BOUNDED DEGREE AND PLANAR SPECTRA

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ABSTRACT. The finite spectrum of a first-order sentence is the set of positive integers that are the sizes of its models. The class of finite spectra is known to be the same as the complexity class NE. We consider the spectra obtained by limiting models to be either planar (in the graph-theoretic sense) or by bounding the degree of elements. We show that the class of such spectra is still surprisingly rich by establishing that significant fragments of NE are included among them. At the same time, we establish non-trivial upper bounds showing that not all sets in NE are obtained as planar or bounded-degree spectra.

1. INTRODUCTION

The *spectrum* of a sentence ϕ of some logic, denoted $\mathrm{spec}(\phi)$, is the set of positive integers n such that ϕ has a model of cardinality n. In this paper we are solely concerned with first-order logic and we use the word spectrum to mean a set of integers that is is the spectrum of some first-order sentence. Scholz in [Sch52] posed the question of characterizing those sets of integers which are spectra. This question has spawned a large amount of research. In particular, Fagin's attempt to answer Asser's question of whether the complement of a spectrum is itself a spectrum launched the field of descriptive complexity theory. An excellent summary of the history of the spectrum question and methods used to approach it are given in the survey by Durand et al. [DJMM12].

An exact characterization of the sets of spectra of first-order sentences can be given in terms of complexity theory. This result, obtained by Fagin [Fag74] and by Jones and Selman [JS74] states that a set $S \subseteq \mathbb{N}$ is a spectrum if, and only if, the binary representations of integers in S can be recognised by a non-deterministic Turing machine running in time $2^{O(n)}$. In the language of complexity theory, we say that the class of spectra is exactly the complexity class **NE**.

Besides characterizations of the class of spectra, researchers have also investigated how the class is changed by restrictions, either on the form of the sentence ϕ or on the class of finite models that we consider. A particularly fruitful line of research in the former direction has investigated the spectra of sentences in a restricted vocabulary. It is an easy observation

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that if the vocabulary includes only unary relations, then the spectrum of any sentence is either finite or co-finite. Durand et al. [DFL98] investigate the spectra of sentences in vocabularies with unary relations and one unary function and show that the class of such spectra is exactly the ultimately periodic (or semilinear) sets of natural numbers. On the other hand, if we allow two unary functions or one binary relation in the vocabulary, there are spectra that are **NEXPTIME**-complete [Fag74, DR96]. However, since the class of these spectra is not closed under polynomial-time reductions, it does not follow that it includes all of **NEXPTIME** or indeed even all of **NE**. Whether every spectrum spectrum S is the spectrum of a sentence in such a vocabulary remains an open question. The results of [DFL98] have also been extended beyond first-order logic to monadic second-order logic by Gurevich and Shelah [GS03].

In the present paper, we investigate the spectra that can be obtained by restricting the class of models considered. To be precise, we consider restrictions on the Gaifman (or adjacency) graph of the structure. We want to explore what reasonable restrictions on such graphs make for a more tractable class of spectra. This is in the spirit of a recent trend in finite model theory aimed at investigating it on *tame* classes of structures (see [Daw07]).

Fischer and Makowsky [FM04] show that if K is a class of structures generated by a certain class of graph grammars known as $eNCE$-grammars (see [Kim97]) and all models of ϕ are in K, then $\text{spec}(\phi)$ is ultimately periodic. Here, ϕ can be a formula of monadic second order logic with counting (CMSO). Classes of graphs generated by $eNCE$-grammars include TW_d, the class of structures of tree-width at most d, and CW_d, the structures of clique width at most d. It is not difficult to show that every semilinear set of natural numbers can be obtained as such a spectrum.

The restricted classes of structures we investigate in this paper are those of bounded degree and those whose Gaifman graph is planar. We also consider the two restrictions in combination. There are two ways that we can define spectra of first-order sentences restricted to a class of structures $\mathcal{C}$. We can define the class of $\mathcal{C}$-spectra as the spectra of those sentences ϕ all of whose models are in $\mathcal{C}$. Alternatively, we can define the class of $\mathcal{C}$-spectra as the class of sets $S \subseteq \mathbb{N}$ such that there is a first-order sentence ϕ and $n \in S$ iff there is a model of ϕ in $\mathcal{C}$ of cardinality n. If $\mathcal{C}$ is itself first-order definable (for instance if it is the class of graphs of degree at most k for some fixed integer k), then the two notions coincide. Otherwise, the first is more restrictive than the second.

Our lower bound results show that the class of spectra obtained by restricting to bounded degree and to planar graphs are still quite rich. In particular, Theorem 4.1 shows that the class of spectra of structures of degree at most 3 still contains all sets S of integers for which $N \in S$ is decidable by a nondeterministic machine running in time $O(N)$ (which is time exponential in the binary representation of N). Similar (though somewhat weaker) lower bounds are established for classes of structures which are required to be both planar and of bounded degree in Theorems 6.1 and 6.2.

On the other hand, the upper bounds show that the class of spectra we obtain is strictly weaker than the class of all spectra. In particular we establish, for the bounded degree restriction, in Theorem 5.2 a complexity upper bound on the class of spectra obtained. This, along with the non-deterministic time hierarchy theorem [SFM78] establishes that this class of spectra is not the whole class **NE**. Theorem 6.3 establishes a related complexity upper bounds on the class of spectra for bounded degree planar graphs. Dropping the degree restriction, we can obtain a complexity upper bound on planar spectra from results of

Frick and Grohe [FG01] from which again it follows that this is not the class of all spectra, answering negatively Open Question 6 from [DJMM12].

2. Preliminaries

For a relational vocabulary τ (i.e. a finite collection of constant and relation symbols), a τ-structure $\mathfrak{A}$ is a set A along with an interpretation in A of every symbol in τ. We will confine ourselves to vocabularies τ in which all symbols are binary or unary. The Gaifman graph $\Gamma(\mathfrak{A})$ of $\mathfrak{A}$ is the graph on the vertex set A such that two distinct vertices u and v are adjacent if, and only if, either (u, v) or (v, u) occurs in the interpretation in $\mathfrak{A}$ of some binary relation in τ. We say that the degree of $\mathfrak{A}$ is bounded by d if, in $\Gamma(\mathfrak{A})$ no vertex is adjacent to more than d others. Similarly, we say that $\mathfrak{A}$ is planar if $\Gamma(\mathfrak{A})$ is.

2.1. **Partial function symbols.** For the most part, we will consider structures in which every binary relation in τ is interpreted as the graph of a *partial injective unary function* (PIF). To emphasise this, we use functional notation and write, for instance, $t_1(x) = t_2(y)$ as shorthand for the formula that says that the terms $t_1(x)$ and $t_2(y)$ are defined and their interpretations are equal. Thus, for example, $f(x) = f(x)$ is equivalent to saying that $f(x)$ is defined.

A structure interpreting d such partial injective function symbols is always of degree at most $2d$. Conversely, if we have a graph of degree d, by Vizing's theorem [Viz64] we can colour the edges with $d + 1$ colours in a way such that all edges incident with a given vertex are of different colours. We can then represent such structures using $2d - 1$ partial injective function symbols. This establishes that in considering the spectra of first-order sentences over bounded-degree structures, there is no loss of generality in considering structures in which all binary relations are PIFs.

Definition 2.1. Let $\mathbf{PIFSpec}_d$ denote the set of $S \subseteq \mathbb{N}$ such that S is a spectrum of some sentence ϕ using only unary relations and d PIF symbols.

Proposition 2.2. *If $S \in \mathbf{PIFSpec}_d$, then the following are also in $\mathbf{PIFSpec}_d$:*
(1) $S \cup \{n\}$,
(2) $S - \{n\}$,
(3) $\{x + 1 : x \in S\}$,
(4) $\{x \in \mathbb{N} : x + 1 \in S\}$.

Proof. Let ϕ be a sentence whose spectrum is S, witnessing that $S \in \mathbf{PIFSpec}_d$. Also let η_n be the first-order sentence that states that there are exactly n distinct elements in the universe. Then, (1) is obtained as the spectrum of the sentence $\phi \vee \eta_n$ and (2) is obtained as the spectrum of $\phi \wedge \neg\eta_n$.

To establish (3), extend the vocabulary by a new unary relation symbol R and let ρ be the sentence that says that there is exactly one element not in R. Now, consider the spectrum of the sentence $\rho \wedge \phi^R$, where ϕ^R denotes the sentence obtained by relativising all quantifiers in ϕ to R.

