CONFORMAL ANALYSIS OF ANTI-DE SITTER-LIKE SPACETIMES

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Declaration

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Abstract

In this thesis, several aspects of anti-de Sitter-like spacetimes are treated under conformal methods. More specifically, the analysis is based on Friedrich's metric conformal formulation of the Einstein field equations. First, it is proved that the conformal Einstein equations coupled to a tracefree matter model imply a system of wave equations for the conformal fields. Under an appropriate gauge choice, these relations are cast as a system of quasilinear wave equations. The analysis is supplemented with a set of homogeneous wave equations for the subsidiary variables.

The problem of the existence of continuous symmetries in vacuum anti-de Sitter-like spacetimes is also considered. Following an approach based on the construction of wave equations for the relevant fields, the problem is reduced to the existence of a Killing vector on the conformal boundary. A necessary and sufficient condition is found to be given by the so-called obstruction tensor. More specifically, the spacetime possesses a Killing vector if and only if the conformal boundary has vanishing obstruction tensor.

Next, a systematic construction of vacuum anti-de Sitter-like spacetimes is carried out by means of the quasilinear system previously obtained. Suitable initial and boundary data for this system are constructed via the conformal constraints. An analysis of the geometric subsidiary variables yields a local result for the existence of this class of spacetimes.

The previous analysis serves as a prelude to the tracefree matter case. Following an analogous construction, three explicit matter models are considered: the conformally invariant scalar field,

the Maxwell field and the Yang-Mills field. For each one of these, suitable boundary data sets are constructed and their relation to the corresponding subsidiary variables is established. This leads to a local result for the existence of anti-de Sitter-like spacetimes coupled to any of the aforementioned matter models.

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Chapter 1

Introduction

1.1 The Einstein field equations

The best current theory to describe gravitational phenomena is Einstein's theory of General Relativity. It postulates the existence of a 4-dimensional manifold $\tilde{\mathcal{M}}$ endowed with a symmetric Lorentzian metric $\tilde{\boldsymbol{g}}$. The pair $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ defines a *physical spacetime*. From now on, objects with an upper tilde will represent physical objects, in the sense that they are defined on $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$.

Throughout this work, the convention for the signature of the spacetime metric is that it possesses three positive eigenvalues and one negative; this is represented as $(-,+,+,+)$. Latin indices $a, b, c, \ldots$ will be used to indicate abstract spacetime objects; $i, j, k, \ldots$ will symbolise abstract ones on a 3-dimensional hypersurface. Greek indices $\alpha, \beta, \gamma, \ldots$ will denote the components in a particular coordinate system $x = (x^\mu)$. Let $\tilde{\nabla}$ be the Levi-Civita connection on $\tilde{\mathcal{M}}$ and demand that it be compatible with $\tilde{\boldsymbol{g}}$, that is to say, $\tilde{\nabla}_a \tilde{g}_{bc} = 0$. Consider a tangent vector $\tilde{V}^a$ on $\tilde{\mathcal{M}}$; then our convention for the curvature is the following: we define the *Riemann curvature tensor* $\tilde{R}^a{}_{bcd}$ as

$$\tilde{\nabla}_a \tilde{\nabla}_b \tilde{V}^c - \tilde{\nabla}_b \tilde{\nabla}_a \tilde{V}^c \equiv \tilde{R}^c{}_{dab} \tilde{V}^d. \tag{1.1}$$

Resulting from contractions of the Riemann tensor, the Ricci tensor and

Ricci scalar are defined, respectively, as

$$\tilde{R}_{ab} \equiv \tilde{R}^c{}_{acb}, \quad \tilde{R} \equiv \tilde{g}^{ab}\tilde{R}_{ab}. \tag{1.2}$$

A further relevant curvature object is the *Weyl tensor* $\tilde{C}^a{}_{bcd}$, defined as the tracefree part of the Riemann tensor. Explicitly this is:

$$\tilde{C}_{abcd} \equiv \tilde{R}_{abcd} + \tilde{g}_{b[c}\tilde{R}_{d]a} - \tilde{g}_{a[c}\tilde{R}_{d]b} + \frac{\tilde{R}}{3}\tilde{g}_{a[c}\tilde{g}_{d]b}. \tag{1.3}$$

A set of field equations for the theory can be deduced from a Lagrangian approach via the least action principle. Under this approach we consider the Einstein-Hilbert action for the gravitational field

$$\tilde{S}_g = \int \tilde{R}\sqrt{-\tilde{g}}\mathrm{d}V, \tag{1.4}$$

where $\tilde{g}$ is the determinant of $\tilde{\boldsymbol{g}}$ and $\mathrm{d}V$ is the volume element. In the presence of an energy-matter field we must also take into account the corresponding Lagrangian density $\tilde{\mathcal{L}}_m$ which, in turn, has an associated action $\tilde{S}_m$. Using units such that $8\pi G = c = 1$, we define the *energy-momentum tensor* of the matter field as

$$\tilde{T}_{ab} \equiv -\frac{1}{\sqrt{-\tilde{g}}}\frac{\delta\tilde{S}_m}{\delta\tilde{g}^{ab}}. \tag{1.5}$$

Remark 1. The explicit form of $\tilde{T}_{ab}$ will depend on the specific matter model under consideration. In this thesis, several concrete examples of interest will be examined in detail.

Finding the extremal of the total action $\tilde{S} = \tilde{S}_g + \tilde{S}_m$ leads to the *Einstein Field Equations* (EFE). After introducing the *cosmological constant* λ these equations take the form

$$\tilde{R}_{ab} - \frac{1}{2}\tilde{R}\tilde{g}_{ab} + \lambda\tilde{g}_{ab} = \tilde{T}_{ab}. \tag{1.6}$$

A detailed derivation can be found in [61]. As a result of the diffeomorphism-invariance of the theory, one has that $\tilde{T}_{ab}$ satisfies the conservation law

$$\tilde{\nabla}^b\tilde{T}_{ab} = 0, \tag{1.7}$$

which is compatible with the second Bianchi identity. It is worth remarking that this identity will play an important role in this work as it will impose some restrictions on the class of matter models we study under a conformal approach.

Remark 2. In the absence of matter, the trace of (1.6) is simply $\tilde{R} = 4\lambda$. This enables us to write the vacuum EFE as

$$\tilde{R}_{ab} = \lambda \tilde{g}_{ab}. \tag{1.8}$$

Manifolds for which the Ricci tensor satisfies the above condition are known as *Einstein manifolds*.

1.2 The Cauchy problem

Adopting a suitable choice of coordinates, the EFE (1.6) represent a system of coupled second order partial differential equations for the components of the metric $\tilde{g}_{ab}$, making it possible to apply the tools from the theory of Partial Differential Equations (PDEs) to analyse solutions to this system. The well-posedness of this problem requires us to provide the system with initial data. In this context, the EFE imply a set of conditions that potential solutions must satisfy on an initial spacelike hypersurface $\tilde{\mathcal{S}}_\star$ with intrinsic Riemannian 3-metric $\tilde{h}_{ij}$ and unit normal vector $\tilde{n}^a$. These are represented by the so-called *constraint equations.* Let $\tilde{D}_i$ and $\tilde{r}$ be, respectively, the covariant derivative and the Ricci scalar of $\tilde{h}_{ij}$, and $\tilde{K}_{ij}$ the extrinsic curvature of the hypersurface. Also, let $\tilde{\rho}$ and $\tilde{j}_i$ denote the pull-backs of the projections $\tilde{n}^a\tilde{n}^b\tilde{T}_{ab}$ and $\tilde{n}^a\tilde{h}_b{}^c\tilde{T}_{ac}$, respectively, to $\tilde{\mathcal{S}}_\star$. Then the constraint equations take the form [5]:

$$\tilde{r} - \tilde{K}_{ij}\tilde{K}^{ij} + \tilde{K}^2 = 16\pi\tilde{\rho}, \tag{1.9a}$$
$$\tilde{D}_j\tilde{K}^j{}_i - \tilde{D}_i\tilde{K} = 8\pi\tilde{j}_i. \tag{1.9b}$$

Here, $\tilde{K} \equiv \tilde{h}^{ij}\tilde{K}_{ij}$ and the fields $\tilde{\rho}$ and $\tilde{j}_i$ must also satisfy the conditions imposed by the conservation law (1.7).

Fourès-Bruhat [25] showed that under an adequate choice of coordinates it is possible to express the EFE as a system of second order quasilinear wave equations for the components of the metric; in Chapter 3 this will be further discussed and exploited. The resulting system requires the prescription of a 3-dimensional manifold $\tilde{\mathcal{S}}_\star$, a Riemannian 3-metric $\tilde{\boldsymbol{h}}$ and a symmetric 2-tensor $\tilde{\boldsymbol{K}}$ satisfying the constraint equations. The triplet $(\tilde{\mathcal{S}}_\star, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ will be called an *initial data set.* As it will be convenient, we introduce the following concept:

Definition 1. *A spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is a development of an initial data set $(\tilde{\mathcal{S}}_\star, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ if under an embedding $\varphi : \tilde{\mathcal{S}}_\star \to \tilde{\mathcal{M}}$ we have that*

(i) $\tilde{\boldsymbol{h}} = \varphi^*(\tilde{\boldsymbol{g}})$,

(ii) the image of the tensor $\tilde{\boldsymbol{K}}$ under φ corresponds to the extrinsic curvature of $\tilde{\mathcal{S}}_\star$.

The question of how solutions to the constraint equations relate to solutions to the EFE was first addressed by Choquet-Bruhat and Geroch in [16], whose main result can be stated as:

Theorem 1. *Let $(\tilde{\mathcal{S}}_\star, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ be an initial data set satisfying the constraint equations. Then there exists a unique maximal development which is a solution to the EFE.*

Spacetimes which can be uniquely constructed by prescribing an initial data set are called *globally hyperbolic.* This notion is closely linked to the causality requirements imposed by the theory [37]. As will be seen in the next section, there exist spacetimes which are solutions to the EFE but do not accept a Cauchy hypersurface, causing this formulation to be incomplete to study their construction. For an extensive discussion of the Cauchy problem in General Relativity see [15].

1.3 The anti-de Sitter spacetime

In this section a well-known exact solution to the EFE will be presented: the *anti-de Sitter* spacetime. The discussion about the basic properties will be based on [35, 59], while a more detailed review on its conformal representation can be found in [60]. Consider an Einstein manifold satisfying (1.8) which is also maximally symmetric and, therefore, has constant curvature. It is easy to see that under these conditions the corresponding metric is characterised by $\tilde{C}^{a}{}_{bcd} = 0$, i.e. it is conformally flat. From now on we will focus on the case $\lambda < 0$, the anti-de Sitter solution.

In order to give a more explicit description of this spacetime, consider a flat 5-dimensional space $\mathbb{R}^{3,2}$ with Cartesian coordinates (U, V, X, Y, Z) such that U and V are timelike. The corresponding line element is simply

$$\tilde{\boldsymbol{g}}_{\mathrm{adS}} = -\mathbf{d}U \otimes \mathbf{d}U - \mathbf{d}V \otimes \mathbf{d}V + \mathbf{d}X \otimes \mathbf{d}X + \mathbf{d}Y \otimes \mathbf{d}Y + \mathbf{d}Z \otimes \mathbf{d}Z.$$

Embedded into this space, the anti-de Sitter solution corresponds to the 4-dimensional hyperboloid defined by

$$-U^2 - V^2 + X^2 + Y^2 + Z^2 = -a^2, \tag{1.10}$$

where $a \equiv \sqrt{-3/\lambda}$ is a constant. This can be re-expressed in a spherically symmetric form by introducing a set of naturally adapted coordinates (t, r, θ, ϕ) which parametrise the Cartesian ones as

$$U = a\cosh r \sin(t/a), \quad V = a\cosh r\cos(t/a), \quad X = a\sinh r \sin\theta\cos\phi,$$
$$Y = a\sinh r\sin\theta\sin\phi, \quad Z = a\sinh r\cos\theta.$$

Here, $t/a \in (-\pi, \pi)$, $r \in [0, \infty)$, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$ are spherical coordinates. In terms of these variables, the metric of the anti-de Sitter space takes the form

$$\tilde{\boldsymbol{g}}_{\mathrm{adS}} = -\cosh^2 r\mathbf{d}t \otimes \mathbf{d}t + a^2(\mathbf{d}r \otimes \mathbf{d}r + \sinh^2 r\boldsymbol{\sigma}),$$

with $\boldsymbol{\sigma} \equiv \mathbf{d}\theta \otimes \mathbf{d}\theta + \sin^2\theta\mathbf{d}\phi \otimes \mathbf{d}\phi$ the metric of the 2-sphere. The time coordinate t deserves further comment: under the transformation $t \mapsto t + 2a\pi$, it can be seen that U and V remain invariant and thus correspond to the same point. The fact that t is a periodic coordinate gives rise to the appearance of closed timelike curves. This issue can be addressed by unfolding the hyperboloid so $t \in \mathbb{R}$, resulting in a representation known as the universal covering of the anti-de Sitter space. In the following, though, we will refer to this simply as the anti-de Sitter space.

Some key properties inherent to this spacetime can be unveiled by studying its conformal structure. To this aim we first introduce a new radial coordinate $R \equiv a \sinh r$, $R \in [0, \infty)$. The metric then becomes

$$\tilde{\boldsymbol{g}}_{\text{adS}} = -\left(1 - \frac{1}{3}\lambda R^2\right)\mathbf{d}t \otimes \mathbf{d}t + \left(1 - \frac{1}{3}\lambda R^2\right)^{-1}\mathbf{d}R \otimes \mathbf{d}R + R^2\boldsymbol{\sigma}. \quad (1.11)$$

Expressed in this way, it is evident that the space does not possess horizons as $\lambda < 0$. Lastly, we perform a further change of variables given by $R = a \tan \chi$ and $T = t/a$. After a direct calculation the metric takes the simpler form

$$\begin{aligned}\tilde{\boldsymbol{g}}_{\text{adS}} &= \frac{a^2}{\cos^2 \chi}\left(- \mathbf{d}T \otimes \mathbf{d}T + \mathbf{d}\chi \otimes \mathbf{d}\chi + \sin^2 \chi \boldsymbol{\sigma}\right) \\ &\equiv \frac{a^2}{\cos^2 \chi}\left(- \mathbf{d}T \otimes \mathbf{d}T + \boldsymbol{\pi}\right) \equiv \frac{a^2}{\cos^2 \chi}\boldsymbol{g}_{\mathcal{E}}, \qquad (1.12)\end{aligned}$$

where $\boldsymbol{\pi}$ and $\boldsymbol{g}_{\mathcal{E}}$ are, respectively, the metrics of the 3-sphere and the Einstein cylinder. Defining the scalar function $\Xi = \cos \chi / a$, we can write

$$\boldsymbol{g}_{\mathcal{E}} = \Xi^2 \tilde{\boldsymbol{g}}_{\text{adS}}. \quad (1.13)$$

We say that $\tilde{\boldsymbol{g}}_{\text{adS}}$ is conformal to $\boldsymbol{g}_{\mathcal{E}}$. More precisely, the fact that $r \to \infty$ corresponds to $\chi = \frac{\pi}{2}$ shows that the metric of the anti-de Sitter space is conformal to only half of the Einstein cylinder. The hypersurface defined by this condition is called the *conformal boundary* at spatial infinity — see Figure 1.1. Subsequently, $\mathscr{I}$ will denote the conformal boundary of a general spacetime. This representation then enables us to study the properties of the space at infinity via local computations. This approach represents the core of this work so its tools and methods will be discussed in more detail in Chapter 2.

The anti-de Sitter spacetime has attracted a considerable amount of interest in the last decades in view of its connections to other areas of Theoretical Physics. In particular, Maldacena [47] showed that some conformal field theories defined on the conformal boundary of the 5-dimensional anti-de Sitter spacetime are equivalent to a higher-dimensional gravitational theory on its bulk. In this sense, this proposal works as a dictionary between both theories, making the conformal boundary of great importance.

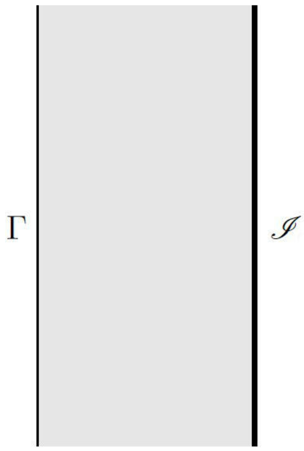

Figure 1.1: Penrose diagram of the anti-de Sitter spacetime. The angular coordinates are omitted by taking the quotient $\mathcal{M}/SO(3)$. The conformal boundary corresponds to the line $\mathscr{I}$ while Γ is the centre of symmetry.

See [42] for a detailed and specialised review.

One of the most relevant facts about the global structure of the anti-de Sitter space is the character of its conformal boundary. From its definition it is easy to see that $\mathscr{I}$ is a timelike hypersurface. Considering, however, a spacelike hypersurface Σ, it is possible to construct non-spacelike curves such that they intersect $\mathscr{I}$ but not Σ. In other words, the spacetime does not contain Cauchy hypersurfaces and consequently is not globally hyperbolic. For this reason, suitable additional data must be provided on the conformal boundary in order to recover the whole spacetime.

Unlike the case for globally hyperbolic spacetimes, it is not clear *a priori* what the basic data on the conformal boundary are. This problem has acquired a great importance in view of its relation to its instability under non-linear perturbations. Using reflective boundary conditions and a massless scalar field under spherical symmetry, Bizoń and Rostworowski [9] showed, via numerical methods, the instability of the space for arbitrarily small perturbations. This can be understood as a consequence of the boundary acting like a mirror in such a way that the perturbations become trapped, interact with each other and generate instabilities. This has been supported by evidence in different settings such as a purely gravitational analysis [23] and a study in higher dimensions [8]. Remarkably, the conjectured instability has been proved in [49] for the Einstein-massless Vlasov system under spherical symmetry. In view of the above, the role the conformal boundary plays becomes evident.

The anti-de Sitter spacetime represents an exact vacuum solution to the EFE with negative cosmological constant. Nevertheless, one can consider a more general class of solutions to the EFE with $\lambda < 0$ which admit a timelike conformal boundary. These solutions are the so-called *anti-de Sitter-like* spacetimes (or adS-like for short) and will be the main object of study in this thesis.

1.4 Alternative constructions

Regarding the construction of adS-like spacetimes, Friedrich [29] established the local existence of this class of spacetimes in the absence of matter under an approach based on conformal methods — see sections 2.2 and 5.1 for a detailed discussion. In this context, it is worth mentioning that matter models arising from a conformally invariant Lagrangian have, in general, a tracefree energy momentum tensor [54]. This will serve as a first motivation to put particular attention on matter models with this property. A vacuum construction for $(n + 1)$-dimensional asymptotically anti-de Sitter solutions has been obtained by Enciso and Kamran [24] via proving the convergence of an iterative process for a quasilinear hyperbolic system. This work contrasts with Friedrich's one — and with the construction aimed to be developed in this thesis — as the boundary conditions for the metric are constructed based on an existence result, impractical for a numerical implementation. Moreover, the conformal approach enables us to analyse the boundary by means of local computations. In the same vein of construction of asymptotically anti-de Sitter spacetimes, in [40] the local well-posedness of the semi-linear Einstein-Klein-Gordon system under the assumption of spherical symmetry was proved by means of L^2 energy estimates and Dirichlet boundary conditions. This result was extended in [41] for Neumann and Robin boundary conditions which, in turn, give rise to non-linear terms in the system which need to be controlled.

1.5 Main results of the thesis

Throughout this work, results about uniqueness and existence of solutions to equations will be in the context of PDEs, that is to say, it will refer to coordinate, but not geometric, uniqueness and existence.

Chapter 2 is devoted to reviewing the relevant concepts and tools from conformal geometry. The notion of conformal rescaling is introduced, from where a collection of transformation formulae for the various geometric objects can be obtained. It will be seen that, apart from being not invariant, the EFE result in a singular system of equations under these type of transformations. A regular conformal representation of the EFE will be discussed, the so-called metric conformal Einstein field equations; this will be the basis of the subsequent analysis. Importantly, some key results regarding the relation of this system to the EFE will be stated. In addition, the conformal version of the constraint equations is presented as it will be extensively used in later chapters.

Chapter 3 investigates some properties of the conformal Einstein field equations; specifically, the system of geometric wave equations for the conformal fields that is implied by them. The main goal is to generalise the result obtained by Paetz in [51] to the tracefree matter case. Moreover, exploiting the conformal and coordinate gauge freedom, this system is cast as a quasilinear system of wave equations. In order to relate solutions to the system of wave equations to the EFE, we carry out the so-called propagation of the constraints for a set of zero-quantities. This is done via obtaining a series of integrability conditions from where a further system of wave equations naturally emerges. Three relevant tracefree matter models are analysed in detail: the conformally invariant scalar field, the Maxwell field, and the Yang-Mills field.

In Chapter 4, the problem of existence of continuous symmetries in adS-like spacetimes is investigated under the light of conformal methods. Taking as a starting point the analyses by Paetz [50, 52] carried out in the context of null hypersurface and spacelike conformal boundaries, we study the problem in the presence of a timelike conformal boundary. The issue of existence is first formulated in terms of a system of wave equations for a set of conformal fields which, in turn, implies a similar problem on the conformal boundary. From analysing the relation between the intrinsic system on $\mathscr{I}$ and the Killing equation, an object obstructing the existence of symmetries arises. It is then shown that a necessary and sufficient condition for the existence of a Killing vector is the vanishing of such an obstruction.

Chapter 5 is centred on the problem of the construction of adS-like spacetimes as an initial-boundary problem in the vacuum case by using the results from Chapter 3. Then we proceed to focus on the identification and construction of suitable initial and boundary data for the system, and discuss the corner compatibility conditions these data must satisfy. The adequate propagation of the constraints is proved to follow from the properties of the zero-quantities. With this we obtain a local result of uniqueness and existence of vacuum adS-like solutions to the EFE, alternative to the one in [29].

As a complement, in Chapter 6 the previous analysis is extended to the tracefree matter case. Using results from Chapter 3, a suitable system of quasilinear wave equations is obtained which then serve to prove the local existence and uniqueness of solutions to the EFE with negative cosmological constant coupled to any of the three matter models considered in Chapter 3. Specifically, a detailed analysis of the identification of the basic boundary data for the matter fields is presented.

Finally, in Chapter 7 we briefly discuss the main results obtained as well as possible further extensions and the new potential problems they pose.

Chapter 2

Methods of conformal geometry

The purpose of this section is to provide a brief overview of the methods and tools that conformal geometry offers, especially how they can be implemented into General Relativity to study global properties of the EFE. The difficulties towards this goal will be discussed, as well as how they motivate the construction of a suitable conformal formulation which will serve as the basis for subsequent analyses in this thesis. The presentation here is inspired by [60] where these methods are extensively discussed.

2.1 Conformal transformations

The study of conformal methods applied to General Relativity started with Penrose's seminal work [53], in which he introduced some of the techniques to study the asymptotic properties of spacetimes by considering instead an auxiliary unphysical metric. More specifically, the concept of a conformal transformation is central to the discussion:

Definition 2. *Let $\tilde{\mathcal{M}}$ and $\mathcal{M}$ be two manifolds with metrics $\tilde{\boldsymbol{g}}$ and $\boldsymbol{g}$, respectively. A conformal transformation consists of a diffeomorphism $\varphi : \tilde{\mathcal{M}} \to \mathcal{M}$ such that*

$$\varphi^*(\boldsymbol{g}) = \Xi^2 \tilde{\boldsymbol{g}}, \tag{2.1}$$

with Ξ a smooth scalar function defined on $\tilde{\mathcal{M}}$.

Moreover, if $\tilde{\boldsymbol{g}}$ and $\boldsymbol{g}$ satisfy relation (2.1) they are said to be conformally related. The scalar function Ξ denotes the so-called *conformal factor*.

Remark 3. Observe that a conformal transformation φ with $\Xi > 0$ induces the existence of a conformal class of the metric $[\tilde{\boldsymbol{g}}]$, defined as the set of metrics conformally related to $\tilde{\boldsymbol{g}}$.

Conformal transformations (2.1) which are not surjective represent a more specific class known as conformal extensions of $\tilde{\mathcal{M}}$. The fact that only a subset of $\mathcal{M}$ becomes relevant leads to the following central definition:

Definition 3. *Let $\tilde{\mathcal{M}}$ and $\mathcal{M}$ be two manifolds and let $\mathcal{U} \subset \mathcal{M}$ be an open, connected submanifold with compact closure. We say that a conformal transformation $\varphi : \tilde{\mathcal{M}} \to \mathcal{U}$ is a conformal compactification if*

$$\boldsymbol{g} = (\varphi^*)^{-1}(\Xi^2)\tilde{\boldsymbol{g}} \tag{2.2}$$

and the following conditions hold:

(i) $\Xi > 0$ *on* $\mathcal{U}$*,*

(ii) $\Xi = 0$ *and* $\mathbf{d}\Xi \neq 0$ on $\partial\mathcal{U}$*.*

Hereafter, the pair $(\mathcal{M}, \boldsymbol{g})$ will denote an *unphysical spacetime* and the set of points defined by $\Xi = 0$ is called the *conformal boundary*. Furthermore, despite the physical and unphysical metrics being defined on different manifolds, with a slight abuse of notation we will obviate the action of the pull-back φ^* and express the relation between the two metrics simply as

$$\boldsymbol{g} = \Xi^2\tilde{\boldsymbol{g}}. \tag{2.3}$$

Transformation (2.3) by itself is not enough to obtain the formulae relating the different curvature objects associated to the metric. For this we need to introduce the covariant derivative operator associated to the unphysical metric ∇, which will be assumed to have zero-torsion and be compatible with $\boldsymbol{g}$. For a scalar function $\nabla_a f = \tilde{\nabla}_a f$, while for a mixed $(1,1)$ tensor $A_a{}^b$, we have that

$$\nabla_a A_b{}^c - \tilde{\nabla}_a A_b{}^c = -Q_a{}^d{}_b A_d{}^c + Q_a{}^c{}_d A_b{}^d, \tag{2.4}$$

with $Q_a{}^b{}_c = \Xi^{-1}(\nabla_a\Xi\delta_b{}^c + \nabla_b\Xi\delta_a{}^c - \nabla^c\Xi g_{ab})$ the transition tensor between $\tilde{\nabla}$ and ∇; generalisations to higher order tensors are direct. Exploiting the last relation, a series of long calculations provide the following expressions:

$$R_{ab} - \tilde{R}_{ab} = -2\Xi^{-1}\nabla_a\nabla_b\Xi - g_{ab}(\Xi^{-1}\nabla^c\nabla_c\Xi, -3\Xi^{-2}\nabla^c\Xi\nabla_c\Xi), \quad (2.5a)$$

$$R - \Xi^{-2}\tilde{R} = -6\Xi^{-1}\nabla^c\nabla_c\Xi + 12\Xi^{-2}\nabla^c\Xi\nabla_c\Xi, \quad (2.5b)$$

$$C^a{}_{bcd} - \tilde{C}^a{}_{bcd} = 0. \quad (2.5c)$$

Notice, in particular, that the Weyl tensor turns out to remain invariant.

With this in hand we can now proceed to investigate how the EFE transform under a conformal rescaling. For simplicity, consider the vacuum EFE with vanishing cosmological constant $\tilde{R}_{ab} = 0$. Transformation (2.5a) implies that

$$R_{ab} = -2\Xi^{-1}\nabla_a\nabla_b\Xi - g_{ab}(\Xi^{-1}\nabla^c\nabla_c\Xi - 3\Xi^{-2}\nabla^c\Xi\nabla_c\Xi),$$

which is singular for $\Xi = 0$. Attempting to overcome this problem by multiplying by Ξ^2 results, however, in the vanishing of the principal part when evaluated on $\mathscr{I}$. Therefore, the last equation is not convenient if we desire to adopt an approach based on results from the theory of PDEs.

2.2 The conformal Einstein field equations

In view of the bad conformal properties of the EFE, their study requires the construction of an alternative and not-so-straightforward system. A successful conformal formulation was first obtained by Friedrich [26] via the introduction of a number of additional conformal fields. In the remainder of this section, we will consider two conformally related spacetimes $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ and $(\mathcal{M}, \boldsymbol{g})$ where the former satisfies the EFE with matter (1.6). In order to deal with non-trivial matter models, we first need to define the basic properties of their unphysical counterparts.

2.2.1 The unphysical energy-momentum tensor

Since equation (2.3) does not determine the way the energy-momentum of a matter field $\tilde{T}_{ab}$ transforms under a conformal rescaling, it will be convenient to define the *unphysical energy-momentum tensor* via the following homogeneous transformation rule:

$$T_{ab} \equiv \Xi^{-2}\tilde{T}_{ab}. \tag{2.6}$$

Remark 4. It is worth emphasising that the conformal rescaling (2.6) has been chosen as it is suitable for our purposes but, in principle, different transformation rules may be more adequate in other contexts.

Using the transformation rule between the Levi-Civita covariant derivatives of conformally related metrics (2.4), it readily follows that equation (1.7) takes the form

$$\nabla^b T_{ab} = \Xi^{-1} T \nabla_a \Xi,$$

where $T \equiv g^{cd}T_{cd}$. This relation represents a challenge as, for a general matter field, becomes singular at $\Xi = 0$. Nevertheless, transformation (2.6) implies that $T = \Xi^{-4}\tilde{T}$. Based on this observation we make the following assumption:

Assumption 1. From here onwards we restrict our attention to matter models for which $\tilde{T} = 0$, so that the corresponding unphysical energy-momentum tensor T_{ab} satisfies

$$\nabla^b T_{ab} = 0. \tag{2.7}$$

2.2.2 Basic relations

Having defined the class of matter models we will deal with, our attention will now be focused on the introduction of a number of conformal fields defined on $\mathcal{M}$ that will become central to the analysis of the EFE. Let the *Friedrich scalar*, the *Schouten Tensor*, the *rescaled Weyl tensor* and the *rescaled Cotton tensor* be defined, respectively, as

$$s \equiv \tfrac{1}{4}\nabla^c\nabla_c\Xi + \tfrac{1}{24}R\Xi, \tag{2.8a}$$

$$L_{ab} \equiv \tfrac{1}{2}R_{ab} - \tfrac{1}{12}Rg_{ab}, \tag{2.8b}$$

$$d^a{}_{bcd} \equiv \Xi^{-1}C^a{}_{bcd}, \tag{2.8c}$$

$$T_{abc} \equiv \Xi\nabla_{[a}T_{b]c} + 3\nabla_{[a}\Xi T_{b]c} - g_{c[a}T_{b]e}\nabla^e\Xi. \tag{2.8d}$$

Observe that $L_{ab} = L_{ba}$, $d^a{}_{bcd}$ inherits the symmetries of the Weyl tensor and T_{abc} possesses the following ones:

$$T_{abc} = T_{[ab]c}, \quad T_{[abc]} = 0. \tag{2.9}$$

Relevant for the subsequent discussion is the well-known fact that the rescaled Weyl tensor accepts two associated Hodge dual tensors, namely

$$^*d_{abcd} \equiv \tfrac{1}{2}\epsilon_{ab}{}^{ef}d_{efcd}, \quad d^*_{abcd} \equiv \tfrac{1}{2}\epsilon_{cd}{}^{ef}d_{abef},$$

where ϵ_{abcd} is the 4-volume form of the metric g_{ab}. Furthermore, one can check the following auxiliary properties:

$$^*d_{abcd} = d^*_{abcd}, \quad ^{**}d_{abcd} = d^{**}_{abcd} = {}^*d^*_{abcd} = -d_{abcd}.$$

Likewise, we define the Hodge dual of T_{abc} as

$$^*T_{abc} \equiv \tfrac{1}{2}\epsilon_{ab}{}^{de}T_{dec}. \tag{2.10}$$

In terms of the notation and conventions used in this work, the *metric tracefree conformal Einstein field equations* (MTCEFE) are given by

$$\nabla_a\nabla_b\Xi = -\Xi L_{ab} + sg_{ab} + \tfrac{1}{2}\Xi^3T_{ab}, \tag{2.11a}$$

$$\nabla_a s = -L_{ab}\nabla^b\Xi + \tfrac{1}{2}\Xi^2\nabla^b\Xi T_{ab}, \tag{2.11b}$$

$$\nabla_a L_{bc} - \nabla_b L_{ac} = \nabla_e\Xi d^e{}_{cab} + \Xi T_{abc}, \tag{2.11c}$$

$$\nabla_e d^e{}_{abc} = T_{bca}, \tag{2.11d}$$

$$6\Xi s - 3\nabla_c\Xi\nabla^c\Xi = \lambda, \tag{2.11e}$$

$$R^a{}_{bcd} = \Xi d^a{}_{bcd} + 2\delta_{[c}{}^aL_{d]b} + 2L_{[c}{}^ag_{d]b}. \tag{2.11f}$$

This system was first obtained and applied to study the initial value problem by Friedrich [26, 27] — see [60] for a detailed derivation. It is worth mentioning that as this system represents the foundation of the upcoming analyses in this thesis, some of their properties and consequences are further explored in the next chapter.

Remark 5. Equations (2.11a)-(2.11d) will be regarded as a set of differential conditions for the fields Ξ, s, L_{ab} and $d^a{}_{bcd}$. Equation (2.11e) can be shown to play the role of a constraint which only needs to be verified at a single point — see e.g. [60], Lemma 8.1. A differential equation for the unphysical metric g_{ab} results from taking the trace of (2.11f), where L_{ab} and R_{ab} are treated as independent fields; this will be further discussed in Chapter 3.

Remark 6. A solution to the conformal EFE with tracefree matter will be understood to be a collection of fields $(g_{ab}, \Xi, s, L_{ab}, d^a{}_{bcd}, T_{ab})$ satisfying equations (2.11a)-(2.11f) and the conservation law (2.7).

