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STRUCTURE OF MULTICORRELATION SEQUENCES WITH INTEGER PART POLYNOMIAL ITERATES ALONG PRIMES

ANDREAS KOUTSOGIANNIS, ANH N. LE, JOEL MOREIRA, AND FLORIAN K. RICHTER

ABSTRACT. Let T be a measure preserving \mathbb{Z}^ℓ -action on the probability space (X, \mathcal{B}, μ) , $q_1, \dots, q_m: \mathbb{R} \rightarrow \mathbb{R}^\ell$ vector polynomials, and $f_0, \dots, f_m \in L^\infty(X)$. For any $\epsilon > 0$ and multicorrelation sequences of the form $\alpha(n) = \int_X f_0 \cdot T^{\lfloor q_1(n) \rfloor} f_1 \cdots T^{\lfloor q_m(n) \rfloor} f_m d\mu$ we

show that there exists a nilsequence ψ for which $\lim_{N-M \rightarrow \infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |\alpha(n) - \psi(n)| \leq \epsilon$

and $\lim_{N \rightarrow \infty} \frac{1}{\pi(N)} \sum_{p \in \mathbb{P} \cap [1, N]} |\alpha(p) - \psi(p)| \leq \epsilon$. This result simultaneously generalizes previous results of Frantzikinakis [2] and the authors [11, 13].

1. INTRODUCTION AND MAIN RESULT

Since Furstenberg's ergodic theoretic proof of Szemerédi's theorem [5], there has been much interest in understanding the structure of *multicorrelation sequences*, i.e., sequences of the form

$$\alpha(n) = \int_X f_0 \cdot T^n f_1 \cdots T^{kn} f_k d\mu, \quad n \in \mathbb{N}, \quad (1)$$

where (X, \mathcal{B}, μ, T) is a measure preserving system and $f_0, \dots, f_k \in L^\infty(X)$. The first to provide deeper insight into the algebraic structure of such sequences were Bergelson, Host, and Kra, who showed in [1] that if the system (X, μ, T) is ergodic then for any multicorrelation sequence α as in (1) there exists a uniform limit of k -step nilsequences ϕ such that

$$\lim_{N-M \rightarrow \infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |\alpha(n) - \phi(n)| = 0. \quad (2)$$

Here, a *k-step nilsequence* is a sequence of the form $\psi(n) = F(g^n x)$, $n \in \mathbb{N}$, where F is a continuous function on a k -step nilmanifold $X = G/\Gamma$,¹ $g \in G$, $x \in X$. A *uniform limit of k-step nilsequences* is a sequence ϕ such that for every $\epsilon > 0$ there exists a k -step nilsequence ψ with $\sup_{n \in \mathbb{N}} |\phi(n) - \psi(n)| \leq \epsilon$.

Later, Leibman extended the result of Bergelson, Host and Kra to polynomial iterates in [14], and removed the ergodicity assumption in [15]. Another extension was obtained by the second author in [12], and independently by Tao and Teräväinen in [17], answering a question raised in [3]. There, it was shown that in addition to (2) one also has

$$\lim_{N \rightarrow \infty} \frac{1}{\pi(N)} \sum_{p \in \mathbb{P} \cap [1, N]} |\alpha(p) - \phi(p)| = 0, \quad (3)$$

where \mathbb{P} denotes the set of prime numbers, $[1, N] := \{1, \dots, N\}$, and $\pi(N) := |\mathbb{P} \cap [1, N]|$.

The proofs of all the aforementioned results depend crucially on the structure theory of Host and Kra, who established in [7] that the building blocks of the factors that control

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¹A *k-step nilmanifold* is a homogeneous space $X = G/\Gamma$, where G is a k -step nilpotent Lie group and Γ is a discrete and co-compact subgroup of G .

multiple ergodic averages are nilsystems. Since the analogous factors for \mathbb{Z}^ℓ -actions are unknown, extending the results above from \mathbb{Z} -actions to \mathbb{Z}^ℓ -actions proved to be a challenge. Nevertheless, in [2] Frantzikinakis concocted a different approach and gave a description of the structure of multicorrelation sequences of \mathbb{Z}^ℓ -actions, which we now explain.