Finally, for a structure $\mathfrak{A}$ and an element a in its universe, we can define a structure $\mathfrak{A}^*$ by removing the element a and expanding the structure with $2d$ additional unary relations that code all the various ways that elements could be connected to a. That is, for each PIF symbol f, we have unary relations $R_{\vec{f}}$ and $R_{\overleftarrow{f}}$ so that $R_{\vec{f}}(b)$ if, and only if, $f(b) = a$ and

$R_f^{\leftarrow}(b)$ if, and only if, $f(a) = b$. It is now not difficult to translate the sentence ϕ into a sentence ϕ^* so that $\mathfrak{A} \models \phi$ iff $\mathfrak{A}^* \models \phi^*$. In particular, ϕ^* asserts that $R_f^{\leftarrow}$ has at most one element. $\square$

We now establish the relationship between $\mathbf{PIFSpec}_d$ and spectra over *graphs* of bounded degree.

Definition 2.3. Let $\mathbf{BDSpec}_d$ denote the class of sets S such that S is a spectrum of some sentence ϕ in a vocabulary with one binary relation E and some number of unary relations such that every model of ϕ interprets E as a symmetric relation and has degree at most d.

The following theorem allows us to simplify formulae using lots of symbols to formulae using only a single symmetric binary relation of degree 3.

Theorem 2.4. *If $S \in \mathbf{PIFSpec}_d$ then $\{2dn + l : n \in S\}$ is in $\mathbf{BDSpec}_3$ for any l.*

Proof. Let ϕ be a formula using d PIF symbols $f_1, \ldots, f_d$ such that S is the spectrum of ϕ and let $\mathfrak{A}$ be a structure of the same vocabulary. Define the vocabulary τ to contain the binary relation E and d additional unary predicates $P_1, \ldots, P_d$. We define a τ-structure G by replacing each element a of $\mathfrak{A}$ with a gadget consisting of a simple E-path of vertices $p_1(a), \ldots, p_d(a), q_1(a), \ldots, q_d(a)$, where $p_i(a)$ and $q_i(a)$ both satisfy P_i and no other predicate from $P_1, \ldots, P_d$. If $f_i(a) = a'$, there is an edge between $p_i(a)$ and $q_i(a')$. In addition G contains l isolated vertices. It is easily verified that G has degree bounded by 3.

It is easy to write a formula ϕ_g of first-order logic whose models are exactly the coloured graphs G obtained in this way. And, it is also easy to see that $\mathfrak{A}$ can be interpreted in G. Combining these, we get a sentence whose spectrum is exactly $\{2dn + l : n \in S\}$. $\square$

We are also interested in spectra obtainable by models which are planar graphs. We write $\mathbf{PPifSpec}_d$ to denote the set of $S \subseteq \mathbb{N}$ such that S is a *planar* spectrum of some formula ϕ using only unary relations and d PIF symbols. That is to say, $n \in S$ iff there is some planar structure $\mathfrak{A}$ such that $\mathfrak{A} \models \phi$. Note that there is no obvious inclusion either way between $\mathbf{PPifSpec}_d$ and $\mathbf{PIFSpec}_d$. Some $S \in \mathbf{PIFSpec}_d$ may be witnessed by a formula ϕ which in some cardinalities has only non-planar models. On the other hand, as planarity is not itself definable in first-order logic, the set S' of those n for which ϕ has planar models of size n may not be itself the spectrum of a first-order sentence. We write $\mathbf{FPPifSpec}_d$ to denote the set of $S \subseteq \mathbb{N}$ such that S is a spectrum of some formula ϕ using only unary relations and d PIF symbols, and such that all models of ϕ are planar sets. $\mathbf{FPPifSpec}_d$ is a subset of both $\mathbf{PIFSpec}_d$ and $\mathbf{PPifSpec}_d$.

2.2. **Complexity classes.** Our aim is to characterize spectra using complexity theory. For this purpose, we define complexity classes of sets of numbers in terms of resource bounds on machines accepting the binary representations of numbers.

We write $\mathbf{NTIME}_2(f(N))$ to denote the class of sets $S \subseteq \mathbb{N}$ such that there is a nondeterministic multi-tape Turing machine that for any N accepts the binary representation of N iff $N \in S$ and runs in time $O(f(N))$. If we interpret the classes $\mathbf{NP}$ and $\mathbf{NE}$ as sets of numbers in this way, then $\mathbf{NP} = \bigcup_k \mathbf{NTIME}_2(\log^k(N))$ and $\mathbf{NE} = \bigcup_k \mathbf{NTIME}_2(N^k)$. By results of Fagin [Fag74] and Jones and Selman [JS74], we know that $\mathbf{NE}$ is exactly the class of first-order spectra. In the present paper we will relate bounded-degree and planar spectra to the classes $\mathbf{NTIME}_2(N \log^k(N))$, which are between $\mathbf{NP}$ and $\mathbf{NE}$. Indeed, by the time

hierarchy theorem for nondeterministic machines [SFM78], the classes are *strictly* between **NP** and **NE**.

We also write $\mathbf{NTISP}_2(T(N), S(N))$ to denote the class of sets $S \subseteq \mathbb{N}$ such that there is a nondeterministic Turing machine accepting the binary representation of N iff $N \in S$ and running in time $O(T(N))$ and space $O(S(N))$. We write $\mathbf{NTS}_2(f(N))$ to denote the union of $\mathbf{NTISP}_2(T(N), S(N))$ over all functions T and S such that $T(N) \cdot S(N) = O(f(N))$.

These classes are less robust than the well known complexity classes such as **P**, **NP**, or **NEXPTIME** because the functions bounding resources are not closed under composition with polynomials. The classes are then sensitive to the exact computation model (e.g. single tape Turing machine, an automaton with two stacks, multi-tape Turing machine, RAM), and some of our results are stated for the specific model of automata with two stacks, or equivalently, single tape Turing machines which can insert or delete symbols from the tape; we use the symbol $\mathbf{NTIME}_2^S(f(N))$ instead of $\mathbf{NTIME}_2(f(N))$ in this case.

However, the classes are somewhat robust in a weaker sense. The following is established by standard methods of speed-up theorems by considering a larger tape alphabet.

Proposition 2.5. *If a machine M recognizes the binary representations of elements of S in time $O(f(N)) = \Omega(\log^3(N))$, then for any $\alpha > 0$ there is a machine M' recognizing the same set and running in time $\alpha f(N)$ for all $N > N_0$, for some $N_0 \in \mathbb{N}$.*

Proposition 2.5 allows us to assume that a machine solving a problem in $\mathbf{NTIME}_2(f(N))$ actually solves it in time $\alpha f(N)$ for all $N > N_0$. When relating complexity to spectra, the finitely many exceptions $N \leq N_0$ are not important as a result of Proposition 2.2.

3. Basic structure

In this section, we describe a basic construction, consisting of the description of a structure of each finite cardinality N, along with a first-order axiomatisation of this class of structures. The structures in question are planar and of bounded degree and play an important role in our results in encoding machine computations.

Let $\mathfrak{A}_N$ be the following structure over the vocabulary τ_0 consisting of a single partial function symbol f_{+1}:

- the universe $A_N = \{1, \ldots, N\}$,
- $f_{+1}(n) = n + 1$ for $n < N$,
- $f_{+1}(N)$ is not defined.

Theorem 3.1. *There is a FO formula ϕ_M over vocabulary $\tau_1 \supset \tau_0$ containing 2 PIF symbols, such that ϕ_M has models of all non-negative integer cardinalities, and all models of ϕ_M restricted to $\{f_{+1}\}$ are isomorphic to $\mathfrak{A}_N$ for some N. Moreover, all models of ϕ_M are planar.*

Proof. The vocabulary τ_1 consists of two PIF symbols f_{+1} and f_{2x}. Instead of writing $f_{+1}(x)$ and $f_{2x}(x)$, we just write $x + 1$ and $2x$, respectively. We also use $x + k$ for the k-th iteration of $x + 1$.

The formula ϕ_M is a conjunction of the following statements:

(1) $x + 1$ is defined for all values of x except one. In the sequel, we write N for the unique x for which $x + 1$ is not defined.

(2) For all values of x except one, there is an element y such that $y + 1 = x$. We write $x - 1$ to denote this y. The unique element x for which $x - 1$ is not defined is denoted 1.
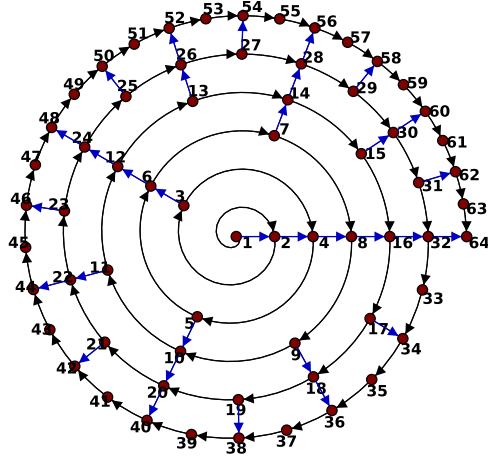
Figure 1: Basic spiral structure.

