor E.C.Zeeman,Mathematics Institute,and Professor N.Stern,Department of Economics,as supervisors.(During the first 12 months Professor A.P.Kirman and Professor A.Dixit supervised me in economics).

The thesis consists of an introduction,four chapters and a conclusion.Chapter I,which has been circulated as a Warwick Economic Research Paper,no.128,is joint work with Kirman.Chapter II, presented at the Econometric Society's Winter Meeting,1979, will be published in the Journal of Mathematical Economics in a slightly different version.Finally Chapter IV,submitted for the European Meeting of the Econometric Society,1979,is joint work with Zeeman.

I would like to draw the reader's attention to the following:

- references are given after each chapter
- the use of footnotes has been avoided
- the notation in Chapter I is different from the remainder of the thesis.

It is a great pleasure for me to express my deepest gratitude to E.C.Zeeman and N.Stern for their invaluable suggestions, discussions,and encouragement during my research.

It is also a pleasure for me to thank A.P.Kirman and A. Dixit for many valuable communications.

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SUMMARY.

The thesis offers various contributions to the formulation of dynamical disequilibrium models in economic theory.

Until now static disequilibrium situations have usually been analyzed by assuming prices fixed so that equilibrium is obtained only by quantity adjustments.We consider in this work dynamical models,where the evolution of a rationed equilibrium economy through time is described by varations in both prices and quantities.

Using Malinvaud's static macromodel as the starting point, we introduce in Chapter I a simple dynamization by requiring quantities to adjust infinitely fast to the equilibrium values. This implies that the evolution of the economy is determined by the evolution in prices.Explicit differential equations for the changes in prices are formulated,and a global stability theorem for the Walrasian (no rationing at all) equilibrium is obtained by assuming a specific behaviour on the boundary between the regions of Repressed inflation and Keynesian equilibrium.

In Chapter II the model is changed by supposing quantities to adjust quickly,though not instantaneously,compared with prices. Assuming the differential equation describing the fast adjustment in quantities to depend smoothly on prices,we show that equilibria exhibit "exchange of stability" whenever the prices cross the boundary between two equilibrium regions in the parameter plane.Using methods from catastrophe theory a generic description of the long term evolution is obtained by introducing new "dual" equilibria in the quantity space and increasing the dimension of the parameter space by introducing parameters,which are usually "hidden" in the standard description.

Chapter III is devoted to an extension of this result to a situation,where consumers may gain by saving money from one period to the next,a case excluded in Chapter II by an assumption on the consumer's money holdings being unchanged from period to period.The result is obtained by dividing the consumers into unemployed,spending the same fixed amount of unemployment benefit every period,and employed consumers,who are allowed to save. The results of changes in either the unemployment benefit or the government's demand are also considered.

Finally Chapter IV is devoted to a specific analysis of the long term evolution in the neighbourhood of the Repressed inflation - Keynesian boundary.This analysis is necessary because of a discontinuity of the vector field describing the adjustments in prices on this boundary.Introducing timelags in agents' response to changes in prices we show how the discontinuous vector field gives rise to a smooth evolution around this boundary. Further,depending on the distribution of timelags,the economy exhibits either decreasing oscillations around the boundary eventually converging to zero,or (semi-)stable steady oscillations between the two regions.In this way the results contribute to a clarification of the claim that a basic feature of modern economies is the cycling motion between Repressed inflation and Keynesian equilibria.

INTRODUCTION.

As an introduction to the thesis we shall here first describe the general area of our work. We shall next give a short review of the existing literature concerning the subject of the thesis. This is followed by a more detailed description of the content of the thesis, where we also indicate how our work and results fit into the already existing literature. Finally the introduction gives a description of the way in which the thesis is organized.

The subject of the thesis is the analysis of dynamical disequilibrium models. A basic feature of much disequilibrium theory is that of sticky prices: whereas equilibrium in "classical" equilibrium theory is supposed to be obtained by instantaneous adjustment in prices making the excess demand in every market equal to zero, the idea in the disequilibrium models that we shall examine, is that, in the short run, quantities may respond faster to discrepancies between demands and supplies than prices. This viewpoint can be introduced into economic theory in a simple way by considering a static one period equilibrium model with fixed prices, where equilibrium is obtained only by quantity rationing. When imposing a dynamic description on the simple static disequilibrium model, there are two main points of concern: one of these is the short term dynamics described by the adjustment processes in the quantities for fixed prices. The other is the long term dynamics describing the changes in prices, which eventually will occur due to excess demands in the different markets. A natural approach to obtain a dynamic description of the long term evolution of a disequilibrium economy is therefore to model the evolution of the economy as the result of an interaction between a relatively "fast" adjustment process in the quantities and a relatively "slow" adjustment process in the prices.

The basic model we use is the simple static macroeconomic model introduced by Malinvaud, [1977]. In this model the economy consists of finitely many identical consumers (consuming one "good" and supplying "labour"), a production sector and a government sector, whose behaviour is exogenous. Depending on the rationing of consumers/producers on the good/labour market three types of equilibria are defined. These are named Keynesian, Classical and Repressed inflation equilibrium. Corresponding to these the price-wage plane splits into three regions of equilibria with only the Walrasian equilibrium, where there is no rationing at all, common to all three. It would appear that the combination of the two dichotomies (consumer/producer and good/labour) might produce 4 regions. The fourth is ruled out by assumption - see below.

As we are concerned with the long term evolution of the economy, we shall specifically look at the evolution in prices and wages. As they change through time, the nature of the rationing conditions on consumers and producers changes, and this happens exactly as the path traced out by prices and wages crosses a boundary between two of the equilibrium regions. As long as prices and wages stay in the interior of one of the equilibrium regions the long term description of the evolution of the economy is fairly straightforward. However, when prices and wages are in a neighbourhood of one of the boundaries, the description becomes rather complex, which is due to the abrupt change in consumers/producers rationing conditions, when a boundary is crossed.

By introducing into economic theory the concept of "exchange of stability", however, we obtain a complete description of the long term evolution around the Walrasian equilibrium, without restrictions on prices' and wages' movements between different equilibrium regions. The "exchange of stability" concept has been used extensively in physics in connections with elastic stability, see e.g. Thompson & Hunt, [1973].

The results in the first part of the thesis are obtained by introducing some rather restrictive assumptions on the behaviour of the single consumer. Later in the thesis we show, how these assumptions can be relaxed without destroying the results on the description of the long term evolution of the economy.

The fundamental principles in a general equilibrium model are clearly exposed in the basic static Arrow-Debreu model (see Debreu, [1959]). Here markets are supposed to exist for all present and future commodities, and all contracts are made once and for all at the very beginning of the functioning of the economy. An equilibrium is defined as a set of prices for which the excess demand on every market is zero. The question of existence is examined but not how or if an equilibrium is established. Comparative static results may be obtained by assuming prices to respond instantaneously to changes in the economy in order to maintain equality between demand and supply on each market. This Arrow-Debreu model presents a very general description of an economy, which of course contains several idealizations compared to the real world. On the other hand its generality makes a number of interpretations possible. For a recent exposition of general equilibrium theory, where stability is also discussed - see Arrow & Hahn, [1971].

Recent discussion on temporary competitive equilibrium analysis (surveyed by Grandmont, [1977]) has taken this model as the starting point and then developed models without the strong assumption of the existence of all future markets. This has led to a series of models all assuming the existence of sequences of spot markets, where agents at each point in time engage in contracts, having formed expectations about the future based on past and present observations of the state of the economy. Assuming also in these models prices to react instantaneously to discrepancies between demand and supply at each point in time,one obtains descriptions of the evolution of an economy through time as sequences of short term equilibria.

At the same time as the development of the general equilibrium theory took place, quite a different area of the discussion of economic theory was occupied with that part of macroeconomics which was aimed at discussing the Keynesian viewpoint of possible persistent unemployment. Works like Clower, [1965], Hicks, [1965], Leijonhufvud, [1965], and Barro & Grossmann, [1971], are basic contributions to a more formal presentation of Keynes' theory as opposed to the already well established "classical" theory.

It was soon obvious, however, that this analysis of situations with unemployment could be discussed more clearly in the framework of the newly developed formal models of general equilibrium theory. By choosing the single agent's behaviour as the basis for the consideration of situations with "disequilibrium", a new line of research, usually called "disequilibrium theory" was initiated. In general equilibrium theory situations with excess demand/supply are never discussed, because the implicit assumption on instantaneous adjustments in prices makes them irrelevant. The new approach incorporates as a basic assumption the observation that in the short run quantities may be the more flexible, by supposing prices and wages fixed. The equilibrium concept is now defined such that quantity rationing is effective on those markets, where demand differs from supply at the prevailing prices. Even though therefore different, the new approach should still be considered as part of general equilibrium theory with the new types of equilibria, named rationed equilibria, being defined by an implicit assumption on instantaneous adjustment in quantities instead of prices. Slightly different approaches with varying assumptions on the way that the quantity rationing enters into the description are Glustoff, [1968], Benassy, [1975], Drèze, [1975], and Younès, [1975]. The main idea in these models may be sketched as follows.

The economy consists of N consumers and ℓ goods with prices $p = (p_1, \dots p_\ell)$, money being good no. 1. Each of the $\ell - 1$ goods other than money is traded on a separate market against money. The i'th agents total actual transaction z^i (an element in the agent's feasible set of transactions) satisfies the budget constraint $p \cdot z^i = 0$. Besides the price system the agents are given upper and lower limits on their actual transactions. These quantity constraints enter the description of the agent's decision making process in varying ways in the different models, but here we shall confine ourselves to the representative Drèze formulation, where each agent forms simultaneously effective (i.e. constrained) demands for each good (other than money), given the rationing on all markets. An equilibrium with rationing is then defined as a set of prices and quantity constraints such that the sum of net trades are zero and for each good only traders on one side of the market for that good perceive binding constraints. This definition of the rationing does not determine how the actual rationing among agents takes place. In order to get a more specific theory, a rationing scheme must be chosen. A typical choice, which is consistent with Drèze's general specification, is a uniform rationing, where each agent obtains the same amount of a rationed good.

The development of these general models of rationed equilibria meant that the micro foundations on which to construct a macro theory now existed. This was used by Barro & Grossman, [1976], Grandmont & Laroque, [1976], Malinvaud, [1977], and Hildenbrand & Hildenbrand, [1976], (a revised version Hildenbrand & Hildenbrand, [1978], has just been published), which became main contributions in this field. As Malinvaud's book forms the starting point for our work, we shall here give a short introduction to his model.

The model is a one period static equilibrium model with rationing. The economy consists of N consumers, all described by the same preference characteristics, a production sector and a government sector with exogenous behaviour. There are three goods in the economy, a consumption good, labour and money. The price of the good (labour) is denoted p (w), the price of money is one. At the beginning of the period p and w are given, and the consumers, each endowed with the initial money holdings m_o, determine how much they want to consume (x), how much they want to supply of labour (ℓ), and how much money (m) to carry forward to the next period, by maximization of the utility function $U(x,\ell,m)$, subject to a budget constraint. The result of this maximization gives the notional demands. The production sector is given by an aggregated production function $y = F(\ell)$, $F' > 0$, $F'' < 0$. As prices and wages are fixed, however, notional supply and demand on the two markets may differ. If so Malinvaud assumes that the short side of the market determines the realized transactions. Given the rationing, the notional demand/supply functions are substituted by effective demand/supply functions, which are obtained by utility/profit maximization subject to the rationing constraints. (p, w) is called a rationed equilibrium, if actual production (realized by the actual employed consumers) is equal to actual demand. Depending on the combinations of rationing of consumers and producers, three types of rationed equilibria are defined. ("Producers rationed on both markets" is ruled out by assumption. It is not plausible to suppose that at (p, w) producers are unable to sell their desired output whilst at the same time wishing they could buy more labour.)

The meaning of the expression that "some agents are rationed on a market" is that their notional demand/supply is larger than the actual supply/demand. Notice that it is an assumption that both consumers and producers are never rationed at the same market at the same time. Corresponding to the three types of rationed equilibria the (p, w) plane is divided into three regions:

1) The Keynesian equilibrium region, where consumers are rationed on the labour market, and producers are rationed on the good market.
2) The Classical equilibrium region, where consumers are rationed on both markets.
3) The Repressed inflation equilibrium region, where consumers are rationed on the good market, and producers are rationed on the labour market.

In this framework Malinvaud discusses varying comparative static results and their implications for practical economic policy. The general validity of these results is limited, as pointed out in Hildenbrand & Hildenbrand, [1976], because Malinvaud's results all depend heavily on the implicit assumptions on the consumers' expectations on the future: if every consumer has positive expectations on employment (i.e. he expects to be employed next period) or negative expectations on employment, Malinvaud's results are valid, provided $\partial\varphi/\partial p < 0$, $\partial\hat{\varphi}/\partial p < 0$ and $\partial\ell/\partial p$ is not "too negative". φ is notional demand, $\hat{\varphi}$ is effective demand and ℓ is notional supply of labour. However, if employed consumers expect to be employed next period and unemployed consumers expect to remain unemployed the comparative static results in Malinvaud's book are not generally valid.

Malinvaud does not introduce a dynamic description directly into this model, but he offers quite a few useful general comments on the way it eventually ought to be done.

In order to obtain comparative static results, just as for the general equilibrium models, one must first make sure that there is uniqueness of the rationed equilibrium, given prices and wages.

In the simple Malinvaud model this follows immediately. In more general models, however, this is not the case. (In the Drèze model, for example until a specific rationing scheme has been chosen, there are infinitely many equilibria). In an important series of papers Laroque has addressed himself to a discussion, in the framework of the Drèze disequilibrium model, of the set of rationed equilibria, analogous to the existing results in the general Arrow-Debreu model. Laroque [1978 I and II], obtains necessary and sufficient conditions on the local uniqueness of equilibria in a neighbourhood of the Walrasian equilibrium. Further he shows that the fixed price equilibrium locally is a continuous function of prices. Laroque and Polemarchakis, [1978], show that for an open subset of prices and initial endowments in PxX^m, which is of full measure (i.e. Lebesgue measure one), the set of fixed price equilibria is finite and varies continuously with variations in prices and initial endowments, as long as prices do not move from one equilibrium region to another. Here $P = \{p \in \mathbb{R}^{\ell+1} | p_o = 1, p_h > o, h = 1, \ldots, \ell\}, X = \mathbb{R}^{\ell+1}_{++}$ and m is the number of consumers.

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In this series of papers Laroque is also concerned with possible ways of introducing dynamics into the static fixed price theory. In Laroque, [1978, I], an adjustment process in the binding quantity constraints is formulated and sufficient conditions for local asymptotic stability of this process are stated. In Laroque,[1978, II], an economy with two agents and two commodities is considered. It is shown for states in a small neighbourhood of the Walrasian equilibrium that if the economy is at a fixed price allocation, then an increase in the price of one commodity will be to the advantage (i.e. increases the utility) of the "seller" of this commodity and to the disadvantage of the "buyer" of the commodity. Laroque therefore suggests that it might be reasonable more generally to model the long term evolution in prices as a struggle between "buyers" and "sellers" in each market (see also Laroque, [1978, III].)

There is also a recent paper by Hahn, [1978], which should be mentioned in relation to Laroque's work. Hahn also works in the framework suggested by Drèze. He is particularly concerned with showing the possibility that "an economy can settle into a quantity constrained equilibrium, even when prices are not fixed a priori and in particular a Walrasian equilibrium is available". (p. 2)

As mentioned above Laroque's papers contain important insights into the dynamic description of rationed equilibrium models. Using some type of non-tâtonnement process the main contributions, following a more "classical" approach in this area, have been particularily concerned with the modelling of the long term evolution of prices, supposing quantity adjustment to be instantaneous. Grossman, [1971], generalizing Clower's dual decision hypothesis to a multi-market situation, points out that changes in prices should be described as guided not by notional, but by effective excess demands. Using this Veendorp, [1975], shows in a two sector model that the "spill over" effect from one market to another may have a stabilizing effect on the price adjustment process guided by effective excess demands.

Another approach to a dynamic description has been given by Böhm, [1978], and Dehez & Gabszewicz, [1977]. They consider simple macro models over time, where prices and wages are fixed such that changes in money holdings constitute the dynamic link between periods. Böhm assumes constant government demand and shows that the long term evolution will lead to states of either unstable Repressed inflation or stable Keynesian equilibrium. Dehz & Gabszewicz introduce the government as a specific economic agent. They show that by allowing this agent to operate some policy measures (i.e. increasing/decreasing exogenous demand) in an appropiate way, the sequence of disequilibrium states will converge to a stationary market equilibrium - a situation, where all agents repeat the same decision over time.

Finally Honkapohja, [1977], combining several approches considers the long term evolution of an economy over time, where consumers' money holdings and prices vary. As the Walrasian equilibrium only by exceptional circumstances is a stationary point for the dynamical system, Honkapohja instead considers the stability of quasi equilibria, defind as steady states, where the real balances of money and the real wage rage remain constant over time.

Having briefly reviewed the existing literature we shall next state, how we intend to organize our analysis in this work. We begin our analysis by considering the static Malinvaud model and its generalization by Hildenbrand & Hildenbrand. Into this framework we introduce a long term dynamic description of the changes in prices and wages. Supposing instantaneous quantity adjustments for fixed prices and wages, a description of the long term evolution of the economy follows, and the stability of the Walrasian equilibrium can be discussed.

By combining infinitely fast adjustments in the quantities with a description of the evolution of prices and wages governed by excess demands on the separate markets, we obtain a model analogous to the classical tâtonnement processes. (In these models it is the quantities which remain fixed until prices have adjusted to the equilibrium values, where all markets are cleared). The typical feature of sticky prices in the short run is expressed in a rather crude way by supposing instantaneous adjustments in the quantities for fixed prices. We therefore next try to relax this assumption by expressing the evolution as the result of an interaction between a relatively fast adjustment process in the quantities and a relatively slow adjustment process in prices and wages. The basic description of the economy is still given by the Malinvaud model, but by making a few minor changes in the formulation we obtain a characterization of his three types of rationed equilibria, which will prove useful for our dynamic analysis. This forms the starting point for the analysis. First in a simplified example, later more generally, we develop our dynamic disequilibrium model, using the concept of "exchange of stability".

Whereas an extensive analysis of the variation in rationed equilibria with prices and wages varying inside one of the equilibrium regions has been given by Laroque & Polemarchakis, [1978], our approach leads to a discussion of the evolution of the economy, when prices and wages cross the boundary between two different equilibrium regions.

In order to obtain our results we introduce a simplifying assumption in the description of the single consumer: in each period the consumer faces the following decision problem. He must decide how much to consume, how much to work and how much money to carry forward to the next period, given prices, wages and his initial money holdings. It is assumed that the consumer has the same fixed amount of money at the beginning of each period. This is unappealing, and we shall therefore discuss what happens, when we try to relax this assumption. We do this in the following chapter by analyzing a situation, where the unemployed consumers are endowed with a fixed amount of money at the beginning of each period, whereas the employed consumers may vary their money holdings by saving. This analysis leads to another problem, concerning the influence of changes in the size of this fixed "unemployment benefit" on the evolution of the economy. Another problem along the same line is the impact of changes in government's exogenous demand. Both these problems will be dealt with in this chapter.

Finally we shall specifically have to consider the evolution of the economy around the Repressed inflation - Keynesian boundary. As the vector field defining the changes in prices and wages is discontinuous on this boundary, the evolution is not well defined here. We shall analyze the result of introducing an assumption of inertia in the agents' decision making process into the model.

There is still one main point left to be mentioned. As soon as one introduces a description of the changes in prices and wages over time, one encounters a problem, which has already been discussed at length in relation to the Arrow-Debreu model (see e.g. Fisher, [1970], Diamond, [1971], and Fisher, [1972]): who are the economic agents that make these price changes? The classical interpretation relies on a fictitious auctioneer, who makes the changes after having computed the discrepancies between demand and supply. Even though this is not a satisfying description, we keep to this type of interpretation here, as there still exists no completely satisfying formulation of an economy, where actual price changes are the result of decisions taken by some of the agents in the economy. Recent work of Fisher, [1978], and Laroque, [1978, II], could perhaps in the future be used as the basis for a more reasonable price adjustment mechanism.

The plan of the thesis is as follows. Chapter I (joint work with Kirman) provides a discussion of the basic long term adjustment process in prices and wages. In Chapter II we introduce the relative fast adjustment process in quantities, and present a long term description by using the concept of "exchange of stability". In an appendix to this chapter we discuss the uniqueness of the rationed equilibrium for (p, w) belonging to each of the three equilibrium regions. Chapter III contains the extension of the analysis to the case where employed consumers may benefit from savings. We also discuss variations in unemployment benefits and government's demand. Finally in Chapter IV (joint work with Zeeman) we present an analysis of the evolution around the Repressed inflation - Keynesian boundary. In the final chapter we summarize the results we have obtained.

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Chapter 1

THE LONG RUN EVOLUTION OF A RATIONED EQUILIBRIUM MODEL

Introduction

In his recent book "The Theory of Unemployment Reconsidered" Malinvaud, [1977], presents a model in which prices and wages are fixed and equilibria are achieved by quantity adjustments in the goods and labour markets. These adjustments are effected by rationing schemes. This model, although it provides a framework for macroeconomic analysis, has the drawback that it is wholly static. The notion that prices and wages are completely rigid in the short run is acceptable as an idealisation of the real situation. However, in the longer term market forces must influence prices and wages, and the latter will respond to excess demands and supplies in different markets. Our aim is to specify such an adjustment process, which will be slow relative to the quantity adjustments in each market, and then to examine the evolution and stability of prices and wages. Although it is possible with suitable assumptions to obtain stability results, the framework inherited from Malinvaud and developed by Hildenbrand & Hildenbrand, [1976], is not a wholly satisfactory one in which to study the dynamics of prices and wages. In particular the simple imposition of a plausible adjustment process on the Malinvaud model does not produce the sort of phenomena one might naturally expect to observe.

This raises some interesting, and we think, fundamental problems, which will not be described here, but will be the subject of the following chapters.

The model.

The model we shall use in this paper is a "generalized" Malinvaud model presented by K. Hildenbrand and W. Hildenbrand in their discussion paper "On Malinvaud's 'Reconsideration of the Theory of Unemployment'". We present the model here, and for a more general discussion the reader is referred to that paper. For comparison we follow their notation, although for the analysis from Chapter II onwards alternative notation will be more convenient.

There are three goods in the economy: a consumption good, labour, and money. The price of the good (labour) is denoted by p, (w), while the price of money is 1. There are N consumers. Consumer α is endowed with some initial moneyholdings e_α. The consumer must then at each point in time decide, how much he wants to supply of work (ℓ_α), how much to consume (q_α), and how much money (m_α) to carry forward to the next period. This decision is derived from a multiperiod utility maximization (given the budget constraint), which is based on the consumer's expectations on quantity rationing and prices on the two markets. We shall here adopt the Hildenbrand notation:

Let (p, w) be given. If the consumer expects no rationing at all in the current or future periods, his decisions (i.e. his notional demand (supply) functions) are denoted $q^*_\alpha(p, w)$, $\ell^*_\alpha(p, w)$, and $m^*_\alpha(p, w)$.

If the consumer expects no rationing on the good market, and he is in the current period unrationed on the labour market, we denote his demand (supply) functions by

$$q_\alpha(p, w),\ \ell_\alpha(p, w),\ \text{and } m_\alpha(p, w),$$

given his employment expectations.

If he is rationed on the labour market in the current period, which means that he is completely unemployed, the functions are denoted

$$\bar{q}_\alpha(p, w),\ \bar{\ell}_\alpha(p, w),\ \text{and } \bar{m}_\alpha(p, w).$$

Rationing on the good market is assumed to be described by an upper bound x on the amount of the good available to the consumer. So corresponding to the two cases above, if the consumer is rationed on the good market in the current period (i.e. $x < +\infty$), we get the functions

$$q_\alpha(p, w, x),\ \ell_\alpha(p, w, x)\ \text{and } m_\alpha(p, w, x)$$

and

$$\bar{q}_\alpha(p, w, x),\ \bar{\ell}_\alpha(p, w, x)\ \text{and } \bar{m}_\alpha(p, w, x).$$

The production side of the economy is described by an aggregated production function F, relating labour input (z) to output of good. The assumptions on F are:

$$F(0) = 0,\ F'(z) > 0,\ F''(z) < 0.$$

Let $y^*(p, w)$ denote the profit maximising output, given the price and wage rate. Finally we denote the exogenous demand in the economy by g. So g corresponds to government expenses on good and services plus business investments, which here are treated autonomous.

As in Malinvaud we shall ignore the effects of the distribution of profits all through our analysis, i.e. we assume demand to be independent of profits. Whereas this is a reasonable simplification in Malinvaud's one-period static model, the assumption is unacceptable in a dynamic analysis of the long term evolution of the economy. However, when we consider the evolution through time, we are not thinking of years, and we are not aiming at presenting a growth path of the economy. We are definitely concerned with shorter periods, e.g. months, where it still is plausible, at least as a first approximation to assume profits to be without influence on the demand in these periods.

Equilibrium concepts

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new equilibrium concept, which specifically takes into account the possibility of rationing on one or both markets. Three types of rationed equilibria are considered here:

A. The Keynesian equilibrium, where consumers are rationed on the labour market, and producers are rationed on the good market.

B. The Classical equilibrium, where consumers are rationed on both markets.

C. The Repressed inflation equilibrium, where consumers are rationed on the good market, and producers are rationed on the labour market.

(For precise definitions see below.)

By u we denote the unemployment rate, i.e. (1-u) is the ratio of actual employment to effective labour supply.

The rationing scheme of the labour market is assumed to be such that a consumer is either fully employed or totally unemployed. By allowing consumers to differ in characteristics, we shall have to assume that unemployment is uniformly distributed over that part of the population, who wants to work. For N large (i.e. many consumers of each type of consumption characteristics) the mean effective labour supply is then independent of which particular individuals, who are rationed.

Let μ denote the distribution of consumer characteristics. Corresponding to the demand (supply) functions defined above, we define the mean demand (supply) functions by

$$\bar{q} = \int_N \bar{q}_\alpha d\mu, \; q = \int_N q_\alpha d\mu, \quad \ell = \int_N \ell_\alpha d\mu, \text{ and } \ell^* = \int_N \ell^*_\alpha d\mu \; .$$

We then have the following definition.

Definition:

Let the price level p, the wage rate w, and the exogenous demand g be given.

A. (p, w, g, y, u, x), $x = \infty$ (i.e. consumers not rationed on goods market) is called a <u>Keynesian</u> equilibrium, if

1° $\quad y = g + u\bar{q}(p, w) + (1-u)q(p, w)$

2° $\quad F^{-1}(y) = (1-u)\ell(p, w)$

where y is production and u the corresponding unemployment rate.

B. (p, w, g, y, u, x), $y = y^*$ is called a <u>Classical</u> equilibrium, if

3° $\quad y^*(p, w) = g + u\bar{q}(p, w, x) + (1-u)q(p, w, x)$

4° $\quad F^{-1}(y^*(p, w)) = (1-u)\ell(p, w, x)$

where x is the rationing level and u the unemployment rate.

C. (p, w, g, y, u, x), u = 0 is called a <u>Repressed inflation</u> equilibrium, if

5^o $y = g + q(p, w, x)$

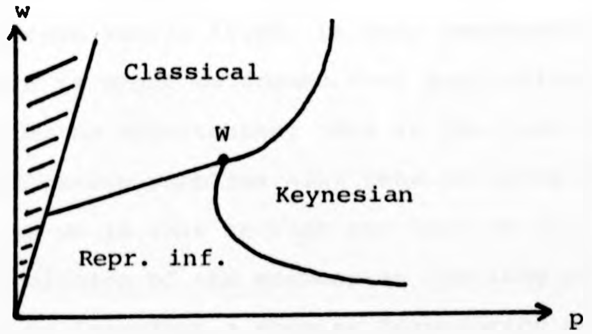
6^o $F^{-1}(y) = \ell(p, w, x)$

where y is production and x the rationing level.

<u>A long run adjustment process.</u>

For fixed g and μ we get the (p, w) plane divided into regions of either Keynesian, Classical, or Repressed inflation equilibrium. The following picture is presented in the Hildenbrand paper. (W denotes the Walrasian equilibrium).

<u>Fig. 1.</u>



The Classical-Keynesian boundary is determined as the set of (p, w) for which $y = y^*$ and $u \geq o$.

The Repressed inflation-Keynesian boundary is determined as the set of (p, w) for which $y^* \geq y$ and u = o. The Classical-Repressed inflation boundary is determined as the set of (p, w) for which $x \geq o$ and u=0. Here we have assumed that a proportional increase in p and w leads to a decrease in the demand q. (This assumption is always fulfilled in Malinvaud's model.)

