



Athorne, C. (2020) Equivariance and algebraic relations for curves. *Journal of Geometry and Physics*, 155, 103748.

(doi: [10.1016/j.geomphys.2020.103748](https://doi.org/10.1016/j.geomphys.2020.103748))

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Deposited on: 5 June 2020

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Counting relations for curves using equivariance

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Abstract

We generalise a classical argument for deducing algebraic models of Riemann surfaces from the Riemann-Roch theorem. The method involves counting arguments based on equivariant resolutions.

1 Introduction

The relationship between curves and their Jacobians is of fundamental importance in a range of topics in mathematical physics from classical nonlinear oscillators to integrable systems on both continuous and discrete spaces. [1, 9, 15, 17]

The simplest nontrivial example is the Weierstraß \wp function defined on the Jacobian of a nonsingular cubic curve. In this instance the curve and its Jacobian are isomorphic and the \wp function and its derivative parametrise the curve itself [8, 18]. The \wp function describes both the oscillations of the classical pendulum as well as quasiperiodic solutions to soliton equations such as the Korteweg-de Vries equation [12, 11].

For nonsingular higher genus curves, the role of the \wp function is played by a multiplicity of functions, $\wp_{ij} = \wp_{ji}$ for $1 \leq i, j \leq g$, g being the genus of the curve [4, 5]. These functions live on the Jacobian and satisfy complicated differential relations analogous to the Weierstraß differential equation which can, however, be greatly simplified by noting an equivariant property which follows from simple coordinate changes for the underlying curve. At root this follows from the fact that the g holomorphic differentials on the Jacobian are a \mathbb{C} basis for an irreducible g dimensional \mathfrak{sl}_2 representation (at least in the hyperelliptic case). The differential relations for the \wp_{ij} functions decompose into finite dimensional irreducibles that can be generated from their highest (lowest) weight elements by lowering and raising operators [2, 3].

An important consideration is to understand the (differential) ideal of generating relations for the \wp_{ij} and their higher derivatives [14].

As a precursor to such a description we consider here relations between functions on the curve itself.

The Riemann-Roch theorem imposes constraints on the dimensions of linear spaces of functions with prescribed poles on a compact, nonsingular Riemann surface [8, 13]. Consequently the module of such functions is not free and the question of describing all possible relations arises. Each function is a coordinate on the surface and the relations constitute models of the surface in projective spaces.

This paper starts by rehearsing a classical argument for the genus one surface with a set of divisors of the form nP for integer $n \geq 0$ to arrive at the cubic model [16]. We then generalise this to the case of two-point divisors where a more complicated set of relations obtains, setting up an equivariant resolution of the coordinate ring in order to show that an explicit, finite set of quadratic and cubic relations generates all relations.

The equivariant property in this case differs from that mentioned earlier in the introduction and corresponds to creation and annihilation of poles.

Finally we generalise the argument to two-point divisors on curves of the (n, s) type for n and s coprime.

2 Riemann-Roch

We start with a statement of the classical Riemann-Roch theorem for a compact, nonsingular Riemann surface X of genus g .

A *divisor* D on X is a formal, finite sum of points, $P \in X$ each with an associated integer order, $n_P \in \mathbb{Z}$:

$$D = \sum_{P \in X}^{\text{finite}} n_P P, n_P \in \mathbb{Z}.$$

The *degree* of D is

$$\deg(D) = \sum_{P \in X} n_P.$$

A divisor is *effective* if $n_P \geq 0$ for all $P \in X$.

A meromorphic function on X : $f : X \rightarrow \mathbb{P}C^1$ has a finite number of poles and zeros with associated negative and non-negative orders and hence a divisor:

$$(f) = \sum_{P \in X} \text{ord}_P(f) P.$$

Such divisors are called *principal*.

Likewise a one-form ω on X is meromorphic if, in local coordinates on X , it is holomorphic except at a finite number of zeros or poles. Again it has divisor

$$(\omega) = \sum_{P \in X} \text{ord}_P(\omega) P.$$

\mathbb{C} -vector spaces of meromorphic functions are associated with any effective divisor D ,

$$L(D) = \{f : (f) + D \geq 0\}$$

and \mathbb{C} -vector spaces of holomorphic one-forms

$$R(D) = \{\omega : (\omega) - D \geq 0\}$$

Let $l(D)$ and $r(D)$ be the \mathbb{C} -dimensions of $L(D)$ and $R(D)$. Then the statement of the Riemann-Roch theorem for any effective D of degree n is

$$l(D) - r(D) = n - g + 1.$$

The number $r(D)$ lies between 0 and g , the genus of the Riemann surface. As n increases by one so either $l(D)$ increases by one or $r(D)$ decreases by one.

One application of this result is to determine the algebraic forms of curves representing the Riemann surface.

Consider the simplest non trivial case of genus one and divisors nP for $n \geq 0$. For $n = 0$ there is only the one-dimensional space of constants so $l(0P) = 1$ and $r(0P) = 1$. For $n = 1$, $l(1P) = 1$ still, as there are no functions with but a single pole, and hence $r(1P) = 0$. For $n \geq 2$, $r(nP)$ must remain at 0 and $l(nP)$ increments with n . There must be a function with a double pole, x_2 , one with a third order pole, y_3 and so on.