Remark 7. If Assumption 1 and equation (2.11a) are taken into account, one obtains two additional identities for the rescaled Cotton tensor, namely

$$\nabla_c T_{ab}{}^c = 0, \tag{2.12a}$$
$$\nabla_c {}^*T_{ab}{}^c = 0, \tag{2.12b}$$
$$\nabla_c {}^*T_a{}^c{}_b = \nabla_c {}^*T_{(a}{}^c{}_{b)}. \tag{2.12c}$$

The relation between the conformal Einstein field equations (2.11a)-(2.11e) and the EFE is given in the following statement — see [60], Proposition 8.1, for a proof:

Proposition 1. *Let* $(g_{ab}, \Xi, s, L_{ab}, d^a{}_{bcd}, T_{ab})$ *denote a solution to the conformal Einstein field equations with matter such that* $\Xi \neq 0$ *on an open set* $\mathcal{U} \subset \mathcal{M}$*. Then the metric* $\tilde{g}_{ab} = \Xi^{-2} g_{ab}$ *is a solution to the Einstein field equations* (1.6) *with energy momentum tensor given by* $\tilde{T}_{ab} = \Xi^2 T_{ab}$ *on* $\mathcal{U}$*.*

Recalling that $\nabla^a \Xi$ is normal to $\mathscr{I}$, the next result follows immediately from equation (2.11e):

Corollary 1. *Suppose that the Friedrich scalar is regular on $\mathscr{I}$. Then $\mathscr{I}$ is a null, spacelike or timelike hypersurface of $\mathcal{M}$, respectively, depending on whether $\lambda = 0$, $\lambda > 0$ or $\lambda < 0$.*

2.2.2.1 An alternative equation for $d^a{}_{bcd}$

For our purposes, it will be convenient to consider an alternative version of the conformal field equation for the rescaled Weyl tensor — see [51] for the calculation in the vacuum case. Multiplying (2.11d) by ϵ_{abfg} and exploiting the identity ${}^*d_{abcd} = d^*_{abcd}$ results in

$$2\nabla_a{}^*d_{fgc}{}^a = 2\nabla_a d^*_{fgc}{}^a = -2{}^*T_{fgc}.$$

From here it follows that

$$3\nabla_{[e}d_{ab]cd} + \epsilon_{eabf}{}^*T_{cd}{}^f = 0. \tag{2.13}$$

Remark 8. Equation (2.13) is equivalent to (2.11d) and will be essential in Chapter 3 where a system of wave equations for the geometric fields and the zero-quantities associated to the equations (2.11a)-(2.11e) is obtained.

2.2.2.2 An equation for the components of the metric g_{ab}

Taking the natural trace of equation (2.11f) leads to the relation

$$R_{ab} = 2L_{ab} + \tfrac{1}{6}Rg_{ab}. \tag{2.14}$$

Here, the Ricci tensor R_{ab} is assumed to be expressed in terms of first and second derivatives of the components of the metric, whilst L_{ab} is a field satisfying equations (2.11a)-(2.11e). This will be further discussed in Section 3.5 where a suitable wave equation for the components of the metric is constructed.

Remark 9. As pointed out in [30], equation (2.14) can be regarded as an Einstein field equation for the unphysical metric g_{ab}. In this sense, the geometric fields Ξ, s, L_{ab} and d_{abcd} can be regarded as unphysical matter fields. Accordingly, in the following we refer to equation (2.14) as the *unphysical Einstein equation*. This point of view should allow us to adapt well-tested numerical methods for the Einstein field equations in the case of the conformal field equations.

2.3 The conformal constraint equations

In the same spirit as in Section 1.2, the MTCEFE impose a number of restrictions that their solutions must satisfy on 3-dimensional hypersurfaces. In the following, let $\mathcal{H} \subset \mathcal{M}$ denote a (spacelike or timelike) hypersurface with unit normal vector n_a. We define the norm of n_a as

$$\epsilon \equiv n_a n^a,$$

so that ϵ takes the values 1 or -1 for timelike or spacelike hypersurfaces, respectively. The normal vector induces a decomposition via the projector to $\mathcal{H}$:

$$h_a{}^b \equiv \delta_a{}^b - \epsilon n_a n^b.$$

Similarly, this defines the intrinsic derivative D_a on $\mathcal{H}$ in the following way. Let f be a scalar function and $A_a{}^b$ be a tensor field on $\mathcal{M}$. Then

$$D_a f \equiv h_a{}^b \nabla_b f,$$
$$D_e A_a{}^b \equiv h_e{}^f h_a{}^c h_d{}^b \nabla_f A_c{}^d.$$

Expressions involving higher order tensors follow an analogous rule. On the other hand, the derivative in the direction of n^a (simply called the normal derivative) is given by

$$D \equiv n^a \nabla_a.$$

Clearly, these differential operators inherit the properties of ∇_a. Moreover, the extrinsic curvature of $\mathcal{H}$ is defined as the symmetric tensor

$$K_{ab} \equiv h_a{}^c h_b{}^d \nabla_c n_d.$$

In the following, let

$$\Sigma, \quad s, \quad h_{ij}, \quad K_{ij}, \quad L_i, \quad L_{ij}, \quad d_{ij}, \quad d_{ijk}, \quad d_{ijkl}$$

denote, respectively, the pull-backs of the following geometric objects

$$n^a \nabla_a \Xi, \quad s, \quad g_{ab}, \quad K_{ab}, \quad n^c h_a{}^d L_{cd}, \quad h_a{}^c h_b{}^d L_{cd},$$
$$n^b n^d h_e{}^a h_f{}^c d_{abcd}, \quad n^b h_e{}^a h_f{}^c h_g{}^d d_{abcd}, \quad h_e{}^a h_f{}^b h_g{}^c h_h{}^d d_{abcd}$$

to $\mathcal{H}$. In order to take into account the contributions from the matter field, let ρ, j_i, T_{ij}, J_i, J_{ij} and T_{ijk} stand, respectively, for the pull-backs of the projections

$$n^a n^b T_{ab}, \quad n^b h_c{}^a T_{ab}, \quad h_c{}^a h_d{}^b T_{ab},$$
$$n^b n^c h_d{}^a T_{abc}, \quad n^c h_d{}^a h_e{}^b T_{abc}, \quad h_d{}^a h_e{}^b h_f{}^c T_{abc}.$$

Remark 10. The tensor h_{ij} corresponds to the 3-metric induced by g_{ab} on $\mathcal{H}$ and will be either Lorentzian if $\mathcal{H}$ is timelike or Riemannian if $\mathcal{H}$ is spacelike.

Remark 11. The fields d_{ij} and d_{ijk} encode, respectively, the *electric* and *magnetic parts* of the rescaled Weyl tensor d_{abcd} with respect to the normal n_a. It can be verified that

$$d_{ij} = d_{ji}, \quad d_i{}^i = 0, \quad d_{ijk} = -d_{ikj}, \quad d_{[ijk]} = 0,$$
$$d_{ijkl} = 2\epsilon(h_{i[l} d_{k]j} + h_{j[k} d_{l]i}).$$

From this point onwards, the restriction of the conformal factor Ξ to $\mathcal{H}$ will be denoted by Ω. That is,

$$\Omega \equiv \Xi|_{\mathcal{H}}.$$

Next, we define the Schouten tensor of the intrinsic 3-metric h_{ij}:

$$l_{ij} \equiv r_{ij} - \frac{1}{4} r h_{ij},$$

where r_{ij} and r are the corresponding intrinsic Ricci tensor and scalar. In terms of the fields defined above, a set of constraints can be obtained from the different projections of the MTCEFE involving derivatives that are intrinsic to $\mathcal{H}$. This results in the following:

$$D_i D_j \Omega = -\epsilon \Sigma K_{ij} - \Omega L_{ij} + s h_{ij} + \frac{1}{2}\Omega^3 T_{ij}, \tag{2.15a}$$

$$D_i \Sigma = K_i{}^k D_k \Omega - \Omega L_i + \frac{1}{2}\Omega^3 j_i, \tag{2.15b}$$

$$D_i s = -\epsilon L_i \Sigma - L_{ik} D^k \Omega + \frac{1}{2}\Omega^2(\epsilon \Sigma j_i + T_{ij} D^j \Omega), \tag{2.15c}$$

$$D_i L_{jk} - D_j L_{ik} = -\epsilon \Sigma d_{kij} + D^l \Omega d_{lkij} - \epsilon(K_{ik} L_j - K_{jk} L_i) + \Omega T_{ijk}, \tag{2.15d}$$

$$D_i L_j - D_j L_i = D^l \Omega d_{lij} + K_i{}^k L_{jk} - K_j{}^k L_{ik} + \Omega J_{ij}, \tag{2.15e}$$

$$D^k d_{kij} = \epsilon\big(K^k{}_i d_{jk} - K^k{}_j d_{ik}\big) + J_{ij}, \tag{2.15f}$$

$$D^i d_{ij} = K^{ik} d_{ijk} + J_i, \tag{2.15g}$$

$$\lambda = 6\Omega s - 3\epsilon \Sigma^2 - 3 D_k \Omega D^k \Omega. \tag{2.15h}$$

This system is supplemented by two further purely geometric constraints arising from equation (2.11f), namely the conformal Codazzi-Mainardi and Gauss-Codazzi equations:

$$D_j K_{ki} - D_k K_{ji} = \Omega d_{ijk} + h_{ij} L_k - h_{ik} L_j, \tag{2.16a}$$

$$l_{ij} = -\epsilon \Omega d_{ij} + L_{ij} + \epsilon\bigg(K\big(K_{ij} - \frac{1}{4} K h_{ij}\big) - K_{ki} K_j{}^k + \frac{1}{4} K_{kl} K^{kl} h_{ij}\bigg), \tag{2.16b}$$

with $K \equiv h^{ij} K_{ij}$. Expressions (2.15a)-(2.15h), (2.16a) and (2.16b) are called the *tracefree conformal constraint equations*. A derivation of this system can be found in [60] along with an extensive discussion about its properties.

2.3.1 The conformal constraints on $\mathscr{I}$

Assume that the hypersurface $\mathcal{H}$ is a timelike conformal boundary with Lorentzian 3-metric ℓ_{ab} and normal vector $\not{n}^a$. Hereinafter, objects and operators crossed by a line / will represent projections obtained via $\ell_a{}^b$ and $\not{n}^a$. Also, $\simeq$ will denote an equality valid on the conformal boundary. Evaluating the conformal constraints on this hypersurface ($\Omega = 0,\ \epsilon = 1$), we obtain the following simplified system:

$$\slashed{\Sigma}\slashed{K}_{ij} \simeq s\ell_{ij}, \tag{2.17a}$$

$$\slashed{D}_i\slashed{\Sigma} \simeq 0, \tag{2.17b}$$

$$\slashed{D}_i s \simeq -\slashed{L}_i\slashed{\Sigma}, \tag{2.17c}$$

$$\slashed{D}_i\slashed{L}_{jk} - \slashed{D}_j\slashed{L}_{ik} \simeq -\slashed{\Sigma}\slashed{d}_{kij} - (\slashed{K}_{ik}\slashed{L}_j - \slashed{K}_{jk}\slashed{L}_i), \tag{2.17d}$$

$$\slashed{D}_i\slashed{L}_j - \slashed{D}_j\slashed{L}_i \simeq \slashed{K}_i{}^k\slashed{L}_{jk} - \slashed{K}_j{}^k\slashed{L}_{ik}, \tag{2.17e}$$

$$\slashed{D}^k\slashed{d}_{kij} \simeq \slashed{K}^k{}_i\slashed{d}_{jk} - \slashed{K}^k{}_j\slashed{d}_{ik} + \slashed{J}_{ij}, \tag{2.17f}$$

$$\slashed{D}^j\slashed{d}_{ij} \simeq \slashed{K}^{jk}\slashed{d}_{jik} + \slashed{J}_i, \tag{2.17g}$$

$$\lambda \simeq -3\slashed{\Sigma}^2, \tag{2.17h}$$

$$\slashed{D}_j\slashed{K}_{ki} - \slashed{D}_k\slashed{K}_{ji} \simeq \ell_{ij}\slashed{L}_k - \ell_{ik}\slashed{L}_j, \tag{2.17i}$$

$$\slashed{l}_{ij} \simeq \slashed{L}_{ij} + \slashed{K}\big(\slashed{K}_{ij} - \frac{1}{4}\slashed{K}\ell_{ij}\big) - \slashed{K}_{ki}\slashed{K}_j{}^k + \frac{1}{4}\slashed{K}_{kl}\slashed{K}^{kl}\ell_{ij}. \tag{2.17j}$$

A procedure to solve these equations in the vacuum case has been discussed in [29], where the solution is given in terms of a gauge quantity related to the Friedrich scalar and the rescaled Cotton tensor associated to ℓ_{ij}, the latter defined as

$$y_{ijk} \equiv \slashed{D}_i\slashed{l}_{jk} - \slashed{D}_j\slashed{l}_{ik}.$$

Using this to deal with the tracefree matter case, the following result can be stated:

Proposition 2. *Let* $(\mathcal{M}, \boldsymbol{g})$ *be a 4-dimensional manifold and* $\mathscr{I} \subset \mathcal{M}$ *a timelike conformal boundary with intrinsic 3-dimensional Lorentzian metric* ℓ_{ij} *and normal* $\slashed{n}^a$*. Consider a tracefree energy-momentum tensor* T_{ab} *with* $\slashed{j}_i$ *its orthogonal-normal projection with respect to* $\slashed{n}^a$*. Let* $\varkappa(x)$ *be a smooth scalar gauge function defined on* $\mathscr{I}$*. Then a solution to the tracefree* *conformal constraint equations* (2.17a)-(2.17j) *on* $\mathscr{I}$ *is given by the fields*

$$\not{\Sigma} \simeq \sqrt{\frac{|\lambda|}{3}}, \tag{2.18a}$$
$$s \simeq \not{\Sigma}\varkappa, \tag{2.18b}$$
$$\not{K}_{ij} \simeq \varkappa\ell_{ij}, \tag{2.18c}$$
$$\not{L}_i \simeq -\not{D}_i\varkappa, \tag{2.18d}$$
$$\not{L}_{ij} \simeq \not{l}_{ij} - \frac{1}{2}\varkappa^2\ell_{ij}, \tag{2.18e}$$
$$\not{d}_{kij} \simeq -\not{\Sigma}^{-1}y_{ijk}, \tag{2.18f}$$

along with a tracefree symmetric tensor field $\not{d}_{ij}$ *satisfying*

$$\not{D}^j\not{d}_{ij} \simeq -\not{\Sigma}\not{j}_i. \tag{2.19}$$

Proof. Firstly, $\not{\Sigma}$ is given by (2.17h). As mentioned above, s is a gauge quantity on $\mathscr{I}$ and expressed by equation (2.18b). Direct substitution into constraints (2.17a), (2.17c), (2.17j) and (2.17d) readily leads to the solutions for $\not{K}_{ij}$, $\not{L}_i$, $\not{L}_{ij}$ and $\not{d}_{ijk}$, respectively. Using these, equations (2.17b), (2.17e) and (2.17i) are trivially satisfied. Regarding the equations with matter terms, when the fields $\not{J}_{ij}$ and $\not{J}_i$ are written explicitly in terms of the energy-momentum tensor via equation (2.8d), a straightforward calculation yields

$$\not{J}_{ij} \simeq 0, \quad \not{J}_i \simeq -\not{\Sigma}\not{j}_i.$$

On the other hand, by virtue of the definition of y_{ijk}, it follows that $\not{D}^k y_{ijk} \simeq 0$. Using this and the expressions for $\not{J}_i$ and $\not{J}_{ij}$ stated above, it is found that (2.17f) is trivially satisfied and (2.17g) corresponds to equation (2.19). □

Having obtained the solutions for the constraint equations on the conformal boundary, a converse-like result can be formulated with the addition of an auxiliary assumption:

Proposition 3. *Let* $\mathcal{T} \subset \mathcal{M}$ *be a timelike hypersurface such that conditions* (2.18a)-(2.18e) *hold. If* $\Omega = 0$ *on some fiduciary spacelike hypersurface* $\mathcal{C}_\star$ *of* $\mathcal{T}$, *then one has that* $\Omega = 0$ *on* $\mathcal{T}$.

Proof. Consider first the case when $\varkappa \neq 0$ on $\mathcal{C}_\star$. Using equations (2.18b), (2.18c) and (2.18e), the trace of the conformal constraint (2.15a) provides us with the following wave equation for Ω to be satisfied on $\mathcal{T}$:

$$\not{D}_i\not{D}^i\Omega \equiv \Box_\ell\Omega = -\Omega\left(\frac{r}{4} - \frac{3}{2}\varkappa^2\right) - \frac{1}{2}\Omega^3\rho, \tag{2.20}$$

where it has been used that for a tracefree unphysical energy-momentum tensor $T_i{}^i = -\epsilon\rho$. On the other hand, when (2.18a), (2.18c) and (2.18d) are substituted into constraint (2.15b) we have

$$\varkappa\not{D}_i\Omega = -\Omega\not{D}_i\varkappa - \frac{1}{2}\Omega^3 j_i. \tag{2.21}$$

As $\varkappa \neq 0$ and $\Omega = 0$ on $\mathcal{C}_\star$, it follows from the last equation that $\not{D}_i\Omega = 0$ on $\mathcal{C}_\star$, which represents a first order initial condition for Ω. Due to the homogeneity of (2.20) along with the uniqueness of its solutions, we conclude that $\Omega = 0$ on $\mathcal{T}$; that is to say, it corresponds to the conformal boundary.

To deal with the case $\varkappa = 0$ we observe that it is always possible to carry out a rescaling $\Xi \mapsto \Xi' \equiv \vartheta\Xi$ of the spacetime conformal factor Ξ with $\vartheta \simeq 1$ and $\mathbf{d}\vartheta \neq 0$, such that if $s \not\simeq 0$ on $\mathscr{I}$ then $s' \simeq 0$ — see [60] Section 11.4.4, page 268. Thus, if $\varkappa \not\simeq 0$ initially, the above rescaling and relation (2.18b) for s' imply that $\varkappa' \simeq 0$. Furthermore, the rescaling $\Xi \mapsto \Xi' \equiv \vartheta\Xi$ does not change the value of Ξ on $\mathcal{T}$. Accordingly, one also has that $\Omega = 0$ on $\mathcal{T}$ if $\varkappa = 0$. $\Box$

2.4 Conformal properties of some tracefree matter models

At the end of this chapter, three tracefree matter models are introduced along with their relevant conformal properties. These models are the conformally invariant scalar field, the Maxwell field and the Yang-Mills field. In Chapters 3 and 6 the problem of their coupling to the MTCEFE will be further explored.

2.4.1 The conformally invariant scalar field

The conformally invariant scalar field constitutes a first example of an explicit tracefree matter model of interest. Let $\tilde{\phi}$ be a scalar field on $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ governed by the equation

$$\tilde{\nabla}_a \tilde{\nabla}^a \tilde{\phi} - \frac{1}{6}\tilde{R}\tilde{\phi} = 0.$$

Defining the unphysical scalar field $\phi \equiv \Xi^{-1}\tilde{\phi}$, it is well-known that this equation remains invariant under a conformal transformation. This means that ϕ satisfies

$$\nabla_a \nabla^a \phi - \frac{1}{6}R\phi = 0. \tag{2.22}$$

Furthermore, the energy-momentum tensor associated to this field takes the form

$$T_{ab} = \nabla_a\phi\nabla_b\phi - \tfrac{1}{2}\phi\nabla_a\nabla_b\phi - \tfrac{1}{4}g_{ab}\nabla_c\phi\nabla^c\phi + \tfrac{1}{2}\phi^2 L_{ab}. \tag{2.23}$$

Remark 12. The scalar field in (2.22) is related to the standard scalar field satisfying the wave equation through a transformation originally due to Bekenstein [7]. Explicitly, in the absence of electromagnetic fields, they are related as $\phi = \zeta^{-1}\coth\zeta\tilde{\phi}$, where $\Omega^{-1} = \sinh\zeta\tilde{\phi}$. Thus, in principle, the theory for the conformally coupled scalar field can be rephrased in terms of its standard counterpart.

2.4.2 The Maxwell field

The next example under consideration is the electromagnetic field. In the physical spacetime the information is encoded in the antisymmetric Faraday tensor $\tilde{F}_{ab}$ which satisfies Maxwell equations

$$\tilde{\nabla}^a\tilde{F}_{ab} = 0,$$
$$\tilde{\nabla}_{[a}\tilde{F}_{bc]} = 0.$$

Defining the unphysical counterpart of the Faraday tensor as $F_{ab} \equiv \tilde{F}_{ab}$ the Maxwell equations remain invariant:

$$\nabla^a F_{ab} = 0, \tag{2.24a}$$
$$\nabla_{[c} F_{ab]} = 0. \tag{2.24b}$$

Alternatively, the second equation can be written in terms of the dual Faraday tensor $F^*_{ab} \equiv -\frac{1}{2}\epsilon_{ab}{}^{cd}F_{cd}$ as

$$\nabla^a F^*_{ab} = 0. \tag{2.25}$$

In addition, the corresponding energy-momentum tensor is given by

$$T_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}. \tag{2.26}$$

A simple calculation shows that equations (2.24a)-(2.24b) imply the conservation of T_{ab}; nevertheless, as they contain more information about the field, they will be employed instead in the following.

2.4.3 The Yang-Mills field

As a third and last example of a tracefree matter field we consider the Yang-Mills field. This consists of a set of physical fields $\tilde{F}^{\mathfrak{a}}{}_{ab}$ and gauge potentials $\tilde{A}^{\mathfrak{a}}{}_a$, where the indices $\mathfrak{a}$, $\mathfrak{b}, \ldots$ take values in the Lie algebra $\mathfrak{g}$ of a group $\mathfrak{G}$. The physical Yang-Mills equations are

$$\begin{aligned}
&\tilde{\nabla}_a \tilde{A}^{\mathfrak{a}}{}_b - \tilde{\nabla}_b \tilde{A}^{\mathfrak{a}}{}_a + C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\tilde{A}^{\mathfrak{b}}{}_a\tilde{A}^{\mathfrak{c}}{}_b - \tilde{F}^{\mathfrak{a}}{}_{ab} = 0,\\
&\tilde{\nabla}^a \tilde{F}^{\mathfrak{a}}{}_{ab} + C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\tilde{A}^{\mathfrak{b}a}\tilde{F}^{\mathfrak{c}}{}_{ab} = 0,\\
&\tilde{\nabla}_{[a}\tilde{F}^{\mathfrak{a}}{}_{bc]} + C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\tilde{A}^{\mathfrak{b}}{}_{[a}\tilde{F}^{\mathfrak{c}}{}_{bc]} = 0.
\end{aligned}$$

Here $C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} = C^{\mathfrak{a}}{}_{[\mathfrak{b}\mathfrak{c}]}$ denote the structure constants of the Lie algebra $\mathfrak{g}$ which satisfy the *Jacobi identity*

$$C^{\mathfrak{a}}{}_{\mathfrak{d}\mathfrak{e}}C^{\mathfrak{b}}{}_{\mathfrak{a}\mathfrak{c}} + C^{\mathfrak{a}}{}_{\mathfrak{e}\mathfrak{c}}C^{\mathfrak{b}}{}_{\mathfrak{a}\mathfrak{d}} + C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{d}}C^{\mathfrak{b}}{}_{\mathfrak{a}\mathfrak{e}} = 0. \tag{2.27}$$

Under the rescalings $F^{\mathfrak{a}}{}_{ab} \equiv \tilde{F}^{\mathfrak{a}}{}_{ab}$ and $A^{\mathfrak{a}}{}_{a} \equiv \tilde{A}^{\mathfrak{a}}{}_{a}$ the unphysical Yang-Mills equations take the form:

$$\nabla_a A^{\mathfrak{a}}{}_b - \nabla_b A^{\mathfrak{a}}{}_a + C^{\mathfrak{a}}{}_{\mathfrak{bc}} A^{\mathfrak{b}}{}_a A^{\mathfrak{c}}{}_b - F^{\mathfrak{a}}{}_{ab} = 0, \tag{2.28a}$$

$$\nabla^a F^{\mathfrak{a}}{}_{ab} + C^{\mathfrak{a}}{}_{\mathfrak{bc}} A^{\mathfrak{b}a} F^{\mathfrak{c}}{}_{ab} = 0, \tag{2.28b}$$

$$\nabla_{[a} F^{\mathfrak{a}}{}_{bc]} + C^{\mathfrak{a}}{}_{\mathfrak{bc}} A^{\mathfrak{b}}{}_{[a} F^{\mathfrak{c}}{}_{bc]} = 0. \tag{2.28c}$$

Motivated by relation (2.25), equation (2.28c) can be written in terms of the dual tensor $F^{*\mathfrak{a}}{}_{ab} \equiv -\frac{1}{2}\epsilon_{ab}{}^{cd} F_{cd}$ as

$$\nabla^b F^{*\mathfrak{a}}{}_{ba} + C^{\mathfrak{a}}{}_{\mathfrak{bc}} A^{\mathfrak{b}a} F^{*\mathfrak{c}}{}_{ab} = 0. \tag{2.29}$$

Lastly, the associated energy-momentum tensor is:

$$T_{ab} = \delta_{\mathfrak{ab}} F^{\mathfrak{a}}{}_{ac} F^{\mathfrak{b}}{}_b{}^c - \frac{1}{4}\delta_{\mathfrak{ab}} F^{\mathfrak{a}}{}_{cd} F^{\mathfrak{b}cd} g_{ab}. \tag{2.30}$$

Chapter 3

Conformal wave equations for the Einstein-tracefree matter system

The material of this chapter is based on [10] and [12].

3.1 Introduction

A key step in the analysis involving a conformal formulation of the EFE is the so-called *procedure of hyperbolic reduction*, in which a subset of the field equations is cast in the form of a hyperbolic evolution system for which known techniques of the theory of PDEs allow to establish well-posedness. An important ingredient in the hyperbolic reduction is the choice of a gauge, which in the case of the conformal Einstein field equations involves not only fixing coordinates (the *coordinate gauge*) but also the representative of the conformal class of the spacetime metric to be considered (the *conformal gauge*). Naturally, gauge choices should bring to the fore the physical and geometric features of the setting under consideration. In order to make contact with the Einstein field equations, the procedure of hyperbolic reduction has to be supplemented by an argument concerning the *propagation of the constraints*, by means of which one identifies the conditions ensuring that a solution to the evolution system implies a solution to the full system of conformal equations, independently of the gauge choice. The propagation of the constraints involves the construction of a *subsidiary evolution system* describing the evolution of the conformal field equations and of the conditions representing the gauge. The construction of the subsidiary system requires lengthy manipulations of the equations which are underpinned by integrability conditions inherent to the field equations.

The MTCEFE constitute a simpler version of a more general conformal formulation of the EFE called the *extended conformal Einstein field equations*. Remarkably, until recently, there was no suitable hyperbolic reduction procedure available for the metric version of the conformal field equations. In [51], Paetz has obtained a satisfactory hyperbolic procedure for the metric vacuum conformal Einstein field equations which is based on the construction of second order wave equations. To round up his analysis, Paetz then proceeds to construct a system of subsidiary wave equations for tensorial fields encoding the conformal Einstein field equations (the so-called *geometric zero-quantities*) showing in this way the propagation of the constraints. The motivation behind Paetz's approach is that the use of second order hyperbolic equations gives access to a different part of the theory of PDEs which complements the results available for first order symmetric hyperbolic systems — see e.g. [20]. Paetz's construction of an evolution system consisting of wave equations has been adapted to the case of the spinorial conformal Einstein field equations in [34]. In addition to its interest in analytic considerations, the construction of wave equations for the metric conformal Einstein field equations is also of relevance in numerical studies, as the gauge fixing procedure and the particular form of the equations is more amenable to implementation in current mainstream numerical codes than other conformal formulations.

The purpose of this chapter is twofold: first, it generalises Paetz's construction of a system of wave equations for the conformal Einstein field equations to the case of tracefree matter models. As discussed in Section 2.2.1, this case is of particular interest since the equation of conservation satisfied by the energy-momentum tensor is conformally invariant. Moreover, the associated equations of motion for the matter fields can, usually, be shown to possess good conformal properties — see [60], Chapter 9. This is achieved by means of a set of integrability conditions for the subsidiary fields which sheds some light on the inner structure of Paetz's original construction. Second, the coupling of the three tracefree matter models introduced in Section 2.4 is analysed in detail following the same strategy as with the geometric fields.

3.2 The evolution system for the geometric fields

In this section we show how to construct an evolution system for the geometric fields appearing in the MTCEFE. These evolution equations take the form of *geometric wave equations* — that is, their principal part involves the D'Alambertian $\square \equiv \nabla_a \nabla^a$ associated to the conformal metric g_{ab}. In [51], Paetz has obtained a system of geometric wave equations for the set of conformal fields $(\Xi, s, L_{ab}, d^a{}_{bcd})$ in the vacuum case. The next statement generalises this result to tracefree matter:

Lemma 1. *The MTCEFE* (2.11a)-(2.11f) *imply the following system of geometric wave equations for the conformal fields:*

$$\square\Xi = 4s - \tfrac{1}{6}\Xi R, \tag{3.1a}$$

$$\square s = -\tfrac{1}{6}sR + \Xi L_{ab}L^{ab} - \tfrac{1}{6}\nabla_a R \nabla^a \Xi + \tfrac{1}{4}\Xi^5 T_{ab}T^{ab} - \Xi^3 L_{ab}T^{ab} + \Xi\nabla^a\Xi\nabla^b\Xi T_{ab}, \tag{3.1b}$$

$$\square L_{ab} = -2\Xi d_{acbd}L^{cd} + 4L_a{}^c L_{bc} - L_{cd}L^{cd}g_{ab} + \tfrac{1}{6}\nabla_a\nabla_b R + \tfrac{1}{2}\Xi^3 d_{acbd}T^{cd} - \Xi\nabla_c T_a{}^c{}_b - 2T_{(a|c|b)}\nabla^c\Xi, \tag{3.1c}$$

$$\begin{aligned}\square d_{abcd} = {} & -4\Xi d_a{}^f{}_{[c}{}^e d_{d]ebf} - 2\Xi d_a{}^f{}_b{}^e d_{cdfe} + \tfrac{1}{2}d_{abcd}R - T_{[a}{}^f \Xi^2 d_{b]fcd} \\ & - \Xi^2 T_{[c}{}^f d_{d]fab} - \Xi^2 g_{a[c}d_{d]gbf}T^{fg} + \Xi^2 g_{b[c}d_{d]gaf}T^{fg} + 2\nabla_{[a}T_{|cd|b]} \\ & + \epsilon_{abef}\nabla^f {}^*T_{cd}{}^e.\end{aligned} \tag{3.1d}$$

Proof. Equation (3.1a) is a direct consequence of (2.11a). Equations (3.1b) and (3.1c) result, respectively, from applying a covariant derivative to (2.11b) and (2.11c), and using the second Bianchi identity. The wave equation for $d^a{}_{bcd}$, on the other hand, requires to consider the alternative conformal field equation (2.13). Applying ∇^e to the latter and using equation (2.11d), together with the first Bianchi identity, a long but straightforward calculation yields the wave equation

$$\begin{aligned}\square d_{abcd} = & -4\Xi d_{a}{}^{f}{}_{[c}{}^{e} d_{d]ebf} - 2\Xi d_{a}{}^{f}{}_{b}{}^{e} d_{cdfe} \tfrac{1}{3} d_{abcd} R - 2 d_{cdf[a} L_{b]}{}^{f} \\ & - 2 d_{abf[c} L_{d]}{}^{f} - 2 g_{a[c} d_{d]ebf} L^{fe} + 2 g_{b[c} d_{d]fae} L^{ef} + 2\nabla_{[a} T_{|cd|b]} \\ & + \epsilon_{abef} \nabla^{f} {}^{*}T_{cd}{}^{e}. \end{aligned} \tag{3.2}$$

It is possible to eliminate terms containing L_{ab} from the wave equation (3.2) through the generalisation of an identity obtained in [51] to the case of tracefree matter. Multiplying equation (2.13) by Ξ, using the definitions of $d^{a}{}_{bcd}$ and ${}^{*}T_{abc}$, equation (2.11c) and the second Bianchi identity to simplify it, one finds that

$$d_{cd[ag} \nabla_{b]} \Xi + d_{de[ag} g_{b]c} \nabla^{e} \Xi - d_{ce[ag} g_{b]d} \nabla^{e} \Xi = 0. \tag{3.3}$$

Applying a further covariant derivative ∇^{g} to the last expression and making use of equations (2.11a), (2.11d) and (2.13) as well as the properties of the rescaled Cotton tensor, the following identity is obtained:

$$\begin{aligned} & 2\Xi d_{cdf[a} L_{b]}{}^{f} + 2\Xi d_{abf[c} L_{d]}{}^{f} + 2 g_{a[c} \Xi d_{d]gbf} L^{fg} - 2\Xi g_{b[c} d_{d]gaf} L^{fg} + \tfrac{1}{6} \Xi d_{abcd} R \\ & - \Xi^{3} d_{cdf[a} T_{b]}{}^{f} - \Xi^{3} d_{abf[c} T_{d]}{}^{f} - \Xi^{3} g_{a[c} d_{d]gbf} T^{fg} + \Xi^{3} g_{b[c} d_{d]gaf} T^{fg} = 0. \end{aligned} \tag{3.4}$$

By substituting this into expression (3.2) we readily get expression (3.1d).
□

Remark 13. In concrete applications it may prove useful to express the Schouten tensor in terms of the tracefree Ricci tensor and the Ricci scalar through the formula

$$L_{ab} = \Phi_{ab} + \tfrac{1}{24} R g_{ab}. \tag{3.5}$$

As will be discussed in Section 3.5.1, the Ricci scalar R is associated to the particular choice of conformal gauge. Thus, the decomposition (3.5) allows us to split the field L_{ab} into a gauge part and a part which is determined through the field equations. Keeping the simplicity of presentation in mind, we do not pursue this approach further as it leads to lengthier expressions.