(3) If $2x$ and $x - 1$ are both defined for a given x, then $(2(x - 1) + 1) + 1$ is also defined, and $(2(x - 1) + 1) + 1 = 2x$.

(4) We have $2 \cdot 1 = 1 + 1$.

(5) Either $x = 2y$ for some y, or $x + 1 = 2y$ for some y.

It is straightforward to check that $\mathfrak{A}_N$ expanded with function $f_{2x}(x) = 2x$ is a model of ϕ_M. Figure 1 shows that such models are indeed planar.

Next, we argue that, up to isomorphism, these are the only models of ϕ_M. Let $\mathfrak{A}$ be a model of ϕ_M. Suppose that the interpretation of f_{+1} has a cycle, i.e., there is an $a \in A$ such that $a + k = a$ for some $k > 0$. Let l be the minimum length of such a cycle. From condition 5 we know that every second element of this cycle is of the form $2x$ for some x. In particular, l must be even. By condition 3 and the injectivity of f_{2x} we know that the elements x such that $2x$ is on the cycle must themselves form a cycle, this time of length $l/2$. This contradicts the assumption that l is minimal.

Since there are no cycles, it follows from conditions 1 and 2 that each element of A is of form $1 + (k - 1)$ for some integer k. Writing a_k for this element, we can prove inductively that $2(a_k) = a_{2k}$. This means that $a_k \mapsto k$ is the required isomorphism to $\mathfrak{A}_N$ (where $N = |A|$). □

The spectrum of ϕ_M is the set of all natural numbers which is not itself very interesting. However, the structures we have axiomatized can be used to describe simple properties of numbers in a simple way.

Example 3.2. Consider the vocabulary τ_1 expanded with an additional unary relation P, and add an axiom: $P(x)$ iff $x = 1$ or $\exists y\, P(y) \wedge x = 2y$. We can prove inductively that in any expansion of $\mathfrak{A}_N$ that is a model of this axiom, P has to be interpreted as the set of powers of two. Adding a further axiom $P(N)$ we obtain a sentence whose spectrum is the set of powers of two. Moreover, all models of this sentence are planar graphs.

We could also achieve a similar effect by adding a function $x \to 2^x$ to our vocabulary, and the following axiom: $2^{x+1} = 2 \cdot 2^x \wedge 2^1 = 1 + 1$. We can prove inductively that this symbol has to be interpreted in the intended way. However, the models are no longer planar, and as we add a new PIF, the degree bound increases to 6.

Example 3.3. Consider the vocabulary τ_1 expanded with an additional constant symbol C and two additional functions that we denote $x + C$ and $x \cdot C$. We axiomatise these PIFs by means of the following axioms: $1 + C = C + 1$, $(x + 1) + C = (x + C) + 1$; $1 \cdot C = C, (x + 1) \cdot C = x \cdot C + C$. We can prove inductively that $x + C$ and $x \cdot C$ are interpreted in the intended way. By adding the following axioms: $C \neq 1, \exists y \, y \neq 1 \wedge N = y \cdot C$, we obtain a sentence whose spectrum is the set of composite numbers, and all of whose models are of degree at most 8.

Example 3.4. Consider the vocabulary τ_1 expanded with an additional PIF F axiomatized by the following axioms: $F(1) = 2$, $F(n + 1) = F(n) + 2$ if $\exists m F(m) = n$, and $F(n + 1) = F(n) + 1$ if no such m exists. We also have an additional unary relation Φ and the axioms: $\Phi(x)$ iff $x = 1$ or $\exists y \, x = F(y) \wedge \Phi(y)$; and $\Phi(N)$. It can be shown (by induction again) that the spectrum of the resulting sentence is the set of Fibonacci numbers.

Our vocabulary here has three PIFs, so the structures have degree bounded by 6. However, the function $2x$ is not necessary in this particular case. We could add the symbols F and Φ to the vocabulary τ_0 instead. We need an axiom similar to condition 5 in the definition of ϕ_M, but for $F(x)$ instead of $2x$. With this $F(x)$ is sufficient to fix the structure. By removing the axioms related to the function $2x$, we obtain a sentence whose spectrum is again the set of Fibonacci numbers, and all of whose models are planar of degree at most 4.

Example 3.5. Consider τ_1 expanded with a function symbol P axiomatized with the following axioms: $P(1) = N$; if $P(x) \neq 1$ then $P(x) = 2P(x + 1)$ or $P(x) = 2P(x + 1) + 1$; and if $P(x + 1)$ is defined then $P(x)$ is also defined. Let $\phi_1(x)$ denote the value 1 if $P(x) = 2P(x + 1) + 1$ and 0 if $P(x) = 2P(x + 1)$. Let l be the unique element such that $P(l) = 1$ and let $\phi_1(l) = 1$. Then $\phi_1(l) \ldots \phi_1(1)$ is the binary representation of N.

Example 3.6. Let $S \in \mathbf{NTS}_2(N)$. We show that $S \in \mathbf{PIFSpec}_4$. This is a consequence of Theorem 4.1 below, but we include the argument here as it is strikingly simple.

Example 3.5 allows us to axiomatise structures $\mathfrak{A}_N$ expanded with the binary representation of N. Example 3.3 then shows how to axiomatise a grid of width C and height C_2, where $C \times C_2 \leq N$ (we only use the functions $x + 1$ and $x + C$). Let $C = S(N)$ and $C_2 = T(N)$. Then on the grid, with additional unary relations, we can describe the computation of a Turing machine in the standard way.

4. Lower bound: Turing machines via spectra

In this section we establish a lower bound, by showing that the class of bounded-degree spectra includes all sets in $\mathbf{NTIME}_2(N)$. The proof proceeds by constructing, from a Turing machine A, a first-order sentence whose models are, in a sense we make precise, codings of accepting computations of A. For simplicity of exposition, in the proof we confine ourselves to machines with one tape, but the construction can easily be extended to machines with two or more tapes.

Theorem 4.1. $\mathbf{NTIME}_2(N) \subseteq \mathbf{PIFSpec}_6$.

Proof. Suppose $S \subseteq \mathbb{N}$ is in $\mathbf{NTIME}_2(N)$ and there is a single tape Turing machine A that accepts the binary representation of N just in case $N \in S$ and runs in time $O(N)$. By Proposition 2.5, we can assume that the running time T of A when the input is the binary representation of N is such that for large enough values of N, $5T + 2 \log T \leq N$.

Our aim is to describe the construction of a first-order sentence ϕ_A which has a model of size N if, and only if, the binary representation w_N of N is accepted by A. The vocabulary includes a unary relation R and the idea is that in any model of ϕ_A the substructure defined by R is an encoding of an accepting computation of A on w_N. The aim is to define this encoding in such a manner that the size of this encoding is bounded by a linear function of T, the running time of A. In particular, we will see that $5T + 2\log T \leq N$ suffices.

Note that it is straightforward, by axiomatising a suitable $T \times T$ grid, to give such an encoding that is quadratic in T. That is, we can axiomatise a model in which the elements are pairs (x, y) representing a tape cell x at time y with unary relations coding the state and tape contents of the machine. The challenge here is to give a linear size construction and we achieve this by having single elements corresponding to a tape cell x over an interval of time during which the content of the cell does not change. This is similar to the idea behind the construction in [Gra84, Gra85] which is used to show that any set $S \subseteq \mathbb{N}$ that can be decided in time $O(N)$ on a nondeterministic random-access machine is the spectrum of a sentence with unary function symbols.

Our vocabulary contains: three PIFs $f_\leftarrow$, $f_\rightarrow$, and $f_\uparrow$; the unary relation R; three additional unary relations M, $T_\leftarrow$, $T_\rightarrow$; a unary relation S_q for each state q of A and a unary relation L_c for each letter c of the tape alphabet of A.

We now describe the intended models of ϕ_A. These are encodings of accepting computations of A on an input w_N. It should be noted that the encoding is not necessarily unique, but ϕ_A does guarantee that it is a valid encoding and it is of size N.

In a model of ϕ_A, the intention is that each element e in the relation R represents a pair $x, [y_1, y_2)$ where x is a position on the tape and $[y_1, y_2)$ (with $y_1 < y_2$) is an interval of time in the computation. The meaning of the unary relation symbols, other than R can then be given as follows. We have $M(e)$ if, and only if, $y_2 = y_1 + 1$ and at time y_1, the machine is reading position x in the tape; $T_\leftarrow(e)$ if, and only if, position x is to the left of the position being read by the machine during the time interval $[y_1, y_2)$ and $T_\rightarrow(e)$ if, and only if, position x is to the right of the position being read by the machine during the time interval $[y_1, y_2)$; $L_c(e)$ if, and only if, the tape cell x during the interval $[y_1, y_2)$ contains the letter c; and $S_q(e)$ if, and only if, $M(e)$ and at time y_1 the machine is in state q.