D	0.P	1.P	2.P	3.P	4.P	5.P	6.P	7.P	8.P	9.P
dim	1	1	2	3	4	5	6	7	8	9
	1	0	x_2	y_3	x_2^2	x_2y_3	x_2^3 y_3^2	$x_2^2y_3$	x_2^4 $x_2y_3^2$	y_3^3 $x_2^3y_3$

At degree six we find two new functions: x_2^3 and y_3^2 . Since the dimension of $l(6P) = 6$ there must be a linear relation over \mathbb{C} between them:

$$y_3^2 - \alpha_6 x_2^3 = \alpha_5 x_2 y_3 + \alpha_4 x_2^2 + \alpha_3 y_3 + \alpha_2 x_2 + \alpha_0,$$

the α_i all being complex constants.

We scale x_2 in order to set $\alpha_6 = 1$ and write the relation

$$\Delta = y_3^2 - x_2^3 \sim 0$$

where by \sim we mean equivalence up to elements of $L(5P) \subset L(6P)$.

At 8.P, 9.P, ... relations arise of the form

$$x_2 y_3^2 - x_2^4 = x_2 \Delta \sim 0$$

where \sim now means equivalence up to elements of $L(7P) \subset L(8P)$ and so on. The single relation, $\Delta \sim 0$, suffices to satisfy the Riemann-Roch constraint at all degrees. We obtain one new function each time the degree increments:

$$x_2^n \text{ at } 2n.P$$

$$y_3 x_2^{n-1} \text{ at } (2n+1).P$$

Thus the coordinate ring of the Riemann surface is $\mathbb{C}[x_2] \oplus y_3 \mathbb{C}[x_2]$.

In order to generalise we redescribe this situation using an exact resolution for the coordinate ring [6, 7]. Let $R = \mathbb{C}[x_2, y_3]$ be graded by weights where x_2 has weight two and y_3 weight three, so the monomial $x_2^p y_3^q$ has weight $2p + 3q$. Then

$$R = \bigoplus_{n=0}^{\infty} R^{[n]}[x_2, y_3].$$

Let $d_n = \dim R^{[n]} = \#$ partitions of n into 2's and 3's. The Hilbert series for R is

$$H(t) = \sum_{n=0}^{\infty} d_n t^n = (1 - t^2)^{-1} (1 - t^3)^{-1}.$$

Let $\tilde{d}_n = \dim(R/(\Delta))^{[n]}$ and

$$\tilde{H}(t) = \sum_{n=0}^{\infty} \tilde{d}_n t^n.$$

There is the exact sequence

$$0 \rightarrow R^{[n-6]} \xrightarrow{\Delta} R^{[n]} \xrightarrow{\pi} (R/(\Delta))^{[n]} \rightarrow 0$$

implying,

$$d_{n-6} - d_n + \tilde{d}_n = 0.$$

Summing over powers of t :

$$\begin{aligned} \tilde{H}(t) &= (1 - t^6)H(t) \\ &= \frac{1 - t + t^2}{1 - t} \\ &= 1 + t^2 + t^3 + t^4 + t^5 + t^6 + t^7 + \dots \end{aligned}$$

as expected from the Riemann-Roch theorem.

We wish to generalise this argument firstly to two-point divisors on the genus one surface.

3 Divisors on the genus one curve at two points

We now have divisors $D = nP + mQ$ for $n, m \geq 0$. At degrees two and three we have linearly independent functions with exactly two or three poles: $x_{ij} \in L(iP + jQ)$, $i + j = 2$; $y_{ij} \in L(iP + jQ)$, $i + j = 3$. Using these we construct functions of higher degree according to the following table:

	$0.P$	$1.P$	$2.P$	$3.P$	$4.P$	$5.P$	$6.P$	$7.P$
$0.Q$	1	0	x_{20}	y_{30}	x_{20}^2	$x_{20}y_{30}$	x_{20}^3, y_{30}^2	$x_{20}^2y_{30}$
$1.Q$	0	x_{11}	y_{21}	$x_{11}x_{20}$	$x_{11}y_{30}, x_{20}y_{21}$	$x_{11}x_{20}^2, y_{30}y_{21}$		
$2.Q$	x_{02}	y_{12}	$x_{11}^2, x_{20}x_{02}$	$x_{02}y_{30}, x_{11}y_{21}, x_{20}y_{12}$	*			
$3.Q$	y_{03}	$x_{11}x_{02}$	$x_{02}y_{21}, x_{11}y_{12}, x_{20}y_{03}$	*				
$4.Q$	x_{02}^2	$x_{11}y_{03}, x_{02}y_{12}$	*					
$5.Q$	$x_{02}y_{03}$							
$6.Q$	x_{02}^3, y_{03}^2	$x_{11}x_{02}^2, y_{12}y_{03}$						
$7.Q$	$x_{02}^2y_{03}$							

At degrees four and five we see there must be quadratic relations in x and y and at degree six relations quadratic in y but cubic in x . We want to show that these exhaust all relations.

Let

$$\begin{aligned} 0 &\rightarrow \{L(rP + sQ) \mid r \leq n, s \leq m, r + s < n + m\} \\ &\rightarrow L(nP + mQ) \xrightarrow{\pi} S^{[n,m]} \rightarrow 0 \end{aligned}$$

be exact. Then the relations are elements of $R^{[n]}$ lying in the kernel of π . Thus we write

$$x_{20}x_{02} - x_{11}^2 \sim 0$$

to mean $\pi(x_{20}x_{02} - x_{11}^2) = 0$.