3.3 Zero-quantities and integrability conditions

In this section we consider a convenient setting for the discussion and book-keeping of the evolution equations implied by the conformal Einstein field equations with matter. Our approach is based on the observation that the MTCEFE constitute an overdetermined system of differential conditions for the various conformal fields. Thus, the equations are related to each other through *integrability conditions*, i.e. necessary conditions for the existence of solutions to the equations.

3.3.1 Definitions and basic properties

First, we proceed to introduce the set of *geometric zero-quantities* (also called *subsidiary variables*) associated to the MTCEFE. These fields are defined as:

$$\Upsilon_{ab} \equiv \nabla_a \nabla_b \Xi + \Xi L_{ab} + s g_{ab} - \tfrac{1}{2}\Xi^3 T_{ab}, \tag{3.6a}$$

$$\Theta_a \equiv \nabla_a s + L_{ac}\nabla^c \Xi - \tfrac{1}{2}\Xi^2 \nabla^c \Xi T_{ac}, \tag{3.6b}$$

$$\Delta_{abc} \equiv \nabla_a L_{bc} - \nabla_b L_{ac} - \nabla_a \Xi d^a{}_{cab} - \Xi T_{abc}, \tag{3.6c}$$

$$\Lambda_{abc} \equiv T_{bca} - \nabla_e d^e{}_{abc}, \tag{3.6d}$$

$$Z \equiv \lambda - 6\Xi s + 3\nabla_c \Xi \nabla^c \Xi, \tag{3.6e}$$

$$P^c{}_{dab} \equiv R^c{}_{dab} - \Xi d^c{}_{dab} - 2(\delta^c{}_{[a} L_{b]d} - g_{d[a} L_{b]}{}^c). \tag{3.6f}$$

In terms of the above, the system (2.11a)-(2.11f) can be expressed as the conditions

$$\Upsilon_{ab} = 0, \qquad \Theta_a = 0, \qquad \Delta_{abc} = 0, \qquad \Lambda_{abc} = 0, \qquad Z = 0, \qquad P^c{}_{dab} = 0,$$

from where these fields take their name.

3.3.1.1 Properties of the zero-quantities

By definition, the zero-quantities possess the following symmetries:

$$\Upsilon_{ab} = \Upsilon_{(ab)}, \quad \Delta_{abc} = \Delta_{[ab]c}, \quad \Delta_{[abc]} = 0, \quad \Lambda_{abc} = \Lambda_{a[bc]}, \quad \Lambda_{[abc]} = 0,$$
$$\Delta_a{}^b{}_b = 0, \quad \Lambda^b{}_{ab} = 0. \tag{3.7}$$

Moreover, one can check that Δ_{abc} and Λ_{abc} satisfy the identities

$$\Delta_{abc} = \tfrac{2}{3}\Delta_{abc} + \tfrac{1}{3}\Delta_{acb} - \tfrac{1}{3}\Delta_{bca}, \qquad \Lambda_{abc} = \tfrac{2}{3}\Lambda_{abc} + \tfrac{1}{3}\Lambda_{bac} - \tfrac{1}{3}\Lambda_{cab}, \tag{3.8}$$

which are useful for simplifying certain combinations of zero-quantities. Regarding $P^a{}_{bcd}$, it inherits the symmetries of the Riemann tensor; in particular, we can define its Hodge dual tensors

$$^*P_{abcd} \equiv \tfrac{1}{2}\epsilon_{ab}{}^{ef}P_{efcd}, \quad P^*_{abcd} \equiv \tfrac{1}{2}\epsilon_{cd}{}^{ef}P_{abef}, \tag{3.9}$$

satisfying $^*P_{abcd} = P^*_{abcd}$.

In addition, it will result useful to introduce a further auxiliary zero-quantity associated to equation (2.13) — see Remark 8:

$$\Lambda_{abcde} \equiv 3\nabla_{[a}d_{bc]de} + \epsilon_{abcf}{}^*T_{de}{}^f = 3\Lambda_{d[ab}g_{c]e} - 3\Lambda_{e[ab}g_{c]d}. \tag{3.10}$$

Here the second equality has been obtained through a calculation similar to the one yielding (2.13). From the above definition it follows that $\Lambda_{ab}{}^d{}_{cd} = \Lambda_{abc}$, as well as

$$\Lambda_{abcde} = \Lambda_{[abc]de}, \qquad \Lambda_{abcde} = \Lambda_{abc[de]}. \tag{3.11}$$

3.3.1.2 Some consequences of the wave equations

Key for our subsequent analysis is the observation that assuming the validity of the geometric wave equations for the conformal fields implies a further set of relations satisfied by the zero-quantities. These are summarised in the following lemma:

Lemma 2. *Assume that the wave equations* (2.14), (3.1a)-(3.1d)*, and Assumption 1 hold. Then the geometric zero-quantities satisfy the identities*

$$\Upsilon_a{}^a = 0, \tag{3.12a}$$

$$P^a{}_{bac} = 0, \tag{3.12b}$$

$$\nabla_b \Upsilon_a{}^b = 3\Theta_a, \tag{3.12c}$$

$$\nabla_a \Theta^a = \Upsilon^{ab} L_{ab} - \tfrac{1}{2}\Xi^2 \Upsilon^{ab} T_{ab}, \tag{3.12d}$$

$$\nabla_c \Delta_a{}^c{}_b = \Upsilon^{cd} d_{acbd} + \Lambda_{abc} \nabla^c \Xi - L^{cd} P_{acbd}, \tag{3.12e}$$

$$\nabla_c \Delta_{ab}{}^c = 2\Xi T_{c[a} \Upsilon_{b]}{}^c - \Lambda_{cab} \nabla^c \Xi, \tag{3.12f}$$

$$\nabla_c \Lambda^c{}_{ab} = d_{[a}{}^{cde} P_{b]cde} - 2T_{c[a} \Upsilon_{b]}{}^c, \tag{3.12g}$$

$$\nabla_c \Lambda_{[ab]}{}^c = 2d_{[a}{}^{cde} P_{b]dec}, \tag{3.12h}$$

$$\nabla_d P_{abc}{}^d = -\Delta_{abc} - \Xi \Lambda_{cab}, \tag{3.12i}$$

$$\begin{aligned}\nabla_c \Lambda_{eg}{}^c{}_{mn} = {} & 2\nabla_{[e} \Lambda_{g]mn} + 2d_{[e}{}^c{}_{|m|}{}^h P_{g]cnh} - 2d_{[e}{}^c{}_{|n|}{}^h P_{g]cmh} \\ & + 2d_{mn}{}^{ch} P_{ecgh}.\end{aligned} \tag{3.12j}$$

Proof. The result follows directly from the definitions of the zero-quantities with the aid of the wave equations for the conformal fields (2.14) and (3.1a)-(3.1d), the second Bianchi identity and the properties of the rescaled Cotton tensor. It is worth mentioning that (3.12j) is obtained by using (3.2) instead of (3.1d) as it greatly simplifies the calculation. □

3.3.2 Integrability conditions

The zero-quantities are not independent of each other but they are related via a set of identities, the so-called *integrability conditions*. These relations are key for the computation of a suitable (subsidiary) system of wave equations for the zero-quantities. The procedure to obtain these relations is to compute suitable antisymmetrised covariant derivatives of the zero-quantities which, in turn, are expressed in terms of lower order objects. Following this general strategy we obtain the following:

Proposition 4. *The zero-quantities,* (3.6a)-(3.6f)*, satisfy the identities*

$$2\nabla_{[a}\Upsilon_{c]b} = 2g_{b[a}\Theta_{c]} + \Xi\Delta_{acb} + P_{acbd}\nabla^d\Xi, \tag{3.13a}$$

$$2\nabla_{[a}\Theta_{b]} = -2L_{[a}{}^c\Upsilon_{b]c} + \Delta_{abc}\nabla^c\Xi + \Xi^2 T_{c[a}\Upsilon_{b]}{}^c, \tag{3.13b}$$

$$3\nabla_{[d}\Delta_{ab]c} = \Lambda_{abdce}\nabla^e\Xi + 3\Upsilon_{[a}{}^e d_{bd]ce} + 3L_{[a}{}^e P_{bd]ce} - \tfrac{3}{2}\Xi^2 P_{[ab|c|}{}^e T_{d]e} + 2\Xi\Upsilon_{[a}{}^e g_{b|c|}T_{d]e} + \Xi\Upsilon_{[a}{}^e g_{|c|b}T_{d]e}, \tag{3.13c}$$

$$\nabla_a Z = -6\Xi\Theta_a + 6\Upsilon_{ab}\nabla^b\Xi, \tag{3.13d}$$

$$3\nabla_{[e}P_{gh]mn} = \Xi\Lambda_{eghnm} - 3\Delta_{[eg|m|}g_{h]n} + 3\Delta_{[eg|n|}g_{h]m}. \tag{3.13e}$$

Proof. Equations (3.13a)-(3.13d) follow from direct calculations employing the definitions of the zero-quantities, the rescaled Cotton tensor and the first Bianchi identity. Equation (3.13e), on the other hand, can be obtained in a similar manner as (2.13): multiplying (3.12i) by $\epsilon_{mn}{}^{cd}$ and exploiting the properties of its Hodge dual tensors — see expressions in (3.9) — yields

$$2\nabla^*_a P_{mnba} = 2\nabla_a P^*_{mnab} = -\Xi\epsilon_{mnac}(\Lambda_b{}^{ac} + \Delta_b{}^{ac}). \tag{3.14}$$

By substituting back the definition of P^*_{mnab}, equation (3.13e) is found after some simplifications. □

Remark 14. Observe that the above relations have right-hand sides consisting of lower order expressions in which one or more positive powers of the zero-quantities appear on each term. In the remainder of the thesis, equations having this property will be said to be *homogeneous* in the zero-quantities. This fact will be key when suitable wave equations for these fields are derived in the next section. Equations (3.13a)-(3.13e) together with (3.12j) constitute the set of integrability conditions for the zero-quantities associated to the MTCEFE.

3.4 The subsidiary evolution system for the zero-quantities

An important aspect of any *hyperbolic reduction procedure* for a conformal formulation of the EFE is the identification of the conditions upon which a solution to the (reduced) evolution equations implies a solution to the full set of field equations — this type of analysis is generically known as the *propagation of the constraints*. In practice, the propagation of the constraints requires the construction of a suitable system of evolution equations for the zero-quantities associated to the field equations.

3.4.1 Construction of the subsidiary system

In this section it is shown how the set of integrability conditions provides a systematic and direct way to obtain wave equations for the zero-quantities — a so-called *subsidiary evolution system*. The propagation of the constraints then follows from the structural properties of the subsidiary system as a consequence of the uniqueness of solutions to systems of wave equations.

3.4.1.1 Equations for Υ_{ab}, Θ_a, Δ_{abc}, Z and P_{abcd}

Equation (3.13a) serves as the starting point to obtain a wave equation for Υ_{ab}. After applying ∇^c and commuting derivatives, equation (3.12c) renders it as a suitable wave equation. Remaining first order derivatives can be rewritten and simplified via equations (3.8), (3.12i), (3.12a), (3.12d) and (3.12e), resulting in:

$$\begin{aligned}\Box\Upsilon_{ab} =& \tfrac{1}{6}\Upsilon_{ab}R - 2\Upsilon^{cd}L_{cd}g_{ab} + \tfrac{1}{2}\Xi^2\Upsilon^{cd}g_{ab}T_{cd} + 4\nabla_{(a}\Upsilon_{b)} - 2\Xi\Upsilon^{cd}d_{(a|c|b)d} \\ &+ 4\Upsilon_{(a}{}^{c}L_{b)c} - 2\Upsilon^{cd}P_{(a|c|b)d} + 2\Xi L^{cd}P_{(a|c|b)d} - \tfrac{1}{2}\Xi^3 P_{(a}{}^{c}{}_{b)}{}^{d}T_{cd}.\end{aligned} \tag{3.15}$$

Regarding Θ_a, an analogous calculation using expression (3.13b) in conjunction with the same equations as in the previous case leads directly to a wave equation for this field. Exploiting (2.11c), (2.8d) and (3.13a) to simplify it, one obtains

$$\begin{aligned}\Box\Theta_c =& 6L_{ca}\Theta^a - 2\Upsilon^{ab}\Delta_{cab} + 2\Xi L^{ab}\Delta_{cab} - \Xi^3\Delta_c{}^{ab}T_{ab} - 2\Xi^2\Theta^a T_{ca} \\ &- 2\Upsilon^{bd}d_{cbad}\nabla^a\Xi + \tfrac{3}{2}\Xi\Upsilon_c{}^{b}T_{ab}\nabla^a\Xi + \tfrac{1}{2}\Xi^2 P_{cbad}T^{bd}\nabla^a\Xi - \tfrac{1}{6}\Upsilon_{ca}\nabla^a R \\ &+ \tfrac{1}{2}\Xi\Upsilon_a{}^{b}T_{cb}\nabla^a\Xi - \tfrac{5}{2}\Xi\Upsilon^{ab}T_{ab}\nabla_c\Xi + 2\Upsilon^{ab}\nabla_c L_{ab} - \Xi^2\Upsilon^{ab}\nabla_c T_{ab}.\end{aligned} \tag{3.16}$$

A wave equation for Δ_{abc} can be obtained by applying ∇^d to the integrability condition (3.13c), commuting derivatives and using (3.12e) to eliminate the second order derivatives. A direct but long calculation exploiting the same relations used in the previous two cases, along with (2.11d) and (3.10), yields

$$\begin{aligned}\square\Delta_{abc} = {} & 2\Lambda_{cab}s - \Upsilon_c{}^d T_{abd} - \Xi\Lambda_{abdce}L^{de} + 3d_{abcd}\Theta^d + \tfrac{1}{3}R\Delta_{abc} + L_c{}^d\Delta_{abd} \\ & + \tfrac{1}{2}\Xi^3\Lambda_{abdce}T^{de} - \Xi P_{abce}T_d{}^e\nabla^d\Xi + \tfrac{1}{6}P_{abcd}\nabla^d R + \nabla^d\Xi\nabla_e\Lambda_{ab}{}^e{}_{cd} \\ & + 2\Upsilon^{de}\nabla_e d_{abcd} + L^{de}\nabla_e P_{abcd} - \tfrac{1}{2}\Xi^2 T^{de}\nabla_e P_{abcd} + 2\Upsilon_{[a}{}^d T_{b]cd} \\ & - \Xi\Upsilon_{[a}{}^d\nabla_{|c|}T_{b]d} - 2\Xi d_{[a}{}^d{}_{b]}{}^e\Delta_{dec} + 2\Xi d_{[a}{}^d{}_{|c|}{}^e\Delta_{b]de} + 2d_{[a}{}^d{}_{|c|}{}^e\nabla_{b]}\Upsilon_{de} \\ & - 2d_{[a}{}^d{}_{|c}{}^e\nabla_{d|}\Upsilon_{b]e} - 2L_{[a}{}^d\Delta_{b]dc} + 2L^{de}\nabla_{[a}P_{b]dce} - 2P_{[a}{}^d{}_{b]}{}^e\Delta_{dec} \\ & + 2P_{[a}{}^d{}_{|c|}{}^e\Delta_{b]de} - 2P_{[a}{}^d{}_{|c|}{}^e\nabla_{b]}L_{de} - 2P_{[a}{}^d{}_{|c}{}^e\nabla_{d|}L_{b]e} + \Xi^2 P_{[a}{}^d{}_{|c}{}^e\nabla_{d|}T_{b]e} \\ & - \Xi^2\Delta_c{}^d{}_{[a}T_{b]d} + \Xi T_{[a}{}^d\nabla_{|c|}\Upsilon_{b]d} - 2\nabla^d\Xi\nabla_{[a}\Lambda_{b]cd} + 2\Upsilon^{de}T_{[a|de|}g_{b]c} \\ & \Xi\Upsilon^{de}g_{[a|c}\nabla_{d|}T_{b]e} - \Upsilon_{[a}{}^d T_{b]d}\nabla_c\Xi - 2L^{de}\Delta_{[a|de|}g_{b]c} + 3\Xi\Upsilon^d g_{[a|c|}T_{b]d} \\ & + 2\Xi P_{[a}{}^d{}_{|c|}{}^e T_{b]e}\nabla_d\Xi - \Xi g_{[a|c}T^{de}\nabla_{d|}\Upsilon_{b]e} + \Upsilon_{[a}{}^d g_{b]c}T_d{}^e\nabla_e\Xi \\ & + \Upsilon^{de}g_{[a|c|}T_{b]d}\nabla_e\Xi. \end{aligned} \tag{3.17}$$

A wave equation for Z is readily found by simply applying ∇^a to equation (3.13d):

$$\square Z = 6\Upsilon_{ab}\Upsilon^{ab} - 12\Xi\Upsilon^{ab}L_{ab} + 6\Xi^3\Upsilon^{ab}T_{ab} + 12\Theta^a\nabla_a\Xi. \tag{3.18}$$

In the case of P_{abcd}, application of ∇^h together with equations (3.12b), (3.12e), (3.12i), as well as the various symmetries of Λ_{abc} and $P^a{}_{bcd}$ results, after a rather direct calculation, in:

$$\begin{aligned}\square P_{egmn} = {} & \tfrac{1}{3}RP_{egmn} - 2L_{[m}{}^h P_{n]heg} + 2\Lambda_{[n|eg|}\nabla_{m]}\Xi + 2\Xi\nabla_{[m}\Lambda_{n]eg} + 2\nabla_{[m}\Delta_{|eg|n]} \\ & + 2\Xi\nabla_{[e}\Lambda_{g]mn} + 2\nabla_{[e}\Delta_{|mn|g]} - 2\Lambda_{[e|mn|}\nabla_{g]}\Xi - 2\Xi d_{[e}{}^h{}_{g]}{}^a P_{mnha} \\ & - 2\Xi d_{[e}{}^h{}_{|m|}{}^a P_{g]hna} + 2\Xi d_{[e}{}^h{}_{|n|}{}^a P_{g]hma} - 2L_{[e}{}^h P_{g]hmn} - 2P_{[e}{}^h{}_{g]}{}^a P_{mnha} \\ & - 4P_{[e}{}^h{}_{|m|}{}^a P_{g]hna} + 2\Xi g_{[e|m}\nabla^h\Lambda_{n|g]h} - 2\Xi g_{[e|n}\nabla^h\Lambda_{m|g]h} \\ & + 2\Upsilon^{ha}d_{[e|hma|}g_{g]n} - 2\Upsilon^{ha}d_{[e|hna|}g_{g]m} + 2\Lambda_{[g|nh|}g_{e]m} + 2\Lambda_{n[g|h|}g_{e]m} \\ & + 2\Lambda_{m[e|h|}g_{g]n} + 2\Lambda_{[e|mh|}g_{g]n} - 4L^{ha}P_{[e|hma|}g_{g]n} + 4L^{ha}P_{[e|hna|}g_{g]m}. \end{aligned} \tag{3.19}$$

3.4.1.2 Equation for Λ_{abc}

Notice that the integrability condition for Λ_{abc}, equation (3.12j), contains derivatives of the zero-quantities on both sides of the equation. This feature seems to hinder our standard approach for the construction of a subsidiary equation. In order to construct a suitable wave equation it will be necessary to use the symmetries of Λ_{abcde}. Applying ∇^e to the integrability condition (3.12j) and commuting derivatives leads to

$$\begin{aligned}
\square\Lambda_{gmn} = {} & \Lambda^c{}_{mn}R_{gc} + \nabla_g\nabla_c\Lambda^c{}_{mn} - 2P_g{}^{ceh}\nabla_h d_{mnce} - 2d_{mn}{}^{ce}\nabla_h P_{gce}{}^h \\
& - \nabla^c\nabla^e\Lambda_{gce[mn]} - 2\Lambda^c{}_{[m}{}^e R_{|gc|n]e} - 2d_{[m}{}^{ceh}\nabla_{|e}P_{gh|n]c} \\
& - 2d_g{}^c{}_{[m}{}^e\nabla^h P_{n]ech} - 2P_{[m}{}^{ceh}\nabla_{|e}d_{gh|n]c} - 2P_g{}^c{}_{[m}{}^e\nabla^h d_{n]ech}.
\end{aligned}$$

Here, the double-derivative terms put at risk the hyperbolicity of the system. For the second derivative of Λ_{abc} one can use (3.12g), while the one involving Λ_{abcde} can be eliminated by recalling that this field is antisymmetric under any interchange of the first three indices — see (3.11). Exploiting this property and commuting derivatives one obtains

$$\begin{aligned}
\square\Lambda_{gmn} = {} & -\Xi\Lambda^c{}_g{}^e d_{mnce} + 4\Lambda^c{}_{mn}L_{gc} + 2d_{mnce}\Delta_g{}^{ce} - 2P_g{}^{ceh}\nabla_h d_{mnce} \\
& + 2\Upsilon_{[m}{}^c\nabla_{|g|}T_{n]c} - 2\Xi\Lambda^c{}_{[m}{}^e d_{|g|n]ce} - 4\Xi\Lambda^c{}_{[m}{}^e d_{|ge|n]c} - 4\Lambda^c{}_{g[m}L_{n]c} \\
& + 2\Lambda_{[m}{}^{ce}P_{|gc|n]e} + 2\Lambda^c{}_g{}^e P_{[m|c|n]e} - 2T_{[m}{}^{ce}P_{|ge|n]c} + 2d_g{}^c{}_{[m}{}^e\Delta_{n]ec} \\
& - 2d_{[m}{}^{ceh}\nabla_{|e}P_{gh|n]c} - 2P_{[m}{}^{ceh}\nabla_{|e}d_{gh|n]c} - 2T_{[m}{}^c\nabla_{|g|}\Upsilon_{n]c} \\
& - \Xi\Lambda^{ceh}d_{[m|ceh}g_{g|n]} - 4\Lambda^c{}_{[m}{}^e L_{|ce}g_{g|n]} - \Lambda^{ceh}P_{[m|ceh}g_{g|n]}. \qquad (3.20)
\end{aligned}$$

Remark 15. The expressions in Lemma 2 and Proposition 4 allow us to show, in particular, that the wave equations (3.1d) and (3.2) differ from each other by a homogeneous combination of zero-quantities. Thus, in arguments involving the propagation of the constraints, both forms of the evolution equation can be used interchangeably.

The results of this section can be summarised in the following lemma:

Lemma 3. *Assume that the conformal fields satisfy equations* (2.14) *and* (3.1a)-(3.1d). *Then, the zero-quantities* (3.6a)-(3.6f) *satisfy the homogeneous system of geometric wave equations* (3.15)-(3.20).

3.4.2 Propagation of the constraints

As it will be discussed in detail in Section 3.5, the system of geometric wave equations (3.15)-(3.20) implies, in turn, a system of proper (hyperbolic) wave equations for which a theory of the existence and uniqueness of solutions is readily available — see e.g. [43]. From the latter, one directly obtains the following result:

Proposition 5. *Assume that the zero-quantities* Υ_{ab}, Θ_a, Λ_{abc}, Δ_{abc}, Z, $P^a{}_{bcd}$ *and their first derivatives vanish on a fiduciary spacelike hypersurface* $\mathcal{S}_\star$ *of an unphysical spacetime* $(\mathcal{M}, \boldsymbol{g})$*. Then, the zero-quantities vanish on the domain of dependence* $D(\mathcal{S}_\star)$ *of* $\mathcal{S}_\star$*.*

Remark 16. Working, for example, with coordinates adapted to the hypersurface $\mathcal{S}_\star$, it can be readily checked that the completely spatial parts of the zero-quantities Υ_{ab}, Θ_a, Λ_{abc}, Δ_{abc}, Z and $P^a{}_{bcd}$ encode the same information as the conformal Einstein constraint equations — see e.g. [60], Chapter 11. Similarly, projections with a transversal (i.e. timelike) component can be read as a first order evolution system for the geometric conformal fields — we ignore null components as these can be obtained as linear combinations of transversal and intrinsic components. Thus, in order to ensure the vanishing of the zero-quantities on the initial hypersurface $\mathcal{S}_\star$, one needs, firstly, to produce a solution to the conformal constraint equations; this ensures the vanishing of the spatial part of the zero-quantities. Secondly, one reads the transversal components of the zero-quantities as definitions for the normal derivatives of the conformal fields which can be readily computed from the solution to the conformal constraints. In this way, the transversal components of the zero-quantities vanish *a fortiori*.

3.5 Gauge considerations

The MTCEFE possess both a coordinate and a conformal freedom which can be exploited to cast the geometric wave equations (2.14) and (3.1a)-(3.1d) as satisfactory hyperbolic evolution equations.

3.5.1 Conformal gauge source functions

In the following, the Ricci scalar R of the metric g_{ab} will be regarded as a *conformal gauge source* specifying the representative in the conformal class $[\tilde{\boldsymbol{g}}]$ one is working with. Recall that given two conformally related metrics g_{ab} and g'_{ab} such that $g'_{ab} = \vartheta^2 g_{ab}$, their respective Ricci scalars are related to each other via

$$R\vartheta - R'\vartheta^3 = 6\nabla^c\nabla_c\vartheta.$$

If the values of R and R' are prescribed, the above transformation law can be recast as a wave equation for the conformal factor relating the two metrics. Namely, one has that

$$\Box\vartheta = \frac{1}{6}\vartheta(R - R'\vartheta^2).$$

Given suitable initial data for this wave equation, it can always be locally solved. Accordingly, it is always possible to find (locally) a conformal rescaling such that the metric g'_{ab} has a prescribed Ricci scalar R'.

Remark 17. Based on the previous discussion, in what follows the Ricci scalar of the metric g_{ab} is regarded as a prescribed function $\mathcal{R}(x)$ of the coordinates, so one writes

$$R = \mathcal{R}(x).$$

3.5.2 Generalised harmonic coordinates and the reduced Ricci operator

The components of the Ricci tensor R_{ab} can be explicitly written in terms of the components of the metric tensor g_{ab} in general coordinates $x = (x^\mu)$ as

$$R_{\mu\nu} = -\frac{1}{2}g^{\lambda\rho}\partial_\lambda\partial_\rho g_{\mu\nu} + g_{\sigma(\mu}\nabla_{\nu)}\Gamma^\sigma + g_{\lambda\rho}g^{\sigma\tau}\Gamma^\lambda{}_{\sigma\mu}\Gamma^\rho{}_{\tau\nu} + 2\Gamma^\sigma{}_{\lambda\rho}g^{\lambda\tau}g_{\sigma(\mu}\Gamma^\rho{}_{\nu)\tau},$$

with

$$\Gamma^\nu{}_{\mu\lambda} \equiv \frac{1}{2}g^{\nu\rho}(\partial_\mu g_{\rho\lambda} + \partial_\lambda g_{\mu\rho} - \partial_\rho g_{\mu\lambda}),$$

where we have defined the *contracted Christoffel symbols* as $\Gamma^\nu \equiv g^{\mu\lambda}\Gamma^\nu{}_{\mu\lambda}$. A direct computation then gives $\Box x^\mu = -\Gamma^\mu$. Following the well-known procedure for the hyperbolic reduction of the EFE, we introduce *coordinate gauge source functions* $\mathcal{F}^\mu(x)$ to prescribe the value of the contracted Christoffel symbols via the condition $\Gamma^\mu = \mathcal{F}^\mu(x)$, This means that the coordinates $x = (x^\mu)$ satisfy the *generalised wave coordinate condition*

$$\square x^\mu = -\mathcal{F}^\mu(x); \tag{3.21}$$

see e.g. [15, 56, 60].

Associated to the latter, it is convenient to define the *reduced Ricci operator* $\mathscr{R}_{\mu\nu}[\boldsymbol{g}]$ as

$$\mathscr{R}_{\mu\nu}[\boldsymbol{g}] \equiv R_{\mu\nu} - g_{\sigma(\mu}\nabla_{\nu)}\Gamma^\sigma + g_{\sigma(\mu}\nabla_{\nu)}\mathcal{F}^\sigma(x). \tag{3.22}$$

More explicitly, one has that

$$\mathscr{R}_{\mu\nu}[\boldsymbol{g}] = -\frac{1}{2}g^{\lambda\rho}\partial_\lambda\partial_\rho g_{\mu\nu} - g_{\sigma(\mu}\nabla_{\nu)}\mathcal{F}^\sigma(x) + g_{\lambda\rho}g^{\sigma\tau}\Gamma^\lambda{}_{\sigma\mu}\Gamma^\rho{}_{\tau\nu} + 2\Gamma^\sigma{}_{\lambda\rho}g^{\lambda\tau}g_{\sigma(\mu}\Gamma^\rho{}_{\nu)\tau}.$$

Thus, by choosing coordinates satisfying the generalised wave coordinate condition (3.21), the unphysical Einstein equation (2.14) takes the form

$$\mathscr{R}_{\mu\nu}[\boldsymbol{g}] = 2L_{\mu\nu} + \frac{1}{6}\mathcal{R}(x)g_{\mu\nu}. \tag{3.23}$$

Assuming that the components $L_{\mu\nu}$ are known, the latter is a quasilinear wave equation for the components of the metric tensor.

3.5.2.1 The reduced wave operator

The geometric wave operator $\square$ acting on tensorial fields contains derivatives of the Christoffel symbols which, in turn, contain second order derivatives of the components of the metric tensor. The presence of these second order derivative terms is problematic as the metric is an unknown in the problem, destroying, in principle, the hyperbolicity of the evolution equations (3.1c) and (3.1d). In what follows, it will be shown how the generalised wave coordinate condition (3.21) can be used to reduce the geometric wave operator $\square$ to a second order hyperbolic operator.

To motivate the procedure, consider a covector ω_a with components ω_μ with respect to a coordinate system $x = (x^\mu)$ satisfying condition (3.21) for some choice of coordinate gauge source functions $\mathcal{F}^\mu(x)$. A direct computation using the expression of the covariant derivative in terms of Christoffel symbols yields

$$\square\omega_\lambda \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu\omega_\lambda = g^{\mu\nu}\partial_\mu\partial_\nu\omega_\lambda - g^{\mu\nu}\partial_\mu\Gamma^\sigma{}_{\nu\lambda}\omega_\sigma + f_\lambda(\boldsymbol{g},\partial\boldsymbol{g},\boldsymbol{\omega},\partial\boldsymbol{\omega}),$$

where $f_\lambda(g,\partial g,\omega,\partial\omega)$ denotes an expression depending on the components $g_{\mu\nu}$, ω_μ and their first order partial derivatives. Now, recall the classical expression for the components of the Riemann tensor in terms of the Christoffel symbols and their derivatives,

$$R^\sigma{}_{\mu\lambda\nu} = \partial_\lambda\Gamma^\sigma{}_{\nu\mu} - \partial_\nu\Gamma^\sigma{}_{\lambda\mu} + \Gamma^\sigma{}_{\lambda\tau}\Gamma^\tau{}_{\nu\mu} - \Gamma^\sigma{}_{\nu\tau}\Gamma^\tau{}_{\lambda\mu},$$

so that

$$R^\sigma{}_\lambda = g^{\mu\nu}R^\sigma{}_{\mu\lambda\nu} = g^{\mu\nu}\partial_\lambda\Gamma^\sigma{}_{\nu\mu} - g^{\mu\nu}\partial_\nu\Gamma^\sigma{}_{\lambda\mu} + g^{\mu\nu}\Gamma^\sigma{}_{\lambda\tau}\Gamma^\tau{}_{\nu\mu} - g^{\mu\nu}\Gamma^\sigma{}_{\nu\tau}\Gamma^\tau{}_{\lambda\mu}.$$

Using this coordinate expression one obtains

$$\begin{aligned}\square\omega_\lambda &= g^{\mu\nu}\partial_\mu\partial_\nu\omega_\lambda + \big(R^\sigma{}_\lambda - g^{\mu\nu}\partial_\lambda\Gamma^\sigma{}_{\nu\mu}\big)\omega_\sigma + f_\lambda(\boldsymbol{g},\partial\boldsymbol{g},\boldsymbol{\omega},\partial\boldsymbol{\omega})\\ &= g^{\mu\nu}\partial_\mu\partial_\nu\omega_\lambda + \big(R^\sigma{}_\lambda - \partial_\lambda\Gamma^\sigma\big)\omega_\sigma + f_\lambda(\boldsymbol{g},\partial\boldsymbol{g},\boldsymbol{\omega},\partial\boldsymbol{\omega})\\ &= g^{\mu\nu}\partial_\mu\partial_\nu\omega_\lambda + \big(R_{\tau\lambda} - g_{\sigma\tau}\partial_\lambda\Gamma^\sigma\big)\omega^\tau + f_\lambda(\boldsymbol{g},\partial\boldsymbol{g},\boldsymbol{\omega},\partial\boldsymbol{\omega}),\end{aligned}$$

and finally

$$\square\omega_\lambda = g^{\mu\nu}\partial_\mu\partial_\nu\omega_\lambda + \big(R_{\tau\lambda} - g_{\sigma\tau}\nabla_\lambda\Gamma^\sigma\big)\omega^\tau + f_\lambda(\boldsymbol{g},\partial\boldsymbol{g},\boldsymbol{\omega},\partial\boldsymbol{\omega}). \tag{3.24}$$

Making the formal replacements

$$R_{\mu\nu} \mapsto 2L_{\mu\nu} + \frac{1}{6}\mathcal{R}(x)g_{\mu\nu}, \qquad \Gamma^\mu \mapsto \mathcal{F}^\mu(x)$$

in equation (3.24), one defines the *reduced wave operator* $\blacksquare$ acting on the components ω_μ as

$$\begin{aligned}\blacksquare\omega_\lambda \equiv g^{\mu\nu}\partial_\mu\partial_\nu\omega_\lambda &+ \left(2L_{\tau\lambda} + \frac{1}{6}\mathcal{R}(x)g_{\tau\lambda} - g_{\sigma\tau}\nabla_\lambda\mathcal{F}^\sigma(x)\right)\omega^\tau\\ &+ f_\lambda(\boldsymbol{g},\partial\boldsymbol{g},\boldsymbol{\omega},\partial\boldsymbol{\omega}),\end{aligned}$$

where $f_\lambda(\boldsymbol{g}, \partial\boldsymbol{g}, \boldsymbol{\omega}, \partial\boldsymbol{\omega})$ denotes lower order terms whose explicit form will not be required. In fact, from the previous discussion it follows that one can write

$$\blacksquare\omega_\lambda = \square\omega_\lambda + \left((2L_{\tau\lambda} + \frac{1}{6}\mathcal{R}(x)g_{\tau\lambda} - R_{\tau\lambda}) - g_{\sigma\tau}\nabla_\lambda(\mathcal{F}^\sigma(x) - \Gamma^\sigma)\right)\omega^\tau.$$

A similar construction for covariant tensors of arbitrary rank results in the following:

Definition 4. *The reduced wave operator* $\blacksquare$ *acting on a covariant tensor field* $T_{\lambda\cdots\rho}$ *is defined as*

$$\blacksquare T_{\lambda\cdots\rho} \equiv \square T_{\lambda\cdots\rho} + \left((2L_{\tau\lambda} + \frac{1}{6}\mathcal{R}(x)g_{\tau\lambda} - R_{\tau\lambda}) - g_{\sigma\tau}\nabla_\lambda(\mathcal{F}^\sigma(x) - \Gamma^\sigma)\right)T^\tau{}_{\cdots\rho} + \cdots$$
$$\cdots + \left((2L_{\tau\rho} + \frac{1}{6}\mathcal{R}(x)g_{\tau\rho} - R_{\tau\rho}) - g_{\sigma\tau}\nabla_\rho(\mathcal{F}^\sigma(x) - \Gamma^\sigma)\right)T_{\lambda\cdots}{}^\tau,$$

where $\square \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$*. The action of* $\blacksquare$ *on a scalar* f *is simply given by*

$$\blacksquare f \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu f.$$

Remark 18. The operator $\blacksquare$ provides a proper second order hyperbolic operator for systems which involve the metric as an unknown, in contrast to $\square$. Accordingly, when working in generalised harmonic coordinates, all the second order derivatives of the metric tensor can be removed from the principal part of geometric wave equations. A system of evolution equations expressed in terms of the reduced wave operator $\blacksquare$ (rather than in terms of the geometric wave operator $\square$) will be said to be *proper*.