Henceforth, let $\ell \in \mathbb{N}$ and let T be a measure preserving \mathbb{Z}^ℓ -action on a probability space (X, \mathcal{B}, μ) . The system (X, \mathcal{B}, μ, T) gives rise to a more general class of multicorrelation sequences,

$$\alpha(n) = \int_X f_0 \cdot T^{q_1(n)} f_1 \cdots T^{q_m(n)} f_m d\mu, \quad n \in \mathbb{N}, \quad (4)$$

where $q_1, \dots, q_m: \mathbb{Z} \rightarrow \mathbb{Z}^\ell$ are integer-valued vector polynomials and $f_0, \dots, f_m \in L^\infty(X)$. Note that (1) corresponds to the special case of (4) when $\ell = 1$ and $q_i(n) = in$. Frantzikinakis showed in [2] that for every α as in (4) and every $\epsilon > 0$ there exists a k -step nilsequence ψ such that

$$\lim_{N-M \rightarrow \infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |\alpha(n) - \psi(n)| \leq \epsilon, \quad (5)$$

where k only depends on ℓ, m , and the maximal degree among the polynomials q_1, \dots, q_m . Moreover, in the special case where each polynomial iterate is linear, it was proved in [2] that one can take $k = m$. It is still an open question whether in (5) one can replace ϵ with 0 after replacing the nilsequence ψ with a uniform limit of such sequences (see Question 2 in Section 3).

For $x \in \mathbb{R}$ we denote by $\lfloor x \rfloor$ the largest integer which is smaller or equal to x , while for $x = (x_1, \dots, x_\ell) \in \mathbb{R}^\ell$ we let $\lfloor x \rfloor := (\lfloor x_1 \rfloor, \dots, \lfloor x_\ell \rfloor)$. In [11], the first author extended Frantzikinakis' results to all multicorrelation sequences of the form

$$\alpha(n) = \int_X f_0 \cdot T^{\lfloor q_1(n) \rfloor} f_1 \cdots T^{\lfloor q_m(n) \rfloor} f_m d\mu, \quad n \in \mathbb{N}, \quad (6)$$

where $q_1, \dots, q_m: \mathbb{R} \rightarrow \mathbb{R}^\ell$ are real-valued vector polynomials.

More recently, the last three authors showed that the conclusion of Frantzikinakis' result also holds along the primes:

Theorem 1 ([13, Theorems A and B]). *For every $\ell, m, d \in \mathbb{N}$ there exists $k \in \mathbb{N}$ with the following property. For any polynomials $q_1, \dots, q_m: \mathbb{Z} \rightarrow \mathbb{Z}^\ell$ with degree at most d , measure preserving \mathbb{Z}^ℓ -action T on a probability space (X, \mathcal{B}, μ) , functions $f_0, f_1, \dots, f_m \in L^\infty(X)$, $\epsilon > 0$, $r \in \mathbb{N}$ and $s \in \mathbb{Z}$, letting α be as in (4), there exists a k -step nilsequence ψ satisfying (5) and*

$$\lim_{N \rightarrow \infty} \frac{1}{\pi(N)} \sum_{p \in \mathbb{P} \cap [1, N]} |\alpha(rp + s) - \psi(rp + s)| \leq \epsilon. \quad (7)$$

In the special case $d = 1$ one can choose $k = m$.

Our main theorem simultaneously generalizes the main results from [11] and [13].

Theorem A. *For every $\ell, m, d \in \mathbb{N}$ there exists $k \in \mathbb{N}$ with the following property. For any polynomials $q_1, \dots, q_m: \mathbb{R} \rightarrow \mathbb{R}^\ell$ with degree at most d , any measure preserving \mathbb{Z}^ℓ -action T on a probability space (X, \mathcal{B}, μ) , functions $f_0, f_1, \dots, f_m \in L^\infty(X)$, $\epsilon > 0$, $r \in \mathbb{N}$ and $s \in \mathbb{Z}$, letting α be as in (6), there exists a k -step nilsequence ψ satisfying (5) and (7). In the special case $d = 1$ one can choose $k = m$.*

The proof of Theorem A, presented in the next section, follows closely the strategy implemented in [11], but uses Theorem 1 instead of Walsh's theorem [18] as a blackbox.