Fixing (p, w, g), adjustments in only (y, u, x) are possible. In this section we assume that adjustments in (y, u, x) do take place. In fact we shall assume that these adjustments are infinitely fast, i.e. we can concentrate attention on the resulting equilibrium values. So for given (p, w) the state of the economy will be an equilibrium point of one of the three types mentioned above, depending on the region to which (p, w) belongs. This sort of assumption is frequently made in economics. As we have said, the assumption made in all the literature on rationed equilibrium theory that prices and wage rate remain fixed, is only reasonable in the short run, where it might be argued that quantities are more likely to perform the adjustments. But in the long run excess demands and excess supplies will tend to force changes in p and w. As we in this section are looking for a description of the evolution of the economy in the long run, we shall now have to introduce a precise formulation of the variation in prices and wages through time.

Before we proceed, a remark on μ is necessary. We have assumed the distribution of consumer characteristics to be fixed, which is an acceptable idealisation, except for the consumers' moneyholdings. As soon as the consumers engage in trades, the distribution of money will change. One can interpret the fixed - μ assumption as arising from the following taxation system: at the end of each period the government redistributes the money such that the original money distribution is obtained. Of course this is not a satisfactory description, and the fixed - μ assumption will be relaxed in Chapter III.

As the economic conditions of the agents vary with the regions, it is reasonable to expect different descriptions of the adjustments in the three regions. It should be stressed again that these adjustment processes are considered to be slow compared to the above adjustments in the quantities. First we state the three processes, and then we discuss their contents and plausibility.

Keynesian region:

$$\dot{p} = K_p((1-u)q(p,w) + u\bar{q}(p,w) + g - y^*(p,w)) \qquad K_p \in \mathbb{R}_+$$

$$\dot{w} = K_w(F^{-1}(y(p,w)) - \ell^*(p,w)) \qquad K_w \in \mathbb{R}_+$$

Classical region:

$$\dot{p} = C_p((1-u)\tilde{q}(p,w) + u\tilde{\bar{q}}(p,w) + g - y(p,w)) \qquad C_p \in \mathbb{R}_+$$

$$\dot{w} = C_w(F^{-1}(y(p,w)) - \ell^*(p,w)) \qquad C_w \in \mathbb{R}_+$$

Repressed inflation region:

$$\dot{p} = R_p(\tilde{q}(p,w) + g - y(p,w)) \qquad R_p \in \mathbb{R}_+$$

$$\dot{w} = R_w(F^{-1}(y^*(p,w)) - \ell(p,w,x)) \qquad R_w \in \mathbb{R}_+$$

where $\tilde{q}$ ($\tilde{\bar{q}}$) denotes the mean demand function from the employed (unemployed) consumers, which will result, when the consumers ignore the actual rationing on the good market.

The first thing to note is that we consider the adjustment processes to take place in continuous time, so they may be formulated by differential equations. As p and w vary, so do y and u as described by the equilibrium equations in the section above. Let us first consider the process in the Keynesian region. $\dot{p}$ is defined to depend on the difference between the total actual demand from consumers plus government and the producers' desired production, while $\dot{w}$ depends on the difference between actual employment and the desired labour supply of the consumers. These equations express the following viewpoint:
Given p and w, the producers want to sell $y^*(p,w)$. When the effective demand is less than y^*, there will be pressure for price decreases according to an excess supply of the good. On the other hand, the wage rate moves with the excess demand on the labour market.

Next we look at the process in the Classical region. In this case $\dot{p}$ depends on the difference between the demand the consumers would have, if they ignored any rationing on the good market, and the actual production (which is equal to desired production, as the producers are not rationed at all). The $\dot{w}$ process is as in the Keynesian region. The idea behind this formulation of the $\dot{p}$ process is as follows: when consumers are rationed on the good market, it implies that total demand cannot be fulfilled. Even though consumers, due to this fact, have to take this restriction into account when making their plans, it is the unrestricted demand which in the long run makes the pressure on the market.

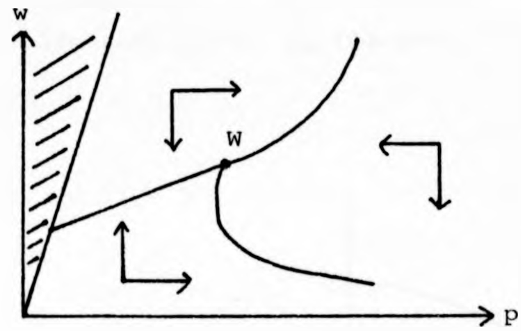
Finally, consider the process in the Repressed inflation region. The $\dot{p}$ process is as in the Classical region with u = 0, as the consumers are unrationed on the labour market, but still rationed on the good market. The $\dot{w}$ process depends on the difference between the desired employment by producers and the actual supply of labour by the consumers. This formulation reflects the viewpoint that when producers are rationed in the labour market that is to say, when they are unable to produce the desired output, there will be a tendency for wages to rise because sufficient labour is unavailable.

<u>Long run stability.</u>

In this section we shall discuss the kind of stability result which may be obtained from the above adjustment process.

It follows immediately from the definition of the three types of disequilibrium that the sign of the changes $(\dot{p},\dot{w})$ can be represented in the following diagram:

Fig. 2.



(This type of diagram is independently presented in Benassy, [1978]).

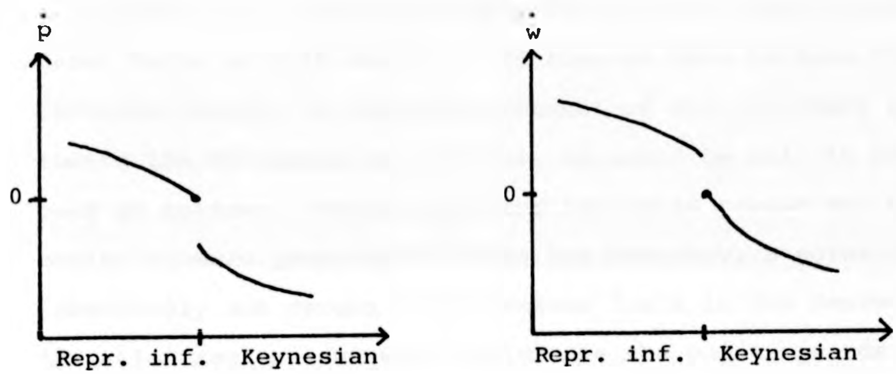
We shall now have to discuss what happens on the boundaries between the three regions:

On the boundary between the Keynesian and the Classical region y is equal to y*, and $\tilde{q} = q$ $(\tilde{\tilde{q}} = \bar{q})$, as the rationing on the good market disappears on this boundary. So this implies that the direction of the vector field describing the adjustment process is continuous on this boundary, although its magnitude may be discontinuous.

Similary, as y = y* and $\ell = \ell^*$ on the boundary between the Classical and the Repressed inflation region, the direction of the vector field describing the adjustment process is continuous on this boundary.

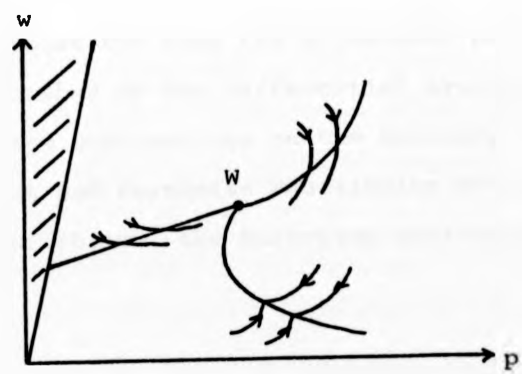
On the other hand we have the following picture for the boundary between Repressed inflation and Keynesian equilibrium, which is associated with the "fourth" region being suppressed. (Behaviour on this boundary is discussed in detail by Malinvaud, [1977] pp.103-104).

Fig. 3.



Altogether this leaves us with the following qualitative picture:

Fig. 4.



We shall therefore have to define the adjustment process on this boundary. On the boundary the consumers are not rationed at all, while producers are rationed on both markets, as they produce less than they want to. If we look at a sequence of points on this boundary, converging to the Walrasian equilibrium, we notice that consumers stay unrationed all the time, while the producers get "closer" to the desired production, i.e. the gap between desired level and actual level falls on both markets. If then we were to make the inviting assumption that the movement on this boundary was toward the Walrasian equilibrium, we would be able to proceed as follows. Moving from the Keynesian region and supposing it were possible to cross the boundary, a point would immediately get caught by the vector field in the Repressed inflation region, and this would take the point upwards towards the boundary. An analogue argument is available, when crossing the boundary from the Repressed inflation region. So we would have the following result:

Theorem

Suppose the long run adjustment processes in p and w are described by the differential equations above. If the price-wage combinations on the boundary between Repressed inflation and Keynesian equilibrium move towards the Walrasian equilibrium, the Walrasian equilibrium is globally stable.

So the movement is as follows: points in the Classical region will either move directly towards the Walrasian equilibrium or they will cross the boundary to the Repressed inflation region or the Keynesian region. Points in these two regions will either converge directly towards the Walrasian equilibrium, or they will hit their common boundary and then, remaining on this boundary, move towards W.

However, we are then led to ask, is the qualitative picture obtained by introduction of this assumption in conformity with what one might expect the evolution of the model to be? The answer must, however, be only a qualified yes. An obvious and important feature that would be lacking in the model would be cycles from the Repressed inflation to the Keynesian region, indeed it has been argued that this is the basic feature of the evolution of modern economies. One possibility would be to suggest that such cycles are generated by changes in autonomous or exogenous variables. Changes in g, for example, move the boundaries between the regions and would produce such cycles. This would, however, suggest that the economy left to its own devices would not exhibit such cycles and would move to the Walrasian equilibrium.

Although the reader will draw his own conclusions, we would suggest that the assumption on the behaviour of price and wage changes of the Keynesian - Repressed inflation boundary is one of convenience and avoids the real question raised by the apparent discontinuity on this line. We would further suggest that far from being of a technical nature, the difficulties presented in defining suitable adjustment processes on either side of this boundary suggest that the model as it stands, is not a satisfactory vehicle for a true dynamic analysis and needs fundamentally reformulating for this purpose.

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CHAPTER II

EXCHANGE OF STABILITY IN A DISEQUILIBRIUM MODEL

Introduction.

Recent discussions on rationed equilibrium economies have focused attention on the short run viewpoint, where prices and wages are supposed fixed such that equilibrium is obtained only by adjustments in the quantities. However, in the long run this fixed price idealization is not acceptable, because one would suppose that prices and wages react to excess demand and supply on the different markets.

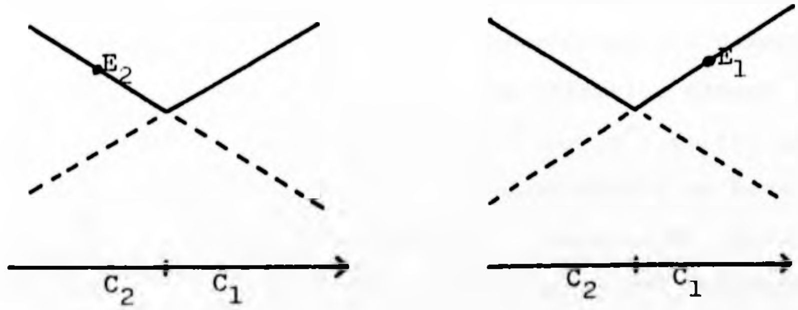
As an introduction to such a description of the long term evolution of a rationed equilibrium economy, we discussed in Chapter I some of the problems which appear, when a long run adjustment process in prices and wages is imposed on the static disequilibrium model, introduced by Malinvaud, [1977]. Assuming that for given prices and wages the adjustments in the quantities are instantaneous, the long term evolution of the economy is completely determined by the changes in prices and wages.

In this chapter we shall propose a slightly different approach by explicitly taking the speed of adjustments in the quantities into account. We assume the quantity adjustments to be fast, though not instantaneous, relative to the adjustments in prices and wages. As long as the price - wage rate stays within one equilibrium region, the total description of the long term evolution remains simple. However, we shall see that the analysis of the crossing of a boundary between two equilibrium regions calls for the introduction of "new" equilibrium points in order to obtain a generic description.

As the analysis is based on the idea that the price and wage rate change through time, we consider the price - wage plane as parameter space C. For given values of price and wage, the state of the economy is described by a point in a three dimensional quantity space X, which we call the state space. (A reference to the basic concepts of dynamical systems is Hirsch & Smale, [1974]). The coordinates of X(x, y, z) will measure three quantities, excess demand of the good, the amount by which actual output falls short of desired output, and excess supply of labour. Applying the Malinvaud notation the description is formulated such that when the parameter c belongs to the Keynesian region C_1 of C, the rationing hypothesis of Keynesian equilibria implies that the corresponding Keynesian equilibrium E_1 lies in the plane $x = 0$. Similarily, when the parameter belongs to the Classical region C_2, the Classical equilibrium E_2 lies in the plane $y = 0$.

At first sight this description implies that if the parameter passes smoothly from C_2 to C_1, the path traced out in X by the equilibrium point would have to have a sharp corner, as it moved from the plane $y = 0$ to the plane $x = 0$. However, our model is based on the fundamental principle that the differential equation that describes the fast adjustment in X depends smoothly on C. By this we mean that if grad V_c, $V: X \times C \to \mathbb{R}$, determines the fast adjustment process in the quantities for fixed $c \in C$, then V is a smooth function of c. This implies that the equilibrium varies smoothly and hence cannot turn sharp corners. Therefore there must be an "exchange of stability" between equilibria: as the parameter moves from C_2 to C_1, the Classical equilibrium E_2 persists in its smooth path, but changes from being stable to unstable.

Fig. 1.



Conversely, on the return path E_1 persists in its smooth path, but changes from being stable to unstable. In other words, over the path traced out by the parameter c lie two equilibrium paths.

Exchange of stability is a well known phenomenon in structural mechanics (see e.g. J.M.T. Thompson and G.W. Hunt, [1973]). This paper studies the more complicated exchange of stability around the Walrasian equilibrium.

The model.

The basic model in the paper is the disequilibrium model, introduced by Malinvaud, [1977], and further developed by Hildenbrand and Hildenbrand, [1976]. In order to keep the presentation simple and yet sufficiently detailed to express our ideas, we shall here use a slightly different formulation. For both consumers and producers there are three different types of demand and supply concepts to consider in a rationed equilibrium economy. If there is no rationing at all, the notional demand and supply functions of consumers are denoted $\varphi(p,w)$ and $\ell(p,w)$. If consumers are rationed on one market, they will have to take this rationing into account, when expressing their wishes on the other market. Letting φ^a and ℓ^a be the planned demand and supply, we denote by $\hat{\varphi}(p,w,\ell^a)$ the effective demand on the good market, given ℓ^a, and we let $\hat{\ell}(p,w,\varphi^a)$ be the effective supply of labour, given φ^a. Correspondingly we have for the producers the following notation: $\psi(p,w)$ and $\nu(p,w)$ denote notional (i.e. profit maximising) output and employment of labour. $\hat{\psi}(p,w,\nu^a)$ and $\hat{\nu}(p,w,\psi^a)$ denote effective supply and demand given the rationing on the other market, where ψ^a and ν^a are the planned supply of the good and demand for labour. Again following Malinvaud we ignore any income which might accrue to individuals from the distribution of profits, and we assume that at each point in time moneyholdings are equally distributed. Both these assumptions are unsatisfactory and will be relaxed in later work (the latter in Chapter III below).

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In a general setting, allowing consumers' characteristics to vary, the above concepts are to be considered as the individual's demand and supply functions. In order to get the total demand and supply functions one would have to aggregate over the entire population, following the methods outlined by Hildenbrand and Hildenbrand, [1976]. Here we shall avoid such further complications by assuming all consumers (and producers) to be described by the same characteristics. We shall maintain the rationing schemes introduced by Malinvaud, and this implies specifically that all consumers, unrationed on the labour market work the same amount of hours, while all consumers, rationed in the labour market, are completely unemployed.

With this notation we have the following equations describing equilibrium:

$\psi^a = \varphi^a + g$ (equilibrium on the good market)

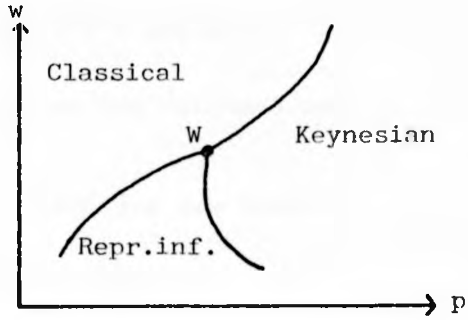
$\nu^a = l^a$ (equilibrium on the labour market)

$\psi^a = F(\nu^a)$ (actual production is feasible)

where g is exogenous (i.e. government) demand, and F is the production function. We shall in the following also consider disequilibrium situations, where these conditions are not satisfied.

Following Malinvaud, the (p,w) plane is partitioned into three regions:

Fig. 2.



named Keynesian, Classical and Repressed inflation, in each of which an equilibrium is defined in the well known way for a rationed equilibrium economy, see Malinvaud, [1977]. We shall here confine ourselves to a description of the three types of equilibria in the new notation.

Keynesian equilibrium:

Consumers, being rationed on the labour market, will take this rationing into account, when forming the demand on the good market, so $\hat{\varphi}(p,w,\ell^a)<\varphi(p,w)$. As they are not rationed on the good market, the effective demand will be realized i.e. $\varphi^a = \hat{\varphi}(p,w,\ell^a)$.

So

$\varphi^a = \hat{\varphi}(p,w,\ell^a)<\varphi(p,w)$.

Analogously

$\ell^a<\hat{\ell}(p,w,\varphi^a) = \ell(p,w)$,

as consumers are not rationed on the good market.

Producers are rationed on the good market, so

$v^a = \hat{v}(p,w,\psi^a)<v(p,w)$,

while

$$\psi^a < \hat{\psi}(p,w,\nu^a) = \psi(p,w),$$

as producers are not rationed on the labour market.

We now introduce new coordinates:

$$x = \hat{\varphi} + g - \psi^a$$

$$y = \psi(p,w) - \psi^a$$

$$z = \hat{\ell} - \nu^a.$$

Here the x-coordinate measures the excess demand of the good, y measures the extent to which notional output exceeds actual output, and z measures the excess supply of labour.

A Keynesian equilibrium is then represented in the product space C x X by a point of the form $(p,w,0,y,z)$ with $y>0, z>0$.

<u>Classical equilibrium:</u>

Consumers are rationed on both markets, so

$$\varphi^a < \hat{\varphi}(p,w,\ell^a) < \varphi(p,w)$$

$$\ell^a < \hat{\ell}(p,w,\varphi^a) < \ell(p,w)$$

while producers are not rationed at all, so

$$\psi^a = \hat{\psi}(p,w,\nu^a) = \psi(p,w)$$

$$\nu^a = \hat{\nu}(p,w,\psi^a) = \nu(p,w).$$

In the new coordinates a Classical equilibrium is represented in the product space C x X by a point of the form (p,w, x,0,z) with $x>0, z>0$.

Repressed inflation equilibrium :

Consumers are rationed on the good market,but unrationed on the labour market, so

$$\varphi^a < \hat{\varphi}(p,w,\ell^a) = \varphi(p,w)$$

$$\ell^a = \hat{\ell}(p,w,\varphi^a) < \ell(p,w).$$

Producers are rationed on the labour market, but unrationed on the good market, so

$$\psi^a = \hat{\psi}(p,w,\nu^a) < \psi(p,w)$$

$$\nu^a < \hat{\nu}(p,w,\psi^a) = \nu(p,w).$$

In the new coordinates a Repressed inflation equilibrium is represented in the product space C x X by a point of the form (p,w,x,y,0) with $x>0, y>0$.

Finally we notice that in this new notation the Walrasian equilibrium, where there is no rationing at all, is represented by a point of the form (p,w,0,0,0).

It is noticed that in an ordinary Walrasian model with three goods the equilibrium conditions are expressed by two equations, stating that supply equals demand on two of the markets. These equations determine the relative prices of the three goods. The notion of an equilibrium in a rationed while

$$\psi^a < \hat{\psi}(p,w,\nu^a) = \psi(p,w),$$

as producers are not rationed on the labour market.

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Consumers are rationed on the good market,but unrationed on the labour market, so

$$\varphi^a < \hat{\varphi}(p,w,\ell^a) = \varphi(p,w)$$

$$\ell^a = \hat{\ell}(p,w,\varphi^a) < \ell(p,w).$$

Producers are rationed on the labour market, but unrationed on the good market, so

$$\psi^a = \hat{\psi}(p,w,\nu^a) < \psi(p,w)$$

$$\nu^a < \hat{\nu}(p,w,\psi^a) = \nu(p,w).$$

In the new coordinates a Repressed inflation equilibrium is represented in the product space C x X by a point of the form (p,w,x,y,0) with $x>0, y>0$.

Finally we notice that in this new notation the Walrasian equilibrium, where there is no rationing at all, is represented by a point of the form (p,w,0,0,0).

It is noticed that in an ordinary Walrasian model with three goods the equilibrium conditions are expressed by two equations, stating that supply equals demand on two of the markets. These equations determine the relative prices of the three goods. The notion of an equilibrium in a rationed equilibrium model requires a characterization, which specifies the type of rationing that prevails. This is usually obtained by including the following three variables in the equilibrium conditions: the rationing level of consumers on the good market, the actual production level, and the unemployment rate (see Chapter I). By defining x,y,and z as above, however, we have in a consistent way allowed a simple characterization of the three types of rationed equilibria. This formulation avoids specifically the inconvenient properties of the usual characterization of rationed equilibria such as e.g. the variable, describing the consumers rationing level on the good market being equal to infinity in a Keynesian equilibrium. At the same time our formulation extends the equilibrium concept in a disequilibrium model by permitting the possibility of rationed equilibria with one or more of the coordinates x, y, and z being negative.

By introducing the coordinates (x,y,z) we get immediately a simple geometrical picture of the equilibrium states. The (p,w) plane is considered as parameter space C. Price and wage rate are supposed to remain fixed in the short run, while the quantities adjust relatively fast to an equilibrium state, determined by c = (p,w). The three dimensional (x,y,z) space X is denoted the state space, because for fixed c, the state of the economy is represented by a point in this space. In this setting the projection of the equilibrium points in the state space is contained in the three faces of the positive quadrant with vertex (0,0,0), representing the Walrasian equilibrium. And corresponding to the partitioning of the parameter plane into a Keynesian, a Classical, and a Repressed inflation region, we get a Keynesian face, a Classical face and a Repressed inflation face. The quadrant is therefore called the equilibrium quadrant.

Before finishing this section we shall make a short remark on our assumption on the consumers' moneyholdings. By not introducing the moneyholdings directly into the model, we have assumed that each consumer is endowed with the same fixed amount of money at the outset of each period. This can be interpreted as the result of a specific taxation system: at the beginning of each period the government undertakes a redistribution of the money, which brings the distribution back to the one it was at the outset of the former period. This is not a satisfying description, and in Chapter III it will be changed. Preliminary investigations of a model where moneyholdings of individuals change as a result of their decisions and experiences, however, suggest that the qualitative configuration of the equilibria does not seem to change. As we are discussing the movement of equilibria, when a(p,w) path crosses from one equilibrium region to another in the parameter plane, introducing "slow" adjustments in consumers' moneyholdings m would imply that we should study a (p,w,m) path, crossing from one equilibrium region to another, and this would bring about no qualitative change of the configuration of equilibria in the state space.

A slow adjustment process in p,w.

In this section we introduce a "slow" adjustment process in price and wage, guided by excess demand/supply, which is the determinant for the long run evolution of the economy. A detailed discussion of such an adjustment process was presented in the previous chapter.

1. (p,w) $\in$ Keynesian region.

$$\dot{p} = K_p(\hat{\varphi} + g - \psi) \qquad K_p \in \mathbb{R}_+$$

$$\dot{w} = K_w(\nu^a - \hat{\ell}) \qquad K_w \in \mathbb{R}_+$$

2. (p,w) $\in$ Classical region.

$$\dot{p} = C_p(\hat{\varphi} + g - \psi^a) \qquad C_p \in \mathbb{R}_+$$

$$\dot{w} = C_w(\nu^a - \hat{\ell}) \qquad C_w \in \mathbb{R}_+$$

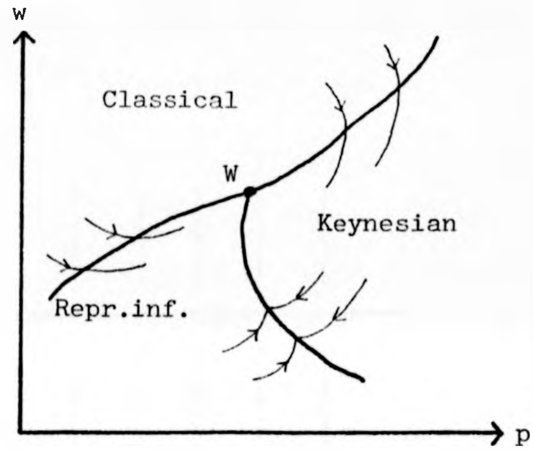
3. (p,w) $\in$ Repressed inflation region.

$$\dot{p} = R_p(\varphi + g - \psi^a) \qquad R_p \in \mathbb{R}_+$$

$$\dot{w} = R_w(\nu - \hat{\ell}) \qquad R_w \in \mathbb{R}_+$$

Though the <u>direction</u> of the corresponding vector field is continuous on the boundaries Classical - Keynesian and Classical - Repressed inflation (and that was all that was necessary for our analysis in Chapter I), it is noticed that in order to obtain the vector field to be continuous on these boundaries, we shall have to assume $K_w = C_w$ and $C_p = R_p$, which we do from now on. On the third boundary, however, both the vector field and its direction are discontinuous. The corresponding flow diagram looks as follows:

Fig. 3.

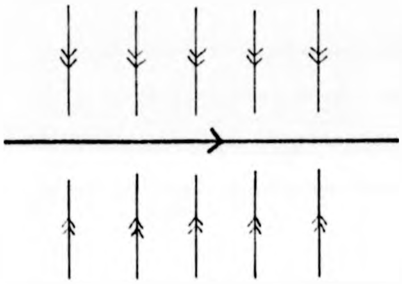


A first approach to a long term evolution.

In the short run the price and wage rate are assumed fixed, such that equilibrium is obtained only by quantity adjustments. We assume that these adjustments do take place, i.e. given an initial quantity combination (x_o, y_o, z_o), there is a "fast" movement towards a point on one of the three faces of the equilibrium quadrant. Here the face in question is determined by the region to which (p,w) belongs.

This description gives the complete picture of the long term evolution of the economy, as long as (p,w) stays within one region: given the intial value (p_o, w_o), a fast adjustment in the quantities will take place from the initial values (x_o, y_o, z_o) towards the corresponding equilibrium point on the equilibrium quadrant. During this fast adjustment the (p,w) value changes very slowly, as described by the slow process above. These slow changes initiate further adjustments in the quantities and so on. The combined process is described continuously, but may be represented graphically in the following way, where the double arrows represent "the fast foliation" and the single arrow "the slow manifold" (see Zeeman, [1973]).

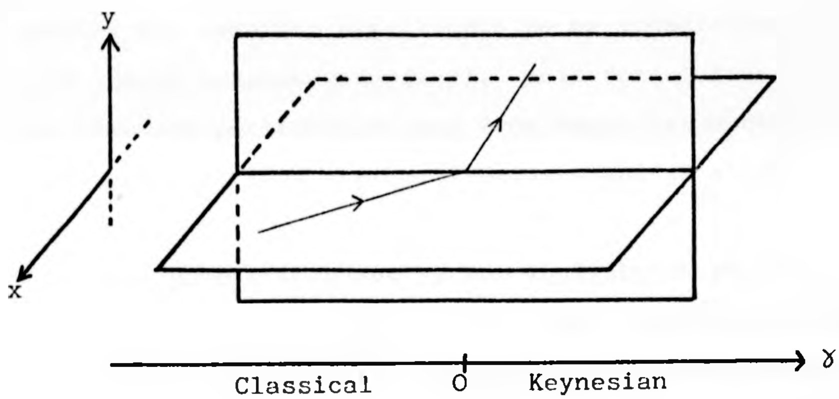
Fig. 4.



As indicated by the picture of the (p,w) trajectories (Fig. 3), we shall now have to discuss the situation when (p,w) moves from one equilibrium region to another, e.g. from the Classical region into the Keynesian region. In order to make the reader familiar with the basic ideas, we shall in this section restrict the analysis to a simplified example, which exhibits all the important features, but at the same time remains simple enough to be represented graphically. The general situation is presented in the next section.