3.5.3 Summary: gauge reduced evolution equations

The previous discussion leads us to consider the following *gauge reduced* system of evolution equations for the components of the conformal fields Ξ, s, L_{ab}, d_{abcd} and g_{ab} with respect to coordinates $x = (x^\mu)$ satisfying the generalised wave coordinate condition (3.21):

$$\blacksquare \Xi = 4s - \frac{1}{6}\Xi \mathcal{R}(x), \tag{3.25a}$$

$$\blacksquare s = -\tfrac{1}{6}s\mathcal{R}(x) + \Xi L_{\mu\nu}L^{\mu\nu} - \tfrac{1}{6}\nabla_\mu \mathcal{R}(x)\nabla^\mu \Xi + \tfrac{1}{4}\Xi^5 T_{\mu\nu}T^{\mu\nu} - \Xi^3 L_{\mu\nu}T^{\mu\nu}$$
$$+\Xi\nabla^\mu\Xi\nabla^\nu\Xi T_{\mu\nu}, \tag{3.25b}$$

$$\blacksquare L_{\mu\nu} = -2\Xi d_{\mu\rho\nu\lambda}L^{\rho\lambda} + 4L_\mu{}^\lambda L_{\nu\lambda} - L_{\lambda\rho}L^{\lambda\rho}g_{\mu\nu} + \tfrac{1}{6}\nabla_\mu\nabla_\nu\mathcal{R}(x)$$
$$+\tfrac{1}{2}\Xi^3 d_{\mu\lambda\nu\rho}T^{\lambda\rho} - \Xi\nabla_\lambda T_\mu{}^\lambda{}_\nu - 2T_{(\mu|\lambda|\nu)}\nabla^\lambda\Xi, \tag{3.25c}$$

$$\blacksquare d_{\mu\nu\lambda\rho} = -4\Xi d_\mu{}^\tau{}_{[\lambda}{}^\sigma d_{\rho]\sigma\nu\tau} - 2\Xi d_\mu{}^\tau{}_\nu{}^\sigma d_{\lambda\rho\tau\sigma} + \tfrac{1}{2}d_{\mu\nu\lambda\rho}R - T_{[\mu}{}^\sigma\Xi^2 d_{\nu]\sigma\lambda\rho}$$
$$-\Xi^2 T_{[\lambda}{}^\sigma d_{\rho]\sigma\mu\nu} - \Xi^2 g_{\mu[\lambda}d_{\rho]\sigma\nu\tau}T^{\tau\sigma} + \Xi^2 g_{\nu[\lambda}d_{\rho]\sigma\mu\tau}T^{\tau\sigma} + 2\nabla_{[\mu}T_{|\lambda\rho|\nu]}$$
$$+\epsilon_{\mu\nu\sigma\tau}\nabla^\tau {}^*T_{\lambda\rho}{}^\sigma, \tag{3.25d}$$

$$\mathscr{R}_{\mu\nu}[\boldsymbol{g}] = 2L_{\mu\nu} + \tfrac{1}{6}\mathcal{R}(x)g_{\mu\nu}. \tag{3.25e}$$

Remark 19. The reduced system of evolution equations (3.25a)-(3.25e) is a system of quasilinear wave equations for the fields Ξ, s, $L_{\mu\nu}$, $d_{\mu\nu\lambda\rho}$ and $g_{\mu\nu}$. Schematically, one has that

$$g^{\sigma\tau}\partial_\sigma\partial_\tau\Xi = X\big(\boldsymbol{g}, \partial\boldsymbol{g}, \Xi, s, \mathcal{R}(x)\big),$$
$$g^{\sigma\tau}\partial_\sigma\partial_\tau s = S\big(\boldsymbol{g}, \partial\boldsymbol{g}, \Xi, \partial\Xi, s, \boldsymbol{L}, \mathcal{R}(x), \partial\mathcal{R}(x), \boldsymbol{T}\big),$$
$$g^{\sigma\tau}\partial_\sigma\partial_\tau L_{\mu\nu} = F_{\mu\nu}\big(\boldsymbol{g}, \partial\boldsymbol{g}, \Xi, \boldsymbol{L}, \boldsymbol{d}, \mathcal{R}(x), \partial^2\mathcal{R}(x), \boldsymbol{T}, \partial\boldsymbol{T}\big),$$
$$g^{\sigma\tau}\partial_\sigma\partial_\tau d_{\mu\nu\lambda\rho} = D_{\mu\nu\lambda\rho}\big(\boldsymbol{g}, \partial\boldsymbol{g}, \Xi, \boldsymbol{d}, \mathcal{R}(x), \partial\boldsymbol{T}\big),$$
$$g^{\sigma\tau}\partial_\sigma\partial_\tau g_{\mu\nu} = G_{\mu\nu}\big(\boldsymbol{g}, \partial\boldsymbol{g}, \boldsymbol{L}, \mathcal{R}(x)\big),$$

where X, S, $F_{\mu\nu}$, $D_{\mu\nu\lambda\rho}$ and $G_{\mu\nu}$ are polynomial expressions of their arguments. Strictly speaking, the system is a system of wave equations only if $g_{\mu\nu}$ is known to be Lorentzian. The basic existence, uniqueness and stability results of systems of the above type have been given in [43] — these results are the second order analogues of the theory developed in [45] for symmetric hyperbolic systems. The basic theory for initial-boundary value problems can be found in [14, 21].

3.6 Propagation of the gauge

This section is devoted to studying the consistency of the conformal and coordinate gauge introduced in Section 3.5 by constructing a system of homogeneous wave equations for the corresponding subsidiary fields.

3.6.1 Basic relations

Consider a set of coordinates $x = (x^\mu)$. Let $g_{\mu\nu}$ denote the components of a metric g_{ab} in these coordinates. Similarly, $R_{\mu\nu}$ denotes the components of the associated Ricci tensor R_{ab}, while R is the corresponding Ricci scalar. We now investigate the requirements for $R_{\mu\nu}$ and R to coincide, respectively, with $\mathscr{R}_{\mu\nu}$ and $\mathcal{R}(x)$. In addition, we also need to investigate the conditions under which $L_{\mu\nu}$ corresponds to the components of the Schouten tensor. This can be expressed as the vanishing of the following fields:

$$Q \equiv R - \mathcal{R}(x), \tag{3.26a}$$

$$Q^\mu \equiv \Gamma^\mu - \mathcal{F}^\mu(x), \tag{3.26b}$$

$$Q_{\mu\nu} \equiv R_{\mu\nu} - \mathscr{R}_{\mu\nu}. \tag{3.26c}$$

Below we make the following assumption:

Assumption 2. *Let $T_{\mu\nu}$ and $T_{\mu\nu\lambda}$ be, respectively, the components of a tracefree energy momentum tensor with vanishing divergence and its associated rescaled Cotton tensor. Let $g_{\mu\nu}$ and $L_{\mu\nu}$ be solutions to the equations:*

$$\mathscr{R}_{\mu\nu} = 2L_{\mu\nu} + \tfrac{1}{6}\mathcal{R}(x)g_{\mu\nu}, \tag{3.27a}$$

$$\begin{aligned}\blacksquare L_{\mu\nu} = {} & -2\Xi d_{\mu\rho\nu\lambda}L^{\rho\lambda} + 4L_\mu{}^\lambda L_{\nu\lambda} - L_{\lambda\rho}L^{\lambda\rho}g_{\mu\nu} + \tfrac{1}{6}\nabla_\mu\nabla_\nu\mathcal{R}(x) \\ & + \tfrac{1}{2}\Xi^3 d_{\mu\lambda\nu\rho}T^{\lambda\rho} - \Xi\nabla_\lambda T_\mu{}^\lambda{}_\nu - 2T_{(\mu|\lambda|\nu)}\nabla^\lambda\Xi.\end{aligned} \tag{3.27b}$$

As a direct consequence of equation (3.27a), one can find that the gauge zero-quantities (3.26a)-(3.26c) are not independent of each other. Simple calculations yield

$$Q_{\mu\nu} = \nabla_{(\mu}Q_{\nu)}, \tag{3.28a}$$

$$Q = Q_{\mu}{}^{\mu} = \nabla_{\mu}Q^{\mu}. \tag{3.28b}$$

Furthermore, equation (3.22) and Definition 4 lead to

$$\mathscr{R}_{\mu\nu}[\boldsymbol{g}] = R_{\mu\nu} - \nabla_{(\mu}Q_{\nu)}, \tag{3.29a}$$

$$\blacksquare L_{\mu\nu} = \square L_{\mu\nu} - (Q_{\mu\sigma} - \nabla_{\mu}Q_{\sigma})L^{\sigma}{}_{\nu} - (Q_{\nu\sigma} - \nabla_{\nu}Q_{\sigma})L^{\sigma}{}_{\mu}. \tag{3.29b}$$

Remark 20. Equations (3.28a)-(3.28b) show that if $Q^{\mu} = 0$ then Q and $Q_{\mu\nu}$ automatically vanish. In this sense, we will consider Q^{μ} as the basic gauge zero-quantity of the system.

3.6.2 The gauge subsidiary evolution system

In this subsection we obtain a system of homogeneous wave equations for the gauge subsidiary variables. This will be achieved via exploiting the properties of the so-called *Bach tensor* which will play the role of an integrability condition for the system.

3.6.2.1 The Bach tensor

Let g_{ab} be a 4-dimensional metric. The Bach tensor is defined as:

$$B_{ab} \equiv \nabla^{c}\nabla_{a}L_{bc} - \nabla^{c}\nabla_{c}L_{ab} - C_{acdb}L^{cd}. \tag{3.30}$$

From this definition it is easy to verify that B_{ab} is symmetric and tracefree. Additionally, it satisfies the following identity, independently of the validity of the Einstein field equations:

$$\nabla^{a}B_{ab} = 0. \tag{3.31}$$

Remark 21. A straightforward calculation shows that the Bach tensor can be expressed in terms of the geometric zero-quantities as

$$B_{ab} = -L^{cd}P_{acbd} - \tfrac{1}{2}\Xi^{3}d_{acbd}T^{cd} + \Xi\nabla_{c}T_{a}{}^{c}{}_{b} + 2T_{(a|c|b)}\nabla^{c}\Xi.$$

Consequently, if g_{ab} is a solution to the MTCEFE then the Bach tensor vanishes if $T_{ab} = 0$.

Remark 22. In view of the fact that trivial initial conditions for the zero-quantities imply the vanishing of $P^a{}_{bcd}$ — see Proposition 5 — throughout the remainder of this chapter, and for the sake of simplicity, our calculations will assume that $P^a{}_{bcd} = 0$.

3.6.2.2 Wave equations for the gauge subsidiary variables

The Bach tensor can be conveniently expressed in terms of the gauge subsidiary quantities. Terms containing $R_{\mu\nu}$ and R can be rewritten according to definitions (3.26a) and (3.26c) along with (3.28a) and (3.29a). This results in:

$$\begin{aligned} B_{\mu\nu} =& - \Xi d_{\mu\lambda\nu\rho} L^{\lambda\rho} - \tfrac{1}{12}\nabla_\mu \nabla_\nu Q + \nabla_\lambda \nabla_\mu L_\nu{}^\lambda + \tfrac{1}{2}\nabla^\lambda \nabla_\mu \nabla_{(\nu} Q_{\lambda)} - \Box L_{\mu\nu} \\ & - \tfrac{1}{2}\nabla_\lambda \nabla^\lambda \nabla_{(\mu} Q_{\nu)} + \tfrac{1}{12} g_{\mu\nu} \nabla_\lambda \nabla^\lambda Q - \tfrac{1}{4}\Xi d_{\mu\lambda\nu\rho}\nabla^\rho Q^\lambda - \tfrac{1}{4}\Xi d_{\mu\rho\nu\lambda}\nabla^\lambda Q^\rho. \end{aligned}$$

An expression for $\Box L_{\mu\nu}$ can be obtained combining (3.27b) and (3.29b). Notice also that this term is the only one containing contributions from the matter field. Direct substitution yields

$$\begin{aligned} B_{\mu\nu} =& L_{\lambda\rho} L^{\lambda\rho} g_{\mu\nu} - 4 L_\mu{}^\nu L_{\nu\lambda} + \tfrac{1}{12} g_{\mu\nu}\nabla_\lambda \nabla^\lambda Q - \tfrac{1}{12}\nabla_\mu \nabla_\nu Q - \tfrac{1}{6}\nabla_\mu \nabla_\nu \mathcal{R} \\ & + \nabla_\lambda \nabla_\mu L_\nu{}^\lambda + \tfrac{1}{2}\nabla^\lambda \nabla_\mu \nabla_{(\nu} Q_{\lambda)} - \tfrac{1}{2}\nabla_\lambda \nabla^\lambda \nabla_{(\mu} Q_{\nu)} - 2 L_{\lambda(\mu} Q_{\nu)}{}^\lambda \\ & + 2 L_{\lambda(\mu}\nabla_{\nu)} Q^\lambda + \Xi d_{\mu\lambda\nu\rho} L^{\lambda\rho} - \tfrac{1}{4}\Xi d_{\mu\lambda\nu\rho}\nabla^\rho Q^\lambda - \tfrac{1}{4} d_{\mu\rho\nu\lambda}\nabla^\rho Q^\lambda + N_{\mu\nu}, \end{aligned}$$

where

$$N_{\mu\nu} \equiv -\tfrac{1}{2}\Xi^3 d_{\mu\lambda\nu\rho} T^{\lambda\rho} + 2 T_{(\mu|\lambda|\nu)}\nabla^\lambda \Xi + \Xi \nabla_\lambda T_\mu{}^\lambda{}_\nu$$

encodes the matter contributions. Commuting covariant derivatives and making further suitable substitutions, a lengthy calculation gives

$$\begin{aligned}B_{\mu\nu} = &- \frac{5}{12}QL_{\mu\nu} - \tfrac{1}{288}Q\mathcal{R}(x)g_{\mu\nu} - -2L_{(\mu}{}^{\lambda}Q_{\nu)\lambda} + \frac{1}{24}Q^2 g_{\mu\nu} - \frac{5}{48}Q\nabla_\mu Q_\nu \\ &+ \frac{1}{48}\mathcal{R}(x)\nabla_\mu Q_\nu + 2\Phi_{\nu\lambda}\nabla_\mu Q^\lambda - \frac{1}{4}\nabla_\mu\nabla_\lambda\nabla^\lambda Q_\nu - \frac{1}{16}Q\nabla_\nu Q_\mu \\ &+ \frac{1}{16}\mathcal{R}(x)\nabla_\nu Q_\mu + \frac{3}{16}\nabla_\mu Q^\lambda\nabla_\nu Q_\lambda + \frac{7}{4}\Phi_{\mu\lambda}\nabla_\nu Q^\lambda + \frac{1}{6}\nabla_\nu\nabla_\mu Q \\ &+ \frac{1}{4}\nabla_\lambda\nabla_\mu\nabla^\lambda Q_\nu + \frac{1}{12}g_{\mu\nu}\nabla_\lambda\nabla^\lambda Q - \frac{1}{2}\nabla_\lambda\nabla^\lambda\nabla_{(\mu}Q_{\nu)} + \frac{3}{4}\Phi_{\nu\lambda}\nabla^\lambda Q_\mu \\ &+ \frac{1}{4}\nabla_{(\mu}Q_{|\lambda|}\nabla^\lambda Q_{\nu)} + \frac{1}{16}\nabla_\lambda Q_\nu\nabla^\lambda Q_\mu + \frac{1}{2}\Phi_{\mu\lambda}\nabla^\lambda Q_\nu - \frac{1}{2}\Xi d_{\mu\lambda\nu\rho}\nabla^\rho Q^\lambda \\ &- \frac{1}{4}\Xi d_{\mu\rho\nu\lambda}\nabla^\rho Q^\lambda - \frac{3}{4}\Phi_{\lambda\rho}g_{\mu\nu}\nabla^\rho Q^\lambda - \frac{1}{16}g_{\mu\nu}\nabla_\lambda Q_\rho\nabla^\rho Q^\lambda \\ &- \frac{1}{16}g_{\mu\nu}\nabla_\rho Q_\lambda\nabla^\rho Q^\lambda + N_{\mu\nu}. \end{aligned} \tag{3.32}$$

Next, we introduce the auxiliary field

$$M_\mu \equiv \Box Q_\mu. \tag{3.33}$$

Taking the divergence of equation (3.32), and after some direct manipulations, equations (3.28a)-(3.28b) and (3.31) imply that

$$\Box M_\nu = H_\nu(\nabla\boldsymbol{M}, \nabla\boldsymbol{Q}, \nabla Q, \boldsymbol{Q}, Q) + 4\nabla^\mu N_{\mu\nu},$$

where $\boldsymbol{Q}$ stands for Q_μ and, for simplicity, H_ν represents a homogeneous function of its arguments. On the other hand, we can rewrite the term $\nabla^\mu N_{\mu\nu}$ in a suitable way by using the symmetries of T_{abc} along with the help of equations (3.26c), (3.29a) and the geometric zero-quantities. A direct calculation shows that

$$\nabla^\nu N_{\nu\mu} = -T_{\mu\nu\lambda}\Upsilon^{\nu\lambda} - \tfrac{1}{2}\Xi^3 T_{\nu\lambda}\Lambda^\nu{}_\mu{}^\lambda,$$

so the wave equation for M_μ takes the schematic form

$$\Box M_\nu = H_\nu(\nabla\boldsymbol{M}, \nabla\boldsymbol{Q}, \nabla Q, \boldsymbol{Q}, Q, \boldsymbol{\Upsilon}, \boldsymbol{\Lambda}). \tag{3.34}$$

Lastly, a wave equation for Q is required to close the system. This can be obtained by direct application of the $\Box$ operator on the definition of Q along with the aid of equations (3.26a), (3.28b) and (3.29a), resulting in

$$\Box Q = -2L_{\mu\nu}\nabla^{\mu}Q^{\nu} - \nabla^{\mu}Q^{\nu}\nabla_{(\mu}Q_{\nu)} - \tfrac{1}{2}Q^{\mu}\nabla_{\mu}Q - \tfrac{1}{2}Q^{\mu}\nabla_{\mu}\mathcal{R}(x) - \tfrac{1}{6}\mathcal{R}(x)Q + \nabla^{\mu}M_{\mu}. \quad (3.35)$$

Remark 23. The gauge subsidiary evolution system, equations (3.33)-(3.35), is homogeneous in M_{μ}, Q_{μ}, Q, $\Upsilon_{\mu\nu}$, $\Lambda_{\mu\nu\lambda}$ and their first derivatives.

The previous discussion leads to the following result:

Lemma 4. *Assume that the hypotheses of Lemma 3 hold. Moreover, let the quantities M_{μ}, Q_{μ}, Q, $\Upsilon_{\mu\nu}$ and $\Lambda_{\mu\nu\lambda}$ along with their first covariant derivatives vanish on a fiduciary hypersurface $\mathcal{S}_{\star}$. Then the unique solution to the system* (3.33)-(3.35) *on a small enough slab of $\mathcal{S}_{\star}$ corresponds to $Q=0$, $Q_{\mu}=0$ and $M_{\mu}=0$, which in turn implies that $Q_{\mu\nu}=0$.*

Remark 24. It must be pointed out that these initial gauge conditions are not equivalent, in the vacuum case, to those considered in [51] which only require the vanishing of the gauge zero-quantities and their first derivatives on the initial hypersurface. In the present case, the conditions require the vanishing of third order derivatives via the definition of M_{μ}.

3.7 Evolution equations for the matter fields

Having settled the analysis of the *geometric* part of the MTCEFE, we now proceed to investigate the evolution of the subsidiary equations associated to the matter models introduced in Section 2.4, namely the conformally coupled scalar field, the Maxwell field and the Yang-Mills field.

3.7.1 The conformally coupled scalar field

First, we consider the conformally invariant scalar field defined in Section 2.4.1. Notice that the second derivatives of ϕ in equation (2.23) will lead to the appearance of second and third order derivatives of the matter field in the expression of the rescaled Cotton tensor — see equation (2.8d)

— which may affect the hyperbolicity of the system (3.25a)-(3.25e). Moreover, T_{ab} is also coupled to the geometric sector via the Schouten tensor. These difficulties will be addressed in the sequel.

3.7.1.1 Auxiliary fields and the evolution equations

We start the analysis by observing that the third order derivative terms in the expression of the rescaled Cotton tensor for the conformally invariant scalar field are of the form $\nabla_{[a}\nabla_{b]}\nabla_c\phi$. Using the commutator of covariant derivatives, these terms can be transformed into first order derivative terms according to the formula

$$\nabla_{[a}\nabla_{b]}\nabla_c\phi = -\tfrac{1}{2}R_{abc}{}^{d}\nabla_d\phi.$$

Thus, one is left with an expression for the Cotton tensor containing, at most, second order derivatives. In order to eliminate second order derivative terms in the rescaled Cotton tensor which, potentially, could destroy the hyperbolic nature of the wave equations, one needs to promote the first and second derivatives of ϕ as further (independent) unknowns. Accordingly, we define

$$\phi_a \equiv \nabla_a\phi, \qquad \phi_{ab} \equiv \nabla_a\nabla_b\phi. \tag{3.36}$$

Following the previous discussion, and exploiting equation (2.11c), one can write the rescaled Cotton tensor for the conformally coupled scalar field as

$$\begin{aligned} T_{abc} = & \left(1 - \tfrac{1}{4}\Xi^2\phi^2\right)^{-1}\Big(\tfrac{3}{2}\Xi\phi L_{c[b}\phi_{a]} + \tfrac{3}{2}\Xi\phi_{[b}\phi_{a]c} - \tfrac{1}{4}\Xi\phi^2 d_{abcd}\nabla^d\Xi - \tfrac{1}{4}\Xi^2\phi d_{abcd}\phi^d \\ & + \tfrac{1}{2}\Xi\phi g_{c[b}L_{a]d} + \tfrac{1}{2}\Xi g_{c[a}\phi_{b]d}\phi^d + g_{c[b}T_{a]d}\nabla^d\Xi + 3T_{c[b}\nabla_{a]}\Xi\Big). \end{aligned} \tag{3.37}$$

We now proceed to construct suitable evolution equations for ϕ_a and ϕ_{ab} by means of a set of integrability conditions for these fields. Firstly, the identity $\nabla_a\phi_b = \nabla_b\phi_a$ represents an integrability condition for ϕ_a. A wave equation then readily follows after applying ∇^b and using equation (2.22):

$$\Box\phi_a = 2\phi^b L_{ab} + \frac{1}{3}R\phi_a + \frac{1}{6}\phi\nabla_a R. \tag{3.38}$$

On the other hand, an integrability condition for ϕ_{ab} can be obtained directly from its definition:

$$2\nabla_{[c}\phi_{a]b} = \phi_d R_{cab}{}^d = -\Xi\phi^d d_{acbd} - 2\phi_{[c}L_{a]b} + 2\phi^d g_{b[c}L_{a]d}.$$

Applying ∇^c to this expression and using equations (2.11c), (2.11d), (2.22) and (3.38), a straightforward calculation leads to:

$$\begin{aligned}\Box\phi_{ab} =& \tfrac{1}{2}\phi_{ab}R - \tfrac{1}{3}R\phi L_{ab} - 2\phi^{cd}L_{cd}g_{ab} - \tfrac{1}{6}\phi^c g_{ab}\nabla_c R + \tfrac{1}{6}\phi\nabla_{(a}\nabla_{b)}R \\ &- 2\Xi\phi^{cd}d_{(a|c|b)d} + 8\phi_{(a}{}^c L_{b)c} + 2\Xi\phi^c T_{(a|c|b)} + \tfrac{2}{3}\phi_{(a}\nabla_{b)}R + 2\phi^c\nabla_{(a}L_{b)c} \\ &- 2\phi^c d_{acb}{}^d\nabla_d\Xi. \end{aligned} \tag{3.39}$$

Remark 25. In equation (3.39) it is understood that the rescaled Cotton tensor T_{abc} is expressed in terms of the auxiliary fields ϕ_a and ϕ_{ab} according to (3.37), so does not contain second or higher derivatives of the fields.

Remark 26. When coupling the wave equations (2.22), (3.38) and (3.39) to the system (3.25a)-(3.25e) satisfied by the geometric conformal fields, it is understood that the geometric wave operator $\Box$ is replaced by its reduced counterpart $\blacksquare$ as discussed in Section 3.5.2.1.

3.7.1.2 Subsidiary equations

To verify the consistency of our approach in dealing with the higher order derivative terms in the rescaled Cotton tensor for the conformally invariant scalar field we introduce the following subsidiary fields:

$$Q_a \equiv \phi_a - \nabla_a\phi, \tag{3.40a}$$

$$Q_{ab} \equiv \phi_{ab} - \nabla_a\nabla_b\phi. \tag{3.40b}$$

A wave equation for Q_a can be obtained in a straightforward way: applying $\Box$ to definition (3.40a) and with the help of relations (2.22) and (3.38), a short calculation yields

$$\Box Q_a = \Box\phi_a - \nabla_a\Box\phi - R_{ab}\nabla^b\phi = \tfrac{1}{3}RQ_a + 2L_a{}^b Q_b. \tag{3.41}$$

Similarly, by applying $\Box$ to equation (3.40b), commuting covariant derivatives and using the definitions of the geometric zero-quantities, one obtains

$$\begin{aligned}\square Q_{ab} =&\tfrac{1}{2}Q_{ab}R - 2Q^{cd}L_{cd}g_{ab} - \tfrac{1}{6}Q^{c}g_{ab}\nabla_{c}R + 2Q^{c}\nabla_{c}L_{ab} - 2\Xi Q^{cd}d_{acbd} \\ &+ 8Q_{(a}{}^{c}L_{b)c} - 2\phi^{c}\Delta_{(a|c|b)} + 4\Xi Q^{c}T_{(a|c|b)} + 4Q^{c}\Delta_{(a|c|b)} + \tfrac{2}{3}Q_{(a}\nabla_{b)}R \\ &- 4Q^{c}d_{(a|c|b)}{}^{d}\nabla_{d}\Xi. \end{aligned} \tag{3.42}$$

Remark 27. The system of wave equations (3.41) and (3.42) is homogeneous in Q_a, Q_{ab} and Δ_{abc}. Thus, it follows from general uniqueness results for solutions to wave equations that if these quantities and their derivatives vanish on an initial hypersurface $\mathcal{S}_\star$, then necessarily $Q_a = 0$ and $Q_{ab} = 0$ at least on a small enough slab around $\mathcal{S}_\star$.

3.7.1.3 Summary

The analysis of the conformally coupled scalar field can be summarised in the following manner:

Proposition 6. *The system of equations* (3.25a)-(3.25e) *with rescaled Cotton tensor given by* (3.37)*, together with the conformally coupled wave equation* (2.22) *and the auxiliary system* (3.38)-(3.39) *written in terms of the reduced wave operator* $\blacksquare$*, constitute a* proper *system of quasilinear wave equations — see Remark 18.*

3.7.2 The Maxwell field

Continuing with the study of tracefree matter models, consider the Maxwell field governed by equations (2.24a)-(2.24b). The strategy is similar to the one in the previous section: first, a set of suitable wave equations for the matter field will be constructed and then the propagation of the corresponding subsidiary variables needs to be proved.

3.7.2.1 Auxiliary field and the evolution equations

Equation (2.24b) represents a direct integrability condition for F_{ab}. Applying ∇^c, commuting covariant derivatives and using equation (2.24a), a calculation yields

$$\square F_{ab} = \tfrac{1}{3}F_{ab}R - 2\Xi F^{cd}d_{acbd}. \tag{3.43}$$

From equation (2.8d) it follows that the rescaled Cotton tensor contains first derivatives of F_{ab}, which puts at risk the hyperbolicity of the system (3.25a)-(3.25d). In order to deal with this problem we introduce the auxiliary variable

$$F_{abc} \equiv \nabla_a F_{bc}, \tag{3.44}$$

satisfying $F_{abc} = F_{a[bc]}$. By virtue of equation (2.24b) it also follows that $F_{[abc]} = 0$. In terms of this quantity, it can be readily checked that the rescaled Cotton tensor for the Maxwell field takes the form

$$\begin{aligned} T_{abc} = \Xi F_{[b}{}^{d} F_{a]cd} - \tfrac{1}{2}\Xi F_{c}{}^{d} F_{dab} + \tfrac{1}{2}\Xi g_{c[a} F^{de} F_{b]de} - 3F_{cd} F_{[a}{}^{d} \nabla_{b]}\Xi \\ + F_{de} F^{de} g_{c[a} \nabla_{b]}\Xi - g_{c[a} F_{b]}{}^{e} F_{de} \nabla^{d}\Xi. \end{aligned} \tag{3.45}$$

From definition (3.44) it follows that F_{abc} possesses two independent divergences: $\nabla^a F_{abc}$ is simply the right-hand side of wave equation (3.43) whilst the other is given by

$$\nabla_c F_{ab}{}^{c} = \Xi F^{cd} d_{acbd} - \frac{1}{6} R F_{ab} + 2F_{[a}{}^{c} L_{b]c}, \tag{3.46}$$

as a direct calculation confirms. In order to obtain an integrability condition for F_{abc}, consider the expression $3\nabla_{[d} F_{|a|bc]}$. Commuting covariant derivatives and using the first Bianchi identity for the Weyl tensor, a straightforward calculation results in:

$$3\nabla_{[d} F_{|a|bc]} = -3\Xi F_{[d}{}^{e} d_{|ae|bc]} + 6F_{[db} L_{c]a} + 6g_{a[d} F_{b}{}^{e} L_{c]e}. \tag{3.47}$$

A geometric wave equation can be obtained by applying ∇^d to the last expression and commuting derivatives. Using equations (2.11c), (2.11d), (2.10), (2.13), (3.46) as well as the symmetries of d_{abcd} and T_{abc} to simplify it, a long but direct calculation yields

$$\begin{aligned} \square F_{abc} = & -2\Xi F_{a}{}^{d} T_{bcd} + 4\Xi F_{[b}{}^{d} T_{|ad|c]} - 2\Xi F_{a}{}^{de} d_{bdce} - 4\Xi F^{d}{}_{[b}{}^{e} d_{c]ead} + \tfrac{1}{2} F_{abc} R \\ & + 4F^{d}{}_{bc} L_{ad} - 4F^{d}{}_{a[b} L_{c]d} - 4F^{d}{}_{[b}{}^{e} g_{c]a} L_{de} + \tfrac{1}{3} F_{bc} \nabla_a R - 2F^{de} d_{ade[b} \nabla_{c]}\Xi \\ & - 4\Xi F^{de} \nabla_{[b} d_{c]ead} - \tfrac{1}{3} F_{a[b} \nabla_{c]} R - 2F_{[b}{}^{e} d_{c]ead} \nabla^{d}\Xi - F_{d}{}^{e} d_{aebc} \nabla^{d}\Xi \\ & - 4F_{[b}{}^{e} d_{c]dae} \nabla^{d}\Xi - F_{a}{}^{e} d_{bcde} \nabla^{d}\Xi + 2F^{ef} g_{a[b} d_{c]edf} \nabla^{d}\Xi + \tfrac{1}{3} g_{a[b} F_{c]d} \nabla^{d} R. \end{aligned} \tag{3.48}$$

This equation can be further simplified via a pair of observations. First, by multiplying equation (3.3) by F^{dg} the following auxiliary identity is found:

$$2F_{[a}{}^{e}d_{b]ecd}\nabla^{d}\Xi - 2F_{[c}{}^{e}d_{d]eab}\nabla^{d}\Xi + 2F^{de}d_{ced[a}\nabla_{b]}\Xi - 2F^{eg}g_{c[a}d_{b]edg}\nabla^{d}\Xi = 0. \tag{3.49}$$

Secondly, from equation (2.13) we have the following relations:

$$4\Xi F^{de}\nabla_{[b}d_{c]ead} = -2\Xi\epsilon_{bcef}F^{de}\,{}^{*}T_{ad}{}^{f} + 2\Xi F^{de}\nabla_{e}d_{adbc},$$
$$\Xi F^{de}\nabla_{e}d_{adbc} = -\tfrac{1}{2}\Xi\epsilon_{adef}F^{de}\,{}^{*}T_{bc}{}^{f} - \tfrac{1}{2}\Xi F^{de}\nabla_{a}d_{bcde}.$$

Combining them we readily obtain the identity

$$4\Xi F^{de}\nabla_{[b}d_{c]ead} = 4\Xi F_{[b}{}^{d}T_{|a|c]d} - 2\Xi F_{a}{}^{d}T_{bcd} + \Xi F^{de}\nabla_{a}d_{bcde}. \tag{3.50}$$

Making use of (3.49) and (3.50), the wave equation for F_{abc} takes a simpler form:

$$\begin{aligned}\square F_{abc} = {} & 4\Xi F_{[b}{}^{d}T_{c]da} - 2\Xi F_{a}{}^{de}d_{bdce} - 4F^{d}{}_{[b}{}^{e}d_{c]ead} + \tfrac{1}{2}F_{abc}R + 4F^{d}{}_{bc}L_{ad} \\ & - 4F^{d}{}_{a[b}L_{c]d} - 4F^{d}{}_{[b}{}^{e}g_{c]a}L_{de} + \tfrac{1}{3}F_{bc}\nabla_{a}R - \tfrac{1}{3}F_{a[b}\nabla_{c]}R + \tfrac{1}{3}g_{a[b}F_{c]d}\nabla^{d}R \\ & - 4F^{de}d_{ade[b}\nabla_{c]}\Xi - 4F_{[b}{}^{e}d_{c]dae}\nabla^{d}\Xi - 2F_{a}{}^{e}d_{bcde}\nabla^{d}\Xi - \Xi F^{de}\nabla_{a}d_{bcde}.\end{aligned} \tag{3.51}$$

As remarked in the case of the conformally invariant scalar field, the geometric operator $\square$ is to be replaced by $\blacksquare$ when equations (3.43) and (3.51) are coupled to the system (3.25a)-(3.25e).