Remark 2. Both Theorems 1 and A are equivalent to seemingly stronger versions involving commuting actions. We say that two actions T_1 and T_2 of a group G commute if for every $g, h \in G$ we have $T_1^g \circ T_2^h = T_2^h \circ T_1^g$. When G is an abelian group, a collection of m commuting G -actions T_1, \dots, T_m can be identified with a single G^m -action T via $T^{(g_1, \dots, g_m)} = T_1^{g_1} \cdots T_m^{g_m}$. Using this observation, and the identification $(\mathbb{Z}^\ell)^m = \mathbb{Z}^{\ell m}$, one sees that, given commuting measure preserving \mathbb{Z}^ℓ -actions T_1, \dots, T_m in a probability space (X, μ, T) , Theorem 1 holds when (4) is replaced by

$$\alpha(n) = \int_X f_0 \cdot T_1^{q_1(n)} f_1 \cdots T_m^{q_m(n)} f_m d\mu, \quad (8)$$

and Theorem A holds when (6) is replaced by

$$\alpha(n) = \int_X f_0 \cdot T_1^{\lfloor q_1(n) \rfloor} f_1 \cdots T_m^{\lfloor q_m(n) \rfloor} f_m d\mu.$$

Remark 3. Let $\lceil x \rceil$ and $\lfloor x \rfloor$ denote the smallest integer $\geq x$ and the closest integer to x , respectively. Using the relations $\lceil x \rceil = -\lfloor -x \rfloor$ and $\lfloor x \rfloor = \lceil x + 1/2 \rceil$, we see that Theorem A remains true if (6) is replaced by

$$\alpha(n) = \int_X f_0 \cdot T^{\lceil q_1(n) \rceil} f_1 \cdots T^{\lceil q_m(n) \rceil} f_m d\mu, \quad n \in \mathbb{N},$$

where $[x]_i = ([x_1]_{i,1}, \dots, [x_\ell]_{i,\ell})$ and $[\cdot]_{i,1}, \dots, [\cdot]_{i,\ell}$ are any of $\lfloor \cdot \rfloor$, $\lceil \cdot \rceil$, or $\lfloor \cdot \rfloor$.

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2. PROOF OF MAIN RESULT

We start by proving a theorem concerning flows, which stands halfway in between Theorems 1 and A. The idea behind this result is that for a real polynomial $q(x) = a_d x^d + \dots + a_1 x + a_0 \in \mathbb{R}[x]$ and a measure presenting flow $(S^t)_{t \in \mathbb{R}}$ we can write $S^{q(n)} = (S^{a_d})^{n^d} \cdots (S^{a_0})^1$, an expression which can be handled by Theorem 1.

Theorem 4. *For every $\ell, m, d \in \mathbb{N}$ there exists $k \in \mathbb{N}$ with the following property. For any polynomials $q_1, \dots, q_m: \mathbb{R} \rightarrow \mathbb{R}^\ell$ with degree at most d , commuting measure preserving \mathbb{R}^ℓ -actions S_1, \dots, S_m on a probability space (X, \mathcal{B}, μ) , functions $f_0, f_1, \dots, f_m \in L^\infty(X)$, $\varepsilon > 0$, $r \in \mathbb{N}$, and $s \in \mathbb{Z}$, letting*

$$\alpha(n) = \int_X f_0 \cdot S_1^{q_1(n)} f_1 \cdots S_m^{q_m(n)} f_m d\mu, \quad (9)$$

there exists a k -step nilsequence ψ satisfying (5) and (7). In the special case $d = 1$ one can choose $k = m$.

Proof. For each $i \in [1, m]$, let $q_i = (q_{i,1}, \dots, q_{i,\ell})$ for some $q_{i,j} \in \mathbb{R}[x]$. Next, for each $j \in [1, \ell]$, write $q_{i,j}(x) = \sum_{h=0}^d a_{i,j,h} x^h$, where the $a_{i,j,h}$'s are real numbers. Also, for each $j \in [1, \ell]$, let e_j be the j -th vector of the canonical basis of \mathbb{R}^ℓ and let $T_{i,j,h}$ be the measure preserving transformation defined by $T_{i,j,h} = S_i^{a_{i,j,h} e_j}$. Next, let $T_{i,h}$ be the composition $T_{i,h} = T_{i,1,h} \cdots T_{i,\ell,h}$, let T_i be the \mathbb{Z}^{d+1} -action defined by $T_i^{(n_0, \dots, n_d)} = T_{i,0}^{n_0} \cdots T_{i,d}^{n_d}$, and let $q: \mathbb{Z} \rightarrow \mathbb{Z}^{d+1}$ be the polynomial $q(n) = (1, n, \dots, n^d)$.