Choose a specific trajectory, which crosses the boundary between the Classical and the Keynesian region, and let γ be the parameter representing the (p,w) value along this trajectory. As consumers are rationed in the labour market in both of these two equilibrium regions, it is an acceptable simplification in this first example to assume the z coordinate, which measures the observed level of unemployment, fixed along the path traced out by the corresponding point on the equilibrium quadrant. By choosing the unit appropriately we can obtain z equal to one. The local picture around the boundary then looks as follows:

Fig. 5.



In order to illustrate the crucial idea of exchange of stability we shall choose the simplest possible example (although the general case of a curved path can, in fact, be reduced to this simple case by suitable differentiable changes of coordinates). Suppose that the equilibrium point is given by the following coordinates:

Classical ($\gamma < 0$):

$x = -\gamma$

$y = 0$

$z = 1$

Keynesian ($\gamma > 0$):

$x = 0$

$y = \gamma$

$z = 1$

The simplest first attempt at describing the fast adjustment process would be:

$\gamma < 0$:

$$\dot{x} = -k(x + \gamma)$$
$$\dot{y} = -ky$$
$$\dot{z} = -k(z - 1) \qquad k > 0$$

$\gamma > 0$:

$$\dot{x} = -kx$$
$$\dot{y} = -k(y - \gamma)$$
$$\dot{z} = -k(z - 1) \qquad k > 0 \, ,$$

expressing the quantity adjustments to be proportional to effective excess demands (supplies). This formulation would ensure that any perturbation away from equilibrium decays exponentially, so that the equilibrium point is stable.

Now the path traced out by the equilibrium point in the state space has a sharp bend at the point where the parameter γ moves from the Classical region C_2 to the Keynesian region C_1. However, by our fundamental principle, <u>the differential equations defining the fast dynamics should vary smoothly with the paramter</u>. Here smoothly means that the partial derivatives with respect to γ of the right hand sides of the differential equations are continuous. But a simple computation shows that

$$\frac{\partial \dot{x}}{\partial \gamma} = \begin{cases} -k & \text{for } \gamma < 0 \\ 0 & \text{for } \gamma > 0. \end{cases}$$

Therefore our simplest first attempt at the fast adjustment process does not satisfy our fundamental principle. Moreover, if there were no other equilibrium points in the neighbourhood, then as a consequence of the fundamental principle the

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$\Upsilon > 0$:

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$$\dot{y} = -k(y - \Upsilon)$$
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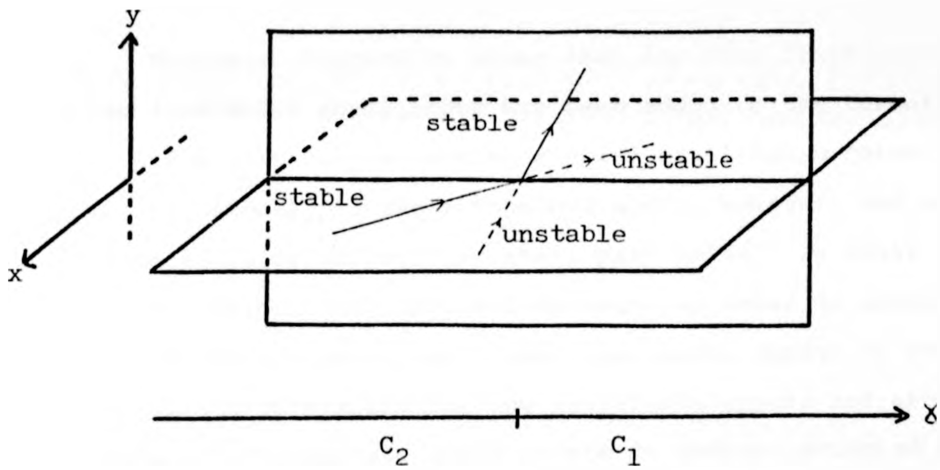
expressing the quantity adjustments to be proportional to effective excess demands (supplies). This formulation would ensure that any perturbation away from equilibrium decays exponentially, so that the equilibrium point is stable.

Now the path traced out by the equilibrium point in the state space has a sharp bend at the point where the parameter Υ moves from the Classical region C_2 to the Keynesian region C_1. However, by our fundamental principle, the differential equations defining the fast dynamics should vary smoothly with the paramter. Here smoothly means that the partial derivatives with respect to Υ of the right hand sides of the differential equations are continuous. But a simple computation shows that

$$\frac{\partial \dot{x}}{\partial \Upsilon} = \begin{cases} -k & \text{for } \Upsilon < 0 \\ 0 & \text{for } \Upsilon > 0. \end{cases}$$

Therefore our simplest first attempt at the fast adjustment process does not satisfy our fundamental principle. Moreover, if there were no other equilibrium points in the neighbourhood, then as a consequence of the fundamental principle the path traced out by the equilibrium point would be smooth. Since this path has a kink, it is not smooth, and so there must be other equilibrium points in the neighbourhood of the kink. In fact the generic case is what Poincaré, [1885], has named an "exchange of stability".

Fig. 6



When γ moves into C_1, the Classical equilibrium point continues to exist, but changes from being stable to being unstable, such that the path traced out by the point is a smooth curve in the plane $y = 0$. At the same time a smooth curve is traced out in the plane $x = 0$ by a Keynesian equilibrium point. This point is unstable as long as $\gamma \in C_2$, and it becomes stable when γ moves into C_1.

The two "new" equilibria are introduced into the above description by continuing exactly the same formula for each of the two previous equilibria across the boundary point $\gamma = 0$ into the opposite region:

New Keynesian ($\gamma < 0$):

$x = 0$

$y = \gamma$

$z = 1$

New Classical ($\gamma > 0$):

$x = -\gamma$

$y = 0$

$z = 1$.

The economic interpretation of these new equilibria will be given later.

The above discussion shows that for each fixed value of γ an (unstable) equilibrium has been added to the former description with only a single (stable) equilibrium point. The fast adjustment process introduced above, however, has only the stable equilibrium point as stationary value. We shall therefore now reformulate the fast dynamics in order to obtain a description, in which the former equilibria appear as stable equilibrium points and the new equilibria appear and are unstable. Unfortunately there exists no general accepted economic theory for such fast quantity adjustments to use as a guideline to the formulation. Accordingly we shall state mathematically convenient equations, which give the required properties, and then discuss the economic contents below. We shall particularly overcome the above failure of obtaining a smooth variation in the differential equations with smooth changes in the parameter by defining the adjustment process by the gradient of a potential function.

In order to obtain simple expressions we introduce new coordinates for the x, y-plane in the state space, even though this makes the economic interpretation of the fast process less obvious:

$\xi = x + y$

$\eta = x - y.$

To describe the simplest dynamics having the required form in this example we utilize a potential function

$$V(\xi,\eta,z) = k[(\xi^3/3 - \gamma^2\xi) + (\eta^2/2 + \gamma\eta) + (z-1)^2/2],\ k > 0.$$

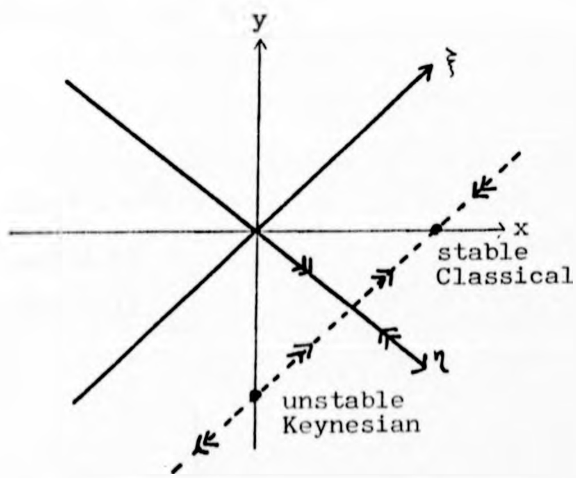
We suppose that the fast dynamics are given by the gradient of V, in others words:

$$\begin{aligned}
\dot{\xi} &= -V_\xi &&= -\partial V/\partial\xi = -k(\xi^2 - \gamma^2)\\
\dot{\eta} &= -V_\eta &&= -\partial V/\partial\eta = -k(\eta + \gamma)\\
\dot{z} &= -V_z &&= -\partial V/\partial z = -k(z - 1) \qquad k > 0.
\end{aligned}$$

In order to get an interpretation of the adjustment process in the quantities, when the new coordinates ξ and η have been introduced, we consider the following graphical representation of the process with the double arrows indicating the direction of the fast dynamics:

Fig. 7

$\gamma < 0$



Thinking of the process in economic terms the adjustment towards an equilibrium point involves changes in the actual production level ψ^a and therefore changes in the effective demand $\hat{\varphi}(p,w,\ell^a)$ by consumers. In the adjustment process first introduced p. 50, it is possible that adjustments take place only in the y coordinate, while the x coordinate remains fixed. This cannot be justified from an economic viewpoint, as a change in $y = \psi(p,w)-\psi^a$ for fixed (p,w) implies a change in $x = \hat{\varphi}(p,w,\ell^a)+g-\psi^a$. The above picture shows, however, that by introducing the new adjustment process in (ξ,η), we have in a simple way obtained that both the original coordinates x and y vary during the fast adjustments as a result of the changes in ψ^a and $\hat{\varphi}$.

Lemma.

If the fast dynamics are given by the gradient of V, then the following is true:
For $\Upsilon < 0$ there is one stable Classical equilibrium and one unstable Keynesian equilibrium.
For $\Upsilon > 0$ there is one stable Keynesian equilibrium and one unstable Classical equilibrium.

Proof.

$$\dot{\xi} = -V_\xi = -k(\xi^2 - \Upsilon^2)$$
$$\dot{\eta} = -V_\eta = -k(\eta + \Upsilon)$$
$$\dot{z} = -V_z = -k(z - 1)$$

This dynamical system has the stationary points:

$$\begin{array}{lcl} \xi = \gamma & & x = 0 \\ \eta = -\gamma & \Leftrightarrow & y = \gamma \\ z = 1 & & z = 1 \end{array}$$

and

$$\begin{array}{lcl} \xi = -\gamma & & x = -\gamma \\ \eta = -\gamma & \Leftrightarrow & y = 0 \\ z = 1 & & z = 1. \end{array}$$

In order to determine the stability of these points, we compute the Hessian H:

$$H = \begin{vmatrix} V_{\xi\xi} & V_{\xi\eta} & V_{\xi z} \\ V_{\eta\xi} & V_{\eta\eta} & V_{\eta z} \\ V_{z\xi} & V_{z\eta} & V_{zz} \end{vmatrix} = k^3 \begin{vmatrix} 2\xi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2\, k^3 \xi .$$

This implies the following:

$\gamma > 0$:

In the Keynesian equilibrium: $H > 0$ and the matrix is positive definite.

In the Classical equilibrium: $H < 0$

$\gamma < 0$:

In the Classical equilibrium: $H > 0$ and the matrix is positive definite.

In the Keynesian equilibrium: $H < 0$.

So for $\gamma > 0$, the Keynesian equilibrium is stable, while the "new" Classical equilibrium is unstable. Symmetrically we get for $\gamma < 0$ that the Classical equilibrium is stable, while the "new" Keynesian equilibrium is unstable.■

In the above formula for the potential function it is noticed that the parameter γ appears only as γ^2 in the first bracket as a coefficient to ξ. As we are looking for the generic description of what happens when the boundary is crossed, we must modify the potential so that it is a universal unfolding of the critical point (for the concepts see Zeeman [1977], Chapter 18). The universal unfolding is obtained by introducing a "hidden" parameter h as follows: (We discuss the meaning of h later)

$$V(\xi,\eta,z) = k[(\xi^3/3 - (\gamma^2 + h)\xi) + (\eta^2/2 + \gamma\eta) + (z-1)^2/2].$$

In order to analyze the influence of this new parameter, we shall specifically consider the situation, where $\eta = -\gamma$, $z = 1$, since these are the equilibrium values of η and z, so that the potential is

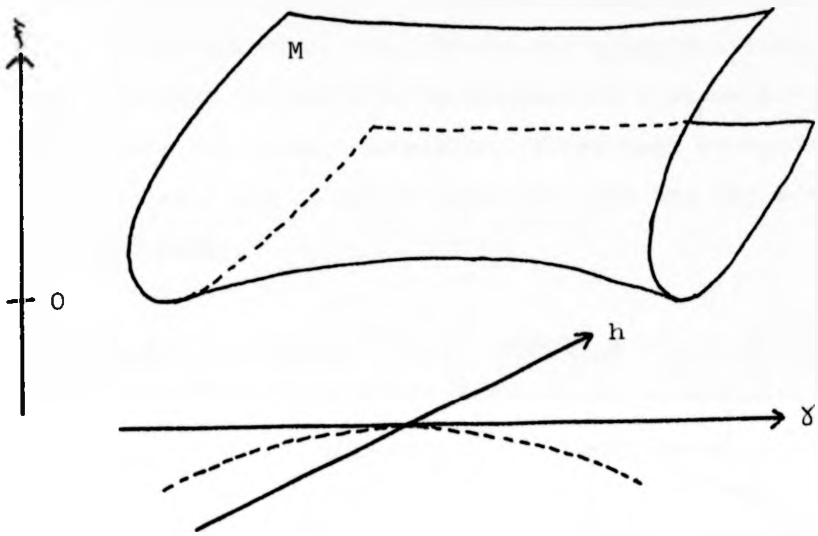
$$V(\xi, -\gamma, 1) = k[(\xi^3/3) - (\gamma^2 + h)\xi] + \text{constant}.$$

The surface M of equilibrium points is determined by the equation

$$V_\xi = 0, \text{i.e. } \xi^2 - (\gamma^2 + h) = 0,$$

which is represented graphically by the following picture:

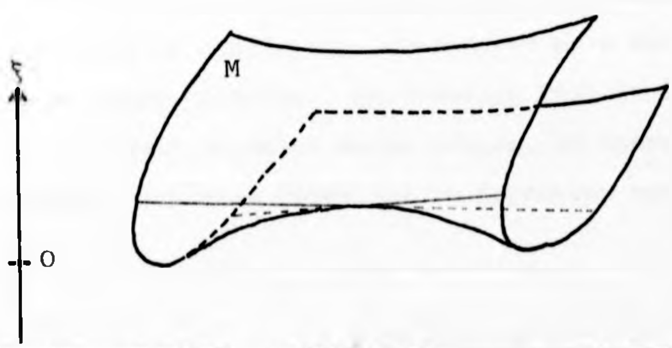
Fig. 8



The set of equilibrium points has a fold along the curve $\xi = 0$, $\gamma^2 = -h$, which projects onto the parabola $\gamma^2 = -h$ in the parameter plane.

Now the plane $h = 0$ is in fact the tangent plane to the equilibrium surface at the critical point; this is why this plane intersects the equilibrium surface in a pair of intersecting lines, which is a very non-generic situation.

Fig. 9



Indeed the non-genericity of this situation is the essential reason behind the exchange of stability, as we explained in the introduction. To obtain the generic situation we have to intersect the equilibrium surface by a plane $h = \text{constant} \neq 0$. There are two cases, according to the sign of h, illustrated by Fig. 10 A and B, where also the fast and the slow dynamics are indicated.

Fig.10 A $\quad h > 0$

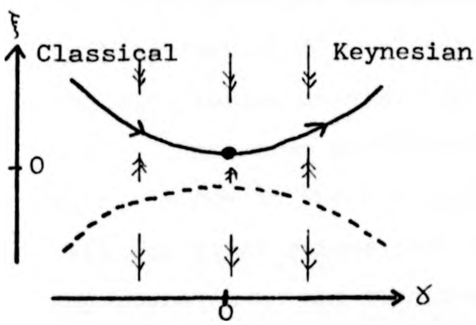
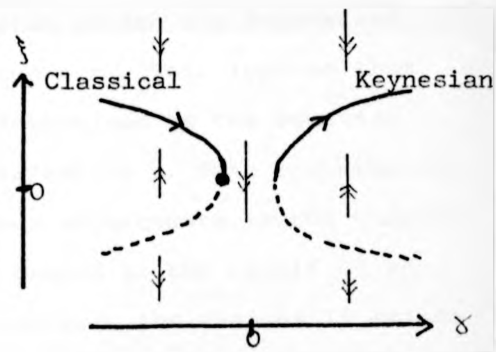


Fig.10 B $\quad h < 0$



From these pictures it is obvious that when a long term adjustment in (p,w) involves the crossing of a boundary between two equilibrium regions, the result heavily depends on the sign of the "hidden" parameter h. If h is positive, Fig. 10 A shows that there is a smooth transition from Classical equilibrium to Keynesian equilibrium, because the effects of the hidden parameter has been to smooth the edge between the two faces of the equilibrium quadrant causing the boundary between the two types of equilibrium (indicated by a dot in Fig. 10 A) to be no longer visible. By contrast Fig. 10 B shows that for $h < 0$ a more dramatic event occurs, as there is no way of leaping the gap from Classical to Keynesian equilibrium. Indeed as γ increases, before γ reaches zero, the equilibrium point reaches a critical point, indicated by a dot, where the Classical equilibrium breaks down and the fast dynamics takes over, causing the state of the economy catastrophically into a new state different from either Classical or Keynesian. We have not shown this state in Fig. 10 B, but we will return to discuss this case in the section below.

In order to understand the impact of the new parameter h we notice that the generic equilibrium points are determined by the equation $\xi^2 = \gamma^2 + h$ for fixed η,z. This implies that the equilibrium points, initially determined by the equation $\xi = \gamma$, have been perturbed by the parameter h. This perturbation , which is due to the non-instantaneous adjustments in the quantities for fixed prices, may be interpreted as the result of an imperfection in the quantity adjustments: the changes in prices and wages, initiated by the quantity adjustments, act as a "barrier", which precludes the quantities from perfect adjustments.

So far this example has shown that even with the above simple formulation of the adjustment processes, a generic description of the long term evolution of the economy involves the introduction of completely new equilibrium states, whenever (p,w) crosses from one equilibrium region to another.

We shall not push this example further, but instead proceed to the general description. This will clarify how the new type of equilibrium, which in the example above would have to be added to the description in order to obtain the full formulation, is a natural part of the complete description. This will also contain an economic interpretetion of the new equilibria and the catastrophic jumps.

<u>The general long term description</u>.

In the previous section we considered a one dimensional parameter γ, which moved smoothly from one equilibrium region into another. In order to obtain the corresponding pair of smooth equilibrium paths traced out in the two faces of the 2 dimensional equilibrium quadrant, (see Fig. 6), we constructed a correspondance between parameter values and equilibrium points: to each value of γ was assigned two equilibria, one stable and one unstable. As we are now considering the general (local) situation around the Walrasian equilibrium, we must allow the parameter space to be two dimensional. Corresponding to paths in the (p,w) plane we shall now consider 3 smooth equilibrium paths traced out on the 3 faces of the 3 dimensional equilibrium quadrant. As above our first major task is to construct a simple correspondance between parameter values and equilibrium points.

Supposing firstly that the angle between each two of the tangents to the boundaries at the Walrasian equilibrium is 120°, mulation, is a natural part of the complete description. This will also contain an economic interpretetion of the new equilibria and the catastrophic jumps.

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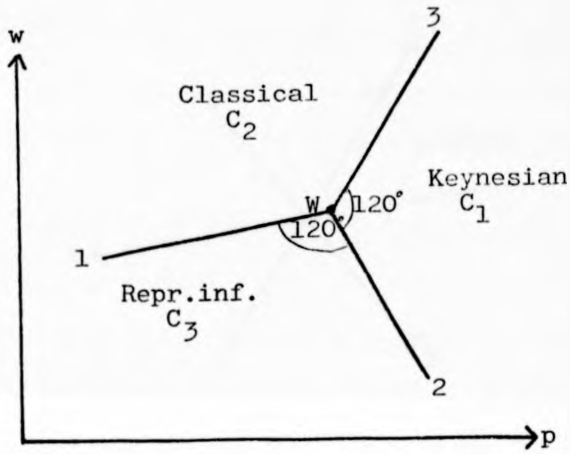
Supposing firstly that the angle between each two of the tangents to the boundaries at the Walrasian equilibrium is 120°,

Fig. 11



we proceed as follows.

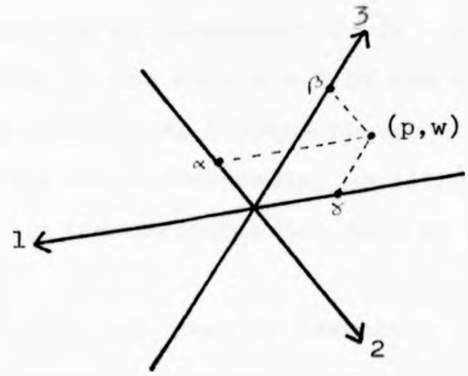
As an equilibrium in the quantities is described by three coordinates, we want to characterize each point in the parameter plane by three coordinates (α,β,γ). This can be done in a 1 - 1 way by imposing the restriction $\alpha + \beta + \gamma = 0$. Choosing the three tangents as coordinate axes, the coordinates to a point (p,w) are defined as follows:

α - coordinate: the coordinate of the projection, parallel to the 1-axis, of (p,w) onto the 2-axis.

β - coordinate: the coordinate of the projection, parallel to the 2-axis, of (p,w) onto the 3-axis.

γ - coordinate: the coordinate of the projection, parallel to the 3-axis, of (p,w) onto the 1-axis.

Fig. 12



$\alpha + \beta + \gamma = 0$ follows from the assumption on the angle between any two axes to be 120^{o}.

Remark.

$(p,w) \in C_1 \Rightarrow \beta > 0, \ \gamma < 0$

$(p,w) \in C_2 \Rightarrow \gamma > 0, \ \alpha < 0$

$(p,w) \in C_3 \Rightarrow \alpha > 0, \ \beta < 0$.

Corresponding to each point (α,β,γ) in the parameter plane we now define three equilibrium points in the state space $X(x,y,z)$ as described in the following scheme. Note that we do not introduce any a priori restriction on the sign of α, β and γ.

	Keynesian	Classical	Repr.inf.
x	0	γ	$-\beta$
y	$-\gamma$	0	α
z	β	$-\alpha$	0

We notice that due to the remark, when (p,w) belongs to the Keynesian region, the corresponding Keynesian equilibrium is characterized by positive y,z - coordinates. Therefore it is a point in the face x = 0 of the equilibrium quadrant. Similarily, when (p,w) belongs to one of the other two equilibrium regions the corresponding equilibrium in the state space is a point on the corresponding face of the equilibrium quadrant.

Furthermore on each of the three coordinate axes two of the equilibrium points coalesce,e.g. if (α,β,γ) belongs to the boundary between two regions C_1 and C_2, then (α,β,γ) belongs to the 3-axis and so the point is of the form $(\alpha,\beta,0),\alpha = -\beta$. Therefore the three corresponding equilibrium points are

$(0,0,\beta),(0,0,-\alpha),(-\beta,\alpha,0)$,

and so the Keynesian and the Classical equilibria coincide at a point on the edge x=y=0 of the equilibrium quadrant.

This correspondance has only been established by supposing the angles between boundaries in the parameter plane to be 120°. In general this is not the case, but we now show that a suitable choice of scales on the three axes may still lead to the above description. In order to deal with this general case, we shall first determine the angles between the boundaries:

Fig. 13 A

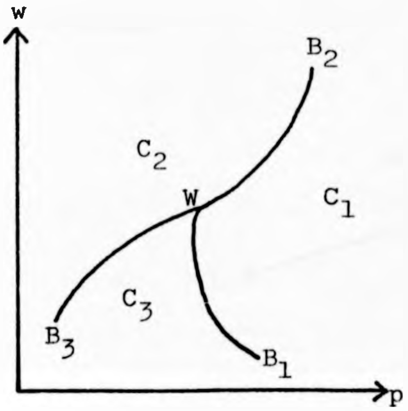
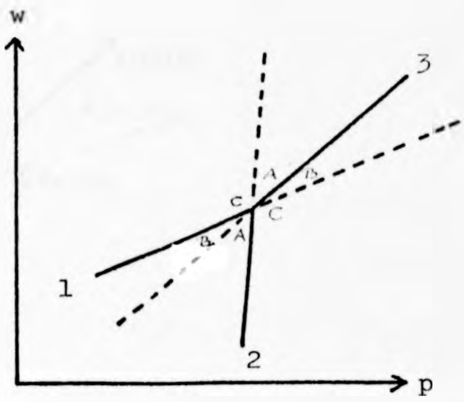


Fig 13 B



Name the angles as on the above picture. They are determined when the slopes of the three lines have been determined. We shall here only sketch how to compute the slope of "2".

The B_1 curve is defined as follows (Hildenbrand and Hildenbrand, [1976]).

$$B_1 = \{(p,w) \mid F(\ell(w)) = g + \varphi(p,w)\} .$$

We compute $dw/dp|_{B_1}$.

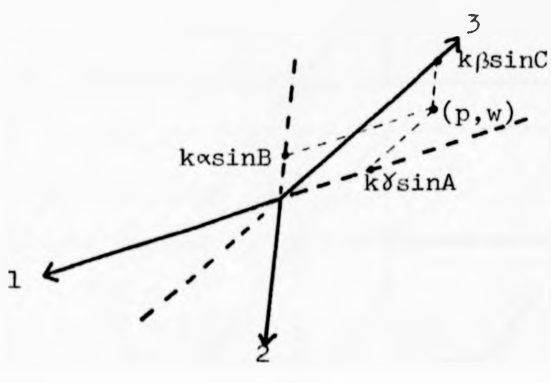
$$F'(\partial\ell/\partial w)\cdot dw = 0 + (\partial\varphi/\partial p)\cdot dp + (\partial\varphi/\partial w)\cdot dw$$

$$dw/dp|_{B_1} = (\partial\varphi/\partial w)/[F'\cdot\partial\ell/\partial w - \partial\varphi/\partial w].$$

The slope at W is obtained by computing this at the Walrasian equilibrium values.

The coordinates in the parameter plane are now as follows:

Fig. 14

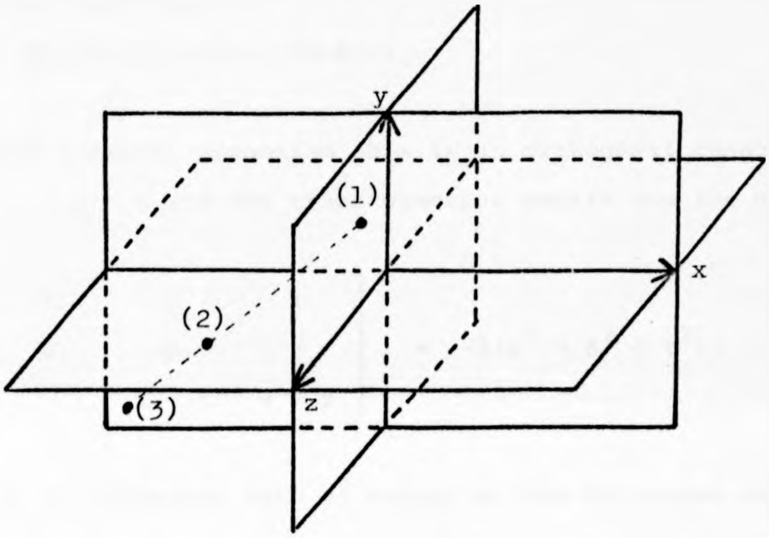


By choosing the units ksin A on the 1-axis, ksin B on the 2-axis, and ksin C on the 3-axis, however, we can choose exactly the same coordinates for the equilibrium points. From now on we assume this has been done. Next we notice the following geometrical description of the equilibrium points.

Lemma.

For a fixed parameter value (α,β,γ) the corresponding three equilibrium points are situated on a line in the (x,y,z) space. The line is determined by one of the points and the vector $(1,1,1)$.

Fig. 15



Proof.