3.7.2.2 Subsidiary equations

In order to complete the discussion of the Maxwell field it is necessary to construct suitable evolution equations for the zero-quantities

$$M_{b} \equiv \nabla^{a}F_{ab}, \tag{3.52a}$$
$$M_{abc} \equiv \nabla_{[a}F_{bc]}, \tag{3.52b}$$
$$Q_{abc} \equiv F_{abc} - \nabla_{a}F_{bc}. \tag{3.52c}$$

Here, M_{abc} possesses the symmetries

$$M_{abc} = M_{a[bc]} = M_{[ab]c} = M_{[abc]}. \tag{3.53a}$$

Also, one can verify the following identities:

$$\nabla^a M_a = 0, \tag{3.54a}$$

$$\nabla^c M_{abc} = -\tfrac{2}{3}\nabla_{[a}M_{b]}. \tag{3.54b}$$

Remark 28. Following the spirit in the discussion of the previous section, the zero-quantities M_a and M_{abc} encode Maxwell equations (2.24a) and (2.24b), respectively, while Q_{abc} does so for the auxiliary field F_{abc}.

Equation for M_a. Observe that equation (3.54b) works as an integrability condition for M_a. Applying ∇^b, using (3.54a) and exploiting the various symmetries of M_{abc}, one obtains

$$\Box M_a = \tfrac{1}{6}M_a R + 2M^b L_{ab}. \tag{3.55}$$

Equation for M_{abc}. In order to avoid lengthy expressions it is simpler to consider the Hodge dual of M_{abc} defined as

$$M^*_a \equiv \nabla^b F^*_{ba} = \frac{1}{2}\epsilon_a{}^{bcd}M_{bcd}. \tag{3.56}$$

Here, the second equality is a consequence of equations (3.44) and (3.52b). From this definition it can be easily checked that M^*_a is divergencefree which, in turn, implies an integrability condition. More explicitly:

$$\nabla^a M^*_a = 0 \iff \nabla_{[d}M_{abc]} = 0. \tag{3.57}$$

Applying ∇^d to (3.57) and commuting derivatives, a straightforward calculation leads to

$$\Box M_{abc} = \tfrac{1}{2}RM_{abc} - 6\Xi d_{[a}{}^d{}_b{}^e M_{c]de} - 6L_{[a}{}^d M_{bc]d}, \tag{3.58}$$

where it has been used that $\nabla_{[a}\nabla_{|d|}M_{bc]}{}^d$ vanishes by virtue of equation (3.54b).

Equation for Q_{abc}. A wave equation for the field Q_{abc} can be obtained by direct application of the $\square$ operator. Employing definitions (3.52a), (3.52c), along with equations (3.6c), (3.6d), (3.43) and (3.51), one obtains the expression

$$\begin{aligned}\square Q_{abc} = {} & 4\Xi F_{[b}{}^{d}\Lambda_{|a|c]d} - 2\Xi Q_{a}{}^{de}d_{bdce} - 2\Xi Q^{d}{}_{[b}{}^{e}d_{c]ead} + \tfrac{1}{2}Q_{abc}R - 4M_{[b}L_{c]a} \\ & + 4Q^{d}{}_{bc}L_{ad} - 4Q^{d}{}_{a[b}L_{c]d} + 6L_{a}{}^{d}M_{bcd} - 4Q^{d}{}_{[b}{}^{e}g_{c]a}L_{de} + 2F^{de}d_{bdce}\nabla_{a}\Xi \\ & - 4F^{de}d_{ade[b}\nabla_{c]}\Xi - 6F_{[a}{}^{e}d_{bc]de}\nabla^{d}\Xi. \qquad (3.59)\end{aligned}$$

In order to show that the terms not containing zero-quantities vanish, observe that the first Bianchi identity implies that

$$2F^{de}d_{bdce}\nabla_{a}\Xi - 4F^{de}d_{ade[b}\nabla_{c]}\Xi = 3F^{de}d_{de[ab}\nabla_{c]}\Xi.$$

On the other hand, multiplying definition (3.10) by F^{de}, a short calculation yields the auxiliary identity

$$3F^{de}d_{de[ab}\nabla_{c]}\Xi - 6F_{[a}{}^{e}d_{bc]de}\nabla^{d}\Xi = 0.$$

From the last two expressions it follows then that

$$\begin{aligned}\square Q_{abc} = {} & 4\Xi F_{[b}{}^{d}\Lambda_{|a|c]d} - 2\Xi Q_{a}{}^{de}d_{bdce} - 2\Xi Q^{d}{}_{[b}{}^{e}d_{c]ead} + \tfrac{1}{2}Q_{abc}R - 4M_{[b}L_{c]a} \\ & + 4Q^{d}{}_{bc}L_{ad} - 4Q^{d}{}_{a[b}L_{c]d} + 6L_{a}{}^{d}M_{bcd} - 4Q^{d}{}_{[b}{}^{e}g_{c]a}L_{de}. \qquad (3.60)\end{aligned}$$

Remark 29. Geometric wave equations (3.55), (3.58) and (3.60) are crucially homogeneous in M_a, M_{abc}, Q_{abc} and Λ_{abc}. Thus, if these quantities and their first covariant derivatives vanish on an initial hypersurface $\mathcal{S}_\star$, it can be guaranteed that there exists a unique solution on a small enough slab of $\mathcal{S}_\star$, and it corresponds to $M_a = 0$, $M_{abc} = 0$ and $Q_{abc} = 0$.

3.7.2.3 Summary

The previous discussion about the coupling of the Maxwell field to the MTCEFE can be summarised as follows:

Proposition 7. *The system of wave equations* (3.25a)-(3.25e) *with rescaled Cotton tensor given by* (3.45)*, together with the wave equations* (3.43) *and*

(3.51) *written in terms of the wave operator* $\blacksquare$*, is a proper quasilinear system of wave equations for the Einstein-Maxwell system.*

3.7.3 The Yang-Mills field

The Yang-Mills field is the last example of an explicit tracefree matter model we consider. Due to its similarities with the Faraday field, some of the calculations will result analogous to the ones performed in the previous subsection. However, one of the distinctive features of the Yang-Mills field is the fact that, in order to obtain a hyperbolic reduction of the equations, one needs to introduce a set of gauge source functions fixing the divergence of the gauge potential. The consistency of this gauge choice will be analysed towards the end of the section.

Remark 30. Due to the form of the energy-momentum tensor given in (2.30), first derivatives of $F^{\mathfrak{a}}{}_{ab}$ will appear in the rescaled Cotton tensor, putting at risk the hyperbolicity of the system (3.25a)-(3.25e). As in the case of the Maxwell field, this will make necessary the introduction of an auxiliary quantity.

3.7.3.1 Evolution equations for the Yang-Mills fields

Suitable wave equations for the Yang-Mills fields can be obtained by a procedure analogous to the one used for the Maxwell field. Accordingly, we introduce the auxiliary field

$$F^{\mathfrak{a}}{}_{abc} \equiv \nabla_a F^{\mathfrak{a}}{}_{bc} + C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} A^{\mathfrak{b}}{}_a F^{\mathfrak{c}}{}_{bc}. \tag{3.61}$$

Moreover, the construction of a geometric wave equation for the Yang-Mills gauge potentials requires the introduction of *gauge source functions* $f^{\mathfrak{a}}(x)$ depending in a smooth way on the coordinates and fixing the value of the divergence of the potential. More precisely, in the following we set

$$\nabla^a A^{\mathfrak{a}}{}_a \equiv f^{\mathfrak{a}}(x). \tag{3.62}$$

Equation for the field strength. The Yang-Mills Bianchi identity, equation (2.28c), represents an integrability condition for the field strength $F^{\mathfrak{a}}{}_{ab}$.

Differentiating it and making use of equations (2.27) and (2.28a)-(2.28c), a straightforward calculation results in

$$
\begin{aligned}
\Box F^{\mathfrak{a}}{}_{ab} = & -2\Xi F^{\mathfrak{a}cd} d_{acbd} + \tfrac{1}{3} F^{\mathfrak{a}}{}_{ab} R + 2C^{\mathfrak{a}}{}_{\mathfrak{bc}} F^{\mathfrak{b}}{}_{a}{}^{c} F^{\mathfrak{c}}{}_{bc} - 2C^{\mathfrak{a}}{}_{\mathfrak{bc}} F^{\mathfrak{c}}{}_{cab} A^{\mathfrak{b}c} \\
& - C^{\mathfrak{a}}{}_{\mathfrak{be}} C^{\mathfrak{e}}{}_{\mathfrak{cd}} F^{\mathfrak{d}}{}_{ab} A^{\mathfrak{b}c} A^{\mathfrak{c}}{}_{c} + C^{\mathfrak{a}}{}_{\mathfrak{bc}} f^{\mathfrak{b}}(x) F^{\mathfrak{c}}{}_{ab}. \qquad (3.63)
\end{aligned}
$$

Equation for the gauge potential. Equation (2.28a) provides a natural integrability condition for the gauge potential field. After applying ∇^b, commuting derivatives and using equation (2.28b), one arrives to:

$$
\begin{aligned}
\Box A^{\mathfrak{a}}{}_{a} = & \tfrac{1}{6} A^{\mathfrak{a}}{}_{a} R + 2A^{\mathfrak{a}b} L_{ab} + C^{\mathfrak{a}}{}_{\mathfrak{bc}} F^{\mathfrak{c}}{}_{ab} A^{\mathfrak{b}b} + C^{\mathfrak{a}}{}_{\mathfrak{bc}} f^{\mathfrak{c}}(x) A^{\mathfrak{b}}{}_{a} - C^{\mathfrak{a}}{}_{\mathfrak{bc}} A^{\mathfrak{b}b} \nabla_b A^{\mathfrak{c}}{}_{a} \\
& + \nabla_a f^{\mathfrak{a}}(x). \qquad (3.64)
\end{aligned}
$$

Equation for the auxiliary field. A suitable integrability condition for the field $F^{\mathfrak{a}}{}_{abc}$ can be obtained from its definition. Using this and equation (2.28c), some manipulations yield

$$
\begin{aligned}
3\nabla_{[d} F^{\mathfrak{a}}{}_{|a|bc]} = & -3\Xi F^{\mathfrak{a}}{}_{[b}{}^{e} d_{|ae|cd]} + 6F^{\mathfrak{a}}{}_{[bc} L_{|a|d]} + 3F^{\mathfrak{b}}{}_{a[bc} A^{\mathfrak{c}}{}_{d]} C^{\mathfrak{a}}{}_{\mathfrak{bc}} \\
& - 3F^{\mathfrak{b}}{}_{a[b} F^{\mathfrak{c}}{}_{cd]} C^{\mathfrak{a}}{}_{\mathfrak{bc}} + 6F^{\mathfrak{a}}{}_{[b}{}^{e} L_{c|e} g_{a|d]}.
\end{aligned}
$$

Proceeding as in the case of the wave equation for $F^{\mathfrak{a}}{}_{abc}$, as well as using the Jacobi identity and definitions (3.61)-(3.62), a lengthy calculation results in

$$
\begin{aligned}
\Box F^{\mathfrak{a}}{}_{abc} = & \tfrac{1}{2} F^{\mathfrak{a}}{}_{abc} R + 4F^{\mathfrak{a}d}{}_{bc} L_{ad} + 2F^{\mathfrak{b}d}{}_{bc} F^{\mathfrak{c}}{}_{ad} C^{\mathfrak{a}}{}_{\mathfrak{bc}} - F^{\mathfrak{c}}{}_{abc} f^{\mathfrak{b}}(x) C^{\mathfrak{a}}{}_{\mathfrak{bc}} \\
& - F^{\mathfrak{d}}{}_{abc} A^{\mathfrak{b}d} A^{\mathfrak{c}}{}_{d} C^{\mathfrak{a}}{}_{\mathfrak{be}} C^{\mathfrak{e}}{}_{\mathfrak{cd}} + \tfrac{1}{3} F^{\mathfrak{a}}{}_{bc} \nabla_a R - 2A^{\mathfrak{b}d} C^{\mathfrak{a}}{}_{\mathfrak{bc}} \nabla_d F^{\mathfrak{c}}{}_{abc} \\
& - F^{\mathfrak{a}}{}_{d}{}^{e} d_{aebc} \nabla^d \Xi - F^{\mathfrak{a}}{}_{a}{}^{e} d_{bcde} \nabla^d \Xi + 2\Xi F^{\mathfrak{a}de} \nabla_e d_{adbc} \\
& - 4\Xi F^{\mathfrak{a}d}{}_{[b}{}^{e} d_{|ad|c]e} - 2\Xi F^{\mathfrak{a}}{}_{a}{}^{de} d_{[b|d|c]e} - 4F^{\mathfrak{a}d}{}_{a[b} L_{c]d} + 4\Xi F^{\mathfrak{a}}{}_{[b}{}^{d} T_{c]da} \\
& + 4\Xi F^{\mathfrak{a}}{}_{[b}{}^{d} T_{|ad|c]} - \tfrac{1}{3} F^{\mathfrak{a}}{}_{a[b} \nabla_{c]} R + 4F^{\mathfrak{b}}{}_{a[b}{}^{d} F^{\mathfrak{c}}{}_{c]d} C^{\mathfrak{a}}{}_{\mathfrak{bc}} - 4F^{\mathfrak{a}d}{}_{[b}{}^{e} L_{|de} g_{a|c]} \\
& - 2\Xi F^{\mathfrak{a}de} T_{[b|de} g_{a|c]} - 4F^{\mathfrak{a}}{}_{[b}{}^{d} d_{|ad|c]}{}^{e} \nabla_e \Xi - 2F^{\mathfrak{a}}{}_{[b}{}^{d} d_{|a|}{}^{e}{}_{c]d} \nabla_e \Xi \\
& + 2F^{\mathfrak{a}de} d_{ad[b|e|} \nabla_{c]} \Xi - \tfrac{1}{3} F^{\mathfrak{a}}{}_{[b}{}^{d} g_{|a|c]} \nabla_d R - 2F^{\mathfrak{a}de} g_{a[b} \nabla_{|d|} L_{c]e}. \qquad (3.65)
\end{aligned}
$$

In a similar manner to the two previous matter models, when equations

(3.63), (3.64) and (3.65) are coupled to the system of wave equations for the geometric conformal fields, the $\square$ operator is to be replaced by its counterpart $\blacksquare$.

3.7.3.2 Subsidiary equations

The next step in the analysis of the Yang-Mills field is the introduction of the corresponding subsidiary quantities and the consequent construction of suitable geometric wave equations for them. For this purpose define the following set of zero-quantities:

$$M^{\mathfrak{a}}{}_{a} \equiv \nabla^{b} F^{\mathfrak{a}}{}_{ba} + C^{\mathfrak{a}}{}_{\mathfrak{bc}} A^{\mathfrak{b}b} F^{\mathfrak{c}}{}_{ba}, \tag{3.66a}$$
$$M^{\mathfrak{a}}{}_{ab} \equiv \nabla_{a} A^{\mathfrak{a}}{}_{b} - \nabla_{b} A^{\mathfrak{a}}{}_{a} + C^{\mathfrak{a}}{}_{\mathfrak{bc}} A^{\mathfrak{b}}{}_{a} A^{\mathfrak{c}}{}_{b} - F^{\mathfrak{a}}{}_{ab}, \tag{3.66b}$$
$$M^{\mathfrak{a}}{}_{abc} \equiv \nabla_{[a} F^{\mathfrak{a}}{}_{bc]} + C^{\mathfrak{a}}{}_{\mathfrak{bc}} A^{\mathfrak{b}}{}_{[a} F^{\mathfrak{c}}{}_{bc]}, \tag{3.66c}$$
$$Q^{\mathfrak{a}}{}_{abc} \equiv F^{\mathfrak{a}}{}_{abc} - \nabla_{a} F^{\mathfrak{a}}{}_{bc} - C^{\mathfrak{a}}{}_{\mathfrak{bc}} A^{\mathfrak{b}}{}_{a} F^{\mathfrak{c}}{}_{bc}. \tag{3.66d}$$

Notice that, unlike the analysis for the Maxwell field, an additional field $M^{\mathfrak{a}}{}_{ab}$ must be considered due to the introduction of the gauge potentials.

From the above definitions, it follows that $M^{\mathfrak{a}}{}_{abc}$ and $M^{\mathfrak{a}}{}_{ab}$ possess the symmetries

$$M^{\mathfrak{a}}{}_{abc} = M^{\mathfrak{a}}{}_{a[bc]} = M^{\mathfrak{a}}{}_{[ab]c} = M^{\mathfrak{a}}{}_{[abc]}, \quad M^{\mathfrak{a}}{}_{ab} = -M^{\mathfrak{a}}{}_{ba}. \tag{3.67}$$

Also, by combining (3.66c) and (3.66d), an auxiliary relation is directly obtained, namely

$$3M^{\mathfrak{a}}{}_{abc} + 3Q^{\mathfrak{a}}{}_{[abc]} - 3F^{\mathfrak{a}}{}_{[abc]} = 0. \tag{3.68}$$

Furthermore, direct calculations show that the Yang-Mills zero-quantities satisfy the relations

$$\nabla_{a} M^{\mathfrak{a}a} = -C^{\mathfrak{a}}{}_{\mathfrak{bc}} A^{\mathfrak{b}a} M^{\mathfrak{c}}{}_{a} + \tfrac{1}{2} C^{\mathfrak{a}}{}_{\mathfrak{bc}} F^{\mathfrak{b}ab} M^{\mathfrak{c}}{}_{ab}, \tag{3.69a}$$
$$\nabla^{b} M^{\mathfrak{a}}{}_{ab} = M^{\mathfrak{a}}{}_{a}, \tag{3.69b}$$
$$\begin{aligned}\nabla_{a} M^{\mathfrak{a}}{}_{bc}{}^{a} = {} & -\tfrac{2}{3} \nabla_{[b} M^{\mathfrak{a}}{}_{c]} - \tfrac{2}{3} C^{\mathfrak{a}}{}_{\mathfrak{bc}} A^{\mathfrak{b}}{}_{[b} M^{\mathfrak{c}}{}_{c]} - C^{\mathfrak{a}}{}_{\mathfrak{bc}} A^{\mathfrak{b}a} M^{\mathfrak{c}}{}_{abc} - \tfrac{2}{3} C^{\mathfrak{a}}{}_{\mathfrak{bc}} A^{\mathfrak{b}a} Q^{\mathfrak{c}}{}_{abc} \\ & -\tfrac{2}{3} C^{\mathfrak{a}}{}_{\mathfrak{bc}} F^{\mathfrak{b}}{}_{[b}{}^{a} M^{\mathfrak{c}}{}_{c]a}. \end{aligned} \tag{3.69c}$$

Equation for $M^{\mathfrak{a}}{}_{ab}$. Consider the expression $3\nabla_{[c}M^{\mathfrak{a}}{}_{ab]}$. Commuting covariant derivatives, substituting expressions (3.66c), (3.66d) and exploiting the Jacobi identity for the structure constants, the following integrability condition is obtained:

$$3\nabla_{[c}M^{\mathfrak{a}}{}_{ab]} = -M^{\mathfrak{a}}{}_{abc} - 3C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}A^{\mathfrak{b}}{}_{[a}M^{\mathfrak{c}}{}_{bc]}. \tag{3.70}$$

Applying ∇^c to the last equation, a short calculation using equations (3.69a) and (3.69c) yields

$$\begin{aligned}\square M^{\mathfrak{a}}{}_{ab} = {}& 3A^{\mathfrak{b}c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}M^{\mathfrak{c}}{}_{abc} + 2A^{\mathfrak{b}c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}Q^{\mathfrak{c}}{}_{cab} + \tfrac{1}{3}RM^{\mathfrak{a}}{}_{ab} - f^{\mathfrak{b}}(x)C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}M^{\mathfrak{c}}{}_{ab}\\ & - A^{\mathfrak{b}c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_c M^{\mathfrak{c}}{}_{ab} - 2\Xi d_{[a}{}^{c}{}_{b]}{}^{d}M^{\mathfrak{a}}{}_{cd} + 2F^{\mathfrak{b}}{}_{[a}{}^{c}C^{\mathfrak{a}}{}_{|\mathfrak{b}\mathfrak{c}|}M^{\mathfrak{c}}{}_{b]c}\\ & - 2C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}M^{\mathfrak{b}}{}_{[a}{}^{c}\nabla_{|c|}A^{\mathfrak{c}}{}_{b]}.\end{aligned} \tag{3.71}$$

Equation for $M^{\mathfrak{a}}{}_{a}$. Equation (3.69c) constitutes an integrability condition for the field $M^{\mathfrak{a}}{}_{a}$. A suitable wave equation can be obtained by first applying ∇^c, commuting derivatives and observing that $\nabla_c\nabla_a M^{\mathfrak{a}}{}_{b}{}^{ac} = \nabla_{[c}\nabla_{a]}M^{\mathfrak{a}}{}_{b}{}^{ac}$. Then, using definitions (3.66a)-(3.66d) along with (3.69a), (3.69b), (3.70), the Jacobi identity, and an appropriate substitution of (3.68), a long but straightforward computation results in

$$\begin{aligned}\square M^{\mathfrak{a}}{}_{b} = {}& 2L_{ba}M^{\mathfrak{a}a} + \tfrac{1}{6}RM^{\mathfrak{a}}{}_{b} + 2F^{\mathfrak{c}}{}_{ba}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}M^{\mathfrak{b}a} - f^{\mathfrak{b}}(x)C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}M^{\mathfrak{c}}{}_{b}\\ & - A^{\mathfrak{b}a}A^{\mathfrak{c}}{}_{a}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}}M^{\mathfrak{d}}{}_{b} - \tfrac{3}{2}F^{\mathfrak{b}ac}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}M^{\mathfrak{c}}{}_{bac} + 3A^{\mathfrak{b}a}A^{\mathfrak{c}c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{d}}C^{\mathfrak{d}}{}_{\mathfrak{c}\mathfrak{e}}M^{\mathfrak{e}}{}_{bac}\\ & + 2A^{\mathfrak{b}a}A^{\mathfrak{c}c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{d}}C^{\mathfrak{d}}{}_{\mathfrak{c}\mathfrak{e}}Q^{\mathfrak{e}}{}_{cba} - \tfrac{3}{2}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}M^{\mathfrak{c}}{}_{bac}M^{\mathfrak{b}ac} + 2F^{\mathfrak{b}a}{}_{b}{}^{c}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}M^{\mathfrak{c}}{}_{ac}\\ & - 2C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}Q^{\mathfrak{b}a}{}_{b}{}^{c}M^{\mathfrak{c}}{}_{ac} + F^{\mathfrak{c}}{}_{b}{}^{c}A^{\mathfrak{b}a}C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{d}}C^{\mathfrak{d}}{}_{\mathfrak{b}\mathfrak{e}}M^{\mathfrak{e}}{}_{ac} - 2A^{\mathfrak{b}a}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_a M^{\mathfrak{c}}{}_{b}\\ & + 2A^{\mathfrak{b}a}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_c Q^{\mathfrak{c}}{}_{ab}{}^{c} - 3C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}M^{\mathfrak{c}}{}_{bac}\nabla^c A^{\mathfrak{b}a} + 2C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}Q^{\mathfrak{c}}{}_{abc}\nabla^c A^{\mathfrak{b}a}.\end{aligned} \tag{3.72}$$

Equation for $M^{\mathfrak{a}}{}_{abc}$. In a similar fashion to the approach adopted for the electromagnetic zero-quantity M_{abc}, and in order to simplify the calculations, we introduce the Hodge dual of $M^{\mathfrak{a}}{}_{abc}$:

$$M^{*\mathfrak{a}}{}_{a} \equiv C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}F^{*\mathfrak{c}}{}_{ba}A^{\mathfrak{b}b} + \nabla^b F^{*\mathfrak{a}}{}_{ba} = \tfrac{1}{2}\epsilon_a{}^{bcd}M^{\mathfrak{a}}{}_{bcd}. \tag{3.73}$$

Here, the second equality has been obtained with help of the definition of

$F^{*\mathfrak{a}}{}_{ab}$ and equation (3.66c). With this expression we compute the divergence of $M^{*\mathfrak{a}}{}_a$. Making use of (3.66b) and the Jacobi identity, a calculation yields

$$\begin{aligned}\nabla_a M^{*\mathfrak{a}a} &= -C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}}F^{*\mathfrak{d}}{}_{ab}A^{\mathfrak{b}a}A^{\mathfrak{c}b} - C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}A^{\mathfrak{b}a}M^{*\mathfrak{c}}{}_a + C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}F^{*\mathfrak{c}}{}_{ab}\nabla^b A^{\mathfrak{b}a}\\ &= -C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}A^{\mathfrak{b}a}M^{*\mathfrak{c}}{}_a - \tfrac{1}{4}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\epsilon_{ab}{}^{cd}F^{*\mathfrak{b}ab}M^{\mathfrak{c}}{}_{cd}.\end{aligned}$$

In terms of non-dual objects this takes the form of an integrability condition:

$$\begin{aligned}\epsilon^{abcd}\nabla_d M^{\mathfrak{a}}{}_{abc} &= C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\epsilon^{abcd}A^{\mathfrak{b}}{}_a M^{\mathfrak{c}}{}_{bcd} + \tfrac{1}{2}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\epsilon^{abcd}F^{\mathfrak{b}}{}_{ab}M^{\mathfrak{c}}{}_{cd}\\ \iff 4\nabla_{[a}M^{\mathfrak{a}}{}_{bcd]} &= 4C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}A_{[a}M^{\mathfrak{a}}{}_{bcd]} + 2C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}F^{\mathfrak{b}}{}_{[ab}M^{\mathfrak{c}}{}_{cd]}.\end{aligned} \tag{3.74}$$

From here, a suitable wave equation can be obtained by applying ∇^d and commuting derivatives. After a long calculation in which definitions (3.66a)-(3.66d), equations (3.68)-(3.70) and the Jacobi identity are employed, one finds that

$$\begin{aligned}\square M^{\mathfrak{a}}{}_{abc} = {}& \tfrac{1}{2}RM^{\mathfrak{a}}{}_{abc} - A^{\mathfrak{b}d}A^{\mathfrak{c}}{}_d C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{d}}C^{\mathfrak{d}}{}_{\mathfrak{c}\mathfrak{e}}M^{\mathfrak{e}}{}_{abc} - C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}f^{\mathfrak{b}}(x)M^{\mathfrak{c}}{}_{abc}\\ &- 2A^{\mathfrak{b}d}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_d M^{\mathfrak{c}}{}_{abc} - 6\Xi d_{[a}{}^{d}{}_{b}{}^{e}M^{\mathfrak{a}}{}_{c]de} - 6L_{[a}{}^{d}M^{\mathfrak{a}}{}_{bc]d}\\ &+ 2F^{\mathfrak{b}d}{}_{[ab}C^{\mathfrak{a}}{}_{|\mathfrak{b}\mathfrak{c}|}M^{\mathfrak{c}}{}_{c]d} - 6F^{\mathfrak{b}}{}_{[a}{}^{d}C^{\mathfrak{a}}{}_{|\mathfrak{b}\mathfrak{c}|}M^{\mathfrak{c}}{}_{bc]d} - 2A^{\mathfrak{b}d}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_{[a}Q^{\mathfrak{c}}{}_{|d|bc]}\\ &+ 2C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}Q^{\mathfrak{b}d}{}_{[ab}\nabla_{c]}A^{\mathfrak{c}}{}_d - 2C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}Q^{\mathfrak{b}d}{}_{[ab}M^{\mathfrak{c}}{}_{c]d} + F^{\mathfrak{b}}{}_{[ab}A^{\mathfrak{c}d}C^{\mathfrak{a}}{}_{|\mathfrak{b}\mathfrak{d}}C^{\mathfrak{d}}{}_{\mathfrak{c}\mathfrak{e}|}M^{\mathfrak{e}}{}_{c]d}\\ &- 2A^{\mathfrak{b}}{}_{[a}A^{\mathfrak{c}d}C^{\mathfrak{a}}{}_{|\mathfrak{b}\mathfrak{d}}C^{\mathfrak{d}}{}_{\mathfrak{c}\mathfrak{e}}Q^{\mathfrak{e}}{}_{d|bc]}.\end{aligned} \tag{3.75}$$

Equation for $Q^{\mathfrak{a}}{}_{abc}$. Similar to the case for the Maxwell field, a wave equation for $Q^{\mathfrak{a}}{}_{abc}$ can be obtained by directly applying the $\square$ operator to its definition. Since the identity used in the deduction of equation (3.60) has the same form for the Yang-Mills strength field, an analogous procedure can be followed. A long computation gives:

$$\begin{aligned}
\square Q^{\mathfrak{a}}{}_{abc} = {} & 6L_a{}^d M^{\mathfrak{a}}{}_{bcd} + \tfrac{1}{2} R Q^{\mathfrak{a}}{}_{abc} + 4L_a{}^d Q^{\mathfrak{a}}{}_{dbc} - f^{\mathfrak{b}}(x) C^{\mathfrak{a}}{}_{\mathfrak{bc}} Q^{\mathfrak{c}}{}_{abc} \\
& - 2F^{\mathfrak{b}}{}_a{}^d C^{\mathfrak{a}}{}_{\mathfrak{bc}} Q^{\mathfrak{c}}{}_{dbc} - A^{\mathfrak{b}d} A^{\mathfrak{c}}{}_d C^{\mathfrak{a}}{}_{\mathfrak{bd}} C^{\mathfrak{d}}{}_{\mathfrak{ce}} Q^{\mathfrak{e}}{}_{abc} + 2A^{\mathfrak{b}}{}_a A^{\mathfrak{c}d} C^{\mathfrak{a}}{}_{\mathfrak{bd}} C^{\mathfrak{d}}{}_{\mathfrak{ce}} Q^{\mathfrak{e}}{}_{dbc} \\
& + F^{\mathfrak{c}}{}_{bc} A^{\mathfrak{b}d} C^{\mathfrak{a}}{}_{\mathfrak{cd}} C^{\mathfrak{d}}{}_{\mathfrak{be}} M^{\mathfrak{e}}{}_{ad} - 2F^{\mathfrak{c}}{}_{bc} A^{\mathfrak{b}d} C^{\mathfrak{a}}{}_{\mathfrak{bd}} C^{\mathfrak{d}}{}_{\mathfrak{ce}} M^{\mathfrak{e}}{}_{ad} + 2C^{\mathfrak{a}}{}_{\mathfrak{bc}} Q^{\mathfrak{c}}{}_{dbc} \nabla_a A^{\mathfrak{b}d} \\
& + 4A^{\mathfrak{b}d} C^{\mathfrak{a}}{}_{\mathfrak{bc}} \nabla_{[a} Q^{\mathfrak{c}}{}_{d]bc} + 2C^{\mathfrak{a}}{}_{\mathfrak{bc}} M^{\mathfrak{b}}{}_a{}^d \nabla_d F^{\mathfrak{c}}{}_{bc} + 4\Xi F^{\mathfrak{a}}{}_{[b}{}^d \Lambda_{c]ad} \\
& + 4\Xi F^{\mathfrak{a}}{}_{[b}{}^d \Lambda_{|a|c]d} - 2\Xi d_{[b}{}^d{}_{c]}{}^e Q^{\mathfrak{a}}{}_{ade} + 4\Xi d_a{}^d{}_{[b}{}^e Q^{\mathfrak{a}}{}_{|d|c]e} + 4L_{a[b} M^{\mathfrak{a}}{}_{c]} \\
& + 4L_{[b}{}^d Q^{\mathfrak{a}}{}_{|da|c]} + \Xi F^{\mathfrak{a}de} \Lambda_{[b|de} g_{a|c]} + 4F^{\mathfrak{b}}{}_{[b}{}^d C^{\mathfrak{a}}{}_{|\mathfrak{bc}} Q^{\mathfrak{c}}{}_{a|c]d} + 4L^{de} g_{a[b} Q^{\mathfrak{a}}{}_{|d|c]e}.
\end{aligned} \tag{3.76}$$

3.7.3.3 Propagation of the gauge

In this subsection we show the consistency of the introduction of the gauge source functions $f^{\mathfrak{a}}(x)$ into the analysis of the propagation of the constraints for the Yang-Mills potential. For this purpose we introduce the zero-quantity $P^{\mathfrak{a}}$ defined as:

$$P^{\mathfrak{a}} \equiv \nabla^a A^{\mathfrak{a}}{}_a - f^{\mathfrak{a}}(x). \tag{3.77}$$

The computation of a wave equation for this field is straightforward: first, a short calculation employing equations (3.64), (3.66a), (3.66b) and (3.69b) gives

$$\nabla_a P^{\mathfrak{a}} = -A^{\mathfrak{b}}{}_b C^{\mathfrak{a}}{}_{\mathfrak{bc}} P^{\mathfrak{c}} - M^{\mathfrak{a}}{}_b + \nabla_a M^{\mathfrak{a}}{}_b{}^a.$$

From here, application of a further covariant derivative directly results in

$$\square P^{\mathfrak{a}} = -f^{\mathfrak{b}} C^{\mathfrak{a}}{}_{\mathfrak{bc}} P^{\mathfrak{c}} + A^{\mathfrak{b}a} C^{\mathfrak{a}}{}_{\mathfrak{bc}} M^{\mathfrak{c}}{}_a - \tfrac{1}{2} F^{\mathfrak{b}ab} C^{\mathfrak{a}}{}_{\mathfrak{bc}} M^{\mathfrak{c}}{}_{ab} - A^{\mathfrak{b}b} C^{\mathfrak{a}}{}_{\mathfrak{bc}} \nabla_b P^{\mathfrak{c}}. \tag{3.78}$$

Remark 31. Geometric wave equations (3.71), (3.72), (3.75), (3.76) and (3.78) are homogeneous in $M^{\mathfrak{a}}{}_a$, $M^{\mathfrak{a}}{}_{ab}$, $M^{\mathfrak{a}}{}_{abc}$, $Q^{\mathfrak{a}}{}_{abc}$, $P^{\mathfrak{a}}$, Λ_{abc} and their first covariant derivatives. Thus, if these fields vanish on an initial hypersurface $\mathcal{S}_\star$, it can be guaranteed that there exists a unique solution on a small enough slab of $\mathcal{S}_\star$, and it corresponds to the trivial one.