With this setup, for each $i \in [1, m]$ and $n \in \mathbb{N}$, we have

$$S_i^{q_i(n)} = \prod_{j=1}^{\ell} S_i^{q_{i,j}(n) e_j} = \prod_{j=1}^{\ell} \prod_{h=0}^d T_{i,j,h}^{n^h} = \prod_{h=0}^d T_{i,h}^{n^h} = T_i^{q(n)}.$$

Since the \mathbb{R}^ℓ -actions S_1, \dots, S_m commute, so do the \mathbb{Z}^{d+1} -actions T_1, \dots, T_m . This implies that the multicorrelation sequence α can be represented by an expression of the form (8). The conclusion now follows directly from Theorem 1 and Remark 2. \square

Next we need a result concerning the distribution of real polynomials.

Lemma 5. *Let $q \in \mathbb{R}[x]$ be a non-constant real polynomial, $r \in \mathbb{N}$ and $s \in \mathbb{Z}$. Then, denoting by $\{\cdot\}$ the fractional part, we have*

$$\lim_{\delta \rightarrow 0^+} \lim_{N \rightarrow \infty} \frac{1}{N - M} \left| \left\{ n \in [M, N) : \{q(n)\} \in [1 - \delta, 1) \right\} \right| = 0,$$

and

$$\lim_{\delta \rightarrow 0^+} \lim_{N \rightarrow \infty} \frac{1}{\pi(N)} \left| \left\{ p \in \mathbb{P} \cap [1, N] : \{q(rp + s)\} \in [1 - \delta, 1) \right\} \right| = 0.$$

Proof. Let

$$A(\delta) = \lim_{N \rightarrow \infty} \frac{1}{N - M} \left| \left\{ n \in [M, N) : \{q(n)\} \in [1 - \delta, 1) \right\} \right|,$$

and

$$B(\delta) = \lim_{N \rightarrow \infty} \frac{1}{\pi(N)} \left| \left\{ p \in \mathbb{P} \cap [1, N] : \{q(rp + s)\} \in [1 - \delta, 1) \right\} \right|.$$

If $q - q(0)$ has an irrational coefficient, then by Weyl's Uniform Distribution Theorem [19] and Rhin's Theorem [16] we have $A(\delta) = B(\delta) = \delta$ which approach 0 as $\delta \rightarrow 0^+$.

Assume otherwise that $q \in \mathbb{R}[x]$ satisfies $q - q(0) \in \mathbb{Q}[x]$, say $q(x) = q(0) + b^{-1} \sum_{j=1}^{\ell} a_j x^j$ where $b \in \mathbb{N}$, $a_j \in \mathbb{Z}$ for $1 \leq j \leq \ell$, and $q(0) \in \mathbb{R}$. It follows that for all $n \in \mathbb{N}$,

$$q(n) - q(0) \bmod 1 \in \left\{ 0, \frac{1}{b}, \frac{2}{b}, \dots, \frac{b-1}{b} \right\}.$$

In particular, the fractional part $\{q(n)\}$ takes only finitely many values. Therefore, if δ is small enough, for every $n \in \mathbb{N}$ we have $\{q(n)\} \notin [1 - \delta, 1)$ and hence $A(\delta) = B(\delta) = 0$, which implies the desired conclusion. \square

For the proof of Theorem A we adapt arguments from [10, 11], i.e., we use a multidimensional suspension flow to approximate α by a multicorrelation sequence of the form (9). The arising error consists of terms of the form $1_{\{n \in \mathbb{N} : q(n) \in [1 - \delta, 1)\}}$ that can be controlled by Lemma 5.

Proof of Theorem A. Given $\ell, m, d \in \mathbb{N}$, let k be as guaranteed by Theorem 4. Let $q_1, \dots, q_m, T, f_0, \dots, f_m, \epsilon > 0, r \in \mathbb{N}, s \in \mathbb{Z}$ and α be as in the statement. By multiplying each function by a constant if needed, we can assume without loss of generality that $\|f_i\|_\infty \leq 1$ for each $i \in [1, m]$.

We start by considering a multidimensional suspension flow with a constant 1 ceiling function. More precisely, let $Y := X \times [0, 1)^{m \times \ell}$ and $\nu = \mu \otimes \lambda$, where λ denotes the Lebesgue measure on $[0, 1)^{m \times \ell}$. For each $i \in [1, m]$ define the measure preserving \mathbb{R}^ℓ -action S_i on (Y, ν) as follows: for any $t \in \mathbb{R}^\ell$ and $(x; b_1, \dots, b_m) \in Y = X \times ([0, 1)^\ell)^m$, let

$$S_i^t(x; b_1, \dots, b_m) := (T^{\lfloor b_i + t \rfloor} x; b_1, \dots, b_{i-1}, \{b_i + t\}, b_{i+1}, \dots, b_m),$$

where $\{u\} := u - \lfloor u \rfloor$ for any $u \in \mathbb{R}^\ell$. Observe that the actions S_1, \dots, S_m commute.