The PIFs $f_\leftarrow$, $f_\rightarrow$, and $f_\uparrow$ then connect these elements. If an element e corresponds to position x and interval $[y_1, y_2)$, then $f_\uparrow(e)$ corresponds to position x and interval $[y_2, y_3)$, for some y_3 with $y_3 > y_2$. Thus, $f_\uparrow$ is used to build a list of contents of the tape at position x during consecutive time intervals. We write $f_\downarrow$ as an abbreviation for $f_\uparrow^{-1}$. This enables us to enforce the requirement that the contents of a tape cell do not change in between successive visits of the tape head: if $L_c(e)$ holds and $M(e)$ does not hold, then $L_C(f_\uparrow(e))$ must hold.

The e is an element corresponding to a pair $x, [y_1, y_2)$ and $f_\leftarrow$ is defined at e, then $f_\leftarrow(e)$ corresponds to a pair $x - 1, [y_1, y_2')$ for some $y_2' \geq y_2$. That is, $f_\leftarrow$ points to the element on the tape immediately to the left for a time interval starting at the same time and extending at least as far. The function $f_\leftarrow$ is defined only at elements e corresponding to positions at or to the left of the tape head. That is, either $M(e)$ or $T_\leftarrow(e)$ must hold for $f_\leftarrow(e)$ to be defined. The idea is that since e corresponds to a position on the tape to the left of the tape head, the tape head must pass through this position to get to to the position immediately to its left. Thus, the position immediately to the left is unchanged for at least as long. Moreover, we require that when $f_\leftarrow(e)$ is defined, and corresponds to the pair $x - 1, [y_1, y_2')$ with $y_2' > y_2$, then $f_\uparrow(e)$ corresponds to the pair $x, [y_2, y_2')$. Thus, from e, we get to the

position immediately to the left during the same interval, either by taking $f_\leftarrow(e)$ if it is defined or $f_\leftarrow(f_\downarrow(e))$ otherwise. For ease of presentation, we introduced some abbreviations. We write $T_\leftarrow^+(e)$ as a shorthand for $M(e) \vee T_\leftarrow(e)$, i.e. to denote that e represents a position at or to the left of the tape head. We also write $F_\leftarrow(e)$ to denote the element $f_\leftarrow(e)$ if it is defined, and $f_\leftarrow(f_\downarrow(e))$ otherwise, i.e. the one that encodes the tape position immediately to the left of the one represented by e, and during the same time interval.

The function $f_\rightarrow$ is to be interpreted symmetrically, taking an element e corresponding to the pair $x, [y_1, y_2)$ to an element coding the pair $x + 1, [y_1, y_2')$ for some $y_2' \geq y_2$, provided that $M(e) \vee T_\rightarrow(e)$ holds. Again, we write $T_\rightarrow^+(e)$ as a shorthand for $M(e) \vee T_\rightarrow(e)$ and we write $F_\rightarrow(e)$ to denote the element $f_\rightarrow(e)$ if it is defined, and $f_\rightarrow(f_\downarrow(e))$ otherwise.

We assume that there are two special elements marking the first and last positions on the tape and that the associated time intervals cover the whole computation (i.e. these tape cells are not changed). In what follows, we refer to these two special elements as l and r respectively.

We now wish to express the following condition we refer to later as $(\star)$:

Suppose that $M(e)$ and $\ldots, F_\leftarrow^2(e), F_\leftarrow(e), e, F_\rightarrow(e), F_\rightarrow^2(e), \ldots$, along with the relations S_q and L_c on these elements, describes a configuration of A. Then for some e' ($e' = f_\uparrow(F_\rightarrow(e))$ if in this configuration A moves right, and $e' = f_\uparrow(F_\leftarrow(e))$ if A moves left), the sequence $\ldots, F_\leftarrow^2(e'), F_\leftarrow(e'), e', F_\rightarrow(e'), F_\rightarrow^2(e'), \ldots$ describes the next configuration of A.

The following axioms ensure this property. In all these, quantifiers are relativized to the set R.

- $M, T_\leftarrow, T_\rightarrow$ is a partition of R;
- $\{L_c : c \in \Sigma\}$ is a partition of R;
- $\{S_q\}$ is a partition of $\{e : M(e)\}$;
- if $T_\rightarrow(e)$ then $f_\leftarrow(e)$ is not defined, otherwise $F_\leftarrow(e)$ is defined except when $e = l$; similarly, if $T_\leftarrow(e)$ then $f_\rightarrow(e)$ is not defined, otherwise $F_\rightarrow(e)$ is defined, unless $e = r$;
- the tape content does not change except under the tape head: if $L_c(e)$ and not $M(e)$ and $f_\uparrow(e)$ is defined then $L_c(f_\uparrow(e))$;
- correctness of the right side of the tape (height 1): if $M(e)$ or $T_\rightarrow(e)$, and $T_\rightarrow(f_\uparrow(e))$, and $f_\rightarrow(e)$ is defined, and $f_\rightarrow(f_\uparrow(e))$ is defined, then $f_\uparrow(f_\rightarrow(e)) = f_\rightarrow(f_\uparrow(e))$;
- correctness of the right side of the tape (height 2): if $M(e)$ or $T_\rightarrow(e)$, and $f_\uparrow(f_\uparrow(e))$ is defined, and $f_\rightarrow(e)$ is defined, and $f_\rightarrow(f_\uparrow(e))$ is not defined, then $f_\uparrow(f_\rightarrow(e)) = f_\rightarrow(f_\uparrow(f_\uparrow(e)))$,
- no greater height: if $f_\uparrow(e)$ is defined and $M(e)$ or $T_\rightarrow(e)$, then $f_\rightarrow(e)$ or $f_\rightarrow(f_\uparrow(e))$ is defined;
- symmetric axioms for correctness of the left side of the tape;
- time zero: there is only one e such that $M(e)$ and $f_\downarrow(e)$ is not defined; if $f_\downarrow(e)$ is not defined, then $f_\downarrow(f_\rightarrow(e))$ and $f_\downarrow(f_\leftarrow(e))$ are also not defined;
- end of computation: there is only one e such that $M(e)$ and $f_\uparrow(e)$ is not defined and for this e, $S_q(e)$ for some final state q; if $f_\uparrow(e)$ is not defined and $F_\leftarrow(e)$ is defined then $f_\uparrow(F_\leftarrow(e))$ is not; if $f_\uparrow(e)$ is not defined and $F_\rightarrow(e)$ is defined then $f_\uparrow(F_\rightarrow(e))$ is not;
- correctness of computation: if $M(e)$ and $S_q(e)$ and q is not a final state, then either $M(f_\uparrow(F_\rightarrow(e)))$ and $f_\uparrow(e) = f_\leftarrow(f_\uparrow(F_\rightarrow(e)))$, or $M(f_\uparrow(F_\leftarrow(e)))$ and $f_\uparrow(e) = f_\rightarrow(f_\uparrow(F_\leftarrow(e)))$; moreover, if $L_c(e)$ and $L_{c'}(f_\uparrow(e))$, then there is a valid transition of A in state q to replace c by c' and move left or right as appropriate.
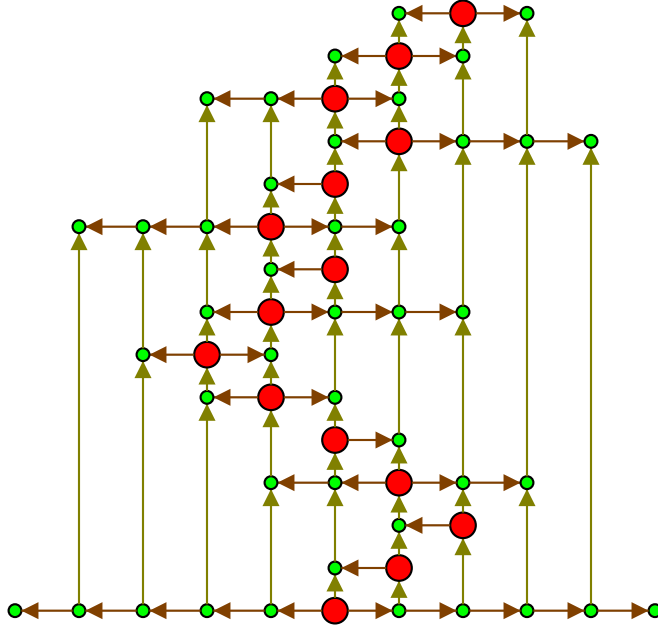
Figure 2: Model coding the computation of a machine.