3.7.3.4 Summary

The previous discussion about the Yang-Mills field coupled to the conformal Einstein field equations leads to the following statement:

Proposition 8. *The system of wave equations* (3.25a)-(3.25e) *with energy-momentum tensor given by* (2.30) *coupled to wave equations* (3.63), (3.64) *and* (3.65) *written in terms of the operator* $\blacksquare$*, is a proper quasilinear system of wave equations for the Einstein-Yang-Mills system.*

Chapter 4

Killing boundary data for anti-de Sitter-like spacetimes

The material of this chapter is based on [13].

The problem of encoding (continuous) symmetries of a spacetime at the level of initial data is an important classical problem in Relativity — see e.g. [48]. A modern presentation of this issue and the related theory can be found in [6, 17]. The key outcome of this theory is the so-called set of *Killing initial data equations*, a system of overdetermined equations for a scalar field and a spatial vector on a spacelike hypersurface — corresponding, respectively, to the *lapse* and *shift* with respect to the normal of the hypersurface of a hypothetical Killing vector of the spacetime. If these Killing equations admit a solution, a so-called *Killing initial data set (KID)*, then the development of the initial data will have a Killing vector. The theory of KID for the Cauchy problem for the Einstein field equations can be also adapted to other settings like the (finite and asymptotic) characteristic initial value problem [19, 50] and, more relevant for the purposes of the present article, to the asymptotic initial value problem for the de Sitter-like spacetimes [52], i.e. solutions to the vacuum Einstein field equations with positive cosmological constant.

The purpose of this chapter is to present a theory of Killing initial and boundary data in the setting of anti-de Sitter-like spacetimes. As Corollary 1 states, this class of solutions have a timelike conformal boundary so, in addition to satisfying the KID equations on some initial hypersurface, one also needs a suitable *Killing boundary data set (KBD)* to ensure the existence of a Killing vector in the spacetime. Additionally, these sets have to satisfy some compatibility conditions at the corner where the initial hypersurface and the conformal boundary meet. The use of a conformal setting allows us to perform the analysis of the boundary conditions for the Killing equations by means of local computations.

An alternative approach to the analysis of continuous symmetries in anti-de Sitter-like spacetimes has been started in [38, 39]. In this work, the objective is to encode the existence of a Killing vector solely through conditions on the conformal boundary — in the spirit of the principle of *holography*. The required analysis, thus, leads to the study of ill-posed initial value problems for wave equations which require the use of methods of the *theory of unique continuation*. Their analysis requires imposing both Dirichlet and Neumann boundary conditions on the conformal boundary while the discussion in the present work requires, as already mentioned, only Dirichlet conditions. The trade-off is that our analysis also requires a solution to the KID equation on a spacelike hypersurface and compatibility conditions between the Killing initial and boundary data.

4.1 Conformal properties of the Killing vector equation

In this section we briefly review the theory of Killing vectors from a conformal point of view, following the presentation in [52]. We begin by recalling the relation between Killing vectors in the physical spacetime $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ and conformal Killing vectors in the unphysical spacetime $(\mathcal{M}, g_{ab})$:

Lemma 5. *A vector field $\tilde{\xi}^a$ is a Killing vector field of $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$, that is*

$$\tilde{\nabla}_a\tilde{\xi}_b + \tilde{\nabla}_b\tilde{\xi}_a = 0,$$

if and only if its push-forward $\xi^a \equiv \varphi_*\tilde{\xi}^a$ is a conformal Killing vector field in $(\mathcal{M}, g_{ab})$, i.e.

$$\nabla_a\xi_b + \nabla_b\xi_a = \frac{1}{2}\nabla_c\xi^c g_{ab} \tag{4.1}$$

and, moreover, one has that

$$\xi^a \nabla_a \Xi = \frac{1}{4} \Xi \nabla_a \xi^a. \tag{4.2}$$

The proof of this result can be found in [52].

Remark 32. In the following we will call equations (4.1) and (4.2) the *unphysical Killing equations*. Observe that if g_{ab} extends smoothly across $\mathscr{I}$, then the unphysical Killing equations are well defined on the conformal boundary.

The above result naturally poses the question about the conditions for the existence of unphysical Killing vectors. This will be addressed in the remainder of this section.

4.1.1 Necessary conditions

For convenience set

$$\eta \equiv \frac{1}{4} \nabla_a \xi^a.$$

Then one has the following result:

Lemma 6. *Any solution to the unphysical Killing equations satisfies the system*

$$\Box \xi_a + R_a{}^b \xi_b + 2\nabla_a \eta = 0, \tag{4.3a}$$

$$\Box \eta + \frac{1}{6} \xi^a \nabla_a R + \frac{1}{3} R \eta = 0. \tag{4.3b}$$

The proof of the above result follows by direct computation from (4.1) and (4.2).

Remark 33. The wave equations (4.3a) and (4.3b) are necessary conditions for a vector ξ^a to be an *unphysical Killing vector*. However, not every solution to these equations is an unphysical Killing vector. In this sense, a vector field satisfying (4.3a)-(4.3b) will be called an *unphysical Killing vector candidate*.

4.1.2 The unphysical Killing equation propagation system

The sufficient conditions are now discussed. It will be convenient to define the following *zero-quantities*:

$$
\begin{aligned}
S_{ab} &\equiv \nabla_a \xi_b + \nabla_b \xi_a - 2\eta g_{ab}, \\
S_{abc} &\equiv \nabla_a S_{bc}, \\
\phi &\equiv \xi^a \nabla_a \Xi - \Xi \eta, \\
\psi &\equiv \eta s + \xi^a \nabla_a s - \nabla_a \eta \nabla^a \Xi, \\
B_{ab} &\equiv \mathcal{L}_\xi L_{ab} + \nabla_a \nabla_b \eta,
\end{aligned}
$$

with $\mathcal{L}_\xi$ denoting the *Lie derivative* along the direction of ξ^a, that is

$$
\mathcal{L}_\xi L_{ab} = \xi^c \nabla_c L_{ab} + L_{cb} \nabla_a \xi^c + L_{ac} \nabla_b \xi^c.
$$

In terms of these quantities, a lengthy computation leads to the following result proved in [52]:

Lemma 7. *Let ξ^a and η be a pair of fields satisfying equations* (4.3a)-(4.3b)*. Then the tensor fields*

$$
S_{ab}, \quad S_{abc}, \quad \phi, \quad \psi, \quad B_{ab},
$$

satisfy a closed system of homogeneous wave equations. Schematically, one has that

$$
\begin{aligned}
\square S_{ab} &= H_{ab}(S, B), && (4.4a) \\
\square S_{abc} &= H_{abc}(S, B, \nabla S, \nabla B), && (4.4b) \\
\square \phi &= H(\phi, \psi, S), && (4.4c) \\
\square \psi &= K(\phi, S, B, \psi, \nabla \phi), && (4.4d) \\
\square B_{ab} &= K_{ab}(S, B, \nabla S, \nabla B, \nabla^2 S), && (4.4e)
\end{aligned}
$$

where ∇S and $\nabla^2 S$ represent $\nabla_a S_{bc}$ and $\nabla_a S_{bcd}$, respectively.

Remark 34. In what follows, the system consisting of equations (4.3a)-(4.3b) together with (4.4a)-(4.4e) will be called the *unphysical Killing equa-*

tion propagation system.

The homogeneity of the evolution system (4.4a)-(4.4e) together with the theory of initial-boundary value problems for systems of wave equations (see e.g. [14, 21]) suggests to consider a Dirichlet problem to ensure the existence of a solution to the unphysical Killing vector equations. Let $\mathcal{S}_\star$ be an initial spacelike hypersurface. The conditions for the problem are:

(i) Initial data

$$S_{ab} = 0, \quad S_{abc} = 0, \quad \phi = 0, \quad \psi = 0, \quad B_{ab} = 0, \tag{4.5a}$$

$$\nabla_e S_{ab} = 0, \quad \nabla_e S_{abc} = 0, \quad \nabla_e \phi = 0, \quad \nabla_e \psi = 0, \quad \nabla_e B_{ab} = 0, \tag{4.5b}$$

(ii) (Dirichlet) boundary data

$$S_{ab} = 0, \quad S_{abc} = 0, \quad \phi = 0, \quad \psi = 0, \quad B_{ab} = 0. \tag{4.6}$$

If the above conditions are satisfied, the homogeneity of the wave equations (4.4a)-(4.4e) guarantees that the only solution of the system is the trivial one. This means, therefore, that the solution to equations (4.3a)-(4.3b) will actually be an unphysical Killing vector. Motivated by this, we will refer to the fields S_{ab}, S_{abc}, ϕ, ψ, B_{ab} as the *Killing vector zero-quantities.*

Remark 35. Strictly speaking, the initial conditions require only the vanishing of the zero-quantities and of their normal derivatives to the initial hypersurface. If these conditions hold then the full covariant derivative of the zero-quantities vanish on $\mathcal{S}_\star$.

4.2 The Killing vector zero-quantities

In order to study the Dirichlet problem, we first investigate the Killing vector zero-quantities and the relations between them. This analysis can be supplemented by the conformal constraint equations (2.15a)-(2.15h), (2.16a) and (2.16b). More specifically, the constraints on the conformal boundary (2.17a)-(2.17j) will become relevant for the subsequent analysis, along with their solution in the particular case of vacuum — see Proposition 2.

Let $\mathcal{H} \subset \mathcal{M}$ be an either spacelike or timelike hypersurface with 3-metric denoted by h_{ab}, normal vector n^a. In this respect, let us define the following relevant quantities:

$$\zeta_i, \quad \zeta, \quad \mathcal{S}_{ij}, \quad \mathcal{S}_i, \quad \mathcal{S}, \quad \mathcal{S}_{ijk}, \quad \mathcal{B}_{ij} \quad \mathcal{B}_i, \quad \mathcal{B}$$

as the respective pull-backs of the following projections of the Killing vector candidate ξ_a and the zero-quantities to $\mathcal{H}$:

$$h_a{}^b\xi_b, \quad n^a\xi_a, \quad h_a{}^c h_b{}^d S_{cd}, \quad n^c h_a{}^b S_{bc}, \quad n^a n^b S_{ab}, \quad h_a{}^d h_b{}^e h_c{}^f S_{def},$$
$$h_a{}^c h_b{}^d B_{cd}, \quad n^c h_a{}^b B_{bc}, \quad n^a n^b B_{ab}.$$

In the next subsection, the vanishing of the zero-quantities on $\mathcal{S}_\star$ and $\mathscr{I}$ will be analysed using these objects.

4.2.1 Decomposition of ϕ and ψ

From their definitions, a straightforward decomposition of the zero-quantities ϕ, ψ, and their normal derivatives, leads to the following expressions:

$$\phi = \zeta^i D_i \Xi + \epsilon\zeta\Sigma - \eta\Xi, \tag{4.7a}$$
$$n^a\nabla_a\phi = -\eta\Sigma - \Xi D\eta + D\zeta^i D_i\Xi + \zeta^i(D_i\Sigma - K_i{}^j D_j\Xi) + \epsilon(\zeta D\Sigma + \Sigma D\zeta), \tag{4.7b}$$

and

$$\psi = \eta s + \zeta^i D_i s + \epsilon\zeta Ds - D_i\eta D^i\Xi - \epsilon\Sigma D\eta, \tag{4.8a}$$
$$n^a\nabla_a\psi = \eta Ds + sD\eta + D\zeta^i D_i s + \zeta^i(D_i Ds - K_i{}^j D_j s) - D^i\eta(D_i\Sigma - K_i{}^j D_j\Xi) - D_i\Xi(D^i D\eta - K^i{}_j D^j\eta) + \epsilon(\zeta D^2 s + D\zeta Ds - D\Sigma D\eta - \Sigma D^2\eta). \tag{4.8b}$$

4.2.2 Decomposition of S_{ab}, B_{ab} and their derivatives

Before performing a decomposition of the remaining zero-quantities, a few observations can be made about the redundancy of some of their components. For this task their explicit decompositions will not be required but expressions will be given in terms of functions which are homogeneous in the zero-quantities and their derivatives; this will prove to be useful when imposing the vanishing initial-boundary data.

Lemma 8. *Let $\mathcal{H} \subset \mathcal{M}$ be either a timelike or spacelike hypersurface. Assume that S_{ij}, $D\mathcal{S}_{ij}$, $\mathcal{B}_{ij}$ and $D\mathcal{B}_{ij}$ are known on $\mathcal{H}$. Then, the remaining components of the zero-quantities and their first-order derivatives can be computed on $\mathcal{H}$.*

Proof. In the following, for ease of presentation, let f denote a generic homogeneous function of its arguments which may change from line to line. As pointed out in [52], equation (4.3a) implies the identity

$$\nabla_a S_b{}^a - \frac{1}{2}\nabla_b S_a{}^a = 0. \tag{4.9}$$

Expressing S_{ab} in terms of its components, a short calculation yields

$$\epsilon D\mathcal{S}_b + \frac{1}{2} n_b D\mathcal{S} = f(\mathcal{S}_{ij},\ D_i\mathcal{S}_{jk},\ D\mathcal{S}_{ij}). \tag{4.10}$$

Multiplying this equation by $h_a{}^b$, an equation for $D\mathcal{S}_i$ is obtained. Similarly, multiplying equation (4.10) by n^b we obtain an analogous expression for $D\mathcal{S}$. Then, all the components of DS_{ab} can be computed on $\mathcal{H}$ and, in consequence, S_{abc} is known.

In order to analyse the fields derived from B_{ab}, consider equation (4.4a) which can be written in a more explicit way as:

$$D^2 S_{ab} = -4\epsilon B_{ab} + f(S_{ab},\ \nabla_c S_{ab},\ D_c D_d S_{ab}). \tag{4.11}$$

As it is assumed that $\mathcal{B}_{ij}$ is known on $\mathcal{H}$, one can solve for $D^2\mathcal{S}_{ij}$ from this last equation; in particular, $D^2\mathcal{S}_i{}^i$ can be computed. On the other hand, by applying ∇_c to (4.9), a lengthy but direct decomposition leads to the following two relations:

$$D^2\mathcal{S}_i = f(S_{ab},\ \nabla_c S_{ab}), \tag{4.12a}$$
$$\epsilon D^2\mathcal{S} = D^2\mathcal{S}_i{}^i + f(S_{ab},\ \nabla_c S_{ab}). \tag{4.12b}$$

From here we observe that their right-hand sides are either known or computable on $\mathcal{H}$ so the components $D^2\mathcal{S}_i$ and $D^2\mathcal{S}$ are determined. Thus, (4.11) implies that the components $\mathcal{B}_i$ and $\mathcal{B}$ can be computed.

Regarding the normal derivatives of B_{ab}, we make use of the identity

$$\nabla_a B_b{}^a - \tfrac{1}{2}\nabla_b B_a{}^a = S_{cd}(\nabla^c L_b{}^d - \tfrac{1}{2}\nabla_b L^{cd}),$$

whose validity is guaranteed by equations (4.3a) and (4.3b) — see [52]. Observe that its left hand side has the same form as equation (4.9), while its right hand side is homogeneous on S_{ab}. Then we conclude that $D\mathcal{B}_i$ and $D\mathcal{B}$ are computable.

Finally, the normal derivative of S_{abc} can be analysed from its definition. Commuting derivatives, a short calculation yields:

$$DS_{abc} = D_a(DS_{bc}) + \epsilon n_a D^2 S_{bc} + f(S_{ab},\ \nabla_c S_{ab}).$$

Since it has been proved that all the terms on the right-hand side are either computable or part of the given data on $\mathcal{H}$, the proof is complete.

□

Remark 36. Lemma 8 is valid either for a spacelike or timelike hypersurface, but given that it assumes certain normal derivatives, it is naturally adapted to a spacelike hypersurface where first-order derivatives are assumed as part of the initial data. If, on the other hand, $\mathcal{H}$ is timelike and Dirichlet conditions are assumed, then $D\mathcal{S}_{ij}$ plays the role of the only necessary component of S_{abc}, while $D\mathcal{B}_{ij}$ is not required.

In view of the previous result, the explicit form of the remaining independent data under a decomposition on $\mathcal{H}$ is given by:

$$\mathcal{S}_{ij} = D_i\zeta_j + D_j\zeta_i + 2\epsilon\zeta K_{ij} - 2\eta h_{ij}, \tag{4.13a}$$

$$\mathcal{S}_i = D\zeta_i + D_i\zeta - \zeta^j K_{ij}, \tag{4.13b}$$

$$\mathcal{S} = 2D\zeta - 2\epsilon\eta, \tag{4.13c}$$

$$D\mathcal{S}_{ij} = 2D_{(i}D\zeta_{j)} - 2K_{(i}{}^k D_{|k|}\zeta_{j)} + 2\zeta^k D_k K_{ij} - 2\zeta^k D_{(i}K_{j)k} + 2\epsilon\zeta DK_{ij} \\ +2\epsilon K_{ij}D\zeta - 2h_{ij}D\eta, \tag{4.13d}$$

$$\mathcal{B}_{ij} = \zeta^k D_k\theta_{ij} + 2\theta_{k(i}D_{j)}\zeta^k + 2\epsilon\zeta K_{(i}{}^k\theta_{j)k} + 2\epsilon\theta_{(i}D_{j)}\zeta + \epsilon\zeta D\theta_{ij} \\ +D_iD_j\eta, \tag{4.13e}$$

$$D\mathcal{B}_{ij} = D_k\theta_{ij}D\zeta^k + K_j{}^m D_m L_{ij} + D_k D\theta_{ij} + 2n^e L_{(i}{}^m R_{j)mek} + 2K_{(i}{}^k\theta_{j)m}D\zeta \\ +2\zeta K_{(i}{}^k D\theta_{j)k} + \zeta\theta_{k(i}DK_{j)}{}^k + 2D_{(j}\zeta^k D\theta_{i)k} + 2\theta_{k(i}(D_jD\zeta^k - K_{b)}{}^e D_e\zeta^c \\ +n^e R_{b)ed}{}^c\zeta^d) + 2D\theta_{(i}D_{j)}\zeta + 2\theta_{(i}D_{j)}D\zeta - 2\theta_{(i}K_{j)}{}^k D_k\zeta + 2\zeta D^2\theta_{ij} \\ +2D\zeta D\theta_{ij} + D_iD_jD\eta - 2K_{(i}{}^k D_{j)}D_k\eta - D_k\eta D_iK_j{}^k \\ -n^e R_{jiek}{}^k D_k\eta. \tag{4.13f}$$

4.3 Boundary analysis

We now proceed to discuss the explicit requirements a well-posed initial-boundary problem with vanishing Dirichlet data imposes on the conformal Killing vector candidate and the related quantities. In this subsection, whenever the symbol $\simeq$ appears — see Section 2.3.1 — the quantities involved will be assumed to be intrinsic to $\mathscr{I}$ despite not being necessarily crossed by a line.

4.3.1 Zero-quantities on $\mathscr{I}$

Having obtained the decomposition of the zero-quantities, one can then study them on the conformal boundary and, in particular, analyse the consequences the vanishing Dirichlet conditions impose. As mentioned in Remark 36, the independent data on $\mathscr{I}$ are given by ϕ, ψ, $\mathcal{S}_{ab}$, $\not{D}\mathcal{S}_{ab}$ and $\mathcal{B}_{ab}$. Evaluating equations (4.7a), (4.8a) and (4.13a)-(4.13e) on $\mathscr{I}$, one obtains

$$\phi \simeq \not{\Sigma}\zeta, \tag{4.14a}$$
$$\psi \simeq \eta s + \zeta^i \not{D}_i s + \zeta \not{D} s - \Sigma \not{D}\eta, \tag{4.14b}$$
$$\mathcal{S}_{ij} \simeq \not{D}_i \zeta_j + \not{D}_j \zeta_i + 2\varkappa\zeta\ell_{ij} - 2\eta\ell_{ij}, \tag{4.14c}$$
$$\mathcal{S}_a \simeq \not{D}_i \zeta + \not{D}\zeta_i - \varkappa\zeta_i, \tag{4.14d}$$
$$\mathcal{S} \simeq 2\not{D}\zeta - 2\eta, \tag{4.14e}$$
$$\begin{aligned}\not{D}\mathcal{S}_{ij} \simeq{} & 2\not{D}_{(i}\not{D}\zeta_{j)} - 2\varkappa\not{D}_{(i}\zeta_{j)} + 2\ell_{ij}\zeta^k\not{D}_k\varkappa - 2\zeta_{(i}\not{D}_{j)}\varkappa + 2\zeta\not{D}K_{ij} \\ & + 2\varkappa\ell_{ij}\not{D}\zeta - 2\ell_{ij}\not{D}\eta, \end{aligned} \tag{4.14f}$$
$$\begin{aligned}\mathcal{B}_{ij} \simeq{} & \zeta^k\not{D}_k l_{ij} + 2l_{k(i}\not{D}_{j)}\zeta^k + 2\varkappa\zeta l_{ij} - 2\not{D}_{(i}\varkappa\not{D}_{j)}\zeta + \zeta\not{D}L_{ij} \\ & + \not{D}_i\not{D}_j\eta. \end{aligned} \tag{4.14g}$$

Imposing Dirichlet vanishing data on $\mathscr{I}$, equations (4.14a)-(4.14g) provide a number of conditions for the fields and their derivatives on the conformal boundary. Using the definition of η and the solution in Proposition 2, it follows that the set of independent conditions is given by:

$$\zeta \simeq 0, \tag{4.15a}$$
$$\not{D}\zeta_i \simeq \varkappa\zeta_i, \tag{4.15b}$$
$$\not{D}_i\zeta_j + \not{D}_j\zeta_i \simeq 2\eta\ell_{ij}, \tag{4.15c}$$
$$\not{D}\eta \simeq \eta\varkappa + \zeta^i\not{D}_i\varkappa, \tag{4.15d}$$
$$\mathcal{L}_\zeta \not{l}_{ij} + \not{D}_i\not{D}_j\eta \simeq 0. \tag{4.15e}$$

Conversely, it is straightforward to check that equations (4.15a)-(4.15e) are sufficient to guarantee the vanishing of the equations (4.14a)-(4.14g). The above discussion leads to the following:

Lemma 9. *Let* $(\mathcal{M}, g_{ab})$ *be a conformal extension of an anti-de Sitter spacetime* $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ *with timelike conformal boundary* $\mathscr{I}$*. Let* ξ^a *be a conformal Killing vector field candidate and* ϕ*,* ψ*,* S_{ab}*,* B_{ab} *and* S_{abc} *be the corresponding zero-quantities. Then, the zero-quantities appearing in equations* (4.14a)-(4.14g) *vanish on* $\mathscr{I}$ *if and only if the components* ζ_i*,* ζ *and* η *satisfy the conditions* (4.15a)-(4.15e)*.*

Remark 37. Equations (4.15a)-(4.15e) will be called the *Killing boundary* *data* equations. In particular, they acquire a simpler form if one makes use of a gauge for which $\varkappa = 0$.

4.3.2 Existence of the intrinsic conformal Killing vector

As stated in Lemma 9, one of the necessary conditions under which the set of zero-quantities vanish on $\mathscr{I}$ is given by (4.15c), i.e. the transversal component ζ_i of the conformal Killing vector candidate has to be a conformal Killing vector with respect to the connection $\not{D}_a$. In order to guarantee the existence of a solution to this equation we consider an initial value problem on $\mathscr{I}$. Following the model of the spacetime problem, we construct a suitable wave equation for ζ_i. More precisely, one has the following:

Lemma 10. *Let ζ_i and η be a pair of fields satisfying equations* (4.15c) *and* (4.15e) *on $\mathscr{I}$. Then, it follows that*

$$\square_\ell \zeta_i \simeq -\not{r}_i{}^j \zeta_j - \not{D}_i \eta, \tag{4.16a}$$

$$\square_\ell \eta \simeq -\tfrac{1}{2}\eta \not{r} - \tfrac{1}{4}\zeta^i \not{D}_i \not{r}. \tag{4.16b}$$

Proof. The result is readily obtained by applying $\not{D}^a$ to (4.15c) and taking the trace of (4.15e). □

Remark 38. Given that this system of wave equations propagates η and ζ_i along the conformal boundary, it must be provided with initial data at the corner $\partial\mathcal{S}_\star$.

To prove that a solution to these wave equations also solves the conformal Killing equation on the boundary, a suitable system of wave equations for the corresponding 3-dimensional zero-quantities has to be constructed. The desired relations are contained in the following lemma:

Lemma 11. *Let $\mathcal{S}_{ij}$, $\mathcal{S}_{ijk}$ and $\mathcal{B}_{ij}$ be the projections of the zero-quantities S_{ab}, S_{abc} and B_{ab} into $\mathscr{I}$, respectively. Assume that there exist fields ζ_i and η on $\mathscr{I}$ satisfying the wave equations* (4.16a) *and* (4.16b)*. Then, one* *has that*

$$
\begin{aligned}
\square_\ell \mathcal{S}_{ij} &\simeq 2\, \slashed{l}_{(i}{}^{k}\mathcal{S}_{j)k} - 2\slashed{d}_{ikjm}\mathcal{S}^{km} - 2\mathcal{B}_{ij}, &(4.17a)\\
\square_\ell \mathcal{S}_{mij} &\simeq \slashed{d}_m{}^{k}\mathcal{S}_{kij} - 4\slashed{d}_{(i|nmk|}\mathcal{S}^{k}{}_{j)}{}^{n} - \tfrac{1}{2}\slashed{d}\mathcal{S}_{mij} + 2\slashed{d}_{(i}{}^{k}\mathcal{S}_{|m|j)k} - 2\slashed{d}_{ikjn}\mathcal{S}_m{}^{kn}\\
&\quad +2\mathcal{S}_{(i}{}^{k}\slashed{D}_{j)}\slashed{d}_{mk} - 2\mathcal{S}_{(i}{}^{k}\slashed{D}_{|k|}\slashed{d}_{j)m} + 2\mathcal{S}_{(i}{}^{k}\slashed{D}_{|m|}\slashed{d}_{j)k} - \tfrac{1}{2}\mathcal{S}_{ij}\slashed{D}_m\slashed{d}\\
&\quad -2\mathcal{S}^{kn}\slashed{D}_m\slashed{d}_{ikjn} - 2\slashed{D}_e\mathcal{B}_{ij}, &(4.17b)\\
\square_\ell \mathcal{B}_{ij} &\simeq \mathcal{O}_{ij} + f(\mathcal{B}_{ij},\ \mathcal{S}_{ij},\ \mathcal{S}_{ijk},\ \slashed{D}_c\mathcal{S}_{ij},\ \slashed{D}_m\mathcal{S}_{ijk}), &(4.17c)
\end{aligned}
$$

where

$$\mathcal{O}_{ij} \equiv \slashed{D}_k\mathcal{L}_\zeta y_i{}^{k}{}_{j} + 2\slashed{D}_k(\eta y_i{}^{k}{}_{j}) \tag{4.18}$$

and f is a homogeneous function of its arguments.

Proof. Relations (4.17a) and (4.17b) are obtained by direct application of the $\square_\ell$ operator on the definitions of $\mathcal{S}_{ij}$ and $\mathcal{S}_{ijk}$, respectively. Regarding the wave equation (4.17c), an analogous approach yields

$$\slashed{\Delta}\mathcal{B}_{ij} \simeq \mathcal{L}_\zeta\slashed{D}_k y_i{}^{k}{}_{j} + 2\eta\slashed{D}_k y_i{}^{k}{}_{j} + 2\slashed{D}_k\eta y_{(i}{}^{k}{}_{j)} + f(\mathcal{B}_{ij},\ \mathcal{S}_{ij},\ \mathcal{S}_{ijk},\ \slashed{D}_c\mathcal{S}_{ij},\ \slashed{D}_m\mathcal{S}_{ijk}).$$

Now, the definition of $\mathcal{S}_{ij}$ implies that the vector field ζ^i satisfies the following identity:

$$\slashed{D}_j\slashed{D}_k\zeta_i = -r^{l}{}_{jki}\zeta_l + 2\ell_{i(j}\slashed{D}_{k)}\eta + \ell_{jk}\slashed{D}_i\eta + f(\slashed{D}_i\mathcal{S}_{jk}).$$

This relation allows us to write

$$\mathcal{L}_\zeta\slashed{D}_k y_i{}^{k}{}_{j} - \slashed{D}_k\mathcal{L}_\zeta y_i{}^{k}{}_{j} = \slashed{y}d_{kij}\slashed{D}^k\eta,$$

from where, the stated result follows. □

Remark 39. The system of wave equations in the previous lemma is homogeneous in the zero-quantities $\mathcal{S}_{ij}$, $\mathcal{S}_{ijk}$ and $\mathcal{B}_{ij}$ provided that the *obstruction tensor* $\mathcal{O}_{ij}$ vanishes on $\mathscr{I}$.

Remark 40. If $\mathscr{I}$ is conformally flat, then the obstruction tensor identically vanishes as $y_{ijk} = 0$.

Lemmas 10 and 11 lead to the following proposition:

Proposition 9. *Let* $(\mathcal{M}, g_{ab})$ *be a conformal extension of an anti-de Sitter-like spacetime. Let* ζ_a *and* η *be fields satisfying* (4.15c) *and* (4.15e)*, and* y_{ijk} *a tensor with the symmetries of the magnetic part of the Weyl tensor. Assume that* $\mathcal{S}_{ij}$*,* $\mathcal{B}_{ij}$ *and* $\mathcal{S}_{ijk}$ *vanish identically at* $\partial\mathcal{S}_\star$*. Then* ζ_i *satisfies the unphysical conformal Killing equation on* $\mathscr{I}$ *if and only if* $\mathcal{O}_{ab} \simeq 0$*.*

Remark 41. We stress that the vanishing of the obstruction tensor $\mathcal{O}_{ij}$ is a necessary and sufficient condition for the existence of a Killing vector on the spacetime. The necessity follows from the fact that if a Killing vector is present in the spacetime then all the zero-quantities associated to the conformal Killing vector evolution system will vanish. This, in turn, implies that the zero-quantities intrinsic to the conformal boundary have to vanish. Equation (4.17c) implies then that $\mathcal{O}_{ij} \simeq 0$.

Remark 42. It should be stressed that the analysis carried out in the previous sections is conformally invariant. More precisely, if the unphysical Killing vector candidate is such that the zero-quantities associated to the Killing equation conformal evolution system vanish for a particular conformal representation, then they will also vanish for any other conformal representation. This follows from the conformal transformation properties for the zero-quantities implied by the change of connection transformation formulae. In particular, the reduced Killing boundary conditions (4.15a)-(4.15e) have similar conformal invariance properties.

4.4 Initial data at $\partial\mathcal{S}_\star$

As mentioned in Remark 38, the system (4.16a)-(4.16b) must be complemented with data at $\partial\mathcal{S}_\star$, that is to say, we have to bring into consideration the conditions implied by the Killing vector zero-quantities on $\mathcal{S}_\star$ and make them consistent with the ones obtained from the boundary analysis in the previous section. The main difference between this section and the preceding ones is the introduction of an adapted system of coordinates suited for studying the corner conditions.