The coordinates of the three equilibrium points are

(1): $(0,-\gamma,\beta)$; (2): $(\gamma,0,-\alpha)$; (3): $(-\beta,\alpha,0)$,

so the vector determined by two of the equilibrium points are:

(2) - (1): $(\gamma,\gamma,-\alpha-\beta) = \gamma(1,1,1)$

(3) - (1): $(-\beta,\alpha+\beta,-\beta) = -\beta(1,1,1)$

(3) - (2): $(-\beta-\gamma,\alpha,\alpha) = \alpha(1,1,1)$. ■

Next we want to introduce the fast dynamics, i.e. the dynamics w.r.t. which the above introduced points are equilibrium (i.e. stationary) points. The fast dynamics describe the adjustment towards these equilibrium points. We are looking for a simple description of the dynamics in x, y, and z for given fixed parameters (α,β,γ). The above lemma indicates that a simple formulation may be obtained, if one of the coordinate axes in the state space is chosen parallel to (1,1,1). We therefore make the following coordinate change:

$$\xi = x + y + z$$
$$\eta = \alpha x + \beta y + \gamma z$$
$$\zeta = (\beta-\gamma)x + (\gamma-\alpha)y + (\alpha-\beta)z.$$

Modulo a scalar expansion this is an orthogonal change, since $\alpha + \beta + \gamma = 0$ and the transformation matrix has the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \beta - \gamma & \gamma - \alpha & \alpha - \beta \end{vmatrix} = -3(\alpha^2 + \beta^2 + \gamma^2),$$

which is different from 0, except at the Walrasian equilibrium.

In the new coordinate system the equilibrium points have the following coordinates:

	Keynesian	Classical	Repr.inf.
ξ	$\beta - \gamma$	$\gamma - \alpha$	$\alpha - \beta$
η	0	0	0
ζ	$-\rho^2$	$-\rho^2$	$-\rho^2$

where $\rho^2 = \alpha^2 + \beta^2 + \gamma^2$.

To describe the simplest dynamics having the required form we utilize the potential function

$$V(\xi,\eta,\zeta) = k[(\xi^4/4 - 3/4\rho^2\xi^2-\Delta\xi)+\eta^2/2+(\zeta^2/2+\rho^2\zeta)], \quad k > 0.$$

where $\Delta = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$.

We suppose that the fast dynamics is given by the gradient of V, in other words

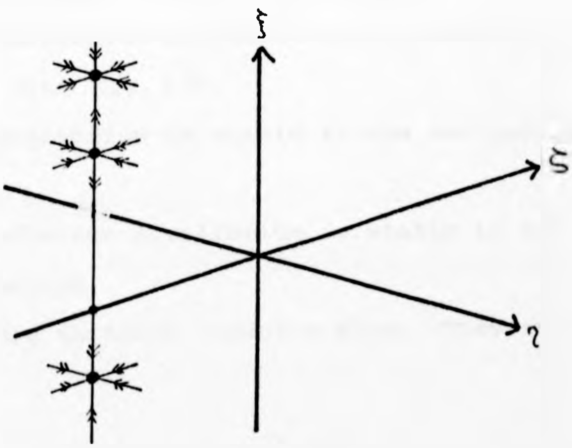
$$\dot{\xi} = -V_\xi = -\partial V/\partial \xi = -k(\xi-\beta+\gamma)(\xi-\gamma+\alpha)(\xi-\alpha+\beta)$$
$$\dot{\eta} = -V_\eta = -\partial V/\partial \eta = -k\eta$$
$$\dot{\zeta} = -V_\zeta = -\partial V/\partial \zeta = -k(\zeta+\rho^2) \qquad k > 0.$$

Note that

$$\dot{\xi} = -k(\xi^3 - 3/2\rho^2\xi-\Delta), \text{ where}$$
$$\Delta = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha).$$

For fixed parameter values (α,β,γ) the fast adjustments in the quantities can be illustrated as follows:

Fig 16



Here the three dots are the equilibria, all situated on the line $\eta = 0$, $\zeta = -\rho^2$. Due to the theorem below the fast dynamics has been drawn such that two of the equilibria are stable and the third one is unstable. As in the introductory example, with the lack of a standard and useful economic theory of fast quantity adjustments in disequilibrium, we have introduced a mathematically simple formulation of the fast dynamics, having the desired property of three stationary points, two stable and one unstable.

The following theorem shows that with this fast dynamics, we have a description of the long term evolution with (p,w) moving around in the parameter plane, which exhibits exchange of stability, each time one of the coordinates α, β, and γ changes sign.

Theorem.

If the fast dynamics are given by the gradient af V, then the following is true.
For each value of (ρ^2, Δ) there are three equilibrium points, a Keynesian, a Classical, and a Repressed inflation equilibrium, of which two are stable and one is unstable.
The Keynesian equilibrium is stable in the two regions C_1 and its reflection. (See Fig. 17).
The Classical equilibrium is stable in the two regions C_2 and its reflection.
The Repressed inflation equilibrium is stable in the two regions C_3 and its reflection.
The equilibria are unstable anywhere else. (They are in fact saddle points).

Proof.

$$\dot{\xi} = -V_\xi = -k(\xi^3 - 3/2\rho^2\xi - \Delta)$$
$$\dot{\eta} = -V_\eta = -k\eta$$
$$\dot{\zeta} = -V_\zeta = -k(\zeta + \rho^2).$$

It follows from above that this dynamical system has the following stationary points:

(1): $(\beta-\gamma,0,-\rho^2)$ (Keynesian)

(2): $(\gamma-\alpha,0,-\rho^2)$ (Classical)

(3): $(\alpha-\beta,0,-\rho^2)$ (Repr.inf.).

In order to prove the stability regions we compute the Hessian.

$$H = k^3 \begin{vmatrix} 3\xi^2 - (3/2)\rho^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = k^3(3\xi^2 - (3/2)\rho^2).$$

(1) stable $\Leftrightarrow 3\xi^2 - (3/2)\rho^2 > 0$

$\Leftrightarrow 3(\beta-\gamma)^2 - 3/2[(\beta+\gamma)^2 + \beta^2 + \gamma^2] > 0$

$\Leftrightarrow 9\beta\gamma < 0$

$\Leftrightarrow \beta>0, \gamma<0$ or $\beta<0, \gamma>0$.

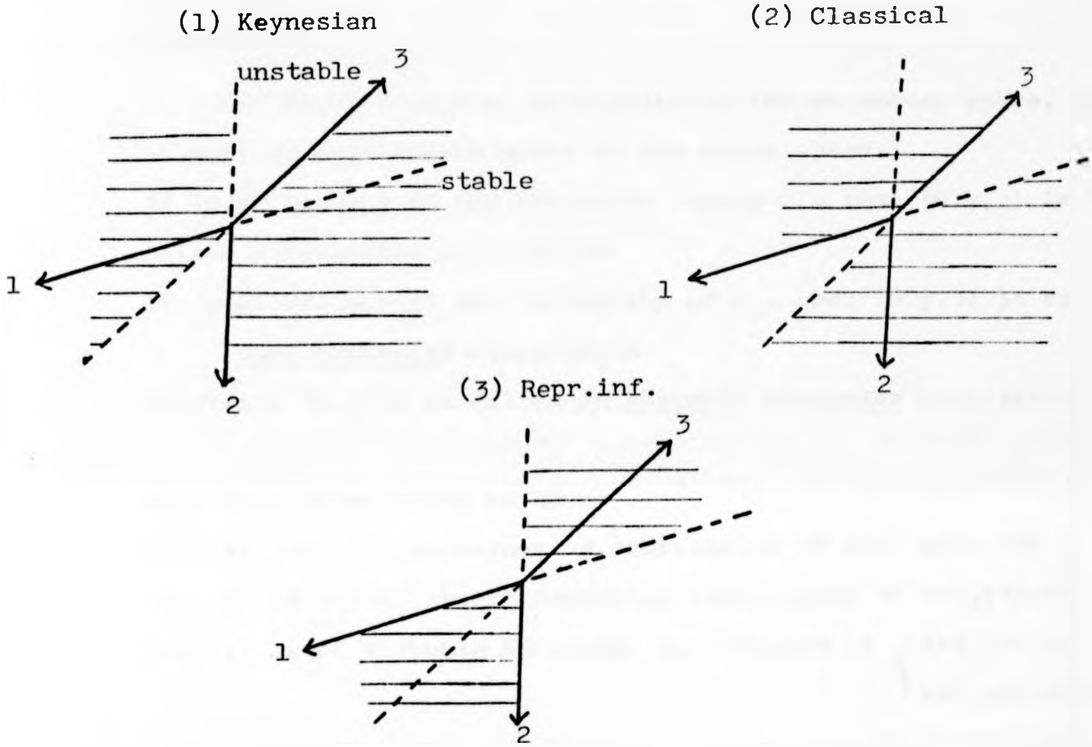
$\Leftrightarrow (\alpha,\beta,\gamma)$ belongs to C_1 or its reflection.

(1) is unstable anywhere else.

Symmetrically we get the other stability regions, mentioned in the theorem.

Graphically the stability regions may be illustrated as follows:

Fig. 17



From these pictures it is obvious that at any point off the coordinate axes in the parameter space, two equilibria are stable and one is unstable. On the three coordinate axes one equilibrium is stable, and the other two equilibria coalesce into an unstable saddle node as they exchange stablities.■

In the above analysis we have named as Keynesian equilibrium every equilibrium point belonging to the plane $x = 0$. However, as we initially only assigned this name to the equilibria of the form $(0,y,z)$ with $y > 0$, $z > 0$, we shall from now on return to this terminology. The above theorem justifies the distinction in the following definition.

Definition.

Let $(p,w) = (\alpha,\beta,\gamma)$ be a point in the parameter space, and $(0,y,z)$ an equilibrium point in the state space.
If (p,w) belongs to the Keynesian region C_1, then $(0,y,z)$ is called a Keynesian equilibrium.
If (p,w) belongs to the reflection of C_1, then $(0,y,z)$ is called a dual Keynesian equilibrium.
Otherwise $(0,y,z)$ is called an unstable Keynesian equilibrium.

With this terminology we have:
(x,y,z) is a stable Keynesian equilibrium if $x=0,\ y>0,\ z>0$.
(x,y,z) is a dual stable Keynesian equilibrium if $x=0, y<0, z<0$.
(x,y,z) is an unstable Keynesian equilibrium if $\begin{cases} x=0, y>0, z<0. \\ x=0, y<0, z>0. \end{cases}$

The analogous terminology is applied to the Classical and Repressed inflation equilibria.

Next we notice that at a dual equilibrium point consumers and/or producers are performing at a level exceeding that associated with utility/profit maximising behaviour. To be more specific, the three types of dual equilibria are characterized by one of the coordinates x,y,and z being equal to zero while the other two are negative. At a dual Keynesian equilibrium $(0,y,z)$, $y<0$, $z<0$, there is equilibrium on the good market, actual production exceeds the profit maximising output, while there is excess demand on the labour market. At a dual Classical equilibrium $(x,0,z)$, $x<0$, $z<0$, there is excess supply on the good market, excess demand on the labour market, while actual production is at the desired level. Finally at a dual Repressed inflation equilibrium $(x,y,0)$, $x<0$, $y<0$, there is excess supply of the good, equilibrium on the labour market, but actual production exceeds the desired level. This characterization of the three types of dual equilibria makes it obvious that in all three cases some external pressure is forcing this behaviour, which differs from agents profit/utility maximising behaviour.

Compared to standard theory, where it is assumed that if a market is not in equilibrium, i.e. supply differs from demand, then the rationed equilibrium point is always determined by the short side of the market, the notion of a dual equilibrium is characterized by the opposite assumption, thus quantities are determined by the long side of the markets. For example in a dual Keynesian or a dual Repressed inflation equilibrium y is negative i.e. firms are producing on a larger scale than the profit maximising, which seems to be a very non-stationary situation. The three types of dual equilibria therefore represent states quite counterintuitive to what is usually thought of as equilibrium situations. However, accepting the smooth variation in the fast dynamics, which we started out with, the above analysis shows that we will have to take these unpleasant equilibria into account.

In order to analyse these dual cases we shall now proceed as in the previous section and modify the potential

$$V(\xi,\eta,\zeta) = k[(\xi^4/4 - 3/4\rho^2\xi^2 - \Delta\xi) + \eta^2/2 + (\zeta^2/2 + \rho^2\zeta)]$$

such that it is a universal unfolding of the critical point (Zeeman [1977], Chapter 18). W.l.o.g. we can assume the constant k equal to 1. Next we notice that in order to get the universal unfolding in this case, two "hidden" parameters h and k have to be added to the coefficients of ξ^2 and ξ in the first bracket, because ρ is quadratic and Δ is cubic in α, β, γ. In order to discuss the introduction of these two new parameters, we shall specifically look at the case with $\eta = 0$, $\zeta = -\rho^2$, since these are the equilibrium values, such that the unfolding is of the form

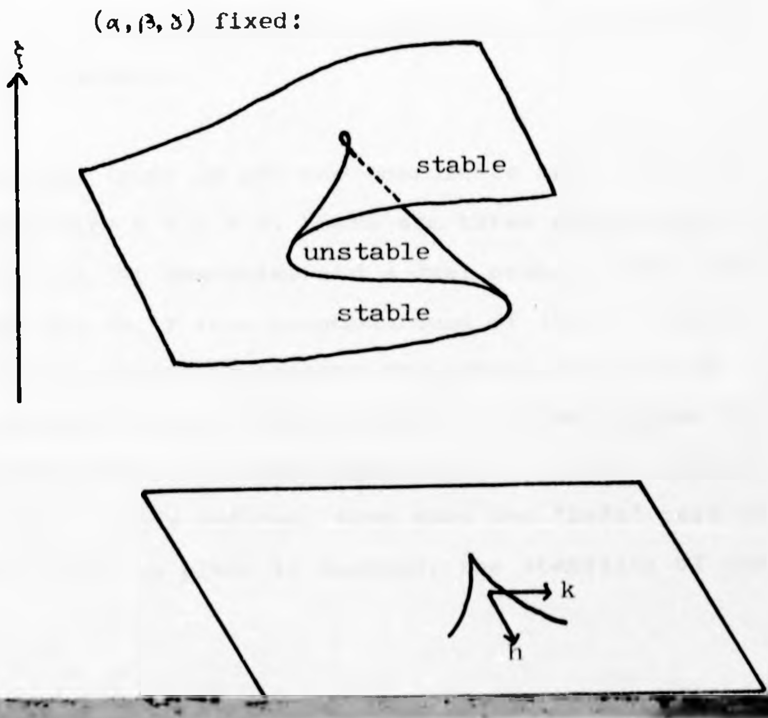
$$V(\xi,\ 0,\ \rho^2) = \xi^4/_4 - 3/4(\rho^2 + h)\xi^2 - (\Delta + k)\xi + \text{constant}$$

with

$$V_\xi = \xi^3 - 3/2(\rho^2 + h)\xi - (\Delta + k).$$

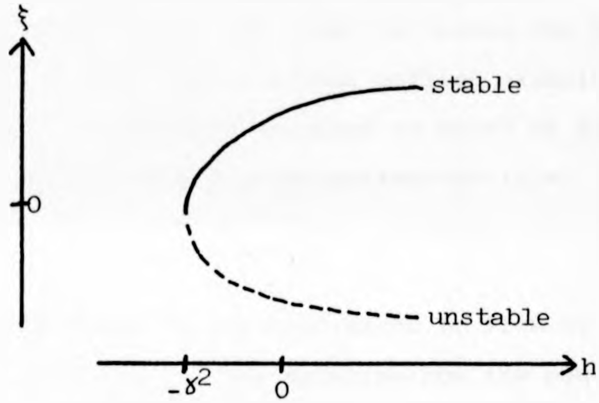
This is the formula for a cusp with h as splitting factor and k as normal factor (see Zeeman, [1976]). This means that for fixed parameter values (α, β,γ), the shape of the equilibrium surface is a cusp with the cusp point translated to the point $(h,k) = (-\rho^2, -\Delta)$ as shown below.

Fig. 18.



This picture corresponds to a section γ = constant of Fig. 8 in our introductory example:

Fig. 19. γ fixed:



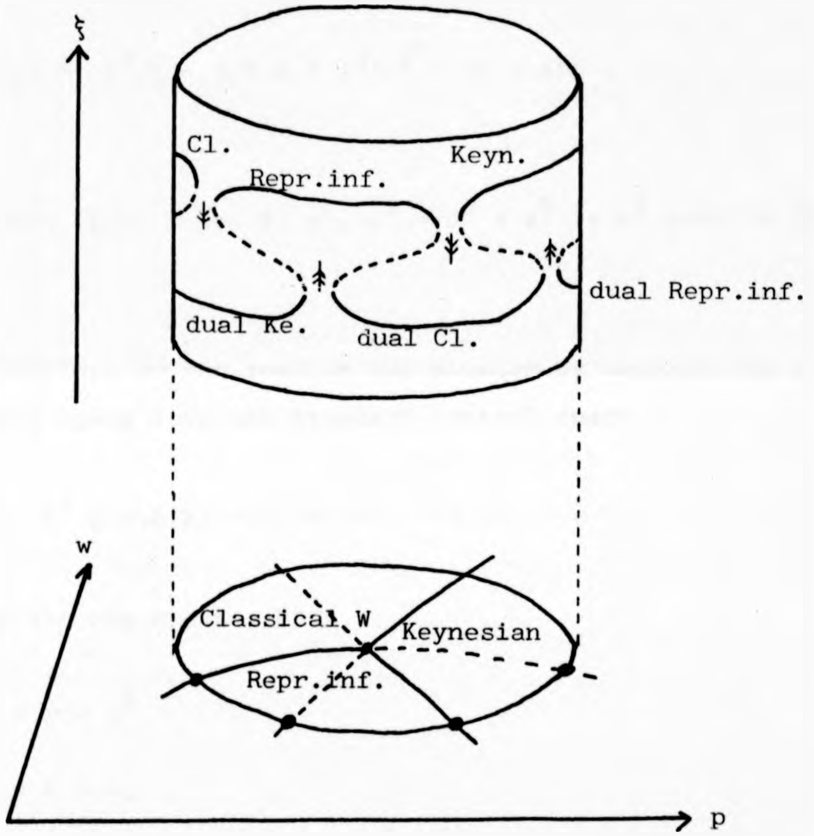
At that stage of the analysis we saw that when the two equilibrium points coincided at the fold point, the equilibrium state would suddenly snap to a new state, but the analysis did not tell anything about the nature of the new equilibrium point. Now the above description immediately gives the answer to this question.

Assume that (p,w) is off the coordinate axes. By the theorem above with h = k = 0, there are three equilibrium points, a stable, an unstable, and a dual stable. The same remains true for (h,k) in a neighbourhood of (0,0). Depending on the history of the economy the actual equilibrium will be on one of the two stable sheets. Let us suppose it is a stable equilibrium on the upper sheet. If the value of the parameter k is now reduced, then when the "left" part of the cusp curve in the plane is reached, the stability of the equilibrium point breaks down, and there is a sudden jump to the dual equilibrium point in the lower sheet.

As shown above for each fixed value of (p,w) the equilibrium surface forms a cusp, i.e. it is a family of cusps, parametrized by (p,w). In order to obtain the pictures corresponding to Fig. 10A, B in our earlier example, we are now going to do the opposite compared to above by fixing the values of h and k and allow the parameters $(p,w) = (\alpha, \beta, \gamma)$ to vary.

In particular we are interested in flowing along the flow lines shown in Fig. 3, and deducing how the equilibrium states will change. We shall approach this problem by examining a section of the equilibrium surface over a circle, centre the Walrasian equilibrium in the parameter plane. For example, Fig. 20 shows such a section for fixed values of (h,k) with $h < 0, k = 0$. As above ξ is the quantity variable, as η and ζ have been fixed at their stable equilibrium values.

Fig. 20.



We will now show how to deduce the surprising shape of Fig.20. In the discussion that follows we take a standard model of the cusp catastrophe given by the potential function

$$V(\xi) = \xi^4/_4 - (3/4)u\,\xi^2 - v\xi$$

with standard 2 dimensional control space C^2 (u,v), splitting factor u, normal factor v, and bifurcation set $u^3 = 2v^2$ with cusp point at the origin (see Fig. 21 A).

Now in our problem we have a 4 dimensional control space $C^4(p,w,h,k)$ and a potential given by

$$V(\xi) = \xi^4/_4 - 3/4(h + \rho^2)\xi^2 - (k + \Delta)\xi ,$$

where $(p,w) = (\alpha, \beta, \gamma)$, $\rho^2 = \alpha^2 + \beta^2 + \gamma^2$ and $\Delta = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$.

Therefore we can realize our problem by mapping the given control space into the standard control space

$$m: \; C^4(p,w,h,k) \to C^2(u,v)$$

by the map m given by

$$u = h + \rho^2$$

$$v = k + \Delta.$$

In particular, when $(h,k) = (0,0)$ m maps the (p,w) plane by folding it into six along the three α, β, γ-axes and mapping the resulting wedge-shape into the interior of the cusp in the (u,v) plane. To see this, introduce polar coordinates (r,θ) centered at the Walrasian equilibrium in the (α, β, γ) plane. Assume first that the α, β, γ-axes lie in canonical directions

$$\theta = \pi,\; 5\pi/3, \pi/3$$

as in Fig. 12. Then a straightforward computation shows that

$$\rho^2 \equiv \alpha^2 + \beta^2 + \gamma^2 = 2r^2$$

$$\Delta \equiv (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha) = 2r^3 \cos 3\theta .$$

Therefore the circle with radius r, centre the Walrasian equilibrium is folded into six and mapped by m into the interval

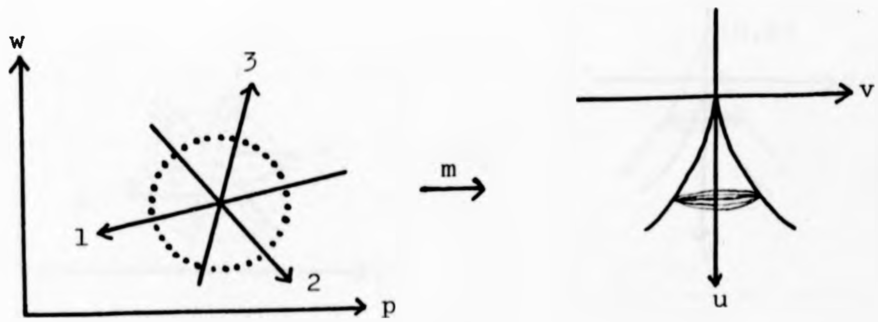
$$u = 2r^2$$

$$|v| \leq 2r^3$$

which is the intersection of the line $u = 2r^2$ with the interior of the cusp (See Fig. 21 A).
Therefore as r varies, the image of m fills out the interior of the cusp.

<u>Fig. 21 A</u>

h=k=0



In the more general case where the α, β, γ axes are not initial at 120°, but at arbitrary angles as in Fig. 13 B, we can map the canonical (α, β, γ) plane by a linear map throwing the canonical polar coordinates into oblique polar coordinates, and throwing the circle with radius r into an ellipse, centre the Walrasian equilibrium. The exactly same formulae remain true and the same result holds. Similarly we map the canonical (α, β, γ) axes into curvilinear axes as in Fig. 13 A, giving curvilinear polar coordinates with respect to which the same result still holds. However, for simplicity in what follows we assume we have the same (α, β, γ) axes in the (p,w) plane.

Now suppose $(h,k) \neq (0,0)$. The map m again folds the (p,w) plane into six along the same fold lines as before, namely the α, β, γ axes; but this time the image is translated by the vector (h,k) as shown in Fig. 21 B, C, D, E.

<u>Fig. 21 B.</u> $h < 0$, $k = 0$.

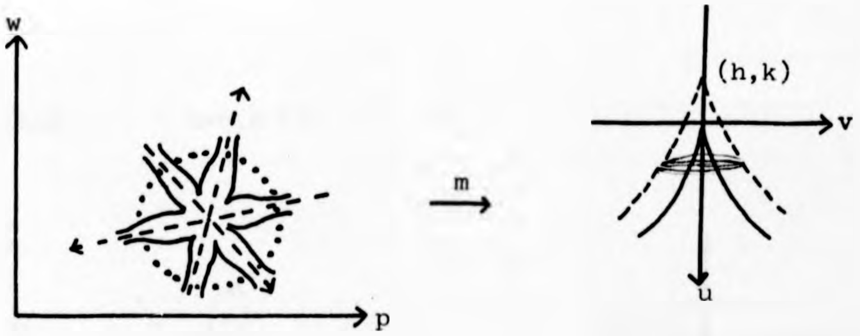


Fig.21 C   h > 0,k=0

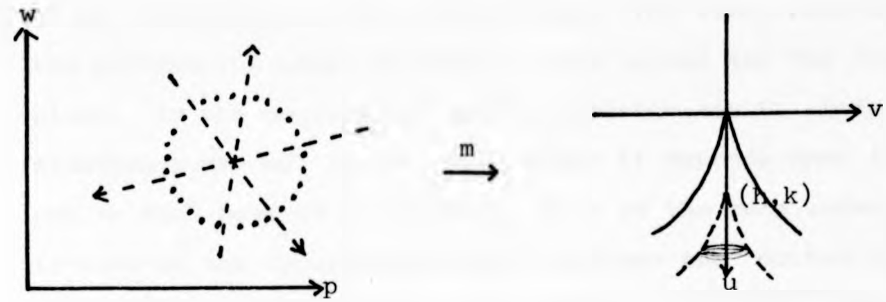


Fig.21 D   h=o,k > 0

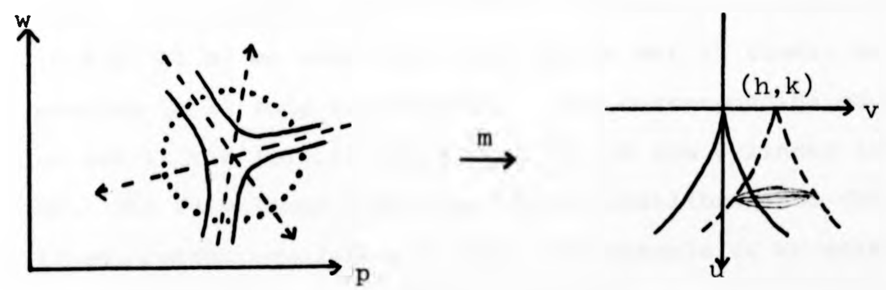
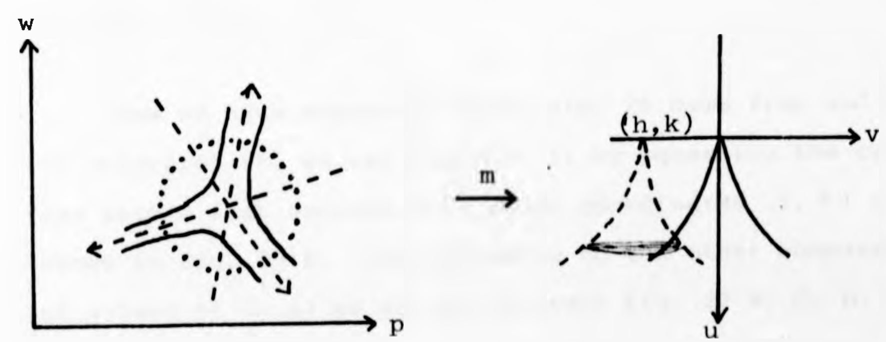


Fig.21 E   h=o,k < 0



In each case the dashed lines show the α, β, γ axes and their image cusps in the (u,v) plane. The firm lines show the bifurcation sets in both the (p,w) plane and the (u,v) plane; in the (u,v) plane the bifurcation set is always the standard cusp, but in the (p,w) plane it depends upon (h,k), and in each case is the inverse image of the cusp under m. In each of the (p,w) planes we have drawn the dotted circle with radius r and centre the Walrasian equilibrium, and in each (u,v) plane we show its image as a horizontal interval covered six times.

We now want to explain how to obtain Fig. 20 from Fig. 21 B. As we move round the dotted circle in the (p,w) plane in Fig. 21 B, we cross the bifurcation set 12 times, corresponding to 12 fold catastrophes. The corresponding equilibrium set in the (p,w,ξ) space is shown in the cylinder in Fig. 20. The firm lines represent stable equilibria and the dotted lines unstable equilibria. Thus, for example if we were in the Repressed inflation equilibrium region, and proceeding anticlockwise round the circle, then just before we reach the Repressed inflation - Keynesian boundary, the equilibrium state would suffer a catastrophic switch into a dual Classical equilibrium state.

Now we have explained where Fig. 20 came from and how to interpret it, we can simplify it by squashing the cylinder into a flat annulus with polar coordinates (ξ, θ) as shown in Fig. 22 B. Corresponding to the other combinations of values of (h,k) we get the picture Fig. 22 A, C, D, E.

Fig. 22 A.