The construction is illustrated in Figure 2. In this figure, the large red circles depict elements e for which $M(e)$ holds while the smaller green circles represent other elements e. The upward pointing arrows depict the partial function $f_\uparrow$, the rightward pointing arrows depict the partial function $f_\rightarrow$ and the leftward pointing arrows depict the partial function $f_\leftarrow$.

In addition to the above axioms (which ensure condition ($\star$)), we can use the construction from Example 3.5 to include in ϕ_A the condition that the initial contents of the tape are the binary encoding of the size of the model itself. This ensures that if ϕ_A has a model of size N then A accepts the binary encoding of N. The vocabulary contains in all six PIF symbols: three from the construction above, two required by the construction from Theorem 3.1 and one additional one for the construction in Example 3.5.

Now, we need to show that if the binary encoding w_N of N is accepted by A, then there is a model of ϕ_A of size N. Suppose that the accepting computation of A on input w_N takes time T and space S. We show that we can construct a model satisfying the axioms above with at most $S + 4T + 2\log T$ elements. Since we can assume $S \leq T$ and by assumption $5T + 2\log T \leq N$, this model has fewer than N elements. We can then obtain a model of ϕ_A with exactly N elements by adding additional elements that are not included in R.

We construct, from the accepting computation of A, a model of the appropriate size by the following iterative procedure:

- We have $S + 2\log T$ elements coding the initial contents of the tape, along with $\log T$ blank cells on either side of it. These are connected through the functions $f_\leftarrow$ and $f_\rightarrow$.
- For each subsequent configuration of A, we add a layer of elements, which are connected to previous layers using $f_\uparrow$. The new layer will always contain two elements corresponding to the positions in the tape where the tape head was in the previous configuration and

the position where it is in the new configuration. In addition to these two, we add new elements to the layer by the following rule: if e has been added to the new layer, $T_\rightarrow^+(e)$ holds and $F_\rightarrow(e)$ is not defined, then we add a new element which will be $f_\rightarrow(e)$. Similarly, if $T_\leftarrow^+(e)$ holds and $F_\leftarrow(e)$ is not defined, we add an element to be $f_\leftarrow(e)$.

This is illustrated in Figure 2 which was obtained by running this procedure. In this case, $S = 5$ and $T = 14$, but two cells of padding on either side suffice.

To obtain a bound on the size of this model, we note that the bottom layer contains $S + 2\log T$ elements. We want to show that the total number of elements in the additional layers is bounded by $4T$. We cannot put an absolute bound on the number of elements added in each layer, but we use an amortized analysis to obtain the required bound. For each element e for which $M(e)$ holds, let the potential $\Phi(e)$ be defined to be the number of elements in the sequence $e, F_\rightarrow(e), F_\rightarrow^2(e), \ldots$ where $f_\rightarrow(e)$ is not defined, plus the number of elements in the sequence $e, F_\leftarrow(e), F_\leftarrow^2(e), \ldots$ where $f_\leftarrow(e)$ is not defined. If we add the number of new elements created in a layer to the change of potential $\Phi(e)$, we can see that the result will be always at most 4. Initially, the potential is 2. Thus the number of positions added in T steps is at most $4T$. It can be also easily checked that the padding of length $\log T$ at each end of the tape is always sufficient. $\square$

The construction in the proof can also be understood as simulating not a single tape machine, but an automaton with two stacks (given by $T_\leftarrow$ and $T_\rightarrow$). It is straightforward to generalize this construction to more stacks, or equivalently, more tapes.

Corollary 4.2. $\mathbf{NTIME}_2(N) \subseteq \mathbf{BDSpec}_3$.

Proof. Let $S \in \mathbf{NTIME}_2(n)$. For each $l = 0, \ldots, 11$, let $S_l = \{n : 12n + l \in S\}$. The set S_l is also in $\mathbf{NTIME}_2(n)$. From Theorem 4.1, S_l is in $\mathbf{PIFSpec}_6$. From Theorem 2.4, the set $S'_l = \{n : n \in S \land n \bmod 12 = l\}$ is in $\mathbf{BDSpec}_3$. Since a union of spectra is also a spectrum (if $\mathrm{spec}(\phi_i) = S_i$, then $\mathrm{spec}(\bigvee \phi_i) = \bigcup S_i$), also $S \in \mathbf{BDSpec}_3$. $\square$

The construction we have given uses a large number of unary relations. Although Corollary 4.2 shows that we can reduce the degree of the models to 3, it is unclear whether we can also reduce the number of unary relations required.

5. Upper bound: spectra via Turing machines

In this section we aim to establish an upper bound on the class of bounded degree spectra. To be precise, we show that any such spectrum can be recognised by a nondeterministic Turing machine in time $O(N(\log N)^2)$. One consequence of this is that there are spectra that are not bounded-degree spectra.

To establish the result, we need to show that for any first-order sentence ϕ, we can construct a nondeterministic machine A_ϕ that given a positive integer N as input (it does not much matter whether N is given in unary or binary as we allow a running time that is greater than the value of N) will decide whether or not ϕ has a model with exactly N elements. We assume, for simplicity, that ϕ is in a vocabulary with one binary relation E and a number of unary relations $R_1, \ldots, R_k$. We can think of structures in this vocabulary as coloured graphs and we use the language of graphs to describe them.

The machine A_ϕ proceeds by nondeterministically guessing a graph G with N vertices and degree bounded by d and then verifying that G is indeed a model of ϕ. The algorithm for deciding whether G satisfies ϕ relies on Hanf's locality theorem [Han65] (see also [Lib04,

Theorem 4.12]). For a vertex v in G and a positive integer r, we write $N_r(v)$ to denote the substructure of G induced by the set of vertices with distance at most r to v, with a distinguished new constant interpreted by v itself. Note that if the degree of G is at most d, then $N_r(v)$ has at most $1 + d(d-1)^{r-1}$ vertices in it. We write S_r to denote the number $1 + d(d-1)^{r-1}$. We denote by $\tau(v)$ the *isomorphism type* of the structure $N_r(v)$. To be precise, we can take $\tau(v)$ to be a canonical structure isomorphic to $N_r(v)$ on the domain $\{1, \ldots, n_\tau\}$ where $n_\tau = |N_r(v)|$. Since $n_\tau \leq S_r$, there are only finitely many distinct types. We write $\mathbb{T}$ for the set of all types. For a fixed positive integer M, let $f_{r,M}(G) : \mathbb{T} \to \{0, \ldots, M\}$ be the function that assigns to each type $\tau \in \mathbb{T}$ the minimum of M and the number of vertices v in G with $\tau(v) = \tau$. For a pair of graphs G and H, we write $G \sim_{r,M} H$ if $f_{r,M}(G) = f_{r,M}(H)$. Then, Hanf's locality theorem can be stated as follows.

Theorem 5.1 (Hanf's locality[Han65]). *Let ϕ be a FO formula. Then there exist integers r and M such that, if $G \sim_{r,M} H$ then $G \models \phi$ if, and only if $H \models \phi$.*

Now, we describe how this can be used to construct the machine A_ϕ as required in the proof of the main theorem.

Theorem 5.2. $\mathbf{BDSpec}_d \subseteq \mathbf{NTIME}_2(N \log^2 N)$.

Proof. Given a first-order sentence ϕ, we describe a multi-tape non-deterministic machine A_ϕ which given an input N decides if ϕ has a model with exactly N elements.

First, A_ϕ non-deterministically guesses a graph G on N vertices by writing its description on tape A. The description of G consists of a list of N *vertex descriptions*, where each vertex description consists of an identifier of a vertex v (which is an integer given by at most $\log N$ bits), a list of the unary predicates satisfied by v and a list of the identifiers of the neighbours of v. Note that each vertex description is $O(\log N)$ bits long, because the number of neighbours of v is bounded by d. Thus, the total length of the description of G is $O(N \log N)$. We can assume that the vertex identifiers are exactly $\{1, \ldots, N\}$ and the vertex descriptions are enumerated in increasing order of the identifier.

Secondly, the machine A_ϕ non-deterministically guesses the type $\tau(v)$ of each vertex. To be precise, it writes on tape B, for each vertex v the type $\tau(v)$ (as there are only a bounded number of types, this can be specified in a constant amount of space) and a list of identifiers of vertices of G that correspond to the elements $\{1, \ldots, n_\tau\}$. Note that the entire list has length $O(N \log N)$. To complete its task, A_ϕ needs to verify two things: (1) that the guessed type of each vertex v on tape B is consistent with the graph G described on tape A; and (2) that the list of types on tape 2 does ensure that $G \models \phi$. Since the list of types determines $f_{r,M}$, and the latter determines whether $G \models \phi$, (2) is a simple table look up. We now describe how A_ϕ can, non-deterministically, in time $O(N \log^2 N)$ perform task (1).