4.4.1 Set up

For simplicity, let us introduce a system of coordinates $(x^\mu) = (x^0, x^1, x^{\mathcal{A}})$ where x^0 and x^1 correspond to the time and radial coordinates, respectively, while the calligraphic indices will represent the angular ones. This system of coordinates is adapted to our problem in the sense that $\mathcal{S}_\star$ and $\mathscr{I}$ are orthogonal and given by

$$\mathcal{S}_\star = \{p \in \mathcal{M} \mid x^0 = 0\} \quad \text{and} \quad \mathscr{I} = \{p \in \mathcal{M} \mid x^1 = 0\}.$$

The corner is determined then by the condition $x^0 = x^1 = 0$.

Once coordinates have been introduced, the metric can be written explicitly in terms of the lapse and shift functions. Adopting a Gaussian gauge we can write

$$\boldsymbol{g} = -\mathbf{d}x^0 \otimes \mathbf{d}x^0 + h_{\alpha\beta}\mathbf{d}x^\alpha \otimes \mathbf{d}x^\beta \quad (\alpha, \beta = 1, 2, 3), \tag{4.19a}$$

$$\boldsymbol{g} = \mathbf{d}x^1 \otimes \mathbf{d}x^1 + \ell_{\gamma\delta}\mathbf{d}x^\gamma \otimes \mathbf{d}x^\delta \quad (\gamma, \delta = 0, 2, 3). \tag{4.19b}$$

where $\boldsymbol{h}$ is the metric of $\mathcal{S}_\star$. From here, we find that the non-zero components of the metric at the corner are:

$$g_{00} = \ell_{00} = -1, \quad g_{11} = h_{11} = 1, \quad g_{\mathcal{AB}} = h_{\mathcal{AB}} = \ell_{\mathcal{AB}}.$$

4.4.2 Corner conditions

Finally, we describe how corner conditions can be obtained from the initial data imposed on η, ζ_a and their first derivatives on the conformal boundary. For convenience, let the symbol $\hat{}$ denote quantities defined on $\mathcal{S}_\star$. Let $\hat{\zeta}_a$ and $\hat{\zeta}$ be, respectively, the pull-backs of $h_a{}^b\xi_b$ and $t^a\xi_a$ to $\mathcal{S}_\star$. Although this decomposition with respect to h_{ab} is clearly different from the one performed on the conformal boundary we can observe that, when expressed in the adapted coordinates, the following relations hold at the corner:

$$\hat{\zeta}_1 = \zeta = 0, \qquad \hat{\zeta} = \zeta_0, \qquad \hat{\zeta}_{\mathcal{A}} = \zeta_{\mathcal{A}}.$$

In this way, the angular components $\zeta_{\mathcal{A}}$ on $\mathscr{I}$ are fixed by the initial data. Similarly, if one requires the conformal factor Ξ to have continuous first derivatives, it follows then that the conditions

$$\hat{\partial}_0\Xi = \partial_0\Xi = 0, \quad \hat{\partial}_1\Xi = \partial_1\Xi = \Sigma, \quad \hat{\partial}_{\mathcal{A}}\Xi = \partial_{\mathcal{A}}\Xi = 0,$$

must be satisfied at $\partial\mathcal{S}_\star$.

Regarding the remaining fields, values for η and the components of ξ_a on $\mathcal{S}_\star$ can be found by solving equations (4.7a)-(4.8b) and (4.13a)-(4.13f) — the KID equations set — with $\epsilon = -1$. Moreover, this system also provides with all their derivatives. In particular, when the limit $\Xi \to 0$ is taken, the corresponding solutions for η, $\hat{\zeta}_0$ and $\hat{\zeta}_{\mathcal{A}}$, along with their time and angular derivatives, serve as initial data at $\partial\mathcal{S}_\star$ for the wave equations (4.3a) and (4.3b).

4.5 Summary

Once the conditions for the existence of a conformal Killing vector on $\mathscr{I}$ have been established, we can link Proposition 9 to the initial-boundary problem in the spacetime via Lemmas 6 and 7. The main result of this work can be formulated as follows:

Theorem 2. *Let* $(\mathcal{M}, \boldsymbol{g})$ *be a conformal extension of an anti de Sitter-like spacetime with conformal boundary* $\mathscr{I}$*. Consider an spacelike hypersurface* $\mathcal{S}_\star \subset \mathcal{M}$ *which intersects* $\mathscr{I}$ *at* $\partial\mathcal{S}_\star$*. Suppose that* $\xi_{a\star}$ *and* $\eta_\star$ *satisfy the conformal KID equations* (4.5a) *and* (4.5b) *on* $\mathcal{S}_\star$*. Let* ζ_i *and* η *be the fields obtained from solving the wave equations* (4.16a) *and* (4.16b) *with initial data given by the restriction of* $\xi_{a\star}$ *and* $\eta_\star$ *to* $\partial\mathcal{S}_\star$*. Assume that the obstruction tensor* $\mathcal{O}_{ab}$ *defined in equation* (4.18) *vanishes. Then the Killing vector candidate* ξ_a *obtained from solving equations* (4.3a) *and* (4.3b) *with initial data* $\xi_{a\star}$*,* $\eta_\star$ *and boundary data* ζ_i*,* η *pull-backs to a Killing vector* $\tilde{\xi}_a$ *on* $\tilde{\mathcal{M}}$*.*

Chapter 5

Construction of anti-de Sitter-like spacetimes using the metric conformal Einstein field equations: the vacuum case

The material of this section is based on [12].

5.1 Introduction

A first analysis of the initial-boundary value problem for 4-dimensional vacuum anti-de Sitter-like spacetimes by means of conformal methods has been carried out by Friedrich in [29] — see also [31] for further discussion of the admissible adS-like boundary conditions. This seminal work makes use of the extended conformal Einstein field equations and a gauge based on the properties of curves with good conformal properties (*conformal geodesics*) to set up an initial-boundary value problem for a first order symmetric hyperbolic system of evolution equations. For this type of evolution equations, one can use the theory of maximally dissipative boundary conditions — see e.g. [36, 55] — to assert the well-posedness of the problem and to ensure the local existence of solutions in a neighbourhood of the corner. The solutions to these evolution equations can be shown, via a further argument, to constitute a solution to the vacuum Einstein field equations with negative cosmological constant.

Friedrich's analysis identifies a large class of maximally dissipative boundary conditions involving the outgoing and incoming components of the Weyl tensor; as such, they can be thought of as prescribing the relation between these components. These conditions are given in a very specific gauge and thus it is difficult to assert their physical/geometric meaning. However, it is possible to identify a subclass of boundary conditions which can be recast in a covariant form. More precisely, they can be shown to be equivalent to prescribing the conformal class of the metric on the conformal boundary — see [29] and also [60]. The question of recasting the whole class of maximally dissipative boundary conditions obtained by Friedrich in a geometric (i.e. covariant) form remains an interesting open problem. An alternative construction of anti-de Sitter-like spacetimes, which does not use the conformal Einstein field equations and holds for spacetimes of dimension greater than four, can be found in [24]. A discussion of global properties of adS-like spacetimes and the issue of their stability can be found in [2].

Numerical simulations involving anti-de Sitter-like spacetimes is an active area of current research — see e.g. [8, 9, 22, 23] which kick-started some of the current flurry of interest. In particular, in [9] the evolution of the spherically symmetric Einstein-scalar field with reflective boundary conditions was considered; different boundary conditions for this system have been considered in [1]. Alternative Cauchy-hyperbolic and characteristic formulations of the spherically symmetric Einstein-scalar field system have been discussed in [57, 58].

Friedrich's results offer a natural and systematic approach to the numerical construction of 4-dimensional vacuum anti-de Sitter-like spacetimes. However, the numerical implementation of this result is not straightforward, among other things, because the equations involved are cast in a form which is not standard for the available numerical codes and, moreover, there is very little intuition about the behaviour of the gauges used to formulate the equations. A further difficulty of Friedrich's approach is that it cannot readily be extended to include matter fields — see [46] for an accomplishment in this direction.

In view of the issues raised in the previous paragraph, this chapter is dedicated to a conformal formulation of the initial-boundary value problem for vacuum anti-de Sitter-like spacetimes, which is closer to the language used in numerical simulations and exploits familiar gauge conditions. This is achieved by means of the system of quasilinear wave equations derived in Section 3.5. The PDEs theory for this type of systems is available in the literature [14, 21]. The conformal constraint equations discussed in Section 2.3 will permit us to construct suitable Dirichlet boundary data for this system. We believe this scheme should be easier for its numerical implementation — see Remark 9. In addition, we analyse the initial and boundary data required for the system of wave equations for the zero-quantities — see Section 3.4.

5.2 General set-up

In Chapter 2 a metric conformal formulation for the Einstein equations coupled to a tracefree matter model was introduced. In particular, in the absence of a matter component we end up with the simplified system

$$\nabla_a \nabla_b \Xi = -\Xi L_{ab} + s g_{ab}, \tag{5.1a}$$

$$\nabla_a s = -L_{ab} \nabla^b \Xi, \tag{5.1b}$$

$$\nabla_a L_{bc} - \nabla_b L_{ac} = \nabla_e \Xi d^e{}_{cab}, \tag{5.1c}$$

$$\nabla_e d^e{}_{abc} = 0, \tag{5.1d}$$

$$6\Xi s - 3\nabla_c \Xi \nabla^c \Xi = \lambda, \tag{5.1e}$$

$$R^a{}_{bcd} = \Xi d^a{}_{bcd} + 2\delta_{[c}{}^a L_{d]b} + 2L_{[c}{}^a g_{d]b}. \tag{5.1f}$$

As discussed in Section 3.2, the above system enables us to construct a system of quasilinear wave equations for the conformal fields. In view of the condition $T_{ab} = 0$, we have the following relations:

$$\blacksquare \Xi = 4s - \tfrac{1}{6}\Xi \mathcal{R}(x), \tag{5.2a}$$

$$\blacksquare s = -\tfrac{1}{6} s \mathcal{R}(x) + \Xi L_{\mu\nu} L^{\mu\nu} - \tfrac{1}{6}\nabla_\mu \mathcal{R}(x) \nabla^\mu \Xi, \tag{5.2b}$$

$$\blacksquare L_{\mu\nu} = -2\Xi d_{\mu\rho\nu\lambda} L^{\rho\lambda} + 4L_\mu{}^\lambda L_{\nu\lambda} - L_{\rho\lambda} L^{\rho\lambda} g_{\mu\nu} + \tfrac{1}{6}\nabla_\mu \nabla_\nu \mathcal{R}(x), \tag{5.2c}$$

$$\blacksquare d_{\mu\nu\lambda\rho} = -4\Xi d_\mu{}^\tau{}_{[\lambda}{}^\sigma d_{\rho]\sigma\nu\tau} - 2\Xi d_\mu{}^\tau{}_\nu{}^\sigma d_{\lambda\rho\tau\sigma} + \tfrac{1}{2} d_{\mu\nu\lambda\rho} \mathcal{R}(x), \tag{5.2d}$$

$$\mathscr{R}_{\mu\nu} = 2L_{\mu\nu} + \tfrac{1}{6}\mathcal{R}(x) g_{\mu\nu}. \tag{5.2e}$$

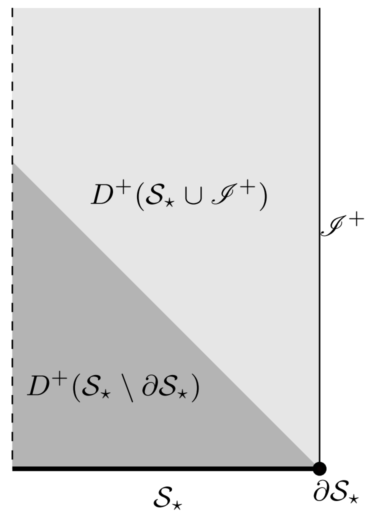

Figure 5.1: Penrose diagram of the set-up for the construction of the anti-de Sitter spacetime as described in the main text. Initial data prescribed on $\mathcal{S}_\star \setminus \partial\mathcal{S}_\star$ allows us to recover the dark shaded region $D^+(\mathcal{S}_\star \setminus \partial\mathcal{S}_\star)$. In order to recover $D^+(\mathcal{S}_\star \cup \mathscr{I}^+)$ it is necessary to prescribe boundary data on $\mathscr{I}^+$. Notice that $D^+(\mathcal{S}_\star \cup \mathscr{I}^+) = J^+(\mathcal{S}_\star)$.

Since we aim to study these equations in the case $\lambda < 0$, they must be supplemented by a set of suitable initial and boundary data in order to be able to make a statement about the existence and uniqueness of a solution to the above equations — see Corollary 1. This problem will be addressed in the sequel.

5.2.1 Coordinates

Let $(\mathcal{M}, g_{ab}, \Xi)$ be a conformal extension of an anti-de Sitter-like spacetime $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ where g_{ab} and $\tilde{g}_{ab}$ are conformally related metrics. Let $\mathcal{S}_\star \subset \mathcal{M}$ be a smooth, compact and oriented spacelike hypersurface with boundary $\partial\mathcal{S}_\star$. Furthermore, $\mathcal{S}_\star \cap \mathscr{I} = \partial\mathcal{S}_\star$ is the so-called *corner*. The portion of $\mathscr{I}$ in the future of $\mathcal{S}_\star$ will be denoted by $\mathscr{I}^+$. In addition, it will be assumed that the causal future $J^+(\mathcal{S}_\star)$ coincides with the future domain of dependence $D^+(\mathcal{S}_\star \cup \mathscr{I}^+)$ and that $\mathcal{S}_\star \cup \mathscr{I}^+ \approx \mathcal{S}_\star \times [0,1)$ so that, in particular, $\mathscr{I}^+ \approx \partial\mathcal{S}_\star \times [0,1)$ — see Figure 5.1.

Using a set of coordinates adapted to $\mathcal{S}_\star$ and $\mathscr{I}$, we have that

$$\mathcal{S}_\star = \{x \in \mathbb{R}^3 \mid x^0 = 0\}, \qquad \mathscr{I} = \{x \in \mathbb{R}^3 \mid x^1 = 0\}.$$

Coordinates are propagated off $\mathcal{S}_\star$ via imposing the generalised wave coordinate condition (3.21). Observe that this can always be locally solved: the expression above provides the value of the coordinates on $\mathcal{S}_\star$ while their normal derivatives are obtained from the requirement that (x^μ) are independent, that is to say, the coordinate differentials $\mathbf{d}x^\mu$ must be linearly independent.

5.3 Initial and boundary data

5.3.1 Solutions to the conformal constraints on a spacelike hypersurface

The conformal constraint equations (2.15a)-(2.16b) enable us to obtain the conformal version of the so-called Hamiltonian and Momentum constraints on a spacelike hypersurface ($\epsilon = -1$). Ignoring the contributions from the matter fields, a straightforward calculation shows that these take the form:

$$\frac{\Omega^2}{2}(r + K^2 - K_{ij}K^{ij}) = 2K\Omega\Sigma - 2\Omega D_i D^i\Omega - 3\Sigma^2 + 3D_i\Omega D^i\Omega + \lambda, \tag{5.3a}$$

$$\Omega(D_j K_i{}^j - D_i K) = 2(K_{ij}D^j\Omega - D_i\Sigma). \tag{5.3b}$$

It follows that under a conformal approach, the collection of fields ($\boldsymbol{h}$, $\boldsymbol{K}$, Ω, Σ) satisfying the previous equations must be prescribed on $\mathcal{S}_\star$. This set of functions will in turn constitute the basic initial data that will completely determine the remaining fields on a spacelike hypersurface. Along with the boundary data, this set will serve to evolve the wave equations for the conformal fields. From the conformal constraints one obtains the following expressions for the initial data:

$$s = \frac{1}{3}\bigg(\Delta\Omega + \frac{1}{4}\Omega\big(r + K^2 - K_{ij}K^{ij}\big) - \Sigma K\bigg), \tag{5.4a}$$

$$L_{ij} = \frac{1}{\Omega}\bigg(sh_{ij} + \Sigma K_{ij} - D_i D_j \Omega\bigg), \tag{5.4b}$$

$$L_i = \frac{1}{\Omega}\big(K_i{}^k D_k\Omega - D_i\Sigma\big), \tag{5.4c}$$

$$\begin{aligned} d_{ij} = \frac{1}{\Omega}\bigg(- L_{ij} + l_{ij} + \big(K\big(K_{ij} - \frac{1}{4}Kh_{ij}\big) - K_{ki}K_j{}^k \\ + \frac{1}{4}K_{kl}K^{kl}h_{ij}\big)\bigg), \end{aligned} \tag{5.4d}$$

$$d_{ijk} = \frac{1}{\Omega}\big(D_j K_{ki} - D_k K_{ji} + h_{ik}L_j - h_{ij}L_k\big). \tag{5.4e}$$

The fact that these expressions are singular at $\Omega = 0$ leads to the following:

Definition 5 (**vacuum anti-de Sitter-like initial data**). *An anti-de Sitter-like initial data set is understood to be a 3-manifold $\mathcal{S}_\star$ with boundary $\partial\mathcal{S}_\star \approx \mathbb{S}^2$ together with a collection of smooth fields $(\Omega, h_{ij}, K_{ij}, \Sigma)$ such that:*

(i) $\Omega > 0$ *on* $\operatorname{int}\mathcal{S}_\star$*;*

(ii) $\Omega = 0$ *and* $|\mathrm{d}\Omega|^2 = \Sigma^2 - \frac{1}{3}\lambda > 0$ *on* $\partial\mathcal{S}_\star$*;*

(iii) the fields s, L_{ij}, L_i, d_{ij} and d_{ijk} computed from relations (6.2a)-(6.2e) *extend smoothly to $\partial\mathcal{S}_\star$.*

Remark 43. Anti-de Sitter-like initial data sets are closely related to so-called hyperboloidal data sets for Minkowski-like spacetimes — see [44]. By means of this correspondence it is possible to adapt the existence results for hyperboloidal initial data sets in [3, 4] to the anti-de Sitter-like setting. In particular, this shows the existence of a large class of time symmetric data, i.e. data for which $K_{ij} = 0$.

5.3.2 Boundary conditions for the conformal evolution equations

In this subsection we discuss the boundary conditions to be imposed on the various conformal fields. In [29] it has been shown that it is possible to formulate an initial-boundary value problem for anti-de Sitter-like spacetimes in which the conformal class of the metric on the conformal boundary is specified freely. In the following, we investigate whether it is possible to make a similar prescription in our scheme. More precisely, we would like to prescribe *Dirichlet boundary data* on $\mathscr{I}$ for the wave equations (3.25a)-(3.25e) from the conformal constraint equations.

5.3.2.1 Boundary data for the conformal factor

The evolution of the conformal factor Ξ is described by the wave equation (3.25a). For this equation one naturally prescribes Dirichlet boundary conditions such that

$$\Xi \simeq 0.$$

In other words, one has that $\Xi = O(x^1)$ close to $\mathscr{I}$. On $\mathcal{S}_\star$ one wants to identify Ξ with some 3-dimensional conformal factor Ω such that $\Omega = 0$ and $\mathbf{d}\Omega \neq 0$ at $\partial\mathcal{S}_\star$, consistent with Definition 5.

5.3.2.2 Boundary data for the Friedrich scalar

The evolution of the Friedrich scalar s is governed by the wave equation (3.25b). As mentioned in Section 2.3.1, the boundary data for s are determined once the scalar function $\varkappa(x)$ has been prescribed according to relations (2.18a)-(2.18b). Notice that the specification of s is independent of the choice of the gauge source function $\mathcal{R}(x)$ associated to the Ricci scalar — see the discussion in Remark 17; furthermore, s contains information about the manner the conformal boundary embeds in the spacetime. In particular, it is possible, say, to have two conformally related representations of the same physical solution with the same spacetime Ricci scalar, one with a conformal boundary which is extrinsically curved and the other is extrinsically flat.

Remark 44. Observe that equation (2.18c) implies that the particular choice $\varkappa(x) = 0$ renders a conformal boundary which is *extrinsically flat* with respect to the ambient spacetime.

5.3.2.3 Boundary data for the components of the conformal metric

In the following it is convenient to make use of the $3+1$ decomposition of the metric g_{ab} with respect to the unit normal to the conformal boundary, namely

$$\boldsymbol{g} = \not{\alpha}^2 \mathbf{d}x^1 \otimes \mathbf{d}x^1 + \ell_{\gamma\delta}\big(\not{\beta}^\gamma \mathbf{d}x^1 + \mathbf{d}x^\gamma\big) \otimes \big(\not{\beta}^\delta \mathbf{d}x^1 + \mathbf{d}x^\delta\big) \quad (\gamma,\, \delta = 0,\, 2,\, 3).$$

Here, $(\ell_{\gamma\delta})$ denote the components of the intrinsic metric ℓ_{ij} of the conformal boundary and $\not{\alpha}$ and $\not{\beta}^\gamma$ are, respectively, the lapse and shift. As $\mathscr{I}$ is timelike, ℓ_{ij} is a 3-dimensional Lorentzian metric with signature $(-,+,+)$. Accordingly, the components $(g_{\mu\nu})$ are given by

$$(g_{\mu\nu}) = \begin{pmatrix} \not{\alpha}^2 + \not{\beta}_\gamma \not{\beta}^\gamma & \not{\beta}_\gamma \\ \not{\beta}_\delta & \ell_{\gamma\delta} \end{pmatrix}, \tag{5.5}$$

so that the ones of the contravariant metric are

$$(g^{\mu\nu}) = \begin{pmatrix} \not{\alpha}^{-2} & -\not{\alpha}^{-2}\not{\beta}^\gamma \\ -\not{\alpha}^{-2}\not{\beta}^\delta & \ell^{\gamma\delta} + \not{\alpha}^{-2}\not{\beta}^\gamma\not{\beta}^\delta \end{pmatrix}.$$

Remark 45. In the sequel we regard the components $(\ell_{\alpha\beta})$ as our basic boundary data.

Without loss of generality, we adopt a *Gaussian gauge* at the conformal boundary so that

$$\not{\alpha} \simeq 1, \qquad \not{\beta}^\gamma \simeq 0, \tag{5.6}$$

and the metric g_{ab} takes the form

$$\boldsymbol{g} \simeq \mathbf{d}x^1 \otimes \mathbf{d}x^1 + \ell_{\alpha\beta}\mathbf{d}x^\alpha \otimes \mathbf{d}x^\beta.$$

Remark 46. The prescription of the gauge conditions on the conformal boundary is independent of the generalised harmonic condition (3.21) and, thus, consistent with each other. Indeed, a calculation shows that for a metric of the form given by (5.5) one has that

$$\Gamma^1 = \frac{1}{\not{\alpha}^3}\big(\partial_1 \not{\alpha} - \not{\beta}^\gamma \partial_\gamma \not{\alpha} + \not{\alpha}^2 \not{K}\big), \tag{5.7a}$$

$$\Gamma^\delta = \gamma^\delta - \frac{\not{\beta}^\delta}{\not{\alpha}^3}\big(\partial_1 \not{\alpha} - \not{\beta}^\gamma \partial_\gamma \not{\alpha} + \not{\alpha}^2 \not{K}\big) + \frac{1}{\not{\alpha}^2}\big(\partial_1 \not{\beta}^\delta - \not{\beta}^\gamma \partial_\gamma \not{\beta}^\delta + \not{\alpha}\partial^\delta \not{\alpha}\big), \tag{5.7b}$$

where $\gamma^\delta \equiv \ell^{\eta\theta}\gamma^\delta{}_{\eta\theta}$ denote the 3-dimensional contracted Christoffel symbols. Therefore, the generalised harmonic condition (3.21) only prescribes the propagation of the gauge fields $\not{\alpha}$ and $\not{\beta}^\gamma$ off the conformal boundary and does not constrain the components of the 3-metric ℓ_{ij}. Notice that in the above expressions $\not{K} \simeq 3\varkappa(x)$ — see (2.18c).

5.3.2.4 Boundary data for the components of the Schouten tensor

Given the 3-metric ℓ_{ij} of the conformal boundary, one can compute the tangential components ($\not{L}_{\alpha\beta}$) and tangential-normal components ($\not{L}_\alpha$) of the spacetime Schouten tensor on the conformal boundary using formulae (2.18d) and (2.18e). To compute the normal-normal component $\not{L}_{11}$, we notice that

$$g^{\mu\nu}\not{L}_{\mu\nu} = \frac{1}{6}R.$$

Thus, one has that

$$\begin{aligned} \not{L}_{11} &\simeq \tfrac{1}{6}\mathcal{R}(x) - \ell^{\alpha\beta}\not{l}_{\alpha\beta} + \tfrac{1}{2}\varkappa^2(x)\ell_{\alpha\beta}\ell^{\alpha\beta} \\ &\simeq \tfrac{1}{6}\mathcal{R}(x) - \tfrac{1}{4}r + \tfrac{3}{2}\varkappa^2(x). \end{aligned} \tag{5.8}$$

5.3.2.5 Boundary data for the rescaled Weyl tensor

The boundary data for the magnetic part of the rescaled Weyl tensor is directly related to the Cotton tensor y_{ijk} of the prescribed metric ℓ_{ij} via formula (2.18f). Regarding its electric part, some information can be extracted by projecting equation (2.11d) with $n^a\ell_b{}^d\ell_c{}^e$ for $T_{ab} = 0$. A calculation shows that one can write

$$\not{D}\not{d}_{ij} \simeq \not{D}_k \not{d}_{ji}{}^k. \tag{5.9}$$

In consequence, Neumann data for d_{ij} on the conformal boundary can be obtained from the prescription of the 3-metric ℓ_{ij}.

5.3.2.6 Summary

The analysis of this section can be summarised as follows:

Proposition 10. *Let* $\mathscr{I}$ *be equipped with a smooth Lorentzian metric* ℓ_{ij} *Moreover, let the fields*

$$\not{\Sigma}, \qquad s, \qquad \not{K}_{ij}, \qquad \not{L}_i, \qquad \not{L}_{ij}, \qquad \not{d}_{ijk}, \qquad \not{D}\not{d}_{ij},$$

be constructed according to formulae (2.18a)-(2.18f) *and* (5.9)*. Finally, let* Υ_{ab}*,* Θ_a*,* Δ_{abc}*,* Λ_{abc}*,* Λ *and* $P^a{}_{bcd}$ *be the zero-quantities defined by relations* (3.6a)-(3.6d)*. Then one has that*

$$\begin{aligned}
\ell_b{}^a\Theta_a \simeq 0, &\quad Z \simeq 0,\\
\ell_c{}^a\ell_d{}^b\Upsilon_{ab} \simeq 0, &\quad \not{n}^a\ell_c{}^b\Upsilon_{ab} \simeq 0,\\
\ell_e{}^c\ell_f{}^d\ell_g{}^b\Delta_{cdb} \simeq 0, &\quad \not{n}^b\ell_e{}^c\ell_f{}^d\Delta_{cdb} \simeq 0,\\
\not{n}^b\ell_e{}^c\ell_f{}^d\Lambda_{bcd} \simeq 0, &\quad \not{n}^c\ell_a{}^b\ell_e{}^d\Lambda_{bcd} \simeq 0,\\
\ell_a{}^e\ell_b{}^f\ell_c{}^g\ell_d{}^h P_{efgh} \simeq 0, &\quad \not{n}^d\ell_a{}^e\ell_b{}^f\ell_c{}^g P_{edfg} \simeq 0,
\end{aligned}$$

where $\not{n}^a$ *and* $\ell_a{}^b$ *denote, respectively, the normal and projector of the conformal boundary.*

5.3.3 Corner conditions

In the previous sections we have discussed the problem of the determination of initial and boundary data. In particular, it is clear that once boundary data have been provided on $\mathscr{I}$, time derivatives of the various conformal fields can be directly calculated. However, these data do not necessarily match smoothly with the ones corresponding to $\mathcal{S}_\star$ at the corner. The purpose of this section is to analyse the compatibility conditions, at different orders, arising from the conformal Einstein field equations and the wave equations — these conditions are commonly known as *corner conditions.* In the following, the subscript $_\odot$ will stand for a quantity evaluated at $\partial\mathcal{S}_\star$.

5.3.3.1 Conditions for the metric

In terms of the adapted coordinates previously introduced, the corner $\partial\mathcal{S}_\star$ is defined by the conditions $x^0 = 0$ and $x^1 = 0$. Exploiting the gauge freedom, we adopt local Gaussian coordinates both on $\mathcal{S}_\star$ and $\mathscr{I}$. Denoting as $h_{\gamma\delta}$ and $\ell_{\mathcal{AB}}$ the intrinsic 3-metrics corresponding to these hypersurfaces, respectively, this condition implies that the spacetime metric at $\partial\mathcal{S}_\star$ can be written in the two following ways:

$$\boldsymbol{g} = -\mathbf{d}x^0 \otimes \mathbf{d}x^0 + h_{\gamma\delta}\mathbf{d}x^\gamma \otimes \mathbf{d}x^\delta \qquad (\gamma, \delta = 1, 2, 3),$$
$$\boldsymbol{g} = \mathbf{d}x^1 \otimes \mathbf{d}x^1 + \ell_{\mathcal{AB}}\mathbf{d}x^\mathcal{A} \otimes \mathbf{d}x^\mathcal{B} \qquad (\mathcal{A}, \mathcal{B} = 0, 2, 3).$$

Zero order conditions. Comparing the two last expressions for the metric, one readily finds that

$$(\ell_{00})_\odot = -1, \quad (h_{11})_\odot = 1, \quad (\ell_{AB})_\odot = (h_{AB})_\odot, \tag{5.10}$$

while the remaining components vanish at $\partial\mathcal{S}_\star$.

First order conditions. In Gaussian coordinates, we can express the normal derivatives of the metric in terms of the corresponding extrinsic curvature. Explicitly, one has:

$$K_{\gamma\delta}|_{\mathcal{S}_\star} = \tfrac{1}{2}\partial_0 h_{\gamma\delta}|_{\mathcal{S}_\star} = \Gamma^0{}_{\gamma\delta}|_{\mathcal{S}_\star}, \tag{5.11a}$$
$$\not{K}_{\mathcal{AB}} \simeq \tfrac{1}{2}\partial_1 \ell_{\mathcal{AB}} \simeq -\Gamma^1{}_{\mathcal{AB}}. \tag{5.11b}$$

As $K_{\gamma\delta}$ is part of the initial data, this establishes a corner condition for $\partial_0 h_{\gamma\delta}$; in particular, the angular components must satisfy the condition $(\partial_0 h_{AB})_\odot = (\partial_0 \ell_{AB})_\odot$.

Recall that in Gaussian coordinates the propagation of the timelike vector $(\partial_0)^a$ along itself implies that $\Gamma^\mu_{00}|_{\mathcal{S}_\star} = 0$; similarly, for the normal to $\mathscr{I}$ one has that $\Gamma^\mu_{11} \simeq 0$. The previous conditions on the Christoffel symbols, along with equations (5.11a) and (5.11b), imply that K_{11} and $\not{K}_{00}$ vanish at the corner. Furthermore, the traces of the extrinsic curvature can be related to the gauge functions $\mathcal{F}^\mu(x)$ as follows:

$$K_\odot = (h^{AB}K_{AB})_\odot = \mathcal{F}^0(x)_\odot, \qquad \not{K}_\odot = (\ell^{AB}\not{K}_{AB})_\odot = -\mathcal{F}^1(x)_\odot.$$

Finally, given that ∇ is a Levi-Civita connection and the acceleration is zero, our coordinate choice determines the remaining partial derivatives: $(\partial_0 g_{0\mu})_\odot = -(\Gamma^0{}_{0\mu})_\odot = 0$.

Second order conditions. Second order conditions can be extracted in a straightforward way from the wave equation for the metric, equation (3.25e), namely

$$\begin{aligned} g^{\lambda\rho}\partial_\lambda\partial_\rho g_{\mu\nu} =& 2\bigg(g_{\lambda\rho}g^{\sigma\tau}\Gamma^\lambda{}_{\sigma\mu}\Gamma^\rho{}_{\tau\nu} + 2\Gamma^\sigma{}_{\lambda\rho}g^{\lambda\tau}g_{\sigma(\mu}\Gamma^\rho{}_{\nu)\tau} - g_{\sigma(\mu}\nabla_{\nu)}\mathcal{F}^\sigma(x) \\ & - 2L_{\mu\nu} - \tfrac{1}{6}g_{\mu\nu}\mathcal{R}(x)\bigg). \end{aligned}$$

Using the conditions discussed above for the first order derivatives, the wave equation for the components $g_{\mu\nu}$ can be written schematically as:

$$(\partial_0^2 \ell_{AB})_\odot = (\partial_1^2 h_{AB})_\odot + (h^{CD}\partial_C\partial_D h_{AB})_\odot + f_{AB}(\boldsymbol{g}, \boldsymbol{K}, \not{\boldsymbol{K}}, \boldsymbol{\mathcal{F}}(x), \boldsymbol{L}, \mathcal{R}(x))_\odot.$$

Apart from the components of the Schouten tensor (to be discussed below), the second order condition can be expressed in terms of the initial data, lower order corner conditions and gauge functions at the corner. Further application of ∂_0 enables us to obtain higher order conditions.

5.3.3.2 Conditions for the conformal factor

As, by definition, $\Xi = 0$ on the conformal boundary, all its intrinsic derivatives of any order will vanish. In particular, $\partial\mathcal{S}_\star$ automatically inherits these conditions. Its normal derivative, on the other hand, is given by (2.18a). When smoothness is imposed, higher order partial derivatives both on $\mathcal{S}_\star$ as well as on $\mathscr{I}$ are forced to coincide at $\partial\mathcal{S}_\star$.