Let $\pi: Y \rightarrow X$ be the natural projection and $\delta > 0$ a small parameter to be determined later. For each $i \in [1, m]$ let $\hat{f}_i \in L^\infty(Y)$ be the composition $\hat{f}_i := f_i \circ \pi$, and $\hat{f}_0 := 1_{X \times [0, \delta]^{m \times \ell}} \cdot f_0 \circ \pi$. Define

$$\tilde{\alpha}(n) = \int_Y \hat{f}_0 \cdot S_1^{q_1(n)} \hat{f}_1 \cdots S_m^{q_m(n)} \hat{f}_m \, d\nu.$$

By Theorem 4 there exists a k -step nilsequence $\tilde{\psi}$ such that

$$\lim_{N \rightarrow \infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |\tilde{\alpha}(n) - \tilde{\psi}(n)| \leq \delta^{\ell m} \epsilon / 2, \quad (10)$$

and

$$\lim_{N \rightarrow \infty} \frac{1}{\pi(N)} \sum_{p \in \mathbb{P} \cap [1, N]} |\tilde{\alpha}(rp + s) - \tilde{\psi}(rp + s)| \leq \delta^{\ell m} \epsilon / 2. \quad (11)$$

On the other hand,

$$\tilde{\alpha}(n) = \int_{[0, \delta]^{\ell m}} \int_X f_0(x) f_1(T^{\lfloor q_1(n) + b_1 \rfloor} x) \cdots f_m(T^{\lfloor q_m(n) + b_m \rfloor} x) d\mu(x) d\lambda(b_1, \dots, b_m),$$

which implies

$$\alpha(n) - \frac{\tilde{\alpha}(n)}{\delta^{\ell m}} = \frac{1}{\delta^{\ell m}} \int_{[0, \delta]^{\ell m}} \int_X f_0(x) \left(\prod_{i=1}^m f_i(T^{\lfloor q_i(n) \rfloor} x) - \prod_{i=1}^m f_i(T^{\lfloor q_i(n) + b_i \rfloor} x) \right) d\mu d\lambda. \quad (12)$$

In particular, it follows from (12) that $|\alpha(n) - \delta^{-\ell m} \tilde{\alpha}(n)| \leq 2$ for all $n \in \mathbb{N}$. If $b_i \in [0, \delta]^\ell$ and $\{q_i(n)\} \in [0, 1 - \delta]^\ell$ then $\lfloor q_i(n) + b_i \rfloor = \lfloor q_i(n) \rfloor$. Therefore (12) also implies that $\alpha(n) = \delta^{-\ell m} \tilde{\alpha}(n)$ whenever

$$n \notin \left\{ n \in \mathbb{N} : \{q_i(n)\} \in [1 - \delta, 1]^\ell \text{ for some } i \in [1, m] \right\}.$$

In view of Lemma 5, by choosing δ small enough, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \left| \alpha(n) - \delta^{-\ell m} \tilde{\alpha}(n) \right| < \frac{\epsilon}{2} \quad (13)$$

and

$$\lim_{N \rightarrow \infty} \frac{1}{\pi(N)} \sum_{p \in \mathbb{P} \cap [1, N]} \left| \alpha(rp + s) - \delta^{-\ell m} \tilde{\alpha}(rp + s) \right| < \frac{\epsilon}{2}. \quad (14)$$

Letting $\psi = \delta^{-m\ell} \tilde{\psi}$ and combining (10) with (13) and (11) with (14) we obtain the desired conclusion. \square

Remark 6. As it was already mentioned in Section 1, it is an open problem whether one can improve upon the approximation in Frantzikinakis' main result in [2] and Theorem 1 and take $\epsilon = 0$ in (5) and (7) (see Question 2 below). However, as the following example shows, in the case of Theorem A it is not possible to improve upon the approximation in that manner.