We start with the quadratic relations. We can normalise the x 's and y 's so that these are all the 2×2 minors of

$$\begin{bmatrix} x_{20} & x_{11} & y_{30} & y_{21} & y_{12} \\ x_{11} & x_{02} & y_{21} & y_{12} & y_{03} \end{bmatrix}$$

and show that if we factor out these $1 + 6 + 3 = 10$ relations we recover the previous situation by a counting argument based on an appropriate exact sequence. we can then employ an equivariance argument to show that the full set of relations are a complete set.

Let Δ now represent the ideal generated by the ten relations

$$x_{20}x_{02} - x_{11}^2, x_{20}y_{21} - x_{11}y_{30}, \dots, y_{30}y_{12} - y_{21}^2, \dots$$

in the polynomial ring $R = \mathbb{C}[x_{20}, x_{11}, x_{02}, y_{30}, y_{21}, y_{12}, y_{03}]$. We wish to construct a resolution of the quotient module R/Δ graded by degree where $\deg x = 2$ and $\deg y = 3$. Such a resolution looks like:

$$\begin{aligned} 0 &\rightarrow \bigoplus^4 R^{[n-13]} \\ &\xrightarrow{\phi_4} \bigoplus^3 \left(\bigoplus^2 R^{[n-11]} \oplus \bigoplus^3 R^{[n-10]} \right) \\ &\xrightarrow{\phi_3} \bigoplus^2 \left(R^{[n-9]} \oplus \bigoplus^6 R^{[n-8]} \oplus \bigoplus^3 R^{[n-7]} \right) \\ &\xrightarrow{\phi_2} R^{[n-4]} \oplus \bigoplus^6 R^{[n-5]} \oplus \bigoplus^3 R^{[n-6]} \\ &\xrightarrow{\phi_1} R^{[n]} \xrightarrow{\pi} (R/\Delta)^{[n]} \rightarrow 0. \end{aligned} \tag{1}$$

In order to define ϕ_1 such that $\text{im } \phi_1 = \ker \pi$, let $\{e_0, e_1\}$ and $\{f_0, f_1, f_2\}$ be bases of two and three dimensional vector spaces, E and F respectively. Let

$$\begin{aligned} \omega_1 &= x_{20}e_0 + x_{11}e_1 + y_{30}f_0 + y_{21}f_1 + y_{12}f_2 \\ \omega_2 &= x_{11}e_0 + x_{02}e_1 + y_{21}f_0 + y_{12}f_1 + y_{03}f_2 \end{aligned}$$

and

$$\Omega = \omega_1 \wedge \omega_2 = \Omega_{e,e}^{[4]} + \Omega_{e,f}^{[5]} + \Omega_{f,f}^{[6]}$$

where

$$\begin{aligned}\Omega_{e,e}^{[4]} &= (x_{20}e_0 + x_{11}e_1) \wedge (x_{20}e_0 + x_{11}e_1) \\ \Omega_{e,f}^{[5]} &= (x_{20}e_0 + x_{11}e_1) \wedge (y_{21}f_0 + y_{12}f_1 + y_{03}f_2) \\ &\quad + (y_{30}f_0 + y_{21}f_1 + y_{12}f_2) \wedge (x_{11}e_0 + x_{02}e_1) \\ \Omega_{f,f}^{[6]} &= (y_{30}f_0 + y_{21}f_1 + y_{12}f_2) \wedge (y_{21}f_0 + y_{12}f_1 + y_{03}f_2).\end{aligned}$$

Then we implement the relations of the ideal by working over the exterior products of $E \oplus F$, with coefficients in R , and taking the wedge product with Ω :

$$\dots \rightarrow R^{[n-4]} \bigwedge_{f,f,f}^3 \oplus R^{[n-5]} \bigwedge_{e,f,f}^3 \oplus R^{[n-6]} \bigwedge_{e,e,f}^3 \xrightarrow{\Omega \wedge \cdot} R^{[n]} \bigwedge_{e,e,f,f,f}^5 \xrightarrow{\pi} \dots$$

Here the subscripts of the \bigwedge symbol denote mixtures of e 's and f 's present in the basis so, for example,

$$R^{[n-5]} \bigwedge_{e,f,f}^3 = \{f(x,y) e_i \wedge f_j \wedge f_k | f \text{ has weight } n-5, i=1,2 \text{ and } j,k=1,2,3\}.$$

This gives our sequence (exact on the right):

$$\dots \xrightarrow{\phi_2} R^{[n-4]} \oplus \bigoplus^6 R^{[n-5]} \oplus \bigoplus^3 R^{[n-6]} \xrightarrow{\phi_1} R^{[n]} \xrightarrow{\pi} \dots$$

Now we will define ϕ_2 such that $\text{im } \phi_2 = \ker \phi_1$. The kernel of $\Omega \wedge \cdot$ in $\bigwedge^3(E \oplus F)$ is $\{\omega_1 \wedge \alpha_1 + \omega_2 \wedge \alpha_1 | \alpha_1, \alpha_2 \in \bigwedge^2\}$. Hence the map

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \rightarrow (\omega_1, \omega_2) \wedge \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

does the job:

$$\begin{aligned}\dots \rightarrow \bigoplus^2 \left(R^{[n-9]} \bigwedge_{e,e}^2 \oplus R^{[n-8]} \bigwedge_{e,f}^2 \oplus R^{[n-7]} \bigwedge_{f,f}^2 \right) \\ \xrightarrow{(\omega_1, \omega_2) \wedge \cdot} R^{[n-4]} \bigwedge_{f,f,f}^3 \oplus R^{[n-5]} \bigwedge_{e,f,f}^3 \oplus R^{[n-6]} \bigwedge_{e,e,f}^3 \xrightarrow{\phi_1} \dots\end{aligned}$$

This gives

$$\bigoplus^2 \left(R^{[n-9]} \oplus \bigoplus^6 R^{[n-8]} \oplus \bigoplus^3 R^{[n-7]} \right)$$

for the domain of ϕ_2 .