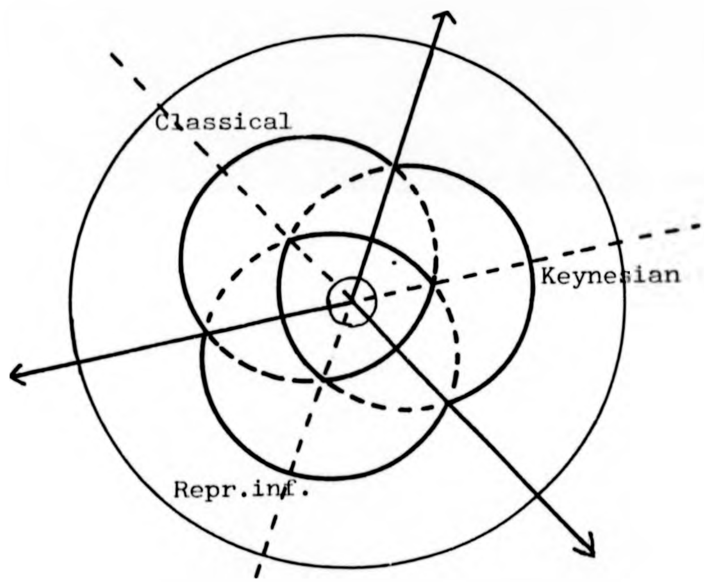


Fig.22 B

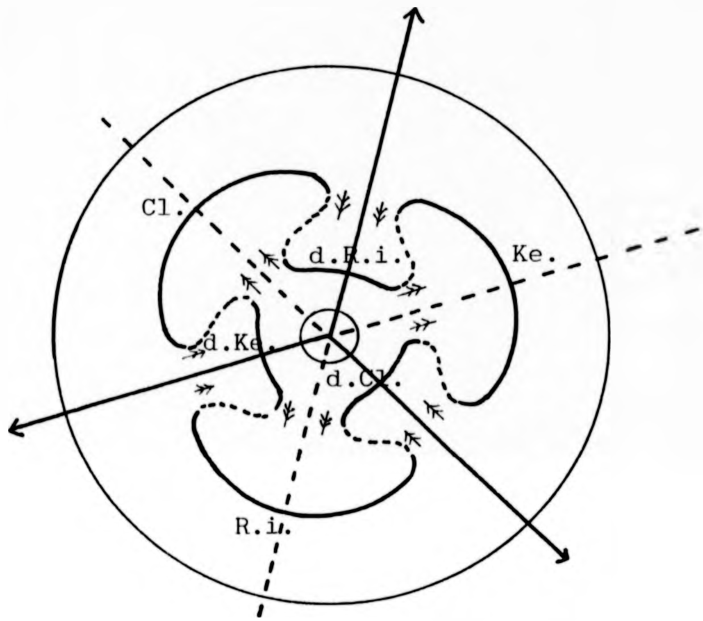


Fig.22 C

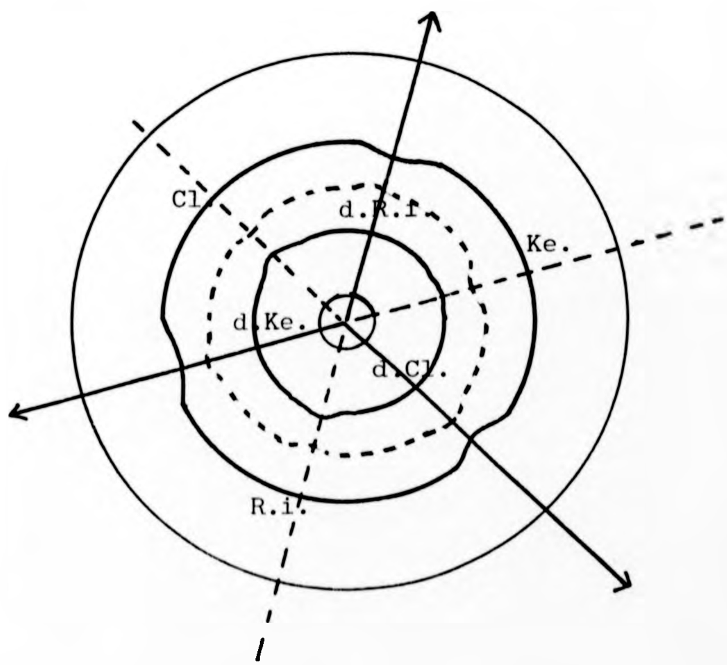


Fig.22 D

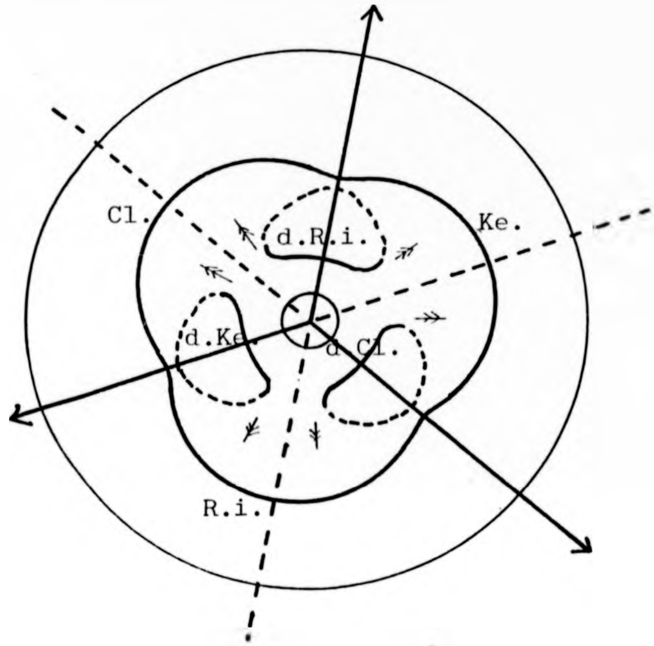
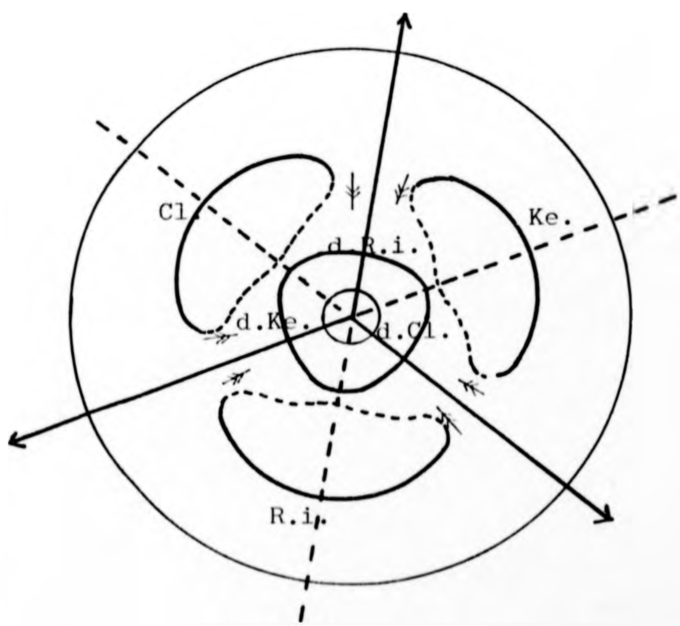


Fig.22 E



Here Fig. 22 A, which at first sight might seem to be the simplest case because h=k=0, in fact turns out to be the most degenerate case, because in Fig. 21 A the two cusps coincide. (The dashed cusp coincides with the firm cusp). Thus instead of smooth curves of equilibria we obtain six crossing points, at each of which there is an exchange of stability. Thus Fig. 22 A represents a return to Fig. 1 at the beginning of the chapter where we first started.

Fig. 22 B, C, D, E are obtained by small pertubations of this highly degenerate situation.

Fig. 22 B represents the case with both the original and the dual equilibria present in the long term evolution. In this case the changes between types of equilibria are usually not smooth, but take the form of sudden jumps between the equilibrium and the dual equilibrium faces. If, however, the radius of the circle is sufficiently small, the interval in Fig. 21 B will never cross the firm drawn cusp and so the changes between original and dual equilibria will be smooth. In this case the equilibrium set in Fig. 22 B consists of one circle. As the radius of the circle decreases the curve of equilibria shown in Fig 22 B with 12 folds pulls out into a convex curve, causing the folds & the dotted parts to disappear.

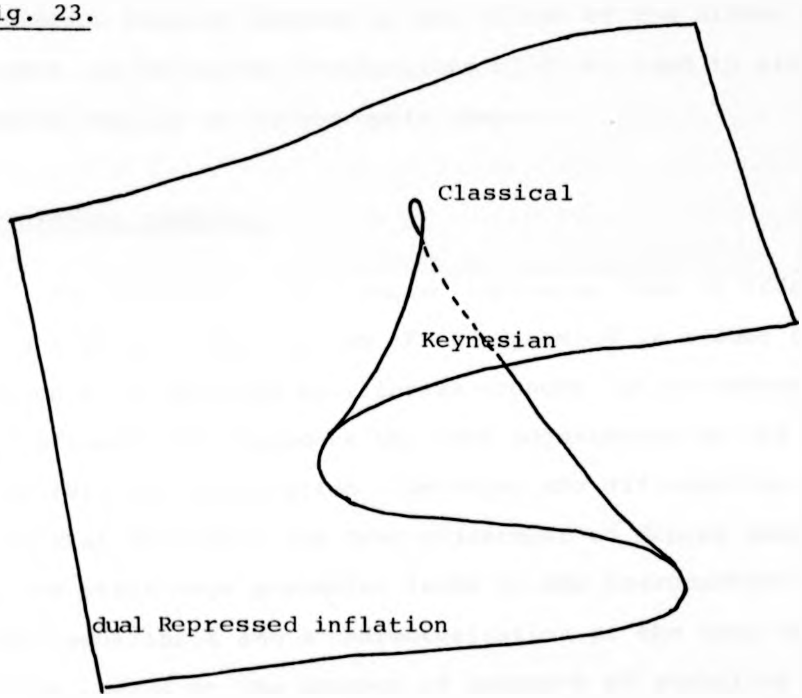
Fig. 22 C represents the situation with smooth changes between states of the original types of equilibria and smooth changes between dual types of equilibria. Depending on the history of the economy the actual state will be either one of the original types or one of the dual types, but no switches between the two types will take place.

Finally Fig. 22 D (E) represents the situation with smooth changes between the original equilibria (dual equilibria). In this case the economy may start out in a dual equilibrium (original equilibrium) situation, but as (p,w) reaches a boundary, there is a catastrophic jump to an original type of equilibrium (dual equilibrium), and the dual equilibria (original equilibria) will not be observed in the long run.

The above pictures show the vital importance of the hidden parameters. As they vary, so does the type of equilibrium that will persist in the long run. In order to obtain one possible economic interpretation of the hidden parameters, we shall now analyse the economic conditions, which initiate the catastrophic jumps between stable equilibria and dual stable equilibria, e.g. from a Classical to a dual Repressed inflation equilibrium.

The initial non-generic crossing of the boundary is depicted in Fig. 22 A (the values of the two hidden parameters being equal to zero). As proved above two new parameters must be introduced in order to obtain the generic description. Four different cases are shown in Fig. 22 B, C, D, E. As in the introductory example the hidden parameters may be interpreted as due to barriers on the perfect adjustments in the quantities. In the simple example one hidden parameter was introduced. In that example z was kept fixed, such that only x and y varied. The effect of the perfect adjustment barrier introduced by h was either to smooth out the edge between Classical and Keynesian equilibria or to separate the two types completely. In the general case z is also allowed to vary, and the generic description requires two hidden parameters.

Fig. 23.



Depending on the relative size of these two parameters there will either be an approximately perfect adjustment in the x variable, taking the state of the economy to a Keynesian equilibrium or an approximately perfect adjustment in z leading to a state of dual Repressed inflation. In the first case the change will be smooth, as x approaches zero when (p,w) approaches the Classical-Keynesian boundary. On the other hand, in case two the change will be sudden and take place exactly as the Classical equilibrium breaks down. The relative size of h and k might for example depend on government's presure on producers and consumers to obtain equilibrium either on the good market or the labour market (e.g. due to social tensions).

Altogether this description shows that the long term evolution heavily depends on the values of the hidden parameters, as different combinations of (h,k) lead to either smooth changes or catastrophic jumps.

Concluding remarks.

The analysis in this chapter indicates that in order to obtain a generic description of the dynamics in a long term evolution of a rationed equilibrium economy, it is necessary specifically to introduce the fast adjustments in the quantities into the description. Assuming the differential equation that describes the fast adjustment to depend smoothly on the price-wage parameter leads to the introduction of "new" equilibria and a characterization of the long term evolution, based on the concept of exchange of stability around the Walrasian equilibrium.

Due to the definition of the long term adjustment process in (p,w), however, the above analysis only covers the crossing of the boundaries Classical - Keynesian and Classical - Repressed inflation. In Chapter IV we shall specifically analyse the behaviour around the Repressed inflation - Keynesian boundary.

Further, when equilibria belong to the dual equilibrium quadrant, the economic conditions which determine the slow adjustment process in (p,w) are different from the situation with equilibria on the original equilibrium quadrant. Therefore a complete description of the long term evolution of the economy also requires the introduction of slow adjustment processes corresponding to the three dual types of equilibria.

Finally, at this state of the analysis the given interpretation of the hidden parameters is to be considered tentative - a more satisfying discussion will only be available, when further investigations of the conditions under which dual equilibria persist, have been carried out.

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APPENDIX TO Chapter II

"EXCHANGE OF STABILITY IN A DISEQUILIBRIUM MODEL"

Uniqueness of the stable equilibrium in each of the 3 equilibrium regions.

In the economic model, which forms the basis in the chapter on exchange of stability, it is tacitly assumed that there is uniqueness of the stable equilibrium point for (p,w) belonging to each of the 3 regions of Keynesian, Classical, and Repressed inflation equilibrium around the Walrasian equilibrium point. In Chapter II we made a reference to Hildenbrand and Hildenbrand, [1976], where a detailed discussion of the uniqueness conditions is presented. As our model, however, differs from theirs, a precise discussion of the uniqueness assumption will be presented in this appendix.

We shall proceed as in the Hildenbrand paper and take a specific look at the determination of the stable equilibrium points in our model for each of the 3 regions of equilibria.

Suppose (p,w) belongs to the Keynesian region. (p,w,x, y,z) is then a Keynesian equilibrium, if

$$x \equiv \hat{\varphi}\,(p,w,\ell^a) + g - \psi^a = 0$$

$$y \equiv \psi\,(p,w) - \psi^a > 0$$

$$z \equiv \hat{\ell}\,(p,w,\varphi^a) - \nu^a > 0.$$

Given (p,w), by varying ℓ^a (or φ^a) we obtain a curve in (x,y,z)-space. The Keynesian equilibrium points are determined by the intersection of this curve with the plane x = 0. Supposing (p,w) belongs to the Keynesian region, a Keynesian equilibrium point can graphically be represented as an intersection point between the 2 curves

(1) $\psi^a = \hat{\varphi}(p,w,\ell^a) + g$ (Aggregated demand)

(2) $\psi^a = F(\ell^a)$ (Aggregated supply)

which gives the following possibilities :

Fig. 1.A

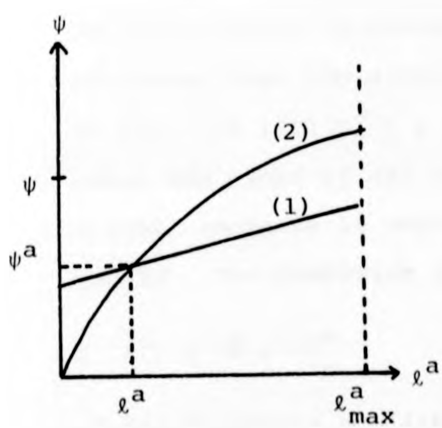


Fig. 1.B.

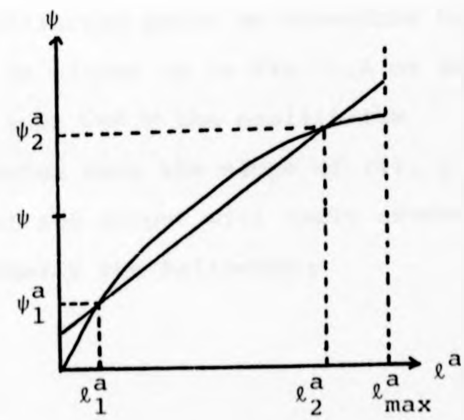


Fig. 1.C.

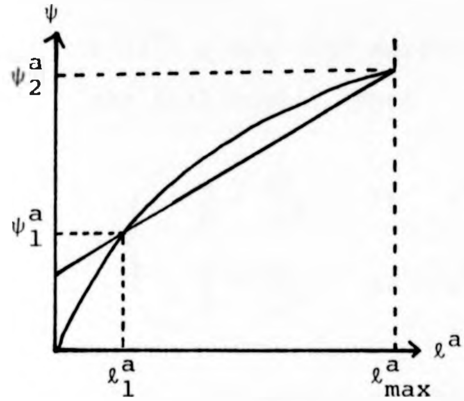
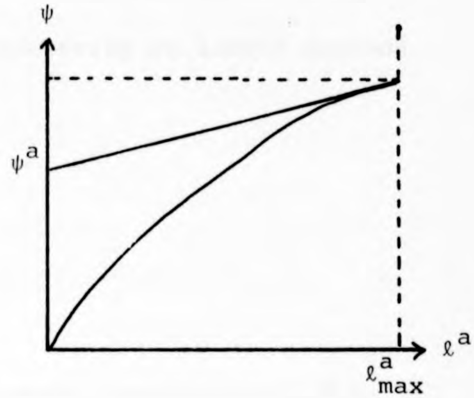


Fig. 1.D.



The function $\hat{\phi}(p,w,\ell^a)$ has been drawn as a linear function, due to the assumption on all agents being described by the same preference characteristics and the rationing scheme on the labour market. This implies that all the employed consumers, working the same amount of time, demand the same amount of the good. We notice that small perturbations of the curves in Fig. 1.C and Fig. 1.D will change these two pictures to the situations shown in Fig. 1.A and Fig. 1.B. (Fig. 1.A and Fig. 1.B are qualitative stable). In order to obtain a unique Keynesian equilibrium point we therefore have to assume that the situation is either as in Fig. 1.A or as in Fig. 1.B with $\psi_2^a \geq \psi \geq \psi_1^a$ such that at the equilibrium point the slope of (2) is greater than the slope of (1), i.e. a small increase in employment and output will imply excess supply. The condition is formally the following:

$$F'(\ell^a) \geq \partial\hat{\phi} \,/\, \partial\ell^a$$

at the Keynesian equilibrium.

Define $\delta = F'(\ell^a) - \partial\hat{\phi} \,/\, \partial\ell^a$. Then the condition for uniqueness of equilibrium is: $\delta \geq 0$. At a Keynesian equilibrium

$\psi^a \leq \psi$, so

$F'(\ell^a) \geq w/p$ (the marginal productivity of labour exceeds the real wage), hence

$$\delta \begin{cases} > \dfrac{w}{p} - \dfrac{\partial\hat{\varphi}}{\partial \ell^a} & \text{if} \quad \psi^a < \psi \\ = \dfrac{w}{p} - \dfrac{\partial\hat{\varphi}}{\partial \ell^a} & \text{if} \quad \psi^a = \psi . \end{cases}$$

Adding the budget constraints across consumers $p \cdot \hat{\varphi} = N \cdot m_o + w \cdot \ell^a$ (N is equal to the number of consumers) we get $p \cdot \dfrac{\partial\hat{\varphi}}{\partial \ell^a} = w$,
which implies

$$\delta \begin{cases} > 0 & \text{if} \quad \psi^a < \psi \\ = 0 & \text{if} \quad \psi^a = \psi . \end{cases}$$

This shows that the condition for uniqueness of Keynesian equilibrium is always fulfilled.

Suppose next that (p,w) belongs to the Classical region. (p,w,x,y,z) is a Classical equilibrium, if

$$x = \hat{\varphi}(p,w,\ell^a) + g - \psi^a > 0$$

$$y = \psi(p,w) - \psi^a = 0$$

$$z = \hat{\ell}(p,w,\varphi^a) - \nu^a > 0.$$

In this case uniqueness of the equilibrium point follows immediately:

$y = 0$ implies that ψ^a is equal to the profitmaximising production. So (ψ^a, ν^a) is determined. This determines $\varphi^a \equiv \psi^a - g$ and $\ell^a \equiv \nu^a$. Then $\hat{\phi}(p,w,\ell^a)$ and $\hat{\ell}(p,w,\varphi^a)$ are determined.

Suppose finally that (p,w) belongs to the Repressed inflation region. (p,w,x,y,z) is a Repressed inflation equilibrium if

$$x \equiv \hat{\phi}(p,w,\ell^a) + g - \psi^a > 0$$

$$y \equiv \psi(p,w) - \psi^a > 0$$

$$z \equiv \hat{\ell}(p,w,\varphi^a) - \nu^a = 0.$$

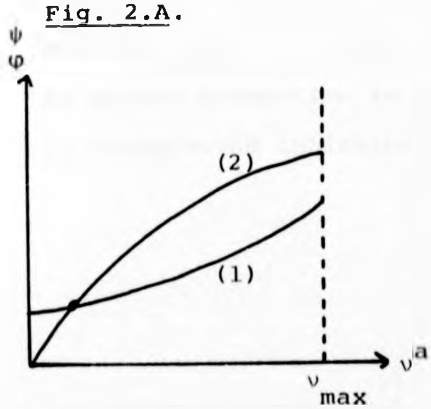
Graphically a Repressed inflation equilibrium may be represented as an intersection point between the 2 curves

(1) $\nu^a = \hat{\ell}(\varphi^a)$

(2) $\psi^a = F(\nu^a)$,

which gives the following possibilities:

<u>Fig. 2.A.</u>



<u>Fig. 2.B.</u>

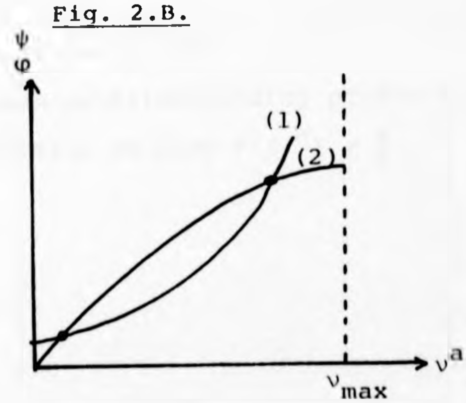


Fig. 2.C.

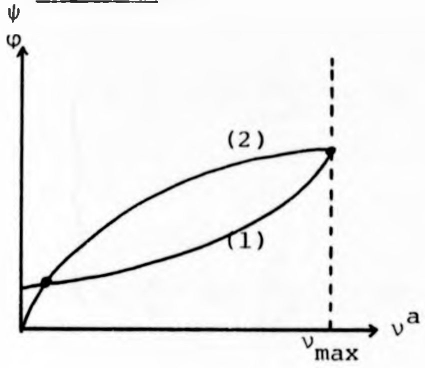
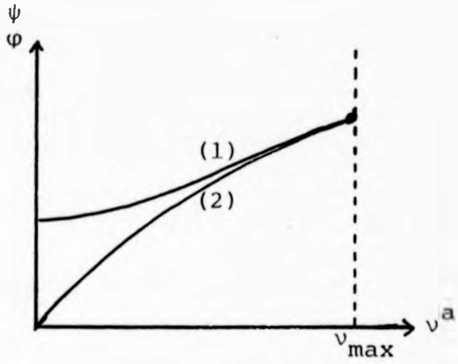


Fig. 2.D.



The function $\hat{\ell}(p,w,\varphi^a)$ has been drawn such that $\partial\hat{\ell}/\partial\varphi^a > 0$, $\partial^2\hat{\ell}/\partial{\varphi^a}^2 < 0$, expressing the idea that an increase in consumers' actual demand of the good implies an increase in effective supply of labour, though this increase becomes smaller when φ^a becomes larger.

As in the Keynesian situation a unique equilibrium corresponding to Fig. 2.A. or Fig. 2.B. is obtained by assuming the slope of (2) greater than the slope of (1) at the intersection point. Formally the condition is

$$\frac{\partial F'}{\partial \nu^a} \geq \frac{1}{\dfrac{\partial \hat{\ell}}{\partial \varphi^a}} .$$

Define $\beta = F'(\nu^a) - 1/\dfrac{\partial\hat{\ell}}{\partial\varphi^a}$.

The condition for unique equilibrium is then $\beta \geq 0$.

As actual production is less than profitmaximising production in a Repressed inflation equilibrium we have $F'(\nu^a) \geq \frac{w}{p}$.

Therefore

$$\beta \begin{cases} > \dfrac{w}{p} - 1/\dfrac{\partial \hat{\ell}}{\partial \varphi^a} & \text{if} \quad \psi^a < \psi \\ \\ = \dfrac{w}{p} - 1/\dfrac{\partial \hat{\ell}}{\partial \varphi^a} & \text{if} \quad \psi^a = \psi \ . \end{cases}$$

From the aggregated budget constraint $p \cdot \varphi^a = N \cdot m_o + w \cdot \ell^a$ we get $p = w. \dfrac{\partial \ell^a}{\partial \varphi^a}$,

which implies

$$\beta \begin{cases} > 0 & \text{if} \quad \psi^a < \psi \\ = 0 & \text{if} \quad \psi^a = \psi \ . \end{cases}$$

So the condition for uniqueness of the Repressed inflation equilibrium is always fulfilled in this model.

Chapter III

EXCHANGE OF STABILITY IN A RATIONED EQUILIBRIUM ECONOMY WITH VARIATIONS IN CONSUMERS' MONEYHOLDINGS AND GOVERNMENT'S DEMAND.

Introduction.

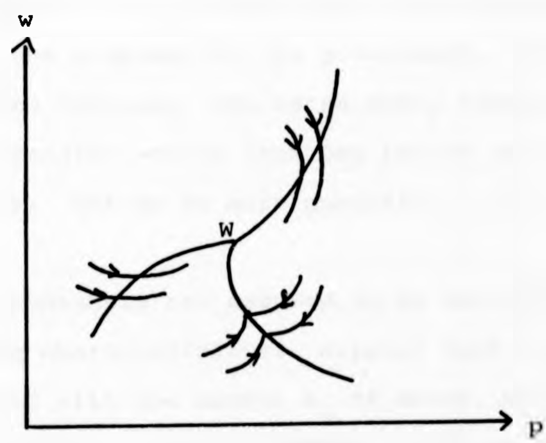
In our first attempt to describe the long term evolution of a rationed equilibrium economy by "exchange of stability" (see preceding chapter), we introduced the simplifying assumption that all consumers have the same fixed amount of money at the beginning of each period. This would be realized by introducing a taxation system, which ensured that at the beginning of a period each consumer was endowed with the same fixed amount of money as he had at the outset of the former period. However, such a system leaves no incentive for consumers to save money from one period to the next, because saving has no effect on next period's wealth. We shall in this chapter show how a more reasonable definition of the government's taxation principles, where at least the employed consumers may gain from saving, may still lead to a long run description of the economy's evolution as presented in our exchange of stability chapter.

To obtain this result we reformulate the model so that all unemployed consumers start the period with the same fixed amount of money. This change leads to the question of how changes in the size of this fixed unemployment benefit may influence the long term evolution. We shall give an answer to this question, and also consider the closely related problem of how changes in government's demand will change the evolution.

The model.

Our recent analysis of the long term evolution of a rationed equilibrium model is based on the introduction of a slow adjustment process in prices and wages, given by a differential equation, see Chapter I. Corresponding to this formulation we obtain the following diagram for the flow lines:

Fig. 1.

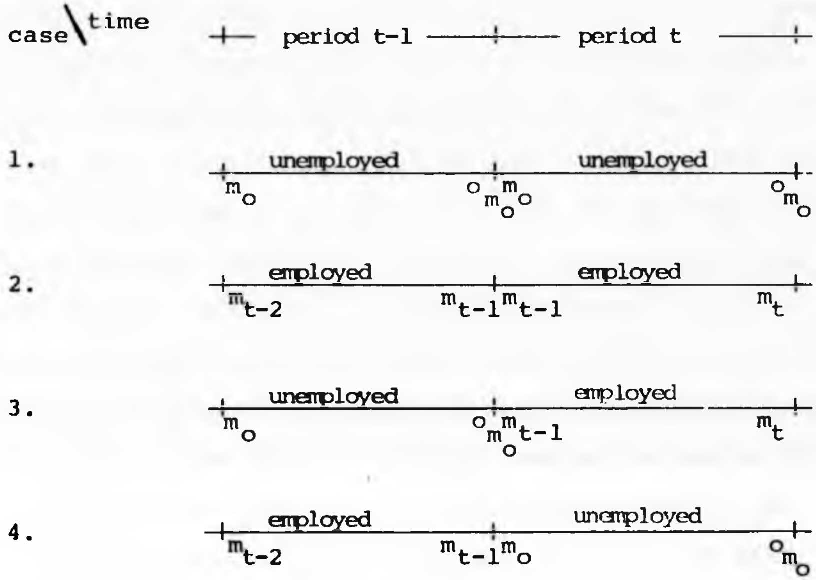


In this setting each consumer has the same fixed amount m_o of money at the beginning of each period. This can be interpreted as the result of a taxation system, which leaves no reason at all for consumers to carry money forward (i.e. to save) to the next period: if a consumer has not spent all of the amount m_o on the consumption good at the end of the period, the residual money will have to be paid to the government as taxes.