The essential idea is that we produce on a series of S_r additional tapes copies of the vertex descriptions indexed in a suitable way so that we have available for each vertex v, not only the vertex identifiers of the vertices in its neighbourhood but their full vertex descriptions. That is, tape i will contain for each vertex v, a complete vertex description of the ith vertex that occurs in $\tau(v)$ and these will be enumerated in increasing order of v. To achieve this, we first scan tape A and for each vertex description v we encounter, we guess the number k (which may be 0 and is at most S_r) of vertices w such that v is vertex number i in the description of w, and copy the vertex description of v on to tape i k times. We then non-deterministically generate a permutation of the vertex descriptions on tape i. This can be done in time proportional to the length of tape i times $\log(N)$, which is $O(N \log^2(N))$.

This is done by guessing a subset S of the set of vertex descriptions, splitting the list into elements of S and non-elements of S (without changing order), and merging them again so that all elements of S are after all non-elements of S. Any permutation can be obtained through $\log(N)$ such operations. This sorting technique is similar to the radix sort algorithm [CLR89, Sec. 9.3]. Finally, we check that the permutation generated is indeed correct by making an additional scan of tape A and checking that the vertex identifier that occurs in position i in the type description of vertex v matches that of the vertex description that occurs in position v on tape i.

Tapes 1 through S_r now provide a suitably indexed table of vertex descriptions and it is straightforward to check that the type $\tau(v)$ ascribed to each vertex v on tape B is consistent with the graph description on tape A. Indeed, as we simultaneously scan tapes 1 through S_r, at position v we have access to the full vertex descriptions of all vertices occurring in the vertex description of v and we can check that $N_r(v)$ is indeed isomorphic to $\tau(v)$. $\square$

We know, as a result of the time hierarchy theorem for non-deterministic machines that $\mathbf{NTIME}_2(N \log^2 N)$ is a proper subset of $\mathbf{NE}$. Since the latter is the class of all spectra, it follows from Theorem 5.2 that there are spectra that are not in $\mathbf{BDSpec}_d$ for any d.

The proof of Theorem 5.2 is based on the construction due to Seese [See96] that shows that, for any fixed first-order sentence ϕ, the problem of deciding whether a given graph on N nodes and degree bounded by a constant d satisfies ϕ is solvable in time linear in N by a deterministic random-access machine. It follows that determining whether ϕ has a model that is a graph on N vertices with degree bounded by d can be done by a nondeterministic random-access machine in time $O(N)$. Such a machine can first guess the description of such a graph in time $O(N)$, since the degree bound ensures that the graph has a description whose length is bounded by $O(N)$. Our proof aims at reconstructing this argument in the context of multi-tape Turing machines. This can also be established by a general result due to Monien [Mon77] which shows that any set decidable in linear time by a nondeterministic random-access machine is in $\mathbf{NTIME}_2(N \log^2 N)$.

6. Assuming planarity

In this section we investigate to what extent upper and lower bounds similar to those in the previous two sections can be established in the case where we only consider planar structures. That is, we are interested in the *planar spectrum* of a formula ϕ—the set of those integers n such that there is a model of ϕ of size n which is also planar.

As we noted earlier, the models of the formula ϕ_M constructed in the proof of Theorem 3.1 are all planar. Moreover, Example 3.2 yields a formula whose spectrum is the set of powers of two, and again all models are planar. The more complicated examples of Example 3.3 and 3.5 do not yield planar models. However, Example 3.4 gives us a simple formula whose planar spectrum is the set of Fibonacci numbers.

The construction of a model encoding the computation of a Turing machine that we used in the proof of Theorem 4.1 yields a planar graph as long as the machine only uses a single tape (or more precisely two stacks). But, the proof also relies on encoding in the formula the statement that the initial contents of the tape encode (in binary) exactly the size of the structure. This relies on the construction in Example 3.5 and the models are no longer planar. Instead, we can modify the construction so that the size of the model is determined by the length S of the tape and the number T of steps taken. Thus, we can
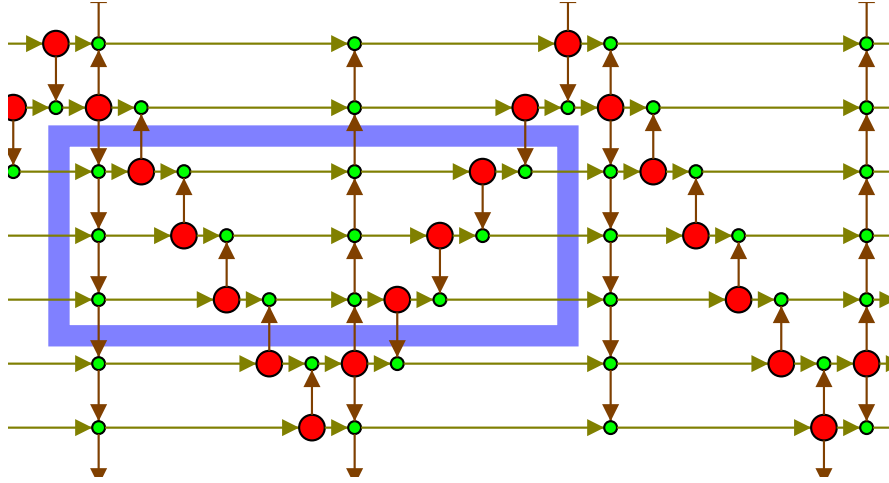
Figure 3: Machine simulation with section for tiling.

easily modify the construction so that at each time step, exactly four additional elements are used, by spreading the amortized cost in the proof of Theorem 4.1 over the time steps. Specifically, we axiomatize the cases where the potential Φ is reduced and ensure there are additional dummy elements in these cases. Thus, as long as all elements in the relation R are part of the coding of the computation, there will be exactly $S + 4T$ of them. As the initial tape contents in the machine form the bottom line (see Figure 2 above), we can add gadgets below this line, without violating planarity, which allow us to formulate statements in the formula about the length S and the contents of the tape.

This idea relies on the assumption that everything in R forms part of the machine computation. This is tricky to enforce as illustrated by Figure 3. Note that this picture has been rotated 90 degrees in order to fit the page and now the time steps run from left to right and a single column connected vertically represents the tape contents. The picture illustrates the coding of a machine that alternately moves six steps to the right on the tape and then six steps to the left.

In this picture, the area bounded by the blue rectangle defines a subgraph which can be used to tile the plane. That is to say, we can take infinitely many copies of this subgraph and connect them by identifying the edges leaving at the top with the incoming edges in the copy above and similarly to the left and right. This yields an infinite graph G which satisfies all the axioms apart from the edge cases which talk of elements that appear only once in the coding of the computation. Hence, the disjoint union of a large enough model of the axioms with G still yields a model (albeit an infinite one). Furthermore if, instead of tiling the plane, we use the rectangle to tile a torus, we obtain a finite graph any number of copies of which can be adjoined to a model to obtain a valid model. This means that the spectrum of our sentence must be ultimately periodic. However, the embedding of the tiling on a torus is necessarily not planar and restricting ourselves to planar graphs, it turns out that we can ensure that all elements in R really are part of the coding of the computation and this allows us to axiomatise valid computations. In the next two subsections, we carry out two such constructions, for two different classes of machines. In terms of complexity, the two results are incomparable.

6.1. **A Turing machine with a clock.**

Theorem 6.1. NTIME$_2^S(N/\log N) \subseteq$ PPifSpec$_3$.

Proof. Let M be a machine such that for any $N \in \mathbb{N}$, it accepts or rejects the binary representation of N in time $O(N/\log N)$.

We consider a machine M' that takes as input a word of the form $w\#u\#b_C\#b_N$, where $w, u \in \{0,1\}^*$ and b_C and b_N are binary strings representing integers C and N respectively. The machine M' performs the following tasks:

- Replace w on the tape with b_s, the binary string representing the number s of occurrences of 1 in the string w. This can be done in time $\Theta(|w| \log s)$. After this step, the tape contains $b_s\#u\#b_C\#b_N$.
- Based on the values of s, u, and C, verify that N contains exactly the size of the model. This point will be explained later. We also erase everything from the tape, except $\#b_C\#b_N$. This can be done in time polynomial in $s + |b_C| + |u|$.
- Verify that the machine M accepts the word b_N in exactly C steps. This is done by simulating the machine M, while keeping the string between the $\#$ signs as a counter that is decremented at each step and which is shifted left or right as necessary along with the head movements of M. This can be done in time $\Theta(|b_C|C)$.

Now, we consider representing the computation of the machine M' as a structure of size exactly N. The structure consists of the following parts.