The kernel of the map $(\omega_1, \omega_2) \wedge \cdot$ is

$$\{(\beta_{11} \wedge \omega_1 + \beta_{12} \wedge \omega_2) \oplus (\beta_{21} \wedge \omega_1 + \beta_{22} \wedge \omega_2) | \beta_{12} = \beta_{21}, \beta_{ij} \in \bigwedge^1\}$$

and we map into this kernel from $\bigoplus^3 \bigwedge^1$ by:

$$\begin{pmatrix} \beta_{11} \\ \beta_{12} \\ \beta_{22} \end{pmatrix} \rightarrow \begin{pmatrix} \omega_1 & \omega_2 & 0 \\ 0 & \omega_1 & \omega_2 \end{pmatrix} \wedge \begin{pmatrix} \beta_{11} \\ \beta_{12} \\ \beta_{22} \end{pmatrix}$$

Thus for the domain of ϕ_3 we have

$$\bigoplus^3 \left(\bigoplus^2 R^{[n-11]} \oplus \bigoplus^3 R^{[n-10]} \right).$$

Continuing in this way we arrive at the resolution of the coordinate ring described in (1).

Using similar notation to earlier, the Hilbert series for R is

$$H(t) = \sum_0^\infty d_n t^n = (1-t^2)^{-3} (1-t^3)^{-4}$$

and the dimension at degree n of the coordinate ring,

$$\begin{aligned} \tilde{d}_n &= d_n - (d_{n-4} + 6d_{n-5} + 3d_{n-6}) + 2(d_{n-9} + 6d_{n-8} + 3d_{n-7}) \\ &\quad - 3(2d_{n-11} + 3d_{n-10}) + 4d_{n-13}. \end{aligned}$$

The Hilbert series for the coordinate ring is therefore

$$\begin{aligned} \tilde{H}(t) &= \frac{1+t+2t^2+4t^3+4t^4}{(1-t)^3(1+t+t^2)(1+t)^2} \\ &= 1 + 3t^2 + 4t^3 + 5t^4 + 6t^5 + \mathbf{14}t^6 + 8t^7 + \mathbf{18}t^8 + \dots \end{aligned}$$

In this expression the coefficients are the number of entries remaining on the anti-diagonals of the $nP + mQ$ diagram once all quadratic relations have been factored out. The relations cubic in x and quadratic in y have not yet been factored out and the magnitudes of the emboldened coefficients reflect this.

4 Equivariance

We note that there is an \mathfrak{sl}_2 action on the divisor diagram which adds and subtracts poles, leaving the degree unaltered,

$$\begin{aligned} \mathbf{e}x_{i,j} &= ix_{i-1,j+1} & \mathbf{f}x_{i,j} &= jx_{i+1,j-1} \\ \mathbf{e}y_{i,j} &= iy_{i-1,j+1} & \mathbf{f}y_{i,j} &= jy_{i+1,j-1} \end{aligned}$$

(where a separator as been added to the indices for clarification) under which the ideal Δ is invariant.

The sequence we have described in the previous section is equivariant in the sense that the following diagram commutes (for appropriate representations, ρ_n etc., of \mathfrak{sl}_2 on the terms of the sequence, A_* of the form $\bigoplus^p \bigwedge^q$)

$$\begin{array}{ccccccc} \dots & \rightarrow & A_{n+1} & \xrightarrow{\phi_{n+1}^+} & A_n & \rightarrow & \dots \\ & & \downarrow \rho_{n+1} & & \downarrow \rho_n & & \\ \dots & \rightarrow & A_{n+1} & \xrightarrow{\phi_{n+1}^+} & A_n & \rightarrow & \dots \end{array}$$

In particular highest weight elements in A_{n+1} map to highest weight elements in A_n . ($\Omega \wedge \cdot$ is an invariant map and so preserves dimensions of irreducibles.)

Each antidiagonal in the original divisor diagram becomes a direct sum of irreducible representations, of which the quadratic relations are cases, and once those quadratics are factored out we are left only with the irreps in $\bigotimes^n \mathbf{x}$ of dimension $2n+1$ and in $\bigotimes^n \mathbf{y}$ of dimension $3n+1$. We apply the earlier argument to factor out the highest weight elements of the form $y_{30}^2 - x_{20}^3$ and hence the remaining identities.

For instance the relation $R_{60} = y_{30}^2 - x_{20}^3 \sim 0$ under the action of \mathbf{e} gives the relation $R_{51} = y_{30}y_{21} - x_{20}^2x_{11} \sim 0$. There is a syzygy $y_{21}R_{60} - y_{30}R_{51} = x_{20}^2(y_{30}x_{11} - y_{21}x_{20}) \sim 0$.

This leaves us with the Hilbert series

$$1 + 3t^2 + 4t^3 + 5t^4 + 6t^5 + 7t^6 + 8t^7 + 9t^8 + \dots$$

consistent with the Riemann-Roch theorem.