We now intend to analyze a situation where this restrictive assumption has been relaxed. In order to keep things simple we describe the variation in consumers' moneyholdings by a one dimensional variable. This is obtained by assuming that an unemployed consumer, who receives no labour income, is endowed with a fixed amount of money by the government in each period. The unemployed consumer will then choose to spend the total amount within that period on the consumption good, as any residual money at the end of the period only will result in a corresponding reduction in the amount paid to the consumer by the government. At the same time an employed consumer, who earns money wages by working, may wish to transfer wealth from one period to the next by storing money. Let us be more specific.

All consumers are assumed to be described by the same preference characteristics. Suppose each consumer is initially endowed with the amount m_o of money, which we also take to be the amount the government transfers to an unemployed consumer. After one period an unemployed consumer has spent all of his money, and he receives the amount m_o from the government. On the other hand, an employed consumer will finish the period with a new amount m_1 of money, which is the wealth he carries forward to the next period. In order to be able to describe the long term evolution of consumers' money-holdings by a one dimensional variable, we still need two further assumptions on the behaviour of consumers, who change from employment to unemployment and vice versa. Firstly,we require that a consumer who changes from being employed to being unemployed, has,via government transfers,a total amount m_o of money at the moment he has become unemployed, such that he is in exactly the same position as any other unemployed consumer when the next period starts off. Secondly,we assume that a consumer who changes from unemployment to employment, is given exactly the amount of money which each of the (former) employed consumers has saved up to this point in time. We would like to add a few remarks to these assumptions. As long as the amount of money saved by an employed consumer is smaller than the unemployment benefit m_o, these two assumptions are quite reasonable: when an agent becomes unemployed his initial moneyholdings are increased to m_o, which we may suppose corresponds to some "poverty line". On the other hand, when an unemployed agent succeeds in getting a job, the government reduces the amount transferred to him to a level equal to that of any other employed consumer, as the agent is now in a situation where he receives moneywages.

In order to clarify the configuration of the evolution of consumers' moneyholdings, we shall below sketch a diagrammatic representation of a few typical situations at time t.



Here the amount to the left of a vertical bar indicates how much money the consumer has at the end of the period. If he has been unemployed (and his moneyholdings therefore are zero), he is given the amount m_o, which is written underneath the bar. The figure on the right hand side of the bar is the amount of money the consumer is endowed with in the next period, when it has been established whether he is employed or not.

Allowing consumers to differ in moneyholdings makes it necessary to reformulate the description of the behaviour of the consumers, inherited from Malinvaud, [1977], and Hildenbrand & Hildenbrand, [1976]. We shall, however, assume that the actual moneyholdings at a given point in time are without any effect on the consumer's unconstrained supply of labour. In order to obtain a description of the consumers' behaviour, consider the general utility maximization problem for the single employed consumer: at a specific point in time the consumer has to determine his demand for the good, q, and his supply of labour, ℓ, over the periods of his planning horizon, given the present prices and expectations on the future. However, as shown in Grandmont, [1977], this intertemporal decision problem can be reduced to a single period problem by a standard dynamic programming technique for given expectations. In this way we can write the consumer's utility function as a function of current level of consumption and supply of labour, and money holdings at the end of this period. So the maximization problem is: max. $U(q, \ell, m)$ subject to the budget constraint $pq + m = w\ell + m_o$.

The first order conditions are

$$(1) \quad \frac{\partial U}{\partial q} = \lambda p; \quad \frac{\partial U}{\partial \ell} = -\lambda w; \quad \frac{\partial U}{\partial m} = \lambda \ ,$$

where λ is the Lagrange multiplier. These three equations combined with the budget constraint allow us to solve for λ, q, ℓ, m. In order to present an example of a utility function, which fulfills the above assumption, suppose U is of the form $U(q, \ell, m) = bq+f(\ell)+g(m)$, f decreasing. Then

$$(2) \quad \frac{\partial U}{\partial q} = b; \quad \frac{\partial U}{\partial \ell} = f'(\ell); \quad \frac{\partial U}{\partial m} = g'(m) \ .$$

This implies

$$(3)\quad \frac{\partial U}{\partial q} \Big/ \frac{\partial U}{\partial \ell} = \frac{b}{f'(\ell)} = -\frac{p}{w} .$$

If a change in m_o could result in a change in ℓ, this would violate the equation, as b, p, w are unchanged. Therefore the supply of labour is independent of m_o, given this specific form of U.

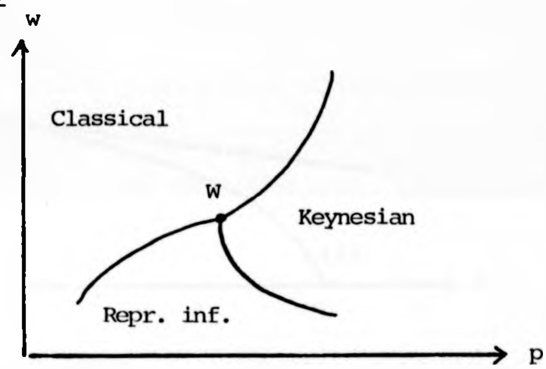
Having assumed the consumers' supply of labour to be independent of the actual money holdings, we need now a new specification of the consumers' demand for the good. In the following we restrict attention to the situation corresponding to an equilibrium in the Keynesian region (in the Malinvaud sense). A similar analysis will apply for the other regions. When consumers' money holdings are allowed to vary through time, the total demand for the good at a given point t in time will depend on u_t, m_{t-1}, and m_o.

Denote by p_t, w_t and u_t the price of the good, the price of labour, and the unemployment rate in period t. We then have that the total demand q_t^1 for the good in period t consists of the sum of the demand $\bar{q}$ from the consumers, who are unemployed in period t and the demand q from the consumers, who are employed in period t, i.e.

$$(4)\quad q_t^1 = u_t \cdot N \cdot \bar{q}(p_t, w_t, m_o) + (1 - u_t) \cdot N \cdot q(p_t, w_t, m_{t-1}) .$$

Having introduced the variable m_t into the analysis, we shall now have to analyze how the former partitioning of the (p,w) plane into a Keynesian, a Classical, and a Repressed inflation region

Fig. 2.



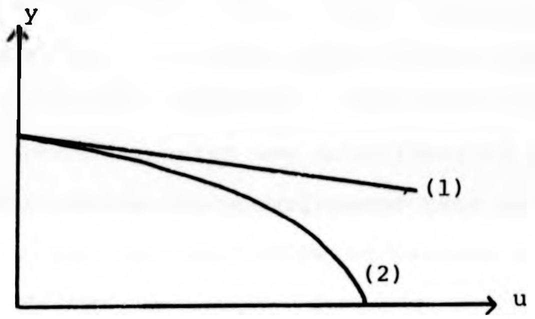
changes, when m_t is added as a third parameter. The new picture is produced by analyzing how the boundaries between the different equilibrium regions move, as m_t varies.

We first analyze what happens to the boundary between the Keynesian and the Repressed inflation region, when m_{t-1} differs from m_o. Let us assume $m_{t-1} < m_o$. Hildenbrand & Hildenbrand, [1976], (where moneyholdings are constant) show that equilibrium values of y and u on this boundary, where the unemployment rate is zero, are determined by the intersection between the following two curves, where g is government demand and all consumers have the same money holdings m_o.

Fig. 3.

Curve (1): $y = g + u\bar{q}(p,w) + (1-u)q(p,w)$ (Aggregated demand)

Curve (2): $y = F((1-u)\ell(p,w))$ (Aggregated supply)



In order to analyze what happens to a boundary point, when $m_{t-1} < m_o$, we shall therefore determine how these two curves move. As curve (2) deals only with the consumers' supply of labour, which we have assumed to be independent of moneyholdings, it is left unchanged. The equation for curve (1) is briefly stated as

$$y_t = q_t + g_t$$

where q_t denotes the total demand for the good from consumers and g_t is government demand. This implies that we must compare the total consumer demand with fixed moneyholdings,

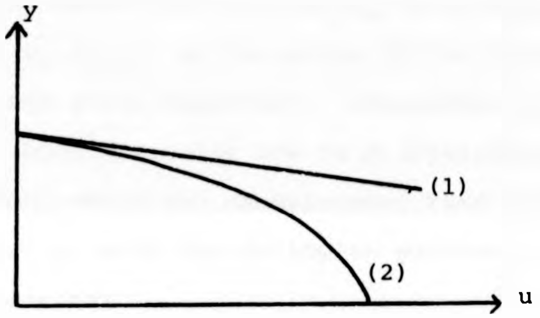
$$(5) \quad u_t \cdot N \cdot (\bar{q}(p_t,w_t,m_o) - q(p_t,w_t,m_o)) + N \cdot q(p_t,w_t,m_o),$$

with the new total demand, computed above,

<u>Fig. 3.</u>

Curve (1): $y = g + u\bar{q}(p,w) + (1-u)q(p,w)$ (Aggregated demand)

Curve (2): $y = F\ ((1-u)\ell(p,w))$ (Aggregated supply)



In order to analyze what happens to a boundary point, when $m_{t-1} < m_o$, we shall therefore determine how these two curves move. As curve (2) deals only with the consumers' supply of labour,which we have assumed to be independent of moneyholdings, it is left unchanged. The equation for curve (1) is briefly stated as

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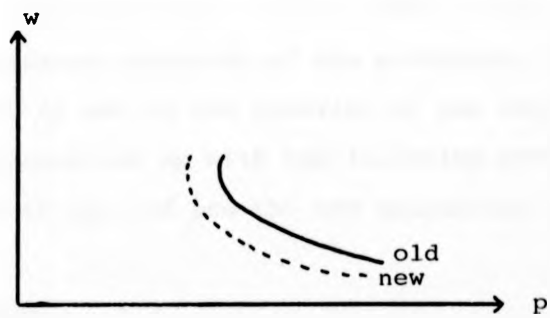
$$(5) \quad u_t \cdot N \cdot (\bar{q}(p_t,w_t,m_o) - q(p_t,w_t,m_o)) + N \cdot q(p_t,w_t,m_o),$$

with the new total demand, computed above,

(6) $u_t \cdot N \cdot (\bar{q}(p_t,w_t,m_o) - q(p_t,w_t,m_{t-1})) + N \cdot q(p_t,w_t,m_{t-1})$.

As $m_{t-1} < m_o$, assuming the consumption good to be normal, $q(p_t,w_t,m_o) > q(p_t,w_t,m_{t-1})$. Therefore the new (1)-curve intersects the y-axis at a lower point. The same type of argument gives that $\bar{q}(p_t,w_t,m_o)-q(p_t,w_t,m_o) < \bar{q}(p_t,w_t,m_o) - q(p_t,w_t,m_{t-1})$, so the slope of the line will be less steep (though still negative). Altogether this implies that a former boundary point now is an interior point in the Keynesian region, where the unemployment rate is positive. Graphically we end up with the following picture of the old and the new boundaries:

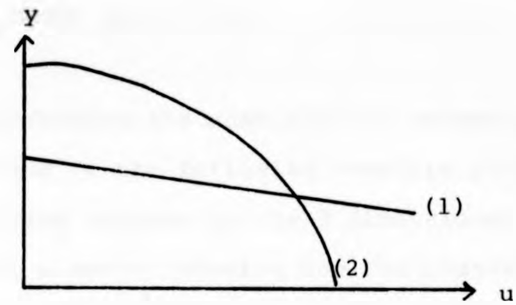
Fig. 4.



Symmetrically we get that this boundary moves to the right, when $m_{t-1} > m_o$.

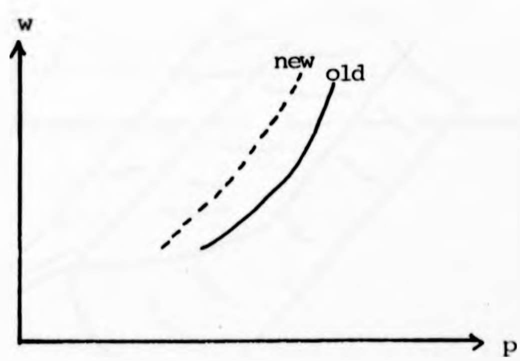
Next we investigate how the boundary between the Keynesian and the Classical region moves, when $m_{t-1} < m_o$. The initial situation looks as follows:

Fig. 5.



There is a positive unemployment rate, whereas producers produce at the unconstrained profit maximising level. Introducing the new money holding m_{t-1} for an employed consumer will push the curve (1) downwards and make it less steep. This implies that the production at the intersection point between the two curves will be less than optimal from the profit maximising viewpoint of the producers, i.e. the equilibrium point is now in the interior of the Keynesian region. We therefore end up with the following graphical representation of the old and the new boundaries.

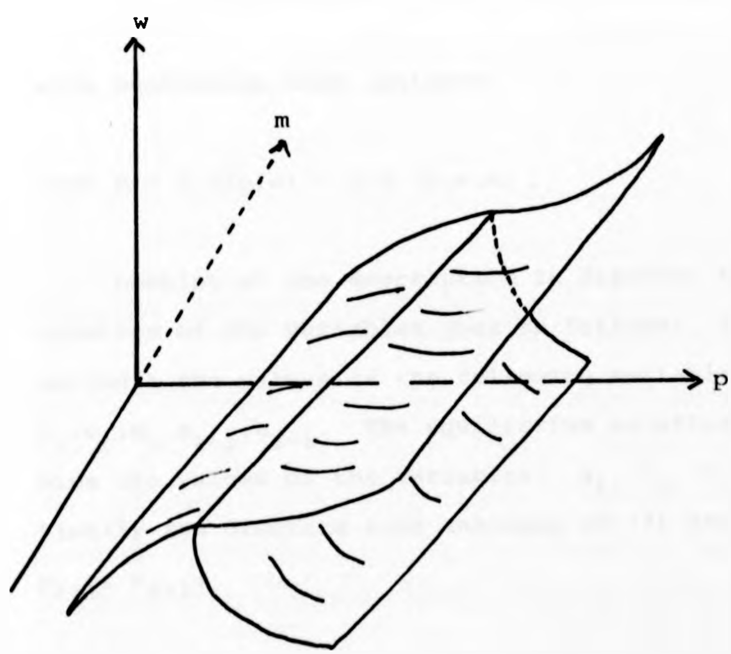
Fig. 6.



Symmetrically we get for $m_{t-1} > m_o$ that the boundary moves to the right hand side.

Performing the same type of argument for the third boundary leads to the following complete picture of the three equilibrium regions in the 3 dimensional space with coordinates p, w and m, showing how the boundaries shift position in accordance with the preceding results, as m varies. (Strictly speaking the picture is obtained only when the above argument has been applied to the general case, where the employed consumers' moneyholdings change from m_{t-1} to m_t. But the argument is unchanged.)

<u>Fig. 7.</u>



We shall now state (in the continuous time version) the equations which describe the evolution in prices, wages and money holdings through time.
As in our earlier analysis, Chapter I, we assume

$$(7)\quad \dot{p} = K_p\,[q(p,w,m_o,m,u) + g - y^*(p,w)]$$

$$(8)\quad \dot{w} = K_w\,[F^{-1}(y(p,w)) - \ell^*(p,w)]$$

for (p,w) belonging to the Keynesian region. To this description we shall add an equation determining the evolution of the employed consumers' moneyholdnings. In discrete time the budget constraint in period t for an employed consumer reads:

$$(9)\quad m_{t-1} + w_t\,\ell(p_t,w_t) = p_t \cdot q_t\,(p_t,w_t,m_{t-1}) + m_t$$

with continuous time analogue

$$(1o)\quad \dot{m} = w\,\ell(p,w) - p\cdot q\,(p,w,m)\,.$$

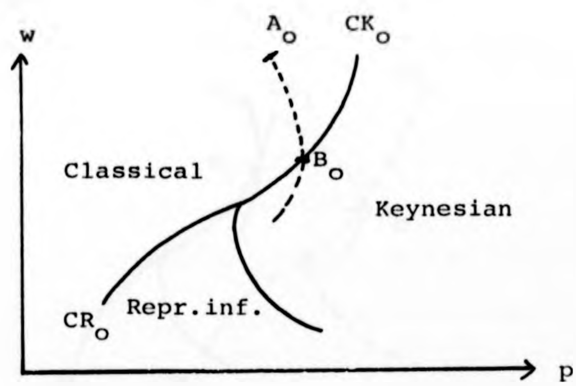
Looking at the description in discrete time, the determination of the variables goes as follows: at the outset of period t the values of the following variables are known: $p_t, w_t, m_o, m_{t-1}, u_{t-1}$. The equilibrium equations then determine the values of the variables: q_t, ℓ_t, u_t, y_t and m_t. Finally the discrete time analogue of (7) and (8) determine p_{t+1}, w_{t+1}.

In a continuous time representation this corresponds to an evolution determined by differential equations, where at a given point in time, the variables' actual values are known together with their changes at that point in time. The above description therefore completely determines the evolution of the system.

The flow diagram.

Next we want to analyze what happens to the basic flow diagram,

Fig. 8 A

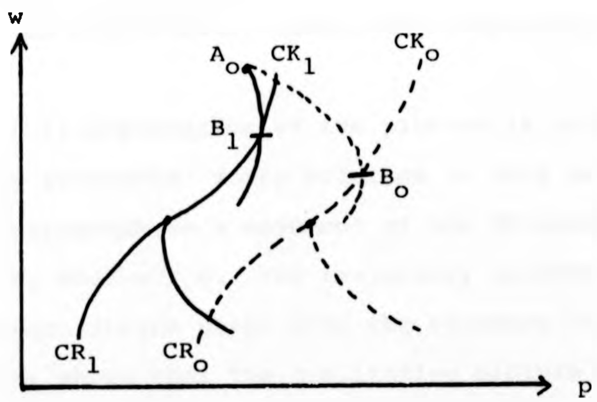


when the m-variable is introduced as a third coordinate. In order to keep the drawings simple we look in the analysis to follow at the projection of the 3 dimensional (p,w,m)-space onto the (p,w)-plane. Let us first consider the trajectory through the point A_o in the Classical region, which crosses the boundary CK_o to the Keynesian region at the point B_o. Depending on the sign of $\dot{m}$ there are two cases to study.

I. Suppose $\dot{m} < 0$.

Then the boundary CK_o moves to the left as the equilibrium point moves along the trajectory. As a result of the changes in consumers' money holdings the trajectory through A_o will change too, such that we end up with the following picture, where the dotted lines describe the initial situation with money holdings kept fixed, and the firm lines represent the picture exactly at the moment when the equilibrium point hits the boundary CK_1 at B_1, with A_o as the initial point (i.e. at time t = 0).

Fig. 9 A.



To see that the new trajectory is steeper than the old one, look at the equations determining the evolution:

$$(7) \quad \dot{p} = K_p[q(p,w,m_o,m,u) + g - y^*(p,w)]$$

$$(8) \quad \dot{w} = K_w \, [F^{-1}(y(p,w)) - \ell^*(p,w)]$$

As the $\dot{w}$-equation is independent of m, the changes in the evolution will be due to changes in $\dot{p}$.

Consider

$$(11)\quad \dot{p}^{(1)} = K_p\,[q(p,w,m_o,m^{(1)}, u) + g - y^*(p,w)]$$

and

$$(12)\quad \dot{p}^{(2)} = K_p\,[q(p,w,m_o,m^{(2)},u) + g - y^*(p,w)]$$

with $m^{(1)} > m^{(2)}$. As

$$(13)\quad q(p,w,m_o,m,u) = u \cdot N \cdot \bar{q}\,(p,w,m_o) + (1-u)\cdot N \cdot q(p,w,m)$$

we have $\dot{p}^{(1)} > \dot{p}^{(2)}$, implying that the changes in p will be smaller as m decreases, making the trajectory steeper.

The interpretation of the picture is as follows: allowing consumers' money holdings to vary as described above correponds to a movement of the CK-boundary towards the left, when $\dot{m} < 0$. The trajectory through Λ_o traced out by the equilibrium point hits the boundary CK_1 at the point B_1. This shows that the qualitative picture of the evolution has not changed.

Suppose now that Λ_o had been situated further to the "southwest" such that the initial trajectory through Λ_o crossed the boundary CR_o to the Repressed inflation region:

Consider

$$(11)\quad \dot{p}^{(1)} = K_p \, [q(p,w,m_o,m^{(1)}, u) + g - y^* \, (p,w)]$$

and

$$(12)\quad \dot{p}^{(2)} = K_p \, [q(p,w,m_o,m^{(2)},u) + g - y^* \, (p,w)]$$

with $m^{(1)} > m^{(2)}$. As

$$(13)\quad q(p,w,m_o,m,u) = u \cdot N \cdot \bar{q} \, (p,w,m_o) + (1-u) \cdot N \cdot q(p,w,m)$$

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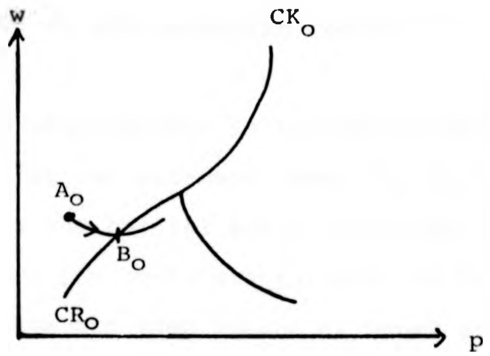
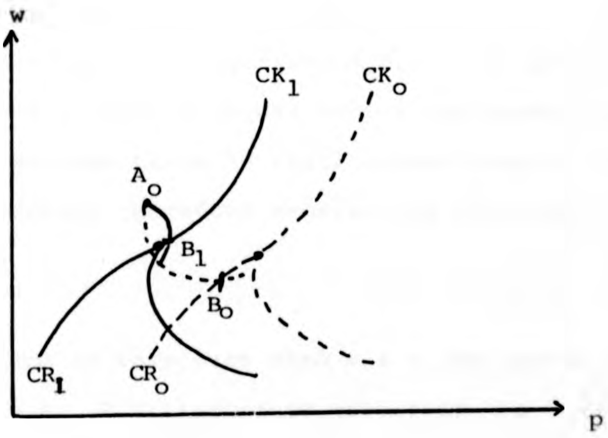
Suppose now that A_o had been situated further to the "southwest" such that the initial trajectory through A_o crossed the boundary CR_o to the Repressed inflation region:

Fig. 8 B.



Still supposing $\dot{m} < 0$ this case would either lead to an analogous situation as above with the trajectory once a-gain crossing the CR-boundary, or the following situation might happen:

Fig. 9 B.



In this case it is seen that the introduction of consumers' money holdings has in fact resulted in a noticeable change of the long run evolution, as the trajectory no longer moves into the Repressed inflation region, but instead crosses the boundary to the Keynesian region.

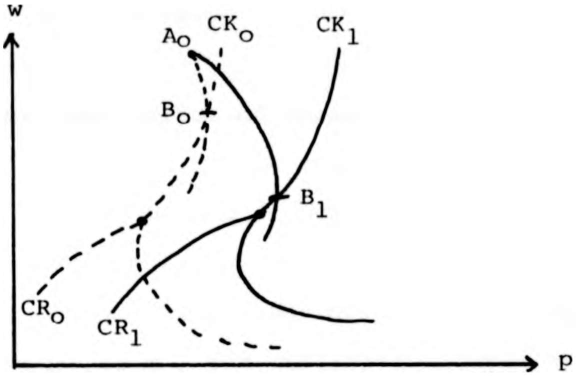
To see what happens in the situation shown in Fig. 9 B consider first the movement along the path A_oB_o in discrete time. (A_o is the initial point, situated in the Classical region, where consumers are rationed on both markets). All consumers have the same amount of money at the outset of each period. The changes in prices and wages lead to the point B_o at the Classical - Repressed inflation boundary, where there is equilibrium on the labour market and excess demand on the good market. Next allow variations in the employed consumers' moneyholdings and suppose $\dot{m}$ negative. This will change both the partitioning of the (p,w)-plane and the (p,w)-trajectory. The increased demand from the unemployed consumers pushes the price of the good upwards. If the price changes are sufficiently large, the (p,w) combination will now move along the trajectory $A_o\dot{B}_1$. At B_1 prices have increased to a level to equal supply and demand on the good market, whereas there is still excess supply of labour. The new trajectory therefore crosses the Classical - Keynesian boundary.

Notice in this case with $\dot{m} < 0$ the opposite could not happen, i.e. if a trajectory goes from the Classical to the Keynesian region with m fixed, it will continue to do so, also after allowing m to vary.

II. Suppose $\dot{m} > 0$.

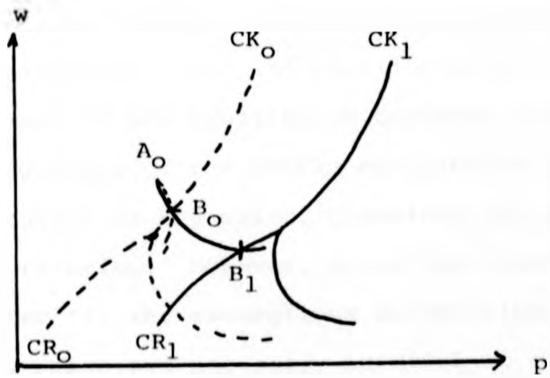
In this case the boundary CK moves to the right as the equilibrium point moves along the trajectory. The picture when the equilibrium crosses the boundary, now looks as follows:

Fig. 10 A.



The interpretation is as above. Again it will be seen that the qualitative picture has not changed. However, the picture might look different as shown below:

Fig. 10 B.



In this case the trajectory initially (i.e. leaving m out of the analysis) crosses the CK_o boundary. Introducing m changes the evolution such that the crossing point now lies on the CR_1-boundary. An economic interpretation of this situation can be given analogous to the one presented above. Again we notice that if the initial trajectory crosses the CR_o-boundary, then so will the trajectory also, when m is allowed to vary.

The implication of these observations is that our initial diagram, Fig. 1, no longer represents the global long term evolution of the economy, independent of the variation in consumers' money holdings. On the other hand, at a given point t in time with money holdings for employed consumers being equal to m_t, the qualitative description of the long term evolution of the economy is still locally (i.e. around the boundaries of the Classical region) represented by a phase portrait as the one given in Fig. 1.

The long term evolution.

In our former analysis of the exchange of stability, Chapter II, we obtain a correspondence between points (p,w) in the parameter plane and equilibrium points (x,y,z) on the surface of the equilibrium quadrant by implicitly assuming uniqueness of the stable equilibrium in each of the three regions of Keynesian, Classical and Repressed inflation equilibrium. However, as we have shown in the appendix to Chapter II, the assumptions on the slope of the curves (1) and (2), which are necessary in order to obtain this uniqueness, are always fulfilled. Therefore no further assumptions have to be added to the earlier formulation in order to obtain the complete description. As we are now analyzing the possibility of extending the former description of the long term evolution of the economy by exchange of stability to a situation where employed consumers' money holdings vary as introduced earlier, we shall investigate how the uniqueness of the stable equilibria can be guaranteed in this more general case. Having uniqueness of the stable equilibrium in each of the three regions, the qualitative configuration of equilibria at a given point in the price-wage plane is unchanged, namely a stable, an unstable and a dual stable equilibrium. The analysis of the long term evolution of the fixed money holdings case therefore applies to this situation too.

As we are now extending the results from Chapter II, we will use the notation introduced in that chapter. Consider the effective demand $\Phi(p,w,\ell^a)$ for the good. It is the total demand from employed and unemployed consumers. Due to the rationing hypothesis it can therefore be written as

$$\hat{\varphi}(p,w,\ell^a) = \hat{\varphi}_1(p,w,\ell^a) + \hat{\varphi}_2(p,w,\ell^a) \tag{14}$$

where $\hat{\varphi}_1$ is total demand from unemployed consumers and $\hat{\varphi}_2$ is total demand from employed consumers. In this formulation all consumers are endowed with the same amount m_o of money, so in fact the equation reads ness, are always fulfilled. Therefore no further assumptions have to be added to the earlier formulation in order to obtain the complete description. As we are now analyzing the possibility of extending the former description of the long term evolution of the economy by exchange of stability to a situation where employed consumers' money holdings vary as introduced earlier, we shall investigate how the uniqueness of the stable equilibria can be guaranteed in this more general case. Having uniqueness of the stable equilibrium in each of the three regions, the qualitative configuration of equilibria at a given point in the price-wage plane is unchanged, namely a stable, an unstable and a dual stable equilibrium. The analysis of the long term evolution of the fixed money holdings case therefore applies to this situation too.