Example 7. Take $X = \mathbb{T} := \mathbb{R}/\mathbb{Z}$, $T(x) = x + 1/\sqrt{2}$, $q(n) = \sqrt{2}n$, $f_0(x) = e(x)$ and $f_1(x) = e(-x)$, where $e(x) := e^{2\pi i x}$. Then we have

$$\begin{aligned} \alpha(n) &= \int f_0 \cdot T^{\lfloor q(n) \rfloor} f_1 d\mu = \int e(x) e\left(-x - \frac{1}{\sqrt{2}} \lfloor \sqrt{2}n \rfloor\right) dx \\ &= e\left(-\frac{1}{\sqrt{2}} \lfloor \sqrt{2}n \rfloor\right) = e\left(\frac{1}{\sqrt{2}} \{\sqrt{2}n\}\right). \end{aligned}$$

In particular, we can write $\alpha(n)$ as $F(T^n x_0)$ with $x_0 = 0 \in \mathbb{T}$ and $F(x) = e(\{x\}/\sqrt{2})$ for $x \in \mathbb{T}$. Assume for the sake of a contradiction that there exists a uniform limit of nilsequences ϕ for which

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N |\alpha(n) - \phi(n)| = 0. \quad (15)$$

By [9, Lemma 18], ϕ can be written as $\phi(n) = G(S^n y_0)$ for all $n \in \mathbb{N}$, where G is a continuous function on an inverse limit of nilsystems (Y, S) and $y_0 \in Y$.

We claim that $\alpha(n) = \phi(n)$ for all $n \in \mathbb{N}$. If not, then there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$|\alpha(n_0) - \phi(n_0)| = |F(T^{n_0} x_0) - G(S^{n_0} y_0)| \geq \delta. \quad (16)$$

Since the system $(X \times Y, T \times S)$ is the product of two distal systems, is a distal system itself. This implies that the point $(T^{n_0} x_0, S^{n_0} y_0)$ is uniformly recurrent, i.e., the sequence $(T^n x_0, S^n y_0)$ visits any neighborhood of $(T^{n_0} x_0, S^{n_0} y_0)$ in a syndetic set. This fact together with (16) and the fact that both the real and imaginary parts of F are almost everywhere continuous and semicontinuous imply that the set

$$\{n \in \mathbb{N} : |F(T^n x_0) - G(S^n y_0)| \geq \delta/2\}$$

is syndetic, which contradicts (15). Hence $\alpha(n) = \phi(n)$ for all $n \in \mathbb{N}$. However, by [6, Proposition 4.2.5], the sequence α is not a distal sequence; in particular, it is not a uniform limit of nilsequences, contradicting our assumption.

3. OPEN QUESTIONS

We close this article with three open questions. Theorem A provides an approximation result of multicorrelation sequences along an integer polynomial of degree one, evaluated at primes. We can ask whether a similar result is true along other classes of sequences.

Question 1. *Let $q \in \mathbb{R}[x]$ be a non-constant real polynomial, $c > 0$, and p_n denote the n -th prime. Suppose $r_n = q(n), q(p_n), \lfloor n^c \rfloor$ or $\lfloor p_n^c \rfloor$ for $n \in \mathbb{N}$. Is it true that for any α as in (6) and $\epsilon > 0$, there exists a nilsequence ψ satisfying*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N |\alpha(r_n) - \psi(r_n)| \leq \epsilon?$$

Variants of the following question have appeared several times in the literature, e.g., [2, Remark after Theorem 1.1], [3, Problem 20], [4, Problem 1], and [8, Page 398].

Question 2. *Let α be as in (4). Does there exist a uniform limit of nilsequences ϕ such that*

$$\lim_{N-M \rightarrow \infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |\alpha(n) - \phi(n)| = 0?$$

As mentioned in Example 7, the answer to Question 2 is negative when α is a multicorrelation sequence as in (6). Nevertheless, it makes sense to ask for the following modification of it.

Question 3. *Let α be as in (6). Does there exist a uniform limit of Riemann integrable nilsequences ϕ satisfying*

$$\lim_{N-M \rightarrow \infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |\alpha(n) - \phi(n)| = 0?$$

Here we say that ϕ is a uniform limit of Riemann integrable nilsequences if for every $\epsilon > 0$ there exists a nilmanifold $X = G/\Gamma$, a point $x \in X$, $g \in G$ and a Riemann integrable function² $F: X \rightarrow \mathbb{C}$ such that $\sup_{n \in \mathbb{N}} |\phi(n) - F(g^n x)| < \epsilon$.

²A function F is Riemann integrable on a nilmanifold if its points of discontinuity is a null set with respect to the Haar measure.

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