To establish the equivariance of the resolution (1) we must find the maps ρ_n and define an action on the vector spaces E and F . We do this in such a way as to make $\Omega \wedge \cdot$ invariant. Namely, let

$$\begin{aligned} \mathbf{e}(e_0) &= 0 & \mathbf{f}(e_0) &= -e_1 \\ \mathbf{e}(e_1) &= -e_0 & \mathbf{f}(e_1) &= 0 \\ \\ \mathbf{e}(f_0) &= 0 & \mathbf{f}(f_0) &= -f_1 \\ \mathbf{e}(f_1) &= -2f_0 & \mathbf{f}(f_1) &= -2f_2 \\ \mathbf{e}(f_2) &= -f_1 & \mathbf{f}(f_2) &= 0. \end{aligned}$$

Then

$$\begin{aligned} \mathbf{e}(\omega_1) &= \omega_2 & \mathbf{e}(\omega_2) &= 0 \\ \mathbf{f}(\omega_1) &= 0 & \mathbf{f}(\omega_2) &= \omega_1 \end{aligned}$$

and consequently

$$\mathbf{e}(\Omega) = \mathbf{f}(\Omega) = 0.$$

Then we have (writing say for \mathbf{e})

$$\begin{array}{ccccc} \rightarrow & \bigwedge^3 & \xrightarrow{\Omega \wedge \cdot} & \bigwedge^5 & \rightarrow \\ \rho_1 = \mathbf{e} & \downarrow & & \downarrow & \rho_0 = \mathbf{e} \\ \rightarrow & \bigwedge^3 & \xrightarrow{\Omega \wedge \cdot} & \bigwedge^5 & \rightarrow \end{array}$$

At the next place

$$\begin{array}{ccccc} \rightarrow & \bigoplus^2 \bigwedge^2 & \xrightarrow{(\omega_1, \omega_2) \wedge \cdot} & \bigwedge^3 & \rightarrow \\ \rho_2 \downarrow & & & \downarrow \rho_1 & \\ \rightarrow & \bigoplus^2 \bigwedge^2 & \xrightarrow{(\omega_1, \omega_2) \wedge \cdot} & \bigwedge^3 & \rightarrow \end{array}$$

for

$$\rho_1 = \mathbf{e}, \quad \rho_2 = \begin{pmatrix} \mathbf{e} & 0 \\ 1 & \mathbf{e} \end{pmatrix}$$

and so on,

$$\begin{array}{ccc}
& \begin{pmatrix} \omega_1 & \omega_2 & 0 \\ 0 & \omega_1 & \omega_2 \end{pmatrix} \wedge \cdot & & \\
\rightarrow & \bigoplus^3 \wedge^1 & \rightarrow & \bigoplus^2 \wedge^2 \rightarrow \\
& \rho_3 \downarrow & & \downarrow \rho_2 \\
\rightarrow & \bigoplus^3 \wedge^1 & \rightarrow & \bigoplus^2 \wedge^2 \rightarrow \\
& \begin{pmatrix} \omega_1 & \omega_2 & 0 \\ 0 & \omega_1 & \omega_2 \end{pmatrix} \wedge \cdot & &
\end{array}$$

for

$$\rho_3 = \begin{pmatrix} \mathbf{e} & 0 & 0 \\ 1 & \mathbf{e} & 0 \\ 0 & 2 & \mathbf{e} \end{pmatrix}$$

This equivariance property reduces the problem to consideration of the top row only of the divisor diagram, which involves the x_{20} and y_{30} functions only and relations $y_{30}^2 - x_{20}^3 \sim 0$.

We can describe the coordinate ring as

$$\begin{aligned}
R/\Delta &= \mathbb{C}[x_{20}, x_{02}] \oplus x_{11}\mathbb{C}[x_{20}, x_{02}] \oplus y_{30}\mathbb{C}[x_{20}] \oplus y_{03}\mathbb{C}[x_{02}] \\
&\oplus y_{12}\mathbb{C}[x_{20}, x_{02}] \oplus y_{21}\mathbb{C}[x_{20}, x_{02}].
\end{aligned}$$

Writing a double Hilbert series for this coordinate ring of the form

$$C(s, t) = \sum_{n, m=0}^{\infty} s^n t^m \dim S^{[n, m]}$$

from the above decomposition, we find,

$$\begin{aligned}
C(s, t) &= \frac{1}{(1-t^2)(1-s^2)} + \frac{st}{(1-t^2)(1-s^2)} + \frac{s^3}{(1-s^2)} + \frac{t^3}{(1-t^2)} \\
&\quad + \frac{st^2}{(1-t^2)(1-s^2)} + \frac{s^2t}{(1-t^2)(1-s^2)} \\
&= \frac{1}{(1-t)(1-s)} - s - t.
\end{aligned}$$

This gives the value unity at all places except $n + m = 1$ which is correct.

5 Explicit relations

We can use the equivariance property to undersatnd the explicit relations between the x 's and the y 's.