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where $\hat{\varphi}_1$ is total demand from unemployed consumers and $\hat{\varphi}_2$ is total demand from employed consumers. In this formulation all consumers are endowed with the same amount m_o of money, so in fact the equation reads

(15) $\hat{\varphi}(p,w,\ell^a,m_o) = \hat{\varphi}_1(p,w,\ell^a,m_o) + \hat{\varphi}_2(p,w,\ell^a,m_o)$

Next we introduce the new assumption on consumers' money holdings. The total demand is now written as

(16) $\hat{\varphi}(p,w,\ell^a,m_o,m) = \hat{\varphi}_1(p,w,\ell^a,m_o) + \hat{\varphi}_3(p,w,\ell^a,m).$

Here a new function has been introduced to represent the employed consumers' demand. This is necessary because the introduction of employed consumers' possibility of transferring money from one period to the next implies a change in the consumers' utility of saving and therefore a change in the consumer's demand function. An unemployed consumer's demand function is unchanged, as he still is supposed to spend all of his money holdings in each period. The equations defining the three types of equilibria are now

$$x = \hat{\varphi}(p,w,\ell^a,m_o,m) + g - \psi^a$$
$$y = \psi(p,w) - \psi^a$$
$$z = \hat{\ell}(p,w,\varphi^a) - \nu^a$$

with the same definition as hitherto of

Keynesian equilibrium:	$(0,y,z)$,	$y > 0$, $z > 0$
Classical equilibrium:	$(x,0,z)$,	$x > 0$, $z > 0$
Repressed inflation equilibrium:	$(x,y,0)$,	$x > 0$, $y > 0$

Let us first consider the condition for uniqueness of the Keynesian equilibrium. As shown in the appendix to Chapter II, the condition is

$$F'(\ell^a) \geq \frac{\partial\hat{\varphi}(p,w,\ell^a,m_o,m)}{\partial\ell^a} .$$

Defining $\delta = F'(\ell^a) - \partial\hat{\varphi}/\partial\ell^a$ the condition is $\delta \geq 0$.
Using that $F'(\ell^a) \geq w/p$ in a Keynesian equilibrium, we have

$$\delta \begin{cases} > & \frac{w}{p} - \frac{\partial\hat{\varphi}}{\partial\ell^a} \qquad \text{if} \quad \psi^a < \psi \\ = & \frac{w}{p} - \frac{\partial\hat{\varphi}}{\partial\ell^a} \qquad \text{if} \quad \psi^a = \psi . \end{cases}$$

Summing the budget constraints we now have the following

$$p \cdot \hat{\varphi} + k(\ell^a) \cdot \dot{m} = (N - k(\ell^a)) \cdot m_o + w \cdot \ell^a ,$$

where $k(\ell^a)$ is the number of employed consumers (k is equal to $\ell^a/\ell(p,w,m)$. As the supply of labour is assumed independent of m, we have for fixed (p,w) that k depends only on ℓ^a). This implies

$$p \frac{\partial\hat{\varphi}}{\partial\ell^a} + \frac{\partial(k(\ell^a) \cdot \dot{m})}{\partial\ell^a} = - \frac{\partial k(\ell^a)}{\partial\ell^a} \cdot m_o + w ,$$

and so

$$\frac{\partial\hat{\varphi}}{\partial\ell^a} = (- \frac{\partial k}{\partial\ell^a} m_o - \frac{\partial(k \cdot \dot{m})}{\partial\ell^a}) \cdot \frac{1}{p} + \frac{w}{p} .$$

Therefore

$$(21)\quad \delta \begin{cases} > \left[\dfrac{\partial(km_o)}{\partial \ell^a} + \dfrac{\partial(k \cdot \dot{m})}{\partial \ell^a}\right] \dfrac{1}{p} & \text{if } \psi^a < \psi \\ = \left[\dfrac{\partial(km_o)}{\partial \ell^a} + \dfrac{\partial(k\dot{m})}{\partial \ell^a}\right] \dfrac{1}{p} & \text{if } \psi^a = \psi . \end{cases}$$

It follows that a sufficient condition for uniqueness of the Keynesian equilibrium is

$$\frac{\partial(km_o)}{\partial \ell^a} + \frac{\partial(k\dot{m})}{\partial \ell^a} \geq 0 ,$$

which says that the result of an increase in employment must be such that the sum of the induced reduction in the aggregated demand by unemployed consumers and the induced change in the aggregated savings by employed consumers is non-negative.

In the Classical region the equilibrium point is uniquely determined, so we now turn to the Repressed inflation case. As in the appendix to Chapter II the condition for uniqueness is $F'(\nu^a) \geq 1/ \dfrac{\partial \hat{\ell}}{\partial \varphi^a}$.

Defining $\beta = F'(\nu^a) - 1/ \dfrac{\partial \hat{\ell}}{\partial \varphi^a}$ the condition is $\beta \geq 0$.

Using $F'(\nu^a) \geq \frac{w}{p}$ in the Repressed inflation equilibrium, we get

$$\beta \begin{cases} > \dfrac{w}{p} - \dfrac{1}{\frac{\partial \hat{\ell}}{\partial \varphi^a}} & \text{if } \psi^a < \psi \\ = \dfrac{w}{p} - \dfrac{1}{\frac{\partial \hat{\ell}}{\partial \varphi^a}} & \text{if } \psi^a = \psi . \end{cases}$$

The sum of consumers budget constraints is $p \cdot \varphi^a + N \cdot \dot{m} = w \cdot \ell^a$, which implies $p + N \cdot \frac{\partial \dot{m}}{\partial \varphi^a} = w \cdot \frac{\partial \hat{\ell}}{\partial \varphi^a}$ and therefore $1/\frac{\partial \hat{\ell}}{\partial \varphi^a} = w/(p + N \cdot \frac{\partial \dot{m}}{\partial \varphi^a})$.
So

$$(22) \quad \beta \begin{cases} > \dfrac{w}{p} - \dfrac{w}{p+N\cdot\dfrac{\partial \dot{m}}{\partial \varphi^a}} & \text{if} \quad \psi^a < \psi \\ = \dfrac{w}{p} - \dfrac{w}{p+N\dfrac{\partial \dot{m}}{\partial \varphi^a}} & \text{if} \quad \psi^a = \psi \end{cases} .$$

Sufficient conditions for uniqueness are therefore

(i) $N \cdot \frac{\partial \dot{m}}{\partial \varphi^a} \geq 0$ or (ii) $N \cdot \frac{\partial \dot{m}}{\partial \varphi^a} < -p$,

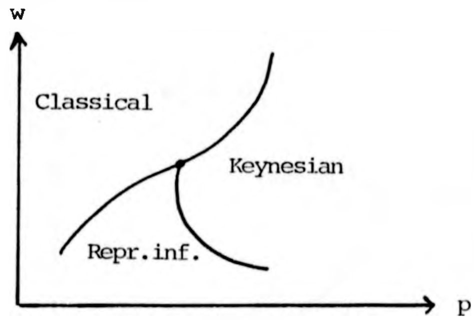
which can be interpreted as expressing the requirement that either savings must increase with increases in the realized demand or savings must strictly decrease, being bounded away from zero by the constant $-\frac{p}{N}$.

Supposing these conditions fulfilled, the description of the long term evolution by exchange of stability immediately applies without further modifications.

Variations in consumers' unemployment benefits.

Having introduced the separation in the description of employed and unemployed consumers' moneyholdings such that the employed consumers may benfit from savings, whereas the unemployed consumers maximise the utility by spending all of their unemployment benefits in each period, the question arises of how the long term evolution will be influenced by changes in the unemployment benefits. Specifically we want to determine the variations in our basic diagram :

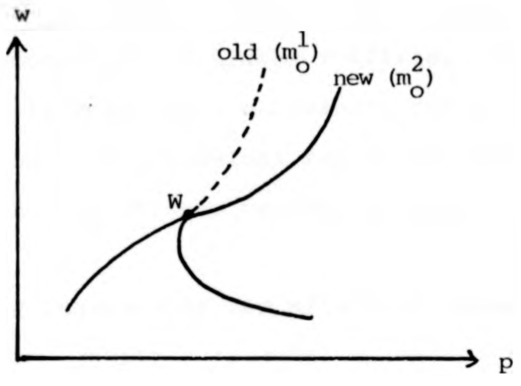
Fig. 2



In Malinvaud's static model, [1977], a simple comparative static argument is given in order to analyze the behaviour of consumers, when they are given two different initial moneyholdings: as every consumer has the same amount m_o of money at the outset of the period, it is easy to see that the diagram in Fig.2 will move in the "north-east" direction, when m_0 is increased.

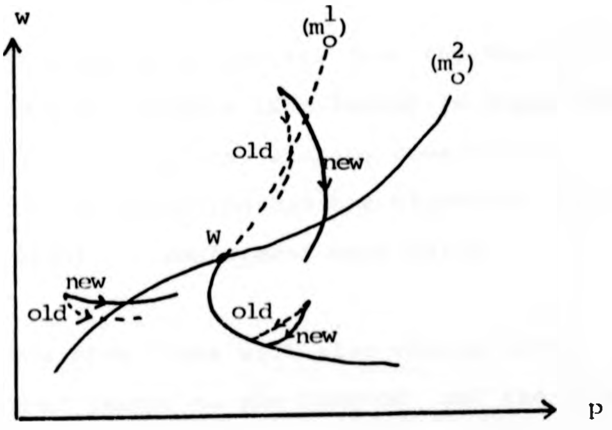
In our model, however, the situation is slightly different. Above we introduced the assumption that changes in the consumers' moneyholdings are without any effect on the supply of labour. Therefore a change in m_o, which is the amount of money transferred to a consumer at the end of a period of unemployment, will only affect the agent's demand of the good. The Repressed inflation region is defined as the region, where consumers are rationed on the good market and unrationed on the labour market. Therefore the above assumption implies that the boundaries defining this region stay fixed under variations in m_o. On the other hand, when m_o is increased, the demand for the good from the unemployed consumers increases. This will make the boundary between the Keynesian and the Classical regions move around clockwise, implying the following picture:

Fig. 11 $\qquad m_o^1 < m_o^2$



Considering the slow adjustment processes in (p,w), a change in m_o induces no change in $\dot{w}$ due to our assumption, whereas an increase in m_o will result in an increase in the value of $\dot{p}$. This corresponds to the following change in the basic flow diagram:

Fig. 12 $\qquad m_o^1 < m_o^2$



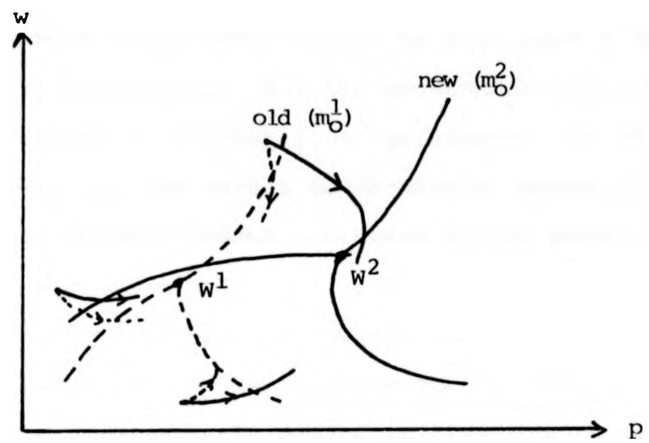
The economic consequences of varying the unemployment benefits are obvious: increasing m_o will imply an increase in the demand of the good by the unemployed consumers. As it is an implicit assumption in this type of model that the demand from the unemployed is always fulfilled, it follows that the employed will more often experience rationing on the good market, such that the Classical region is enlarged, while the Keynesian region shrinks correspondingly.

In the analysis of the effect of allowing employed consumers' money holdings to vary, m_o was given fixed. Therefore the assumption on the unemployed consumers' independence of variation in m_o was never needed. We are now comparing situations with different unemployment benefits. It seems therefore reasonable to replace the above m_o-independence assumption on the supply of labour by the less restrictive assumption that supply of labour $\ell(p,w,m_o)$ by a person, who was unemployed last period, is a decreasing function of the unemployment benefits. We shall shortly discuss the modifications to the analysis that this more realistic requirement necessitates.

Increasing m_o implies that the Repressed inflation region is enlarged, because less labour is supplied by the formerly unemployed consumers, whereby some of the unemployment equilibria in the Keynesian and the Classical regions are transformed into total employment equilibria.

The flow lines will also change because the decreased supply of labour in the Classical and the Keynesian regions will make the changes in w smaller. The analogous picture to Fig.12. is now

Fig. 13. $m_o^1 < m_o^2$



Notice that the new flow lines in the Keynesian region may be either steeper or less steep than the old lines, depending on the relative changes in $\dot{p},\dot{w}$.

The economic consequences of an increase in the unemployment benefits are as follows: part of the consumers' demand for labour will be transferred to a demand for the good, making the region of Keynesian equilibria smaller.

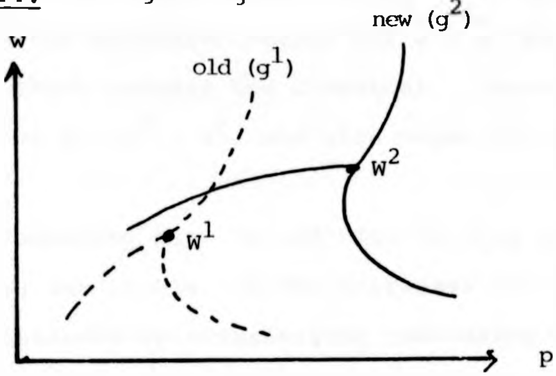
Variations in government demand.

Next we want to consider how changes in government demand q will change the basic diagrams, Fig. 1 and Fig. 2. It is an implicit assumption in the comparative static discussion to follow that variations in q must be kept inside a certain range, which guarantees the existence of rationed equilibria (i.e. g is never allowed to be increased to a size, where demand from unemployed consumers plus government cannot be met).

Before we go into details we will make a small digression in order to consider how the government finances its demand. As mentioned in Chapter I, by government (or exogenous) demand we understand the actual government's demand plus the producers' investment demand. Looking at the macro accounting equalities we have $Y = C + g$, $W = C + S$, $Y = P + W$, $\dot{M} = P + S$ where Y is production, C consumption, W wages, S savings, P profits, and $\dot{M}$ changes in the stock of money. This implies $g = \dot{M}$, so government's demand is financed by the issuing of money.

Increasing g implies that total demand for the good increases. This may lead to increases in the producers' demand for labour and this again to increases in consumers' demand for the good (the multiplier). Therefore a part of the former Keynesian equilibrium region, where consumers are rationed on the labour market and producers are rationed on the good market, will change to equilibria, where either consumers are rationed on both markets (Classical equilibria), or consumers are rationed on the good market and producers are rationed on the labour market (Repressed inflation equilibria). There will also be a region which shifts from Classical to Repressed inflation equilibria. The total result of an increase in g is therefore an upward shift in the diagram, Fig. 2, in the "northeast" direction.

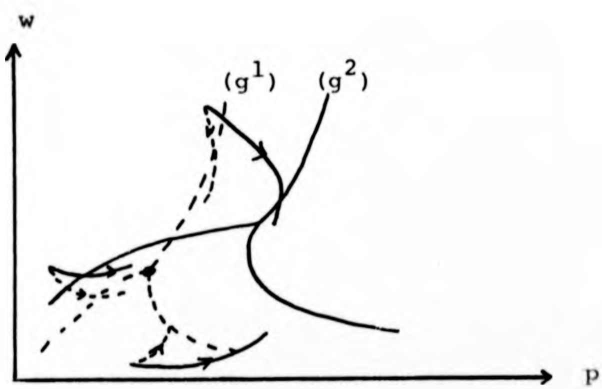
Fig. 14. $g^1 < g^2$



When g is increased, it will directly increase the $\dot{p}$-coordinate of the slow flow, and only indirectly via changes in consumers and producers actual behaviour it will also change the $\dot{w}$-coordinate. It is easy to check that $|\dot{w}|$ becomes smaller in all three regions, while $|\dot{p}|$ decreases in the Keynesian region and increases in the other two regions.

This leaves us with the following changes in the flow diagram:

Fig. 15. $g^1 < g^2$



We notice that a flow line which passes from the Classical to the Keynesian region for $g = g^1$ may change to a flow line, which crosses the Classical - Repressed inflation boundary for $g = g^2 > g^1$, and vice versa for g decreasing.

Comparing Fig. 13 and Fig. 15 they seem very similar. But whereas in Fig. 13 the increases in the total demand have been obtained by transferring increasing amounts of wealth to the unemployed, the changes in Fig. 15 have been realized by increases in the external demand for the good.

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Chapter IV

OSCILLATIONS BETWEEN REPRESSED INFLATION AND KEYNESIAN EQUILIBRIA DUE TO INERTIA IN DECISION MAKING.

Introduction.

As a first step towards a formulation of a dynamic disequilibrium model we add in Chapter I a long term description of changes in prices and wages to the static macro model introduced in Malinvaud's book (see Malinvaud, [1977]). This, however, leads to problems around the boundary between the Repressed inflation equilibrium region and the Keynesian equilibrium region, because the vector field describing the adjustments is discontinuous on this boundary. One way to overcome this difficulty is to assume a specific behaviour on the boundary as suggested in that paper. Another solution is presented in Chapter II, where it is shown that the transition between different regions of equilibria depends on the values of certain "hidden" parameters. Whenever these parameters take on values such that the transition between Repressed inflation and Keynesian equilibria goes via some "new" states of equilibria (called dual Classical equilibria), prices and wages will never reach the boundary, so there is no need to define the vector field in a neighbourhood of this boundary.

However, nothing conclusive can be said about the values of these parameters. Therefore a complete analysis of the long term evolution of the economy should also take into consideration the case where the trajectory describing the evolution in prices and wages hits the boundary between the Repressed inflation and the Keynesian equilibrium regions. We do this in this chapter by introducing timelags in agents' response to changes in prices and wages. Depending on the distribution of timelags we show that if prices and wages approach near the Repressed inflation -Keynesian boundary, then the state of the economy will be characterized either by decreasing oscillations around the boundary (eventually converging to the boundary) or by an equilibrium situation with steady stable (or semi-stable) oscillations between the two types of equilibria. In this way our results provide a specific answer to the suggestion in Malinvaud, [1977], that in the long run the economy will tend to exhibit oscillations between states of Keynesian and Repressed inflation equilibrium.

The model.

We consider a rationed equilibrium economy of the Malinvaud type as described in Chapter II. There are a finite number of consumers, three goods: labour, a consumption good, and money, and a production sector given by an aggregated production function. The prices of the good and the labour are denoted p and w, and the price of money is the unit. In the short run prices and wages are fixed; therefore equilibrium

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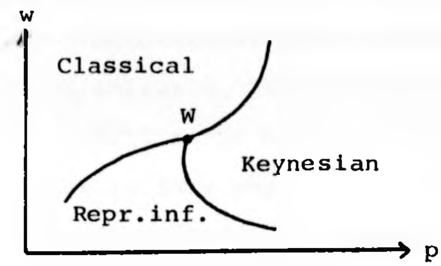
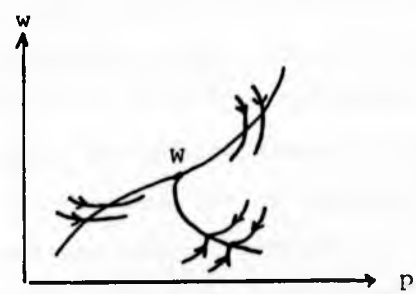
Fig. 1.



However, in the long run we suppose that prices and wages react to excess demands on the different markets. In Chapter I we discuss the introduction of "slow" adjustment processes in prices and wages over time, which give rise to the following flow diagram:

Fig. 2.



The adjustment processes are defined by "classical" derived equations for $\dot{p}$ and $\dot{w}$. The equations are different in each of the three regions, because different rationing hypotheses apply. The explicit equations are presented in Chapter I. However, the crucial qualitative feature that we are concerned with in this paper is that the equations give rise to a discontinuity in the vector field on the Repressed inflation -Keynesian boundary. Therefore if we integrate this discontinuous vector field, we obtain a flow that is both discontinuous and ambiguous on the boundary. But this is unrealistic. In reality the evolution of prices and wages is likely to be both continuous and smooth. Therefore our object in this paper is to refine the model in the neighbourhood of the Repressed inflation - Keynesian boundary so that the discontinuous vector field gives rise to a smooth flow.

The production sector is introduced into the model by an aggregated production function, giving total production as a function of prices and wages (subject to rationing). The latter is implicitly contained in the equations given in Chapter I. This is of course an idealization of the real situation, where total production is the result of several smaller production units' (firms') decisions. Specifying total production in this way, however, implies that at a given point in time, every firm forms its production decisions on the basis of the same set of values of the external parameters (here prices and wages). Although it is an acceptable simplification to characterize production by an aggregated production function as long as one considers a static model with p and w fixed, it would be necessary to introduce an assumption about complete information in order to maintain this formulation, when one is concerned with the evolution of the economy over time, where prices and wages change.

We shall here relax this assumption by postulating the existence of inertia in the process of decision making by firms, since it is a time consuming process to obtain and incorporate correct information on actual prices and wages. Therefore, while some firms will be able to make production decisions at time t based on the actual prices and wages exactly at time t, other firms will base their decisions at time t on the former values of the external parameters at time $(t-\tau)$ for $0 < \tau < \infty$. We say these firms have timelag τ. In the same way we introduce the timelag τ in consumers decision making process. Denoting the timelag distribution by $\theta(\tau)$ we introduce the hypothesis

<u>Hypothesis 1.</u>

The timelag distribution $\theta(\tau)$ defines a probability measure on $[0,\infty)$, i.e.

$$\int_0^\infty \theta(\tau)d\tau = 1.$$

We let μ denote the mean, $\mu = \int_0^\infty \tau\theta(\tau)d\tau$.
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Hypothesis 2.

Let $F(p,w)$ be the vector field defined on the (p,w) plane, except on the Repressed inflation - Keynesian boundary, describing the original change in (p,w). If $\varphi : \mathbb{R} \to \mathbb{R}^2$ is a flow line of the evolution, then φ satisfies the differential delay equation

$$\frac{d\varphi}{dt} = \dot{\varphi}(t) = \int_0^\infty \theta(\tau)F(\varphi(t-\tau))d\tau .$$

Remarks.

1°. The integral is well defined, since F is defined except on a set of measure zero.

2°. The differential delay equation is actually derived from the introduction of timelag in the description of demand/supply by consumers/producers: consider e.g. the equation for $\dot{p}$ in the Keynesian region,

$$\dot{p}(t) = K_p(\hat{\phi}(p(t),w(t)) + g - \psi(p(t),w(t))) .$$

Introducing a timelag τ in the determination of demand and supply we get

$$\hat{\varphi}(p(t),w(t)) = \int_0^\infty \theta(\tau)(\varphi(p(t-\tau),w(t-\tau)))d\tau$$

$$\psi(p(t),w(t)) = \int_0^\infty \theta(\tau)(\psi(p(t-\tau),w(t-\tau)))d\tau .$$

So

$$\dot{p}(t) = \int_0^\infty K_p \theta(\tau)(\varphi(p(t-\tau),w(t-\tau))+g-\psi(p(t-\tau),w(t-\tau)))d\tau$$

$$= \int_0^\infty \theta(\tau)F(p(t-\tau))d\tau \quad .$$

The timelag distribution.

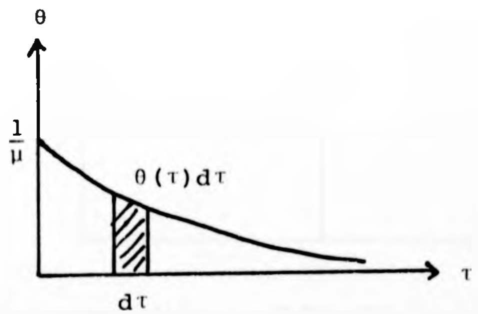
In this chapter we shall discuss three types of timelag distributions.

Example 1. The negative exponential distribution.

Here

$$\theta(\tau) = \frac{1}{\mu} e^{-\frac{\tau}{\mu}}, \tau \geq 0, \ \mu > 0 \text{ (constant)}.$$

Fig. 3.

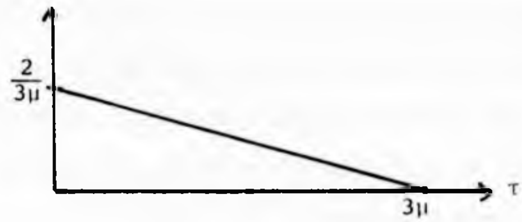


This distribution represents the case, where the timelag is randomly distributed over the interval $[0,\infty)$.

Example 2. The decreasing linear distribution.

$$\text{Here } \theta(\tau) = \begin{cases} \frac{2}{9\mu^2}(3\mu-\tau) & 0 \leq \tau \leq 3\mu \\ 0 & \tau > 3\mu \end{cases}$$

Fig. 4.

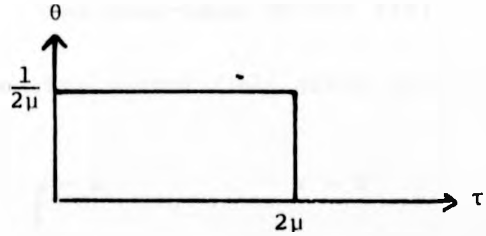


This represents a linear approximation with bounded delay to the distribution in example 1.

Example 3. The bounded uniform distribution.

$$\text{Here } \theta(\tau) = \begin{cases} \frac{1}{2\mu} & 0 \leq \tau \leq 2\mu \\ 0 & \tau > 2\mu \end{cases}$$

Fig. 5.



This distribution will be obtained by assuming that agents respond to information on prices and wages at the fixed interval 2μ, and different agents respond on different days.

The long term evolution.

We next want to describe the vector field determining the evolution. Let B denote the Repressed inflation - Keynesian boundary. Choose new coordinate axes (x,y) with x perpendicular to B, y parallel to B, so that B is given by $x = 0$. It is then an acceptable simplification to assume that in the neighbourhood of B the vector field is of the form $F(x,y) = (f(x),g)$, where f depends only on x and g is constant. The differential delay equation therefore reduces to

$$\begin{cases} \dot{x}(t) = \int_0^\infty \theta(\tau) f(x(t-\tau))d\tau \\ \\ \dot{y}(t) = g \ . \end{cases}$$

Hence all the interesting motion takes place in the variable x, so this is the equation we want to solve.

In this chapter we consider two examples of vector fields.

<u>Example a.</u> The bang-bang vector field.

This is the vector field given by

$$f(x) = \begin{cases} -k & x > 0 \\ & \\ +k & x < 0 \ . \end{cases} \qquad k \text{ (constant)} > 0$$

"Bang-bang" refers to the complete opposite price behaviour in the two regions: in the Keynesian region, where there is excess supply on both markets, prices are reduced by k at every point; in the Repressed inflation region, where there is excess demand on both markets, prices are increased by k at every point.

Example b. The bounded below vector field.

This is an arbitrary vector field f fulfilling the requirement

$$f(x) \begin{cases} \leq -k & x > 0 \\ \geq +k & x < 0 \end{cases} \qquad k > 0.$$

In other words, f is a vector field pointing towards B with a discontinuity at B, bounded away from zero. Notice that f is not defined on $x = 0$, but this does not matter.

We can now state our first main result, which specifies the differential equation describing the evolution in x.

Theorem 1.

If θ is the negative exponential distribution, then x satisfies the ordinary differential equation

$$\mu\ddot{x} + \dot{x} - f(x) = 0 .$$

Proof.