- The *base line* which is a straight horizontal path of length S representing the initial tape contents. We let $S = 3 \cdot 2^s$ for some s, and partition the initial tape into three equal length segments, the first and third of which are blank and the middle one contains the input $w\#u\#b_C\#b_N$, padded with blanks on the right side. Here w is the word of length 2^{s-1} which contains 1s at positions which are powers of two.
- The computation, represented as a subgraph of size exactly $4T$ above the base line, where T is the number of steps taken. We use the construction from Theorem 4.1, improved as outlined above so that the size of the subgraph is exactly $4T$.
- To enforce the length requirement, the spiral structure from Example 3.2 is attached below the base line, but now requiring that the outer edge of the spiral is $3 \cdot 2^s$, instead of checking whether the whole spiral is a power of 2. Additionally, the spiral structure is used to enforce that the occurrences of 1 in w are indeed at the powers of two. There are exactly S elements in the spiral structure.
- One additional element is attached to each occurrence of 1 in u in the baseline. These elements are not attached to anything else so they do not affect the planarity of the construction.

Note that all of this can be axiomatized in first-order logic.

Now, we have to explain how M' verifies that the value N given in the input is indeed exactly the size of our structure. By counting all the elements above, we know that our structure contains $2S + 4T + U$ elements in total, where $S = 3 \cdot 2^s$, U is the number of 1s in u, and $T = \Theta\left(|b_C|C + 2^s \log s + (s + |b_C| + |u|)^{O(1)}\right)$; the machine M' can be designed so that it can compute its own running time T precisely.

Hence, if our formula has a model of size N, this N must indeed be equal to the number encoded as b_N in the initial tape, and this number must be accepted by the machine M.

Now, let N be a number such that the binary representation of N is accepted by the machine M. If we assume that $|u| = s^2$ and $|b_C| = s$, the value of N computed by M' is of

form $\hat{N} = U + f(C, s) + g(s)$, where $f(C, s) = \Theta(C \cdot s)$ and $g(s) = \Theta(2^s \log s)$. Take s such that $S = 3 \cdot 2^s = \Theta(N/\log N)$, and such that if we take the maximum possible values of $C = 2^s - 1$ and $U = s^2$, the obtained $\hat{N}$ is greater than N. By replacing these values by smaller ones (C is obtained by division, and U is used for exact padding), we get $\hat{N} = N$; since $f(C, s)$ is the dominant component of $\hat{N}$, our C will still be $\Theta(N/\log(N))$. Since we have assumed the computation of M can be done in time and space $O(N/\log N)$, these values of $C, S = \Theta(N/\log N)$ will be sufficient (if the machine M actually needs less time for the given input, we can change it to non-deterministically wait, so that there is always a run of the exact length we need). Hence, if the binary representation of N is accepted by the machine M, there is a model of our formula of size exactly N. $\square$

6.2. **Queue machines.** In this section, we show that the class $\mathbf{PPifSpec}_2$ includes the complexity class $\mathbf{NTS}_2(N)$, which is a result incomparable with Theorem 6.1. The result is established by considering *queue machines*, which we define next.

A *queue machine* is a non-deterministic Turing machine with two heads: a read head and a write head. Both heads can only move to the right. The machine starts in an initial state with an input string on the tape; the read head pointing to the first symbol of the string and the write head at the first tape cell to the right of the string. The transitions are given as 4-tuples $(q, w_1, w_2, q') \in Q \times \Sigma^+ \times \Sigma^+ \times Q$, where Q is the set of states of the machine and Σ is the tape alphabet. A transition (q, w_1, w_2, q') is enabled if the machine is in state q and the string starting with the symbol under the read head begins with w_1. If the transition is taken, then the read head moves right to the symbol just after this occurrence of w_1, the string w_2 is added to the write of the current string on the tape, the write head moves to just past the end of this string, and the machine changes state to q'. The machine halts when it enters a final state.

It is clear that queue machines are Turing-universal. However, we are interested in the following measure of complexity: what is the total number of symbols (including the initial tape contents) on the tape at the end of the computation. Note that no symbol is erased from the tape, it is only the heads that move to the right. As an example, consider a machine with two state q and q_F of which the latter is final, and the following transitions: (q, b, a, q), (q, a, ab, q), (q, A, Ab, q), (q, A, AA, q_F). Suppose this machine is started with initial tape contents A. One possible run of the machine ends with the tape containing $AAbAbaAbaabAbaababaAA$. It can be shown that in any computation of this machine the number of symbols at the end of the computation is a Fibonacci number, and conversely, there is such a computation for each Fibonacci number. To be precise, successive As (apart from the last one) appear exactly at positions $F - 1$ where F is a Fibonacci number.

For a queue machine M, let $S_M \subseteq \mathbb{N}$ denote the set of numbers n such that some computation of M ends with exactly n symbols on the tape. Let $\mathbf{QTL}$ denote the collection of sets which are S_M for some queue machine M. We can then establish the following inclusions.

Theorem 6.2. $\mathbf{NTIME}_2(\sqrt{N}) \subseteq \mathbf{NTS}_2(N) \subseteq \mathbf{QTL} \subseteq \mathbf{PPifSpec}_2$

Proof. The first inclusion $\mathbf{NTIME}_2(\sqrt{N}) \subseteq \mathbf{NTS}_2(N)$ follows from definitions, since a Turing machine running in time $\sqrt{N}$ cannot take space more than $\sqrt{N}$ and thus the product of time and space required is at most N.

For the second inclusion, we can use a simulation similar to that in the proof of Theorem 6.1. That is, from a machine M accepting a set in $\mathbf{NTS}_2(N)$, we can first construct a machine M' that clocks the number of steps and also clocks the space required, while simulating M. The machine M' halts when the total space (i.e. the length of the string on the tape, summed over all configurations that have occurred in the computation) is exactly N, the number that was represented in binary in the initial configuration. Now, it is easily checked that a queue machine simulating M' (in the standard way) will halt with exactly N symbols on the tape if the binary representation of N is accepted by M.

For the third inclusion, we show that, given a queue machine M, we can axiomatise in first-order logic a structure W with two PIFs that represents the string that is on the tape at the end of a successful computation of M. One PIF is the successor function, connecting position x on the tape to $x + 1$. The other is a function f_{rw} which maps a position which is the position of the read head at some time t to the position of the write head at the same time. In addition, we have a number of unary relations to code the content of the tape and the state of the machine. To see that the structure described is planar, consider the embedding that places all points in a line, with the edges corresponding to the successor function following the line. The edges corresponding to f_{rw} connect to the source above the line and the target below the line, looping around the left end of the line. Since, moving from time t to time $t + 1$, both the read and the write heads move to the right, it can be seen that such edges do not cross. Axiomatising such a structure in first-order logic is now a straightforward matter: we need to note that if a position x is the source of an f_{rw} edge and x encodes the fact that at this point the machine is in state q, then for some transition (q, w_1, w_2, q'), the string w_1 appears in the cells starting at x, the string w_2 appears in the tape to the right of $f_{rw}(x)$, that f_{rw} is defined at $x + |w_1|$ and nowhere else in between x and $x + |w_1|$, and $f_{rw}(x + |w_1|) = f_{rw}(x) + |w_2|$. $\square$

Note that the assumption of planarity is essential to the proof above. The first-order axioms we construct from a machine M will admit models that, in addition to N elements representing a valid computation of M contain a loop containing a sequence of symbols wx such that M when reading wx outputs xw. This loop will be necessarily non-planar, though embeddable on a torus. Indeed a model of the axioms could admit any number of such additional loops. On the other hand, if M is a machine which (by some measure) always increases the length of the string at each step, then such a loop becomes impossible. The example given above generating sequences of Fibonacci number length is one such machine. Thus, one obtains this class as a planar spectrum without having to assume planarity. Indeed, Example 3.4 (in its planar form) is essentially encoding this idea.

6.3. **Upper bounds.** We now establish an upper bound that shows, in particular, that not every spectrum is a planar spectrum. Let **PSpec** denote the class of *planar spectra*, i.e. those sets $S \subseteq \mathbb{N}$ for which there is a first-order sentence ϕ such that $N \in S$ if, and only if, there is a planar model of ϕ with N elements. In particular, $\mathbf{PPifSpec}_d \subseteq \mathbf{PSpec}$ and $\mathbf{PBDSpec}_d \subseteq \mathbf{PSpec}$ for any d.

Theorem 6.3. $\mathbf{PSpec} \subseteq \mathbf{NTIME}_2(N \log^2 N)$.

Proof. It is known that, for any first-order sentence ϕ there is a linear-time algorithm that decides, for a planar graph G whether or not $G \models \phi$ [FG01], for instance, on a random-access machine (RAM). Since the average degree of a planar graph is at most 6, a planar graph on

N vertices has at most $3N$ edges. Thus, a non-deterministic RAM can, on input N guess a graph G with at most $3N$ edges, check that the graph is planar and then check determine whether $G \models \phi$, all in time $O(N)$. For a linear time algorithm for checking that a graph is planar, we can use the algorithm from [dFdMR06]. Since any set decidable in linear time on a non-deterministic RAM is in $\mathbf{NTIME}_2(N \log^2 N)$ [Mon77], the result follows. $\qquad\square$

From the time hierarchy theorem [SFM78], it follows that $\mathbf{NTIME}_2(N \log^2 N)$ is a proper subset of $\mathbf{NE}$ and since the latter is exactly the class of all spectra, there are spectra which are not planar spectra. This answers negatively Open Question 6 from [DJMM12].