For example, the relation at $D = 2P + 2Q$ must be

$$x_{20}x_{02} - x_{11}^2 = \beta_{12}y_{21} + \beta_{21}y_{12} + \gamma_{20}x_{02} - 2\gamma_{11}x_{11} + \gamma_{02}x_{20} + \delta_{00}.$$

Application of \mathbf{e} or \mathbf{f} yields relations involving $\beta_{21}y_{03}$ or $\beta_{12}y_{30}$ respectively which must be trivial. Hence $\beta_{12} = \beta_{21} = 0$. In addition we get

$$0 = \mathbf{e}(\gamma_{20})x_{02} - 2\gamma_{11}x_{02} - 2\mathbf{e}(\gamma_{11})x_{11} + 2\gamma_{02}x_{11} + \mathbf{e}(\gamma_{02})x_{20} + \mathbf{e}(\delta_{00})$$

$$0 = 2\gamma_{20}x_{11} + \mathbf{f}(\gamma_{20})x_{02} + \mathbf{f}(\gamma_{11})x_{11} + \gamma_{11}x_{20} + \mathbf{f}(\gamma_{02})x_{20} + \mathbf{f}(\delta_{00}).$$

Again, these relations must be trivial and hence the coefficients γ_{ij} form a three dimensional representation,

$$\mathbf{e} : \quad \gamma_{20} \rightarrow 2\gamma_{11}, \quad \gamma_{11} \rightarrow \gamma_{02}, \quad \gamma_{02} \rightarrow 0$$

$$\mathbf{f} : \quad \gamma_{02} \rightarrow 2\gamma_{11}, \quad \gamma_{11} \rightarrow \gamma_{20}, \quad \gamma_{20} \rightarrow 0$$

and δ_{00} is invariant. We can define $\tilde{x}_{ij} = x_{ij} - \gamma_{ij}$ and $\tilde{\delta}_{00} = \delta_{00} + \gamma_{02}\gamma_{20} - \gamma_{11}^2$ to write

$$\tilde{x}_{20}\tilde{x}_{02} - \tilde{x}_{11}^2 = \tilde{\delta}_{00}.$$

Thinking of the x_{ij} as coordinates on the Riemann surface this relation is one of the defining relations for a model of the curve embedded in some projective space of sufficiently high dimension.

On the other hand we may start with any model of the curve and write down explicit functions satisfying these relations.

For example, suppose we start with a pair of points (X_P, Y_P) and (X_Q, Y_Q) on the curve

$$Y^2 = g(X)$$

where $g(X)$ is either cubic or quartic. We define a polar form for the (more general) quartic case:

$$g(X) = g_0X^4 + 4g_1X^3 + 6g_2X^2 + 4g_3X + g_4,$$

namely

$$g_{pol}(X, Z) = g_0X^2Z^2 + 2g_1XZ(X+Z) + g_2(X^2+4XZ+Z^2) + 2g_3(X+Z) + g_4.$$

Now let

$$\begin{aligned} x_{20} &= \frac{Y_P Y - g_{pol}(X, X_P)}{(X_P - X)^2} \\ x_{11} &= \frac{\sqrt{2(Y_P Y_Q + g_{pol}(X_P, X_Q))}}{Y_P + Y_Q} \left(\frac{(Y_P + Y_Q)Y - Y_P Y_Q}{2(X_P - X)(X_Q - X)} \right. \\ &\quad \left. - \frac{g_{pol}(X, X_P) + g_{pol}(X, X_Q) - g_{pol}(X_P, X_Q)}{2(X_P - X)(X_Q - X)} \right) \\ x_{02} &= \frac{Y_Q Y - g_{pol}(X, X_Q)}{(X_Q - X)^2} \end{aligned}$$

Then one verifies that $x_{20} \in L(2P)$, $x_{11} \in L(P+Q)$, $x_{02} \in L(2Q)$ and

$$x_{20}x_{02} - x_{11}^2 + cx_{11} + d = 0$$

for appropriate constants c and d depending on P and Q .

The argument leading to the quadratic relation for the x_{ij} can be simplified slightly. Write \mathbf{x}_3 and \mathbf{y}_4 for the 3 and 4 dimensional irreducible representations associated with the x and the y . Denote by $[\cdot]_n$ the projection onto the n dimensional irrep. Then we have

$$[\mathbf{x}_3 \otimes \mathbf{x}_3]_1 = [\beta_4 \otimes \mathbf{y}_4]_1 + [\gamma_3 \otimes \mathbf{x}_3]_1 + \delta_1$$

for some constant irreps β_4 , γ_3 and δ_1 of dimensions specified by the subscripts. However, since \mathbf{y}_4 has elements in $L(3P)$ and $L(3Q)$ it must be that $\beta_4 = 0$.

Now consider the next set of relations

$$x_{20}y_{12} - 2x_{11}y_{21} + x_{02}y_{30} = \alpha_{31}x_{20}x_{11} + \beta_{11}x_{11}^2 + \gamma_{02}x_{20} + \gamma_{11}x_{11} + \gamma_{20}x_{02} + \delta_{00}$$

$$x_{20}y_{03} - 2x_{11}y_{12} + x_{02}y_{21} = \alpha_{13}x_{02}x_{11} + \beta'_{11}x_{11}^2 + \gamma'_{02}x_{20} + \gamma'_{11}x_{11} + \gamma'_{20}x_{02} + \delta'_{00}$$

Application of \mathbf{f} to the first and \mathbf{e} to the second tells us that $\alpha_{31} = \beta_{11} = \alpha_{13} = \beta'_{11} = 0$.

Considering the terms linear in the x similarly leaves only the possibility,

$$[\mathbf{x}_3 \otimes \mathbf{y}_4]_2 = [\gamma_2 \otimes \mathbf{x}_3]_2 + \delta_2$$

for some set of two constants γ_2 .

Likewise

$$[\mathbf{x}_3 \otimes \mathbf{y}_4]_4 = [\alpha_2 \otimes [\mathbf{x}_3 \otimes \mathbf{x}_3]_5]_4 + [\beta_3 \otimes \mathbf{y}_4]_4 + [\gamma_4 \otimes \mathbf{x}_3]_4 + \delta_4.$$

All the fundamental relations can be so described.