Put $\lambda = 1/\mu$. We then have

$$\begin{aligned} \dot{x}(t+\delta) &= \int_0^\delta \lambda e^{-\lambda\tau} f(x(t+\delta-\tau))d\tau + \int_\delta^\infty \lambda e^{-\lambda\tau} f(x(t+\delta-\tau))d\tau \\ &= \int_0^\delta \lambda e^{-\lambda\tau} f(x(t+\delta-\tau))d\tau + \int_0^\infty \lambda e^{-\lambda(\delta+\sigma)} f(x(t-\sigma)d\sigma \\ &= \int_0^\delta \lambda e^{-\lambda\tau} f(x(t+\delta-\tau))d\tau + e^{-\lambda\delta}\dot{x}(t). \end{aligned}$$

Expanding in terms of δ yields

$$\dot{x}(t) + \delta\ddot{x}\ (t) + O(\delta^2) = \delta\lambda f(x(t)) + (1-\lambda\delta)\dot{x}(t) + O(\delta^2).$$

Dividing by δ and letting $\delta \to 0$ we obtain

$$\ddot{x}(t) = \lambda f(x(t)) - \lambda\dot{x}(t),$$

which proves the result. ■

We now need a few lemmas in order to determine the long term evolution of the economy given by the above differential equation.

<u>Lemma 1.</u>

Suppose the path traced out by the economy crosses the boundary B with velocity u, and recrosses it later with velocity v. Then $|v| < |u|$.

<u>Proof.</u>

Analogous to damped mechanics we can introduce two types of energy:

the kinetic energy $\frac{1}{2}\mu\dot{x}^2$ and

the potential energy $-\int_0^x f(z)dz$.

The total energy is $E(x) = \frac{1}{2}\mu\dot{x}^2 - \int_0^x f(z)dz$. Therefore

$$\dot{E}(x) = \mu\dot{x}\ddot{x} - f(x)\dot{x} = \dot{x}(\mu\ddot{x} - f(x)) = -\dot{x}^2 .$$

So the total energy is monotonic decreasing, and strictly decreasing when $\dot{x} \neq 0$. At each crossing of B the potential energy vanishes. So at the first crossing of the boundary $E_1 = \frac{1}{2}\mu u^2$, and at the second crossing $E_2 = \frac{1}{2}\mu v^2$. Therefore $v^2 < u^2$. ∎

We now want to ensure that wherever the economy starts, the path traced out by the economy will cross the boundary B in finite time. Moreover, we want to estimate the time taken to recross it, given the initial speed of crossing. Throughout the next few lemmas, we continue to assume $\theta(\tau)$ to be the negative exponential distribution.

<u>Lemma 2.</u>

If f is a bang-bang vector field, and if initially $x = \xi$, $\dot{x} = u$ in the Keynesian region ($\xi > 0$), then for x in the Keynesian region we have

$$x = \xi - kt + \mu(k+u)(1-e^{-\frac{t}{\mu}}) .$$

<u>Proof.</u>

Solve the linear differential equation $\mu\ddot{x} + \dot{x} + k = 0$ with the given initial conditions. ∎

<u>Lemma 3.</u>

If f is bounded below, then wherever the economy starts, after a finite time it will cross the boundary B.

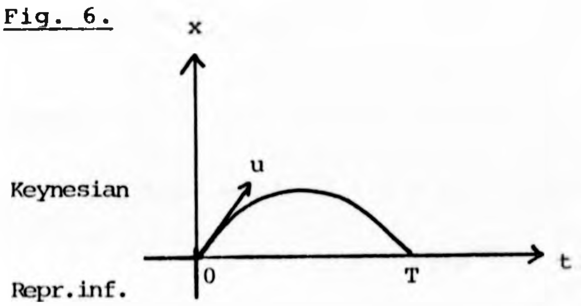
Proof.

If f is bang-bang and starts in the Keynesian region, then $x < \xi - kt + \mu(k+u)$ by lemma 2. Therefore $x < 0$ provided $t > \frac{\xi + \mu(k+u)}{k}$, which implies that the path crosses B before this.

If f is bounded below, then the velocity is less than if f were bang-bang, and hence it crosses the boundary sooner. Similarly if the economy starts out in the Repressed inflation region. ■

Next we shall estimate the period of recrossing the boundary. Suppose at $t = 0 : x = 0, \dot{x} = u > 0$, in other words the path crosses the boundary into the Keynesian region. Let T be the time at which it first recrosses back into the Repressed inflation region.

Fig. 6.



Lemma 4.

If f is bounded below, then $T < \frac{2\mu u}{k}$.

Proof.

It suffices to prove the lemma, when f is bang-bang, because if f is bounded below, it will recross sooner.

In lemma 2 put $\xi = 0$. Then

$$x = -kt + \mu(k+u)(1-e^{-s}), \; s = \frac{t}{\mu}$$

$$= \mu u(1-e^{-s}) - \mu k(e^{-s} - 1+s)$$

$$< \mu(1-e^{-s})(u - \frac{ks}{2}) \text{ by lemma 5 below.}$$

So $x < 0$ if $s \geq \frac{2u}{k}$, i.e. if $t \geq \frac{2\mu u}{k}$.

Therefore the first recrossing must occur before this, i.e.

$T < \frac{2\mu u}{k}$. ∎

Lemma 5.

If $s > 0$, then $e^{-s} - 1 + s > (1-e^{-s})\frac{s}{2}$.

Proof.

Let $h(s) = 2(e^{-s} - 1 + s) - s(1-e^{-s})$

$$= s - 2 + (s+2)e^{-s} .$$

Then $h(0) = 0$ and

$h'(s) = 1 - (s+1)e^{-s} = e^{-s}(e^{s} - 1 - s) > 0, \; s \neq 0$.

Therefore $h(s) > 0, \; s > 0$. ∎

We can now give a complete description of the evolution of the economy around the Repressed inflation - Keynesian boundary.

Theorem 2.

If θ is the negative exponential distribution and f is bounded below, then x converges to zero with oscillations of decreasing amplitude and period.

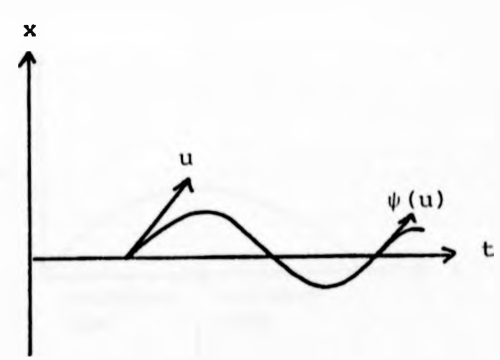
Remark.

It can be shown, further, that the amplitude and period decay exponentially.

Proof of theorem 2.

Whatever the initial position, the path crosses the boundary by lemma 3, and it continues to recross it infinitely many times by lemma 4. Let u be the velocity at the first crossing of B towards the Keynesian region, and let $\psi(u)$ denote the next crossing in the same direction.

Fig. 7.



By theorem 1 $\psi(u)$ is determined by u and depends continuously upon u. By lemma 1, $\psi(u) < u$. Hence $\psi: (\mathbb{R}_+, 0) \to (\mathbb{R}_+, 0)$ is continuous, monotonic, strictly decreasing. The sequence of crossing velocities u, $\psi(u)$, $\psi^2(u)$, ..., $\psi^n(u)$, ... is positive decreasing and hence tends to a limit $\psi^n(u) \to w$ as $n \to \infty$. Therefore $\psi^{n+1}(u) \to \psi(w)$. So $\psi(w) = w$, and by lemma 1, $w = 0$. $\psi^n(u) \to 0$ gives by lemma 1 that $E \to 0$. Therefore the amplitude converges to zero by the definition of energy.

Finally lemma 4 implies that the period converges to zero.■

<u>Example.</u>

Suppose f is bang-bang with $k = 1$, and suppose the path for $t < 0$ has been entirely in the Repressed inflation region. Then it will approach the boundary with unit speed. Therefore we may take initial conditions $t = 0$: $x = 0$, $u = 1$. We calculate the initial overshoot distance, in other words the amplitude of the initial oscillation into the Keynesian region, and the time for overshoot and recovery.

<u>Fig. 8.</u>

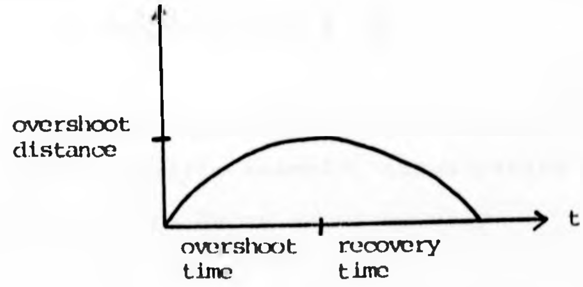
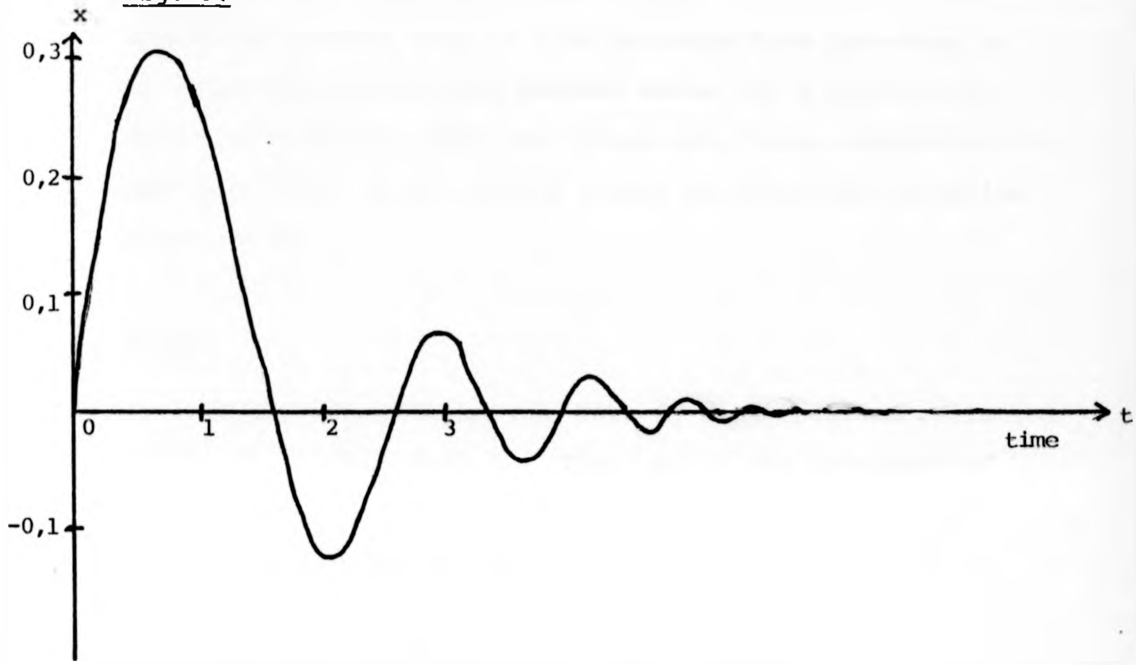


Fig. 9.



We now turn from the negative exponential distribution for the timelags to the bounded uniform distribution (example 3). For simplicity we shall assume f is bang-bang.

Theorem 3.

Suppose θ is the bounded uniform distribution and f is bang-bang. Suppose further that the economy has spent longer than 2μ in one region. Then the path will approach the boundary B and perform stable steady oscillations of period 2μ and magnitude $\frac{k\mu}{2}$.

Remark.

The important word in this theorem is "stable", meaning that if the oscillations are perturbed, then the system will asymptotically return to steady oscillations again. The stability ensures that if f is perturbed from bang-bang to an arbitrary vector field bounded below, by a sufficiently small perturbation, then the system will again asymptotically approach close to the stable steady oscillations of period close to 2μ.

Proof.

Suppose the economy has been in the Repressed inflation region for longer than 2μ. Then, since the integrand of

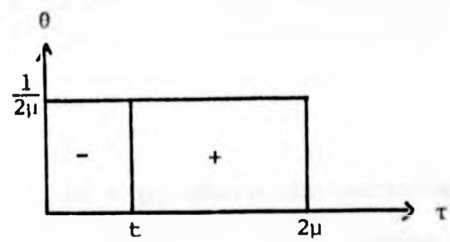
$$\dot{x} = \int_0^\infty \theta(\tau) f(x(t-\tau))d\tau$$

is in effect limited to $0 \leq \tau \leq 2\mu$, we have

$$\dot{x} = k, \text{ where } f = k.$$

Therefore the economy will approach the boundary with constant velocity k. Choose $t = 0$ to be the moment of crossing. Then in the Keynesian region, for $0 \leq t \leq 2\mu$:

Fig. 10.



$$\dot{x} = \frac{k}{2\mu}\ (-t + (2\mu - t)) = k(1-\frac{t}{\mu})\ .$$

So

$$x = k(t - \frac{t^2}{2\mu}) = \frac{kt}{2\mu}(2\mu - t),$$

therefore $x = 0$, when $t = 2\mu$. This shows that x executes a parabolic orbit into the Keynesian region with period 2μ and amplitude $\frac{k\mu}{2}$ and returning velocity $-k$.

By symmetry the same oscillation will persist in alternate regions.

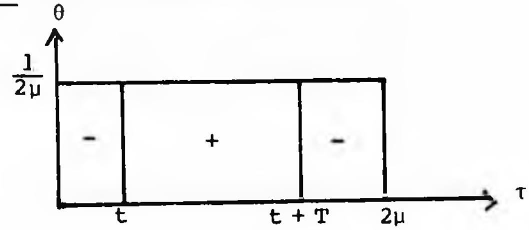
We now investigate the stability by assuming that the period T in the previous region has been perturbed away from 2μ. If $T > 2\mu$, then the above argument shows that the next period returns to 2μ. Therefore suppose $\mu < T < 2\mu$. More precisely suppose that the path was in the Keynesian region during the time $[-2\mu, -T]$ and in the Repressed inflation region during $[-T,0]$. We now investigate the period T^* of the next interval $[0,T^*]$ in the Keynesian region. We shall show in lemma 7 that $T < T^* < 2\mu$ and that T^*depends continuously upon T. Hence the succeding periods increase asymptotically back to 2μ as required, ensuring the stability.

<u>Lemma 7.</u>

If $T = 2\mu - p$, where the pertubation $p = \mu \sin\alpha$, $0 < \alpha < \frac{\pi}{2}$, then $T^* = 2\mu - p^*$, where $p^* = p\tan\frac{\alpha}{2} < p$.

Proof.

Fig. 11.



During the interval $0 \leq t \leq p$ we have

$$\dot{x} = \frac{k}{2\mu}(-t+T-(2\mu-t-T)) = k(1-\sin\alpha)\ .$$

So

$$x = kt\ (1-\sin\alpha) = k\mu\sin\alpha\ (1-\sin\alpha),\ \text{where } t=p.$$

Subsequently, when $t \geq p$ we have

$$\dot{x} = \frac{k}{2\mu}(-t + (2\mu - t)) = k(-\frac{t}{\mu} + 1)\ .$$

So

$x = k(-\frac{t^2}{2\mu} + t + \Lambda)$, when Λ is a constant. We can compute the value of Λ by putting

$$x = k\mu\sin\alpha(1-\sin\alpha),\ \text{when } t=p=\mu\sin\alpha,$$

then

$$\Lambda = -\frac{\mu}{2}\sin^2\alpha.$$

Therefore

$$x = -\frac{k}{2\mu}\,(t^2-2\mu t + \mu^2\sin^2\alpha).$$

So

$$x = 0, \text{ when } t = \mu(1\pm\cos\alpha).$$

As $T^* > p$ we have $T^* = \mu(1+\cos\alpha)$.

Therefore

$$\begin{aligned} p^* &= 2\mu - T^* \\ &= \mu(1-\cos\alpha) \\ &= 2\mu\sin^2(\tfrac{\alpha}{2}) \\ &= p\tan\frac{\alpha}{2} \\ &< p \text{ as required.} \ \blacksquare \end{aligned}$$

<u>Remark.</u>

We assumed a perturbation $p < \mu$ in the above proof. A larger perturbation $\mu < p < \frac{3\mu}{2}$ will destroy the original oscillation and cause the system to approach asymptotically stable steady oscillations of one third of the original period, $\frac{2\mu}{3}$. Similarly there exist stable steady oscillations of period $\frac{2\mu}{n}$ for any odd integer n (although, of course, the associated basin of stability will decrease with increasing n).

Finally we turn to the decreasing linear distribution, which is a compromise between the previous two (see example 2). We state a theorem that shows that the system exhibits a behaviour that is, in a sense, a compromise between the two previous results. We leave the proof to the reader, since it is similar to the proof of the last theorem.

Theorem 4.

Suppose θ is the decreasing linear distribution, and f is bang-bang. Suppose further the economy has spent longer than 3μ in one region. Then the path will approach the boundary B and perform oscillations that approach asymptotically semi-stable steady oscillations of period $\frac{3\mu}{2}$ and magnitude $\frac{3k\mu}{16}$.

Remark.

Here the word "semi-stable" means that if the period is perturbed above $\frac{3\mu}{2}$, then the oscillations asymptotically return to the steady state, but if the period is perturbed below $\frac{3\mu}{2}$, then the period will continue to decrease, and the oscillations will approach asymptotically semi-stable steady oscillations of half the period, $\frac{3\mu}{4}$.

The semi-stability implies that for suitable pertubations of f, the system may either develop stable steady oscillations, as in theorem 3, or else the oscillations may decay to zero as in theorem 2.

Conclusion.

Our analysis has provided specific results on the evolution of the economy around the Repressed inflation - Keynesian region. Depending on the distribution of timelags we have shown that the economy will exhibit either decreasing oscillations across the boundary, eventually converging to zero, or to stable or (semi-stable) steady oscillations between the two regions. In the framework of our model the results therefore clarify the validity of the often made claim that a basic feature of modern economies is the cycling motion between Repressed inflation and Keynesian equilibria.

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CONCLUSION.

In this work we have considered a series of dynamical disequilibrium models, which all are based on Malinvaud's macro economic equilibrium model with rationing. The purpose of our analysis has been to determine the long term evolution of a rationed equilibrium economy under varying conditions on the adjustments in quantities and prices through time. As there is, however, no unique dynamical theory describing the evolution of a disequilibrium economy, we have chosen to begin our analysis by considering a model which we think constitutes the simplest dynamization of the Malinvaud model (Chapter I). This model presented only a partly satisfying description of the evolution of the economy. In order to improve on the description, we elaborated further on the dynamics in this model by changing some assumptions which we thought too restrictive in the initial dynamical model. This led to a complete description of the long term evolution (Chapter II). However, as in the initial model, the analysis rested on a specific behaviour of consumers' moneysavings, which is less than acceptable. We therefore thought it important to show how this assumption could be relaxed without spoiling the obtained characterization of the long term evolution (Chapter III). Finally in Chapter IV we suggested a slight change of the model, developed in Chapter II, in order to obtain one possible answer to the question of the long term evolution around the Repressed inflation - Keynesian boundary; a problem which was present all the way through the analysis.

Chapter I contains the introduction of the basic dynamical model. We have chosen Malinvaud's static model as the starting point, because we consider this model to be a complete formal presentation of a type of aggregated macro model, which has been used extensively in economic literature. At the same time the model is sufficiently simple to make the introduction of the dynamics possible without too many technical complications.

The actual framework in Chapter I is given by the paper by Hildenbrand & Hildenbrand. This paper provides a more general framework in which to apply the Malinvaud model, as agents are allowed to vary in preference characteristics. Even though we later return to the simpler setting with only one type of consumer (as in Malinvaud's book) it is useful to have the basic dynamization formulated in the general set up.

The main point in Chapter I is the introduction and discussion of the adjustment processes in prices and wages in the three regions of Keynesian, Classical and Repressed inflation equilibrium. Having supposed instantaneous adjustments in quantities the evolution of the economy is completely determined by the evolution in prices and wages. We describe the price and wage changes as governed by excess demands/supplies on the different markets. In this way a vector field in the price-wage plane is defined, such that the direction of the vector field is continuous except on the Repressed inflation - Keynesian boundary. This implies that the evolution is well defined except on this boundary. The discontinuity of the vector field is closely related to the dynamization of the model using excess demands. In the following chapters we show how the problem of evolution around the Repressed inflation - Keynesian boundary can be determined without introducing any artificial assumption of behaviour on the boundary. However, in this first attempt we simply assume convergence towards the Walrasian equilibrium along the boundary. This makes a theorem of global stability of the Walrasian equilibrium straightforward.

A completely different approach would be to introduce the possibility of inventories by firms into the static model. As pointed out by Malinvaud, this would imply the introduction of a fourth equilibrium region in the price-wage plane, situated between the Repressed inflation and the Keynesian region. We have not attempted this approach.

Chapter II contains the main dynamical model. Preserving the adjustment process in prices and wages from Chapter I, but dropping the assumption of instantaneous adjustments in the quantities, the model describes the evolution of the economy as the result of an interaction between a relatively fast adjustment process in the quantities and a relatively slow adjustment process in prices and wages.

In order to keep the analysis of this new formulation clear, we wanted to avoid the many technical complications, which would follow from applying a static model with different types of consumers' preference characteristics. In this chapter we therefore return to the basic static macro model, introduced by Malinvaud.

In order to obtain exact equations for the fast adjustment processes in the quantities, we introduce a new three dimensional coordinate system in the quantity space in which to measure excess demand/supply of the consumption good and labour. In this setting the three types of rationed equilibria, defined in chapter I by having one coordinate zero and the other two coordinates positive, have a useful geometrical representation as the set of points on the three faces of the 3 dimensional positive quadrant with vertex (0,0,0), the Walrasian equilibrium. Furthermore this description makes an extension of the rationed equilibrium concept possible: allowing negative coordinates three types of dual equilibria are defined. Each is characterized by one zero-coordinate and two negative coordinates.

The long term evolution is straightforward, as long as prices and wages stay within one equilibrium region: quantities adjust quickly to their equilibrium values, while prices change only slowly, as described by the adjustment processes. When a boundary is crossed, however, a rather more complicated analysis is necessary to obtain a generic description. The analysis has been divided into a simplified example, presenting the basic ideas, followed by the general case. There are two main results. The first is contained in the theorem on the exchange of stability. There exist fast adjustment processes in the quantities such that to each point in the price-wage plane three points of quantity equilibria on the equilibrium quadrant are assigned. Two of these are stable and one is unstable w.r.t. the fast adjustment processes. Furthermore, the crossing of a boundary between two equilibrium regions in the price-wage plane corresponds to an exchange of stability between two of the corresponding points in the equilibrium quadrant. The stability of the third equilibrium point remains unchanged.

The second result, which relies on the exchange of stability, presents the generic evolution in the quantity space as prices and wages approach a boundary. Depending on the values of some "new" parameters, the transition in the quantity space is either given by smooth changes between stable equilibria on two faces of the equilibrium quadrant, or it takes the form of sudden jumps between one of the ordinary types of quantity equilibria and one of the three new dual types of quantity equilibria. In the last case the shift actually happens before prices and wages hit the boundary.

The concept of exchange of stability has been used in physics in connection with elastic stability for low dimensional cases. Our application to economics shows the usefulness of this mathematical concept to a higher dimensional problem too.

Even though the exact formulation of the fast adjustment processes in the quantities should be questioned, the theorem presents the first complete analysis of the long term dynamic evolution of an economy, in which the adjustments in the quantities are taken directly into account. The second result shows the importance of this new description. In contrast with the result in Chapter I, the transition between equilibrium states is shown to depend heavily on the values of some parameters, which were "hidden" in the former description. A satisfying interpretation of the new parameters and the dual equilibria is still left. An answer might follow from further investigations of the conditions under which dual equilibria will be observed. One such possibility would be to analyze situations, where producers and/or consumers because of earlier rationing continue to by/sell whatever is possible, even though this is no longer the "optimal" strategy. Consider e.g. a situation, where producers have been rationed on the labour market (a Repressed inflation equilibrium). When prices and wages slowly change a time lag in producers' perception of the changed constraints in the markets will make them continue, at least for some time, to employ all the workers they can get, taking the economy to a situation of dual Classical equilibrium.

An important quality of the results in this chapter is the genericity. This implies that the evolution of the economy is stable under small pertubations, i.e. the qualitative description is unchanged under small variations in the equations. The result is a consequence of an application of catastrophe theory. In relation to the recent critique on the applications of catastrophe theory to the social sciences (see H.J. Sussman & R.S. Zahler [1978], "Catastrophe Theory as applied to the social and biological sciences: A critique". Synthese, vol. 37) it should be stressed that the use of catastrophe theory here is completely natural - no assumptions have been introduced in order to "prepare" for the application.

The results in this chapter partly provides a solution to the problem of indeterminateness of the evolution of the economy on the Repressed inflation - Keynesian boundary. As long as the values of the hidden parameters are such that the transition from states of Repressed inflation / Keynesian equilibrium goes to states of dual Classical equilibrium, the price-wage trajectory will never reach the boundary. In these cases the discontinuity of the original price-wage adjustment process becomes insignificant. This is due to the fact that in a dual equilibrium situation the economic conditions are completely different. Therefore the processes describing the slow adjustments in prices and wages must be reformulated. A complete characterization of the long term evolution therefore requires the introduction of slow adjustment processes for the three regions of dual Keynesian, dual Classical and dual Repressed inflation equilibrium. This has not been carried out in Chapter II. It is, however, easy to reason by the use of the varying economic environment in the three regions in order to obtain "dual" processes in the sense that the signs of the changes in prices and wages in a dual equilibrium region are the opposite of the signs in the corresponding original equilibrium region. We notice that to desist from giving the exact specification of the dual adjustment processes in prices and wages is unimportant. It follows from Chapter II that the exact formulae for the original slow adjustment processes are never used in the analysis. Only the flow-lines, giving the qualitative evolution are necessary to obtain the results.

The main point in Chapter III has been to show that the description in Chapter II of the evolution of the economy by exchange of stability can be extended to cover a situation, where some consumers may gain from saving. We do this by considering the case where unemployed consumers are endowed with the same fixed amount of money at the beginning of each period, whereas employed consumers' money-holdings may vary from period to period. In order to keep things simple we want to describe the variations in money holdings by a one dimensional variable. Therefore we also introduce specific assumptions on the changes in money holdings of the consumers, who shift from employment to unemployment and vice versa. Depending on the relative size of the unemployment benefit and the employed consumers' saving, these assumptions are either reasonable or rather artificial. It would therefore be interesting to investigate if this formulation could be substituted by a description relying on less restrictive assumptions, considering our approach here to be only the first step towards a generalization of the results in Chapter II.

The main result in Chapter III splits into two parts. The first part is a generalization of the long term evolution, presented in Chapter I. We show that even though the single flow line, starting in the Classical region, may change by the introduction of money holdings from crossing one boundary to crossing the other boundary, the qualitative picture of the evolution in prices and wages around the boundaries of the Classical region is unchanged. The second part of the result is a generalization of the generic description of the long term evolution by exchange of stability to the more general situation in Chapter III. This is obtained by introducing an assumption, generalizing the result from the appendix to Chapter II on the uniqueness of the equilibrium in each of the three regions to the situation in Chapter III.

In the last part of Chapter III we discuss the influence on the evolution of variations in unemployment benefits and government's consumption. In both cases we show that this implies changes in the partitioning of the price-wage plane, whereas the qualitative description of the evolution by flow lines is unchanged. These results, explicitly obtained in the framework of a dynamical model, are the analogues to the results on government's consumption as policy variable, presented in the comparative static analysis in Malinvaud's book.

Finally Chapter IV contains another small change of the model from Chapter II. Our main concern here is once more the evolution of the economy around the Repressed inflation - Keynesian boundary. The important fact, which we aim at in this chapter, is to show that although the vector field defining the changes in prices and wages, is discontinuous on this boundary, this vector field may still give rise to a well defined (and smooth) evolution of the economy in a neighbourhood of the boundary. The result is obtained by introducing an assumption of "inertia" in agents' responses to changes in prices and wages. The assumption expresses the viewpoint that there is a timelag between the actual change in a price and the point in time, when an agent realizes and reacts to the change. The price-change is therefore now determined on the basis of changes in the demand/supply, which the individual agent would express, given the lagged information on prices. Supposing that this always leads to a decrease (increase) in the price in the Keynesian region (the Repressed inflation region), we prove that the long term evolution of the economy will be characterized by oscillations around the boundary between these two regions. Depending on the actual distribution of agents over timelags we obtain either that the oscillations will decrease monotonically towards zero or that the oscillations will describe an equilibrium situation with (semi-) stable steady oscillations of a fixed magnitude. The result therefore proves that the situation with cyclical movements between states of Repressed inflation and Keynesian equilibria can be realized in the long run, a remark we made in the final notes of Chapter I. It should be noticed that the description by exchange of stability, given in Chapter II, is still applicable to the above situation. By this we merely say that whenever the values of the hidden parameters change during the long term evolution such that the transition from the Repressed inflation or Keynesian region is to the region of dual Classical equilibrium, the above oscillations will terminate and the evolution will proceed to be governed by the dual adjustment processes.