With respect to bounded-degree planar spectra, we note that the set of such spectra with degree bounded by 3 is already quite rich.

Theorem 6.4. *For any d, if $S \in \mathbf{PPifSpec}_d$ then $S' = \{2dn \mid n \in S\} \in \mathbf{PBDSpec}_3$.*

Proof. This can be established by an argument similar to that for Corollary 4.2. We cannot literally use the construction from the proof of Theorem 2.4, as there is no guarantee that the structure G obtained in the proof of that theorem is planar, even if $\mathfrak{A}$ is. In that construction we build a gadget corresponding to each element a of $\mathfrak{A}$ consisting of points $p_1, \ldots, p_d, q_1, \ldots, q_d$ connected in an E-path *in that order*. It is easily seen that we can, instead, choose a different order for each a which will guarantee that G is planar if $\mathfrak{A}$. Indeed such an order is given by the order of incidence of edges on a in a planar embedding of $\mathfrak{A}$. Moreover, since we have unary relations P_i identifying the vertices p_i and q_i, the order is not relevant and we are still able to interpret G in $\mathfrak{A}$ and vice versa. $\qquad\square$

7. Forcing planarity

In this section, we turn to considering the classes of spectra of the form $\mathbf{FPPifSpec}_d$ for some d. We have previously seen examples that demonstrate that some sparse sets of numbers (such as the powers of 2 or the Fibonacci numbers) are realized as spectra of sentences ϕ such that all models of ϕ are planar. However, in order to prove the general lower bounds established in Theorems 6.1 and 6.2, we had to explicitly restrict attention to planar models as this is not something that was enforced by the axioms. Indeed, in both cases, the axioms admit non-planar models where there are components disjoint from the machine simulation that can be embedded on a torus. We now show that we are able to construct a machine simulation which can be axiomatised by a formula that enforces planarity, provided that we severely restrict the memory size: it needs to be linear in the size of the input (i.e. the binary representation of the number N). We also need to restrict the time that the machine can run more stringently than in our previous results.

Theorem 7.1. $\mathbf{NTISP}_2(N^{1-\epsilon}, \log N) \subseteq \mathbf{FPPifSpec}_2$ *for all ϵ with $0 < \epsilon < 1$.*

Proof. Suppose we have a nondeterministic Turing machine M which accepts the binary representation of N in time $N^{1-\epsilon}$ and space $\log N$. Let $d > 10$ be an integer such that $d > 2^{\frac{1}{\epsilon}}$. We can assume, by changing alphabet as necessary, that machine M uses space $\log_d N$. Given a structure in a vocabulary with two PIFs that defines a $(N^{1-\epsilon}) \times \log_d N$ grid, we can, in a standard way, give first-order axioms saying that certain unary relations on the grid elements code an accepting computation of M. Thus, to establish the result, it suffices to show that we can axiomatise, in first-order logic, a planar structure of size N containing a definable grid of dimension $(N^{1-\epsilon}) \times \log_d N$. This is what we do next.

We build on the basic spiral structure constructed in the proof of Theorem 3.1. Indeed, the two PIFs are exactly those defined there, we will only add unary relations to the structure. We assume that the elements of the structure are $A = \{1, \ldots, N\}$. Let n be the largest integer such that $2^n < N$ and we divide the set A into layers $L_1, \ldots, L_n, L_{n+1}$ where, for $k \leq n$, L_k contains the elements in the range $[2^{k-1}, 2^k - 1]$ and L_{n+1} contains the elements in the range $[2^n, N]$. We now inductively define two functions $v : A \to \{0, \ldots, 4\}$ and $f : A \to \{0, \ldots, d-1\}$.

For $x \in L_{n+1}$, we let $v(x) = 1$ and $f(x) = 0$.

For $x \in L_n$, we let $v(x) = 2 + v(2x) + v(2x+1)$, where we assume $v(y)$ denotes 0 if $y > N$. Again, we let $f(x) = 0$. Note that $\sum_{x \in L_n} v(x) = N + 1$.

For $m < n$ and $x \in L_m$, we define $f(x) = \sum_{\{y \in L_{m+1} | y \geq 2x\}} v(y) \pmod{d}$. Note that $f(x) = (v(2x) + v(2x+1) + f(x+1)) \mod d$. We then define $v(x)$ by $v(x) = (v(2x) + v(2x+1) + f(x+1) - f(x))/d \in \{0, 1\}$. In other words, we compute the sum of numbers $v(y)$ for all y in the layer L_{m+1}, from right to left, and store the suffix sums as $f(x)$ in the layer L_m. Whenever the calculated suffix sum exceeds d, we subtract d, and set $v(x)$ to 1. Hence, the sum of $v(x)$ for $x \in L_m$ will equal the sum of $v(y)$ for $y \in L_{m+1}$ divided by d, and $f(2^{m-1})$ will be the remainder.

Note that $f(1), f(2), \ldots, f(2^{n-1}), f(2^n)$ then contains the representation of N in base d. Since $d > 2$, there is m_0 such that $f(2^k) = 0$ for all $k < m_0$. Let $\Gamma = \{2^i x : x \in L_{m_0}, i \geq 0\}$. For $x \in \Gamma$, let $g(x)$ be the smallest $y \in \Gamma$ such that $y > x$, but is still in the same layer – if there is none, $g(x)$ is undefined.

We add d unary relation symbols to our vocabulary interpreted by the d sets $f(x) = 0, f(x) = 1, \ldots, f(x) = d - 1$; five unary relations to represent $v(x) = 0, \ldots, v(x) = 4$, and one interpreted by the set Γ. It can be checked that there are first-order axioms that ensure that the interpretation of these relation symbols is exactly that.

Now, the set Γ, together with the two PIFs g and f_{2x} is a grid of dimension $2^{m_0} \times (n - m_0)$. Moreover, $n - m_0$ is just the number of digits in the base d representation of N, i.e. $\lceil \log_d N \rceil$. Thus we have

$$2^{m_0} \sim 2^{log_2 N - log_d N} = 2^{log_2 N(1 - log_d 2)} = N^{1 - log_d 2} > N^{1 - \epsilon},$$

where the last inequality follows from the fact that $d > 2^{\frac{1}{\epsilon}}$. Thus, this gives us the grid we require. However, g is not definable in FO from f_{+1} and f_{2x}. Nonetheless, we can axiomatise the fact that a certain set $\mathcal{R}$ of unary relations codes the computation of the machine M on the grid defined by g and f_{2x} on Γ. We can say that if the interpretation of $\mathcal{R}$ at a position $x = 2^i t$ encodes the contents of tape cell i at time t, then position $2^i t + 1$ encodes it at the next time step. This can then be propagated to the position $2^i(t+1)$ (i.e. $g(x)$) by means of the axiom $\forall_{x \notin \Gamma} R(x+1) \iff R(x)$ for each $R \in \mathcal{R}$. $\qquad \square$

As an example, it follows that the set of prime numbers is the spectrum of a sentence using only two PIFs and all of whose models are planar graphs (of degree at most 4). This is the case because the straightforward primality test of checking all potential divisors from 2 to $\sqrt{N}$ shows that the set of primes is in $\mathbf{NTISP}_2(N^{\frac{1}{2}}, \log N)$.

8. Conclusions and future work

We have shown that, in considering the spectra of first-order sentences, restricting the models to have bounded degree or to be planar (or, indeed, both in combination) is a real

restriction. We can put a complexity upper bound on these spectra that is strictly lower than the **NE** characterization we know for the class of all spectra. At the same time, we have demonstrated that these classes of spectra are still very rich by showing that we can encode certain classes of machine simulations in them. There are, however, gaps between the lower bounds and the upper bounds we have established and an obvious question for future work is whether either set of bounds can be tightened. It would be interesting to obtain an exact characterisation in terms of complexity for the various classes of spectra that we consider. Such tight characterisations seem unlikely with Turing machines, but it might be possible to obtain them with machine models with a more robust notion of linear time such as, for example, Kolmogorov-Uspenski machines (see [Gur88]).

If we consider spectra of sentences which only have planar models, we can still establish a lower bound, as in Theorem 7.1. This is based on a simulation of machines with a severe space restriction, and it is quite possible that this bound is not tight. Thus, a natural question is whether we can construct simulations of machines with higher memory requirements in such cases.

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