The relations quadratic in the y 's, will be

$$\begin{aligned} y_{30}y_{12} - y_{21}^2 &\in \text{Span}\{x_{02}y_{30}, x_{11}y_{21}, x_{20}y_{12}, x_{20}y_{21}, x_{11}y_{30}, \dots\} \\ &\in \text{Span}\{x_{11}y_{21}, x_{11}y_{30}, \dots\} \\ y_{30}y_{12} - y_{21}^2 &= \alpha_{21}x_{11}y_{21} + \alpha_{30}x_{11}y_{30} + \dots \\ 0 &= \alpha_{21}(x_{20}y_{21} + x_{11}y_{30}) + \alpha_{30}x_{20}y_{30} + \mathbf{f}(\alpha_{21})x_{11}y_{21} + \dots \\ 0 &= 2\alpha_{21}x_{20}y_{21} + \alpha_{30}x_{20}y_{30} + \mathbf{f}(\alpha_{21})x_{11}y_{21} + \dots \end{aligned}$$

so that $\alpha_{21} = \alpha_{30} = 0$.

The three quadratic relations in the y therefore must be written

$$[\mathbf{y}_4 \otimes \mathbf{y}_4]_3 = [\alpha_3 \otimes [\mathbf{x}_3 \otimes \mathbf{x}_3]_5]_3 + [\beta_2 \otimes \mathbf{y}_4]_3 + [\gamma_3 \otimes \mathbf{x}_3]_3 + \delta_3$$

Finally using the same ideas we arrive at an expression for the identities cubic in x 's and quadratic in y 's:

$$\begin{aligned} [\mathbf{y}_4 \otimes \mathbf{y}_4 - \mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x}]_7 &= [\alpha_2 \otimes [\mathbf{x}_3 \otimes \mathbf{y}_4]_6]_7 + [\beta_3 \otimes [\mathbf{x}_3 \otimes \mathbf{x}_3]_5]_7 \\ &\quad + [\gamma_4 \otimes \mathbf{y}_4]_7 + [\delta_5 \otimes \mathbf{x}_3]_7 + \epsilon_7 \end{aligned}$$

Alternatively by writing

$$Y = y_{30} + 3ty_{21} + 3t^2y_{12} + t^3y_{03}$$

$$X = x_{20} + 2tx_{11} + t^2x_{02}$$

and $A^{(p)}$ for an arbitrary polynomial of degree p in t , the above seven equations can be succinctly written

$$Y^2 - X^3 = A^{(1)}XY + A^{(2)}X^2 + A^{(3)}Y + A^{(4)}X + A^{(6)}.$$

Using $(\cdot)'$ for differentiation with respect to t we may similarly write the other relations:

$$2XX'' - X'^2 = X''B^{(2)} - X'B^{(2)'} + XB^{(2)''} + B^{(0)}$$

and so on.

6 The (n, s) curve

The (n, s) curve, for n and s coprime, arises when we have a coordinate ring generated by functions belonging to divisor spaces with special divisors of degree n and s . We take these to be the $n + 1$ functions,

$$x_{n,0}, x_{n-1,1}, \dots, x_{0,n}$$

and the $s + 1$ functions,

$$y_{s,0}, y_{s-1,0}, \dots, y_{0,s}$$

generating the polynomial ring

$$R = \mathbb{C}[x_{n,0}, x_{n-1,1}, \dots, x_{0,n}, y_{s,0}, y_{s-1,0}, \dots, y_{0,s}].$$

Let $\{e_0, e_1, \dots, e_{n-1}\}$ and $\{f_0, f_1, \dots, f_{s-1}\}$ be bases of n and s dimensional vector spaces E and F respectively and define one-forms

$$\begin{aligned} \omega_1 &= \sum_{i=0}^{n-1} x_{n-i,i} e_i + \sum_{j=0}^{s-1} y_{s-j,j} f_j \\ \omega_2 &= \sum_{i=0}^{n-1} x_{n-1-i,i+1} e_i + \sum_{j=0}^{s-1} y_{s-1-j,j+1} f_j \end{aligned}$$

and an associated two-form

$$\Omega = \omega_1 \wedge \omega_2.$$

We wish to factor out the relations formed by the 2×2 minors of

$$\begin{bmatrix} x_{n,0} & x_{n-1,1} & \dots & x_{1,n-1} & y_{s,0} & y_{s-1,1} & \dots & y_{1,s-1} \\ x_{n-1,1} & x_{n-2,2} & \dots & x_{0,n} & y_{s-1,1} & y_{s-2,2} & \dots & y_{0,s} \end{bmatrix}.$$

Under the \mathfrak{sl}_2 action

$$\begin{aligned} \mathbf{e}x_{i,j} &= ix_{i-1,j+1} & \mathbf{f}x_{i,j} &= jx_{i+1,j-1} \\ \mathbf{e}y_{i,j} &= iy_{i-1,j+1} & \mathbf{f}y_{i,j} &= jy_{i+1,j-1} \end{aligned}$$

and

$$\begin{aligned} \mathbf{e}e_0 &= 0 & \mathbf{e}f_0 &= 0 \\ \mathbf{e}e_i &= -(n-i)e_{i-1} & \mathbf{f}e_i &= -(i+1)e_{i+1} \\ \mathbf{e}f_i &= -(m-i)f_{i-1} & \mathbf{f}f_i &= -(i+1)f_{i+1} \end{aligned}$$

we find

$$\begin{aligned} \omega_1 &\xrightarrow{\mathbf{e}} \omega_2 \xrightarrow{\mathbf{e}} 0 \\ 0 &\xleftarrow{\mathbf{f}} \omega_1 \xleftarrow{\mathbf{f}} \omega_2 \end{aligned}$$

so that Ω is invariant.