5.3.3.3 Conditions for the Friedrich scalar

Zero order condition. The Friedrich scalar s is determined on the conformal boundary by the gauge function $\varkappa(x)$. Nevertheless, when the 00 component of equation (2.11a) is evaluated at the corner, our choice of Gaussian coordinates implies that

$$s_\odot = 0.$$

First order conditions. Equation (2.15c) determines the intrinsic derivatives of s on the boundary. In particular, one has that

$$(\partial_0 s)_\odot = -\not\Sigma(L_{01})_\odot,$$

which is simply solution (2.18d).

Second order conditions. The second order condition for s can be extracted from the wave equation (3.25b) expressed in Gaussian coordinates. The evaluation of this equation at the corner yields

$$(\partial_0^2 s)_\odot = (\partial_1^2 s)_\odot + (h^{AB}\partial_A\partial_B s)_\odot - (\mathcal{F}^\mu(x)\partial_\mu s + \tfrac{1}{6}(s\mathcal{R}(x) + \not\Sigma\partial_1\mathcal{R}(x))_\odot.$$

Here, the spatial derivatives of s can be computed from the restriction of the initial data to $\partial\mathcal{S}_\star$ while $\partial_0 s$ corresponds to the first order condition. The functions $\mathcal{F}^\mu(x)$ and $\mathcal{R}(x)$ are gauge-dependent prescribed quantities. Furthermore, notice that $\partial_0^2 s$ is written in terms of the first order derivatives, indicating then a recursive procedure. Higher order conditions can be found by further application of ∂_0 to (3.25b).

5.3.3.4 Conditions for the Schouten tensor

Zero order corner conditions. The value of the components L_{AB} and L_{0A} at the corner can be obtained from the initial data (6.2b) and (6.2c) by taking the limit $\Omega \to 0$. Imposing smoothness, they must match the boundary data given by equations (2.18d) and (5.8) at $\partial\mathcal{S}_\star$. An analogous condition is imposed for L_{00}.

First order corner conditions. First time derivatives of the components L_{AB} and L_{0A} can be read from equation (2.11c). More explicitly one has:

$$(\partial_0 L_{AB})_\odot = \not\Sigma({d^1}_{B0A})_\odot + f_{AB}(\boldsymbol{L}, \boldsymbol{h}, \boldsymbol{K}, \not{\boldsymbol{K}})_\odot,$$
$$(\partial_0 L_{A0})_\odot = \not\Sigma({d^1}_{00A})_\odot + f_{A}(\boldsymbol{L}, \boldsymbol{h}, \boldsymbol{K}, \not{\boldsymbol{K}})_\odot.$$

As will be seen below, the components of the Weyl tensor appearing here are part of the data satisfying zero-order conditions, so they must be consistent with the last equations. On the other hand, a condition for $(\partial_0 L_{00})_\odot$ can be obtained via the contracted Bianchi identity.

Second order corner conditions. Second order time derivatives of L_{ab} are to be obtained by evaluating the wave equation (3.25c) at $\partial\mathcal{S}_\star$. For L_{AB} one has

$$(\partial_0^2 L_{AB})_\odot = (\partial_1^2 L_{AB})_\odot + (h^{CD}\partial_C\partial_D L_{AB})_\odot + f_{AB}(\boldsymbol{h}, \boldsymbol{L}, \boldsymbol{K}, \not{\boldsymbol{K}}, \partial\mathcal{F}(x), \mathcal{R}(x))_\odot.$$

Similar expressions can be obtained for the rest of the components.

5.3.3.5 Conditions for the Weyl tensor

Information about the Weyl tensor is encoded in the electric and magnetic parts, which are given on $\mathcal{S}_\star$ by equations (6.2d) and (6.2e). Since these data have been obtained using different projections, their components must be carefully made compatible. One can check that they share the components d_{0101}, d_{010A}, d_{01A1}, d_{01AB} and d_{0A1B} which, when matched, represent the zero-order conditions.

First order corner conditions. Given the structure of equation (2.11d), only certain conditions can be extracted from it. Ultimately, when it is evaluated at the corner it takes the form:

$$(\partial_0 d^0{}_{\lambda\mu\nu})_\odot = f_{\lambda\mu\nu}(\boldsymbol{K}, \not{\boldsymbol{K}}, \boldsymbol{d})_\odot.$$

Second order corner conditions. Second order time derivatives of the rescaled Weyl tensor are given by the wave equation (3.25d). As Ξ vanishes at the corner, the equation is significantly simplified. Expanding the reduced wave operator $\blacksquare$ it takes the schematic form

$$(\partial_0^2 d_{\lambda\mu\nu\sigma})_\odot = (\partial_1^2 d_{\lambda\mu\nu\sigma})_\odot + (\partial_A\partial_B d_{\mu\nu\lambda\sigma})_\odot + f_{\lambda\mu\nu\sigma}(\boldsymbol{g}, \boldsymbol{K}, \not{\boldsymbol{K}}, \boldsymbol{d})_\odot.$$

5.3.3.6 Concluding remarks regarding the corner conditions

The discussion in the previous paragraphs provides a recursive procedure to compute the corner conditions to any required order. Given this procedure, it is natural to ask whether there exist any examples of pairs of initial data and boundary conditions which satisfy the corner conditions to *any arbitrary order*; the inherent difficulties in this task have been discussed in [31]. A way of satisfying corner conditions to an arbitrary order is to make use of the gluing constructions for asymptotically hyperbolic initial data sets in [18]. Given an asymptotically hyperbolic initial data set satisfying certain smallness conditions, these constructions allow to deform the data by a deformation which is supported arbitrarily far in the asymptotic region, to ones which are exactly Schwarzschild-anti de Sitter in the asymptotic region. This class of data is naturally supplemented by Schwarzschild-anti de Sitter boundary initial data, and thus trivially satisfies the corner conditions to any order. The resulting spacetime has, accordingly, a very special behaviour near the corner. In particular, the metric ℓ_{ij} must be conformally flat near the corner. It is of interest to analyse whether it is possible to construct a more general class of initial-boundary data for adS-like spacetimes satisfying the corner conditions at any order.

5.4 Propagation of the constraints

Proposition 10 establishes a link between the boundary data for the conformal fields and a number of components of the geometric zero-quantities. In this sense, the purpose of this section is to analyse the boundary data for the system of wave equations (3.15)-(3.20) in order to establish the uniqueness and existence of its trivial solution.

5.4.1 Boundary data for the subsidiary equations

Boundary data for $P^a{}_{bcd}$. By construction, the field $P^a{}_{bcd}$ inherits the symmetries of the Riemann tensor. This makes it possible to decompose it into three main components:

$$\hat{P}_{abcd} \equiv \ell_a{}^e \ell_b{}^f \ell_c{}^g \ell_d{}^h P_{efgh}, \quad \hat{P}_{abc} \equiv \not{n}^d \ell_a{}^e \ell_b{}^f \ell_c{}^g P_{edfg}, \quad \hat{P}_{ab} \equiv \not{n}^c \not{n}^d \ell_a{}^e \ell_b{}^f P_{ecfd}.$$

The first two vanish by virtue of the constraints (2.17i) and (2.17j), while a calculation shows that $\hat{P}_{ab} \simeq P^c{}_{acb} - \not{n}_a \not{n}_b \not{n}^c \not{n}^d P^e{}_{ced}$. From equation (3.12b) it follows that $\hat{P}_{ab} \simeq 0$.

Boundary data for Υ_{ab}. The zero-quantity Υ_{ab} can be decomposed with respect to $\not{n}^a$ by defining the projections $\gamma_{ab} \equiv \ell_a{}^c \ell_b{}^d \Upsilon_{cd}$, $\gamma_a \equiv \not{n}^b \ell_a{}^c \Upsilon_{bc}$

and $\gamma \equiv \not{n}^a \not{n}^b \Upsilon_{ab}$. Accordingly, we can write

$$\Upsilon_{ab} = \gamma_{ab} + \gamma_a \not{n}_b + \gamma_b \not{n}_a + \gamma \not{n}_a \not{n}_b.$$

The prescription of the boundary data discussed in the previous section implies that $\gamma_{ab} \simeq 0$ and $\gamma_a \simeq 0$. Then, equation (3.12a) implies that $\gamma \simeq \gamma_a{}^a \simeq 0$.

Boundary data for Θ_a. Consider the projections $\theta_a \equiv \ell_a{}^b \Theta_b$ and $\theta \equiv \not{n}^a \Theta_a$. Then we have that

$$\Theta_a = \theta_a + \not{n}_a \theta.$$

The boundary data for $\not{L}_i$ are equivalent to $\theta_a \simeq 0$. In order to prove the vanishing of θ we use the identity (3.12c). Using that $\gamma_a \simeq 0$, a short calculation yields

$$\theta \simeq \tfrac{1}{3} \not{D} \gamma \simeq -\tfrac{1}{3} \not{D} \gamma_a{}^a,$$

where the second equality is readily obtained from taking the normal derivative of equation (3.12a). On the other side, from the definition of γ_{ab}, one can write

$$\not{D} \gamma_a{}^a \simeq \left(\tfrac{1}{4} \not{r} - \tfrac{3}{2} \varkappa^2(x)\right) \not{\Sigma}.$$

Without loss of generality, it is always possible to, under a further conformal rescaling of the form $g'_{ab} = \omega^2 g_{ab}$, choose a conformal representation for which $\varkappa \simeq 0$ — see Proposition 3. Similarly, the Ricci scalar associated to a 3-dimensional hypersurface satisfies the following transformation rule:

$$D_i D^i \omega = \tfrac{1}{8} r \omega - \tfrac{1}{8} r' \omega^5.$$

Providing suitable initial data at $\partial \mathcal{S}_\star$ for this wave equation, it is seen that one can freely prescribe the value of r. In particular, considering the conformal boundary, one can choose a representation for which $r \simeq 0$. This means that $\not{D} \gamma_a{}^a \simeq 0$ and, in turn, implies that $\theta \simeq 0$.

Boundary data for Δ_{abc}. Consider the system of wave equations for the geometric fields (3.25a)-(3.25e). As initial and boundary data sets for the system have already been established, a solution can then be locally obtained in a neighbourhood of $\partial\mathcal{S}_\star$. In particular, $d^a{}_{bcd}$ and its derivatives are well-defined, meaning that all the components of Λ_{abc} are regular. On the other hand, it can be checked that the trivial data for $P^a{}_{bcd}$ imply that $\nabla_d P_{abc}{}^d \simeq 0$. Thus, from equation (3.12i) we conclude that $\Delta_{abc} \simeq 0$.

Boundary data for Λ_{abc}. In the case of Λ_{abc} we introduce its relevant components: $\lambda_{abc} \equiv \ell_a{}^d \ell_b{}^e \ell_c{}^f \Lambda_{def}$, $\lambda_{ab} \equiv \not{n}^c \ell_a{}^d \ell_b{}^e \Lambda_{cde}$, $\Lambda_{ab} \equiv \not{n}^c \ell_a{}^d \ell_b{}^e \Lambda_{dce}$ and $\Lambda_a \equiv \not{n}^b \not{n}^c \ell_a{}^d \Lambda_{bcd}$. In terms of these we have:

$$\Lambda_{abc} = \lambda_{abc} + \lambda_{bc}\not{n}_a + 2\Lambda_{a[c}\not{n}_{b]} + 2\Lambda_{[c}\not{n}_{b]}\not{n}_a. \qquad (5.12)$$

The boundary data for the electric and magnetic parts of $d^a{}_{bcd}$ are equivalent to $\lambda_{ab} \simeq 0$ and $\Lambda_{ab} \simeq 0$. Next, we proceed to prove that the two remaining components vanish as well. First, consider the normal derivative of the identity (3.12i) and project all its free indices onto $\mathscr{I}$. This results in

$$\not{\Sigma}\lambda_{abc} \simeq -\not{D}\delta_{abc},$$

where $\delta_{abc} \equiv \ell_a{}^d \ell_b{}^e \ell_c{}^f \Delta_{def}$. Furthermore, projecting the integrability condition (3.13c) with $\not{n}^a \ell_d{}^a \ell_e{}^b \ell_f{}^c$ and using the vanishing of Υ_{ab} and Δ_{abc} on $\mathscr{I}$, a calculation yields

$$\not{D}\delta_{abc} \simeq 0,$$

which then implies that $\lambda_{abc} \simeq 0$.

To complete the proof, define a further component of Δ_{abc}: $\Delta_a \equiv \not{n}^b \not{n}^c \ell_a{}^d \Delta_{bdc}$. Observe that multiplying (3.12e) by $\not{n}^c$ one readily finds that $\not{D}\Delta_a \simeq 0$. On the other hand, taking the normal derivative of (3.12i) and then multiplying it by $\not{n}^a \not{n}^c \ell_b{}^g$ we obtain

$$\not{D}\Delta_g \simeq -\not{\Sigma}\Lambda_g,$$

from where we conclude that $\Lambda_a \simeq 0$.

The above results can be summarised as:

Lemma 12. *Assume that the wave equations* (3.1a)-(3.1d) *and* (2.14) *are valid. If the boundary data for the geometric fields are given as in Proposition 10, then the zero-quantities vanish on* $\mathscr{I}$.

Remark 47. Regarding the zero-quantities on $\mathcal{S}_\star$, the components corresponding to projections on this hypersurface vanish by the way the anti-de Sitter-like initial data has been constructed. Components with a transversal (i.e., timelike) projection can be read as a first order evolution system for the geometric conformal fields. Thus, in order to ensure the vanishing of the zero-quantities on $\mathcal{S}_\star$, one needs, firstly, to produce a solution to the conformal constraint equations. Secondly, one reads the transversal components of the zero-quantities as definitions for the normal derivatives of the conformal fields which can be readily computed from the solution to the conformal constraints. In this sense, the transversal components of the zero-quantities vanish *a fortiori*. Furthermore, as a consequence of this procedure, the normal derivatives of the zero-quantities trivially vanish on $\mathcal{S}_\star$.

5.4.2 Boundary conditions for the subsidiary gauge evolution system

The final piece in the construction is the analysis of the propagation of the gauge. In Section 3.6, it was shown that the fields Q, Q_μ and M_μ satisfy the system of homogeneous geometric wave equations (3.33)-(3.35). Lemma 4 establishes the vanishing of the gauge fields on the spacetime provided that trivial initial and boundary conditions are imposed. In this regard, this subsection analyses the data

$$
\begin{gathered}
M_\mu = 0, \qquad Q_\mu = 0, \qquad Q = 0, \qquad \nabla_\mu M_\nu = 0, \\
\nabla_\mu Q_\nu = 0, \qquad \nabla_\mu Q = 0 \qquad \text{on} \quad \mathcal{S}_\star,
\end{gathered}
$$

along with

$$
M_\mu = 0, \qquad Q_\mu = 0, \qquad Q = 0 \qquad \text{on} \quad \mathscr{I}.
$$

First, the fundamental subsidiary field Q_μ can be decomposed with respect to $\not{n}^a$ in terms of the projections $\hat{q}_\mu \equiv \ell_\mu{}^\nu Q_\nu$ and $\hat{q} \equiv \not{n}^\nu Q_\nu$. Accordingly,

$$
Q_\mu \simeq \hat{q}_\mu + \hat{q}\not{n}_\mu.
$$

From here we notice that the system (5.7a)-(5.7b) is equivalent to $\hat{q}_\mu \simeq 0$ and $\hat{q} \simeq 0$. Conditions $Q \simeq 0$ and $M_\mu \simeq 0$, on the other hand, involve a number of additional higher order normal derivatives of $\hat{q}$, namely

$$\not{D}^{(n)}\hat{q} \simeq 0, \quad \not{D}^{(n)}\hat{q}_\mu \simeq 0, \qquad n = 1, 2. \tag{5.13}$$

Following the same approach to study the data on $\mathcal{S}_\star$, one finds that adopting a Gaussian system to the initial hypersurface as in Section 5.2, it follows that $Q_\mu = 0$. As vanishing first order derivatives of the subsidiary fields need to be prescribed, a series of straightforward calculations shows that the projections $q \equiv n^a Q_a$ and $q_\mu \equiv h_\mu{}^\nu Q_\nu$ must satisfy

$$q = 0, \quad q_\mu = 0, \quad D^{(n)}q = 0, \quad D^{(n)}q_\mu = 0, \quad n = 1, 2, 3. \tag{5.14}$$

Invoking Lemma 4 with $T_{ab} = 0$, we have the following result:

Lemma 13. *If conditions* (5.13) *and* (5.14) *are satisfied on* $\mathscr{I}$ *and* $\mathcal{S}_\star$*, respectively, then* Q*,* Q_μ *and* $Q_{\mu\nu}$ *vanish identically in a neighbourhood of* $\partial\mathcal{S}_\star$*.*

5.5 The local existence result

We are now in the position of formulating the main result of this chapter: a local in-time existence result for anti-de Sitter-like spacetimes. This result can, in turn, be patched together with the domain of dependence of open subsets of $\mathcal{S}_\star$ away from $\partial\mathcal{S}_\star$ to obtain a solution on a slab around $\mathcal{S}_\star$ — see e.g. [60], Section 12.3.

One has the following:

Theorem 3. *Let* $\mathcal{S}_\star$ *be a 3-dimensional spacelike hypersurface with boundary* $\partial\mathcal{S}_\star$ *and smooth anti-de Sitter-like initial data defined on it. Let* ℓ_{ij} *be a smooth 3-dimensional Lorentzian metric defined on* $\mathscr{I}$*. Assume that the data on* $\mathcal{S}_\star$ *and* $\mathscr{I}$ *satisfy, up to some order, the corner conditions at* $\partial\mathcal{S}_\star$*. Then, there exists a smooth solution to the vacuum Einstein field equations with* $\lambda < 0$ *in a neighbourhood of* $\partial\mathcal{S}_\star$*.*

Proof. Consider initial data on $\mathcal{S}_\star$ given as in Definition 5. Given a 3-dimensional Lorentzian metric ℓ_{ij} on the boundary, the data given by (2.18a)-(2.18f) along with (5.8)-(5.9) can be directly computed. If these two sets of data satisfy the corner conditions at $\partial\mathcal{S}_\star$, then the theory of initial-boundary value problems, as given in e.g. [14, 21], guarantees the existence of a unique solution to the system of wave equations (3.25a)-(3.25e) in a neighbourhood of $\partial\mathcal{S}_\star$.

Given the boundary data described above, Lemma 12 implies that the geometric zero-quantities vanish on $\mathscr{I}$. In addition, we also have vanishing initial data on $\mathcal{S}_\star$ — see Remark 47. Thus, by virtue of Lemma 3, we can guarantee the existence and uniqueness of the trivial solution to the system (3.15)-(3.20). In consequence, any solution to the system of wave equations (5.2a)-(5.2e) is also a solution to the conformal field equations (5.1a)-(5.1f). Therefore, Proposition 1 implies that the metric $\tilde{g}_{ab} = \Xi^{-2} g_{ab}$ is a solution to the vacuum EFE (1.8) with $\lambda < 0$ for $\Xi \neq 0$.

□

Chapter 6

Construction of anti-de Sitter-like spacetimes using the metric conformal Einstein field equations: the tracefree matter case

The material of this section is based on [11]

6.1 Introduction

Having dealt with the problem of the construction of vacuum anti-de Sitter-like spacetimes, we now proceed to generalise the result to the case of trace-free matter. As a preamble, models with a tracefree energy-momentum tensor are preferred given that their conformal properties are more suitable for study in a systematic way. In [46], for example, a result of local existence for the Einstein-Yang-Mills system has been obtained under the assumption of spherical symmetry. Despite the complications from considering non-trivial matter fields, advances have been made under a conformal scheme in a variety of scenarios — see, for example, [28, 32, 33]. Nevertheless, an approach allowing us to encompass more general matter models and whose equations have better structural properties for its numerical analysis is ideal.

The strategy to be pursued is an extension to the one in Chapter 5, with particular emphasis on the three models introduced in Section 2.4. This means that the material and results from chapters 2, 3 and 5 will be intensively exploited. Moreover, given the complications that the coupling of a matter sector brings, some pieces of the construction will need further work. Most significantly, the problem of prescribing boundary data for the matter fields under consideration does not have a straightforward solution.

In the following we will consider the system of quasilinear wave equations for the conformal fields given by (3.25a)-(3.25e). In order to give initial and boundary data, we will use a system of coordinates adapted to $\mathcal{S}_\star$ and $\mathscr{I}$ which satisfies the generalised wave coordinate condition.

6.2 Initial and boundary data

6.2.0.1 Solutions to the conformal constraints on a spacelike hypersurface

In the presence of a tracefree matter component, the Hamiltonian and Momentum constraints need to be generalised to include the relevant matter fields which, in turn, also become part of the basic initial data. Proceeding in the same way as in Section 5.3, it can be shown that the Hamiltonian and Momentum constraints on a spacelike hypersurface take the form

$$\frac{\Omega^2}{2}(r + K^2 - K_{ij}K^{ij}) = 2K\Omega\Sigma - 2\Omega D_i D^i\Omega - 3\Sigma^2 + 3D_i\Omega D^i\Omega + \lambda + \Omega^4\rho, \tag{6.1a}$$

$$\Omega(D_j K_i{}^j - D_i K) = 2(K_{ij}D^j\Omega - D_i\Sigma) + \Omega^3 j_i. \tag{6.1b}$$

Accordingly, the fields $\boldsymbol{h}$, $\boldsymbol{K}$, Ω, Σ, ρ and $\boldsymbol{j}$ satisfying the previous equations represents the initial data. Using the conformal constraints we obtain expressions for them, namely

$$s = \frac{1}{3}\left(\Delta\Omega + \frac{1}{4}\Omega\big(r + K^2 - K_{ij}K^{ij}\big) - \Sigma K + \frac{1}{2}\Omega^3\rho\right), \tag{6.2a}$$

$$L_{ij} = \frac{1}{\Omega}\left(sh_{ij} + \Sigma K_{ij} - D_iD_j\Omega\right) + \frac{1}{2}\Omega^2 T_{ij}, \tag{6.2b}$$

$$L_i = \frac{1}{\Omega}\big(K_i{}^k D_k\Omega - D_i\Sigma\big) + \frac{1}{2}\Omega^2 j_i, \tag{6.2c}$$

$$\begin{aligned} d_{ij} = \frac{1}{\Omega}\bigg(& -L_{ij} + l_{ij} + \Big(K\big(K_{ij} - \frac{1}{4}Kh_{ij}\big) - K_{ki}K_j{}^k \\ & + \frac{1}{4}K_{kl}K^{kl}h_{ij}\Big)\bigg), \end{aligned} \tag{6.2d}$$

$$d_{ijk} = \frac{1}{\Omega}\big(D_jK_{ki} - D_kK_{ji} + h_{ik}L_j - h_{ij}L_k\big). \tag{6.2e}$$

Analogous to the vacuum case, the collection of fields for which the above expressions are regular on $\mathcal{S}_\star$ will be called a tracefree anti-de Sitter-like initial data set.

6.2.1 Boundary conditions for the conformal evolution equations

Regarding the boundary data, it is not difficult to notice that, with the exception of the data for the electric part of the rescaled Weyl tensor, the conformal constraints prescribe identical data for the remaining fields on $\mathscr{I}$. Regarding $\not{d}_{ij}$, the inclusion of matter terms in equation (2.11d) yields the corresponding Neumann data, provided the matter model has been specified:

$$\not{D}\not{d}_{ij} \simeq \not{D}_k \not{d}_{ji}{}^k - T_{ij}. \tag{6.3}$$

Remark 48. Adding the field T_{ij} to the hypotheses of Proposition 10, the same conclusion about the geometric zero-quantities follows immediately in the tracefree matter case. Also notice that the analysis of the boundary data for these fields — see Section 5.4.1 — remains unchanged in the tracefree case.

6.3 Data for the matter fields

Prescription of boundary data for the conformal fields does not result sufficient for the system (3.25a)-(3.25e) due to the appearance of terms depending on T_{ab} and its derivatives. This section will be devoted to analysing the way boundary data for the matter fields can be given on $\mathscr{I}$.

6.3.1 Data for the conformally invariant scalar field

The fields ϕ, ϕ_a and ϕ_{ab} satisfy the system of wave equations (2.22), (3.38), (3.39), for which we prescribe suitable Dirichlet boundary data. Notice that ϕ can be freely prescribed as its value is not constrained by any equation intrinsic to $\mathscr{I}$, which in turn determines its derivatives on the boundary. Moreover, the normal derivative is not independent since (2.22) can be written as an equation constraining $\not{D}\phi$. Alternatively, observe that the prescription of Neumann boundary conditions, instead of Dirichlet ones, also yields a well-posed problem.

6.3.1.1 Boundary data for the evolution systems

In order to analyse the Dirichlet boundary data for the auxiliary fields it is convenient to introduce the following projections

$$\varphi_a \equiv \ell_a{}^b \phi_b, \quad \varphi \equiv \not{n}^a \phi_a, \quad \bar{\phi}_{ab} \equiv \ell_a{}^c \ell_b{}^d \phi_{cd}, \quad \bar{\phi}_a \equiv \not{n}^c \ell_a{}^b \phi_{bc}.$$

From the discussion above, φ_a and φ can be obtained directly once the basic data have been imposed. These represent the boundary data for ϕ_a. On the other hand, observing that ϕ_{ab} satisfies $\phi_a{}^a = \frac{1}{6} R\phi$, we can write

$$\phi_{ab} = \bar{\phi}_{ab} + \not{n}_a \bar{\phi}_b + \not{n}_b \bar{\phi}_a + \left(\tfrac{1}{6} R\phi - \bar{\phi}_a{}^a\right) \not{n}_a \not{n}_b.$$

Since $\bar{\phi}_{ab}$ and $\bar{\phi}_a$ can, via commutation of covariant derivatives, be determined from ϕ on the conformal boundary it follows that the boundary data for ϕ_{ab} is completely determined from the basic data.

6.3.1.2 Data for the subsidiary fields

In the same spirit as in Section 5.4, we now investigate the relation between the boundary data for the conformally invariant field and their associated subsidiary variables (3.40a) and (3.40b). It can be easily seen that the prescription of boundary data for ϕ is equivalent to

$$\ell_a{}^b Q_b \simeq 0, \quad \not{n}^a Q_a \simeq 0, \quad \ell_a{}^c \ell_b{}^d Q_{cd} \simeq 0, \quad \not{n}^b \ell_a{}^c Q_{bc} \simeq 0.$$

Exploiting the fact that $Q_a{}^a = 0$, it readily follows that $\not{n}^a \not{n}^b Q_{ab} \simeq -\ell^{ab} Q_{ab} \simeq 0$, implying the vanishing of Q_a and Q_{ab} on $\mathscr{I}$.

Remark 49. Similarly, we prescribe initial data consisting of ϕ and $D\phi$ on $\mathcal{S}_\star$ in an analogous manner as on $\mathscr{I}$. Accordingly, vanishing initial data for Q_a and Q_{ab} are obtained, which in turn implies that their intrinsic first derivatives vanish. Additionally, from definitions (3.40a) and (3.40b) it can be checked that their normal derivatives vanish too. Hence, $\nabla_a Q_b = 0$ and $\nabla_a Q_{bc} = 0$ on $\mathcal{S}_\star$.

6.3.1.3 Summary

The material of this subsection can be summed up as:

Lemma 14. *Let ϕ be the conformally invariant scalar field satisfying equation* (2.22) *with energy-momentum tensor given by* (2.23)*. Then, ϕ represents the supplementary basic boundary data required by the system* (3.25a)-(3.25e) *coupled to the wave equations for the fields ϕ, ϕ_a and ϕ_{ab}.*

6.3.2 Data for the Maxwell field

The Faraday tensor accepts a simple decomposition with respect to a vector ν^a normal to $\mathcal{S}_\star$. Defining the electric and magnetic parts, respectively, as $F_a \equiv \nu^c h_a{}^b F_{bc}$ and $F^*_a \equiv \nu^c h_a{}^b F^*_{bc}$ we have

$$F_{ab} = 2F_{[b}\nu_{a]} + \epsilon_{ab}{}^c F^*_c, \qquad F^*_{ab} = 2F^*_{[b}\nu_{a]} - \epsilon_{ab}{}^c F_c, \tag{6.4}$$

where $\epsilon_{abc} \equiv \nu^d \epsilon_{dabc}$ is the 3-volume form induced by ν^a. It follows that the Faraday tensor and its dual are completely determined by F_a and F^*_a.

Unlike the conformally invariant scalar field, these components cannot be freely prescribed, but a number of constraints are imposed by Maxwell equations:

$$D^i F_i = 0, \tag{6.5a}$$
$$D^i F_i^* = 0, \tag{6.5b}$$
$$DF_i = \epsilon_{ijk} D^j F^{*k}, \tag{6.5c}$$
$$DF_i^* = -\epsilon_{ijk} D^j F^k. \tag{6.5d}$$

The initial data set for this field consists of fields f_i and f_i^* which are solutions to (6.5a)-(6.5b). These, in turn, imply data for the normal derivatives of the electric and magnetic fields via equations (6.5c)-(6.5d). Regarding the boundary data, by performing a further decomposition with respect to the normal $\not{n}^a$ we can identify that the basic boundary data correspond to $f_i \equiv s_i{}^j F_j$. The remaining data can be obtained as follows. Equation (6.5a) provides Neumann data for the component $f \equiv \not{n}^i F_i$. Relation (6.5d) allows us to compute, along with the corresponding initial data, the field $f^* \equiv \not{n}^i F_i^*$. Finally, using this information and (6.5c) one directly obtains Neumann data for $f_i^* \equiv s_i{}^j F_j^*$. Following an analogous procedure, using (6.5b) instead of (6.5a), shows that one can alternatively prescribe f_i^* instead of f_i.

6.3.2.1 Data for the subsidiary fields

As done with the conformally invariant scalar field, we are now required to prove that the boundary data for the electric and magnetic fields implies trivial data for the subsidiary variables (3.52a)-(3.52c). First, boundary data for the fields obtained from the constraints (6.5a)-(6.5b) is equivalent to

$$\not{n}^a M_a \simeq 0, \quad \ell_a{}^d \ell_b{}^e \ell_c{}^f M_{def} \simeq 0.$$

Additionally, we also have that $Q_{abc} \simeq 0$, which is a direct consequence from the way the data for F_{abc} were constructed. The vanishing of the remaining boundary data for the subsidiary fields is proved as follows:

$$\ell_a{}^b M_b = \ell_a{}^b (F^c{}_{cb} - Q^c{}_{cb}) \simeq f^c{}_{ca} + F_a \simeq 0, \tag{6.6a}$$

$$\not{n}^c \ell_a{}^d \ell_b{}^e M_{cde} = \not{n}^c \ell_a{}^d \ell_b{}^e (F_{[cde]} - Q_{[cde]}) \simeq \tfrac{1}{3}(f_{ab} + 2\hat{f}_{[ba]}) \simeq 0. \tag{6.6b}$$

Remark 50. A direct calculation shows that the vanishing of the subsidiary variables on $\mathcal{S}_\star$ follows directly from the prescription of the initial data. Moreover, their normal derivatives vanish as a consequence of the wave equations (3.43) and (3.51).

6.3.2.2 Summary

Now we sum up the main results from this subsection:

Lemma 15. Let F_{ab} be the Faraday tensor satisfying the Maxwell equations (2.24a)-(2.24b) with energy-momentum tensor given by (2.26). Then, the components of the field f_i represent the basic boundary data required for the systems (3.25a)-(3.25e) coupled to F_{ab}.

6.3.3 Data for the Yang-Mills field

We end this section by working out a similar analysis for the Yang-Mills field. The identification of the corresponding boundary data will result similar to the one for the Maxwell field.

First, by introducing the projections $F^{\mathfrak{a}}{}_a \equiv \nu^c h_a{}^b F^{\mathfrak{a}}{}_{bc}$ and $F^{\mathfrak{a}}{}_a \equiv \nu^c h_a{}^b F^{*\mathfrak{a}}{}_{bc}$, the fields $F^{\mathfrak{a}}{}_{ab}$ and $F^{*\mathfrak{a}}{}_{ab}$ accept decompositions that are analogous to the ones in (6.4). On the other hand, for the gauge potential we define $\mathcal{A}^{\mathfrak{a}}{}_a \equiv h_a{}^b A^{\mathfrak{a}}{}_b$ and $\mathcal{A}^{\mathfrak{a}} \equiv \nu^a A^{\mathfrak{a}}{}_a$. Accordingly, we have

$$A^{\mathfrak{a}}{}_a = \mathcal{A}^{\mathfrak{a}}{}_a - \not{n}^a \mathcal{A}^{\mathfrak{a}}. \tag{6.7}$$

Equations (2.28a)-(2.28c) provide a set of relations from which the basic data can be extracted. The projections defined above enable us to write them as follows:

$$D^i F^{\mathfrak{a}}{}_i = C^{\mathfrak{a}}{}_{\mathfrak{bc}} \mathcal{A}^{\mathfrak{c}}{}_i F^{\mathfrak{b}i}, \tag{6.8a}$$

$$D^i F^{*\mathfrak{a}}{}_i = C^{\mathfrak{a}}{}_{\mathfrak{bc}} \mathcal{A}^{\mathfrak{c}}{}_i F^{*\mathfrak{b}i}, \tag{6.8b}$$

$$DF^{\mathfrak{a}}{}_i = \epsilon_{ijk} D^i F^{*\mathfrak{a}k} - C^{\mathfrak{a}}{}_{\mathfrak{bc}} \epsilon_{ijk} \mathcal{A}^{\mathfrak{b}j} F^{*\mathfrak{c}k} + C^{\mathfrak{a}}{}_{\mathfrak{bc}} \mathcal{A}^{\mathfrak{b}} F^{\mathfrak{c}}{}_i, \tag{6.8c}$$

$$DF^{*\mathfrak{a}}{}_i = -\epsilon_{ijk} D^j F^{\mathfrak{a}k} + 2C^{\mathfrak{a}}{}_{\mathfrak{bc}} \