We get an equivariant exact sequence as before but longer.

We define the map ϕ_1 by considering

$$R^{[p-2n]} \bigwedge_e^{n-2} \bigwedge_f^s \bigoplus R^{[p-n-s]} \bigwedge_e^{n-1} \bigwedge_f^{s-1} \bigoplus R^{[p-2s]} \bigwedge_e^n \bigwedge_f^{s-2} \\ \xrightarrow{\Omega \wedge \cdot} R^{[p]} \bigwedge_e^n \bigwedge_f^s \xrightarrow{\pi} (R/\Delta)^{[p]}$$

where the notation $\bigwedge_e^p \bigwedge_f^q$ denotes $p+q$ forms with p e 's and q f 's.

The general term is

$$\dots \rightarrow \bigoplus_{i+j=q+1}^q \left(\bigoplus R^{[p-ni-sj]} \bigwedge_e^{n-i} \bigwedge_f^{s-j} \right) \xrightarrow{\Omega^{(q)} \wedge \cdot} \bigoplus_{i+j=q}^{q-1} \left(\bigoplus R^{[p-ni-sj]} \bigwedge_e^{n-i} \bigwedge_f^{s-j} \right)$$

for $q = 2, \dots, n+s$ where the map $\Omega^{(q)} \wedge \cdot$ is given by the $(q-1) \times q$ matrix

$$\Omega^{(q)} = \begin{pmatrix} \omega_1 & \omega_2 & 0 & \dots & \dots & 0 \\ 0 & \omega_1 & \omega_2 & 0 & \dots & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & \dots & \dots & \omega_1 & \omega_2 \end{pmatrix}$$

acting on the left.

The expression for $\tilde{H}(t)$ is then

$$\tilde{H}(t) = \left(1 - \sum_{q=2}^{n+s} (-1)^q (q-1) \left(\sum_{i+j=q} \binom{n}{i} \binom{s}{j} t^{ni+sj} \right) \right) H(t)$$

where

$$H(t) = (1-t^n)^{-n-1} (1-t^s)^{-s-1}.$$

We need to simplify this expression to render it intelligible. Start by rewriting it as a sum over all integer $q \geq 0$ putting $t^n = u$ and $t^s = v$:

$$\begin{aligned} \tilde{H}(t) &= \left(\sum_{q=0}^{n+s} (-1)^{q-1} (q-1) \sum_{i+j=q} \binom{n}{i} \binom{s}{j} u^i v^j \right) H(t) \\ &= (u\partial_u + v\partial_v - 1) \left(\sum_{q=0}^{n+s} (-1)^{q-1} \sum_{i+j=q} \binom{n}{i} \binom{s}{j} u^i v^j \right) H(t) \\ &= -(u\partial_u + v\partial_v - 1) ((1-u)^n (1-v)^s) H(t) \\ &= \frac{(1-u)^n (1-v)^s + nu(1-u)^{n-1} (1-v)^s + sv(1-u)^n (1-v)^{s-1}}{(1-u)^{n+1} (1-v)^{s+1}} \\ &= \frac{1}{(1-t^n)(1-t^s)} + \frac{nt^n}{(1-t^n)^2(1-t)^s} + \frac{st^s}{(1-t^n)(1-t^s)^2} \\ &= \frac{d}{dt} \left(\frac{t}{(1-t^n)(1-t^s)} \right) \\ &= \sum_{m=0}^{\infty} (\# \text{ partitions of } m \text{ into } n\text{'s and } s\text{'s}) (m+1) t^m \end{aligned}$$

The maps establishing equivariance are, again for the \mathbf{e} generator,

$$E^{(q)} : \bigoplus^q (\dots) \rightarrow \bigoplus^q (\dots)$$

$$E^{(q)} = \begin{pmatrix} \mathbf{e} & 0 & \dots & 0 \\ 1 & \mathbf{e} & 0 & 0 \\ 0 & 2 & \mathbf{e} & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & \dots & & q-1 & \mathbf{e} \end{pmatrix}$$

and this allows us to conclude the argument as before.

7 Conclusions and directions

We have described the ideal of relations for two point functions following from the Riemann-Roch theorem for a compact, nonsingular Riemann surface and shown it is generated by a small set of quadratic polynomials and by relations of the form $y^n \sim x^s$. This is not a surprising result but it illustrates the role of an equivariance property that is useful in other contexts also.

It will be instructive to see how the same ideas may apply to p -point divisors where, for large enough p , relations due to addition formulae must come into play. In such a case there will exist equivariance under \mathfrak{sl}_p .

The application to the Jacobian, alluded to in the introduction, the classification of all differential relations between \wp_{ij} -functions is more challenging. In this case the divisor, D , is a codimension one variety on the Jacobian and the dimension of the space of meromorphic functions of pole order n is $l(nD) = n^g$. The \wp_{ij}, \wp_{ijk} and so on have (for the hyperelliptic case at least) an equivariant structure where \mathfrak{sl}_2 acts on derivative indices rather than poles and there is added complexity owing to the derivations which map functions in $L(nD)$ to functions in $L((n+1)D)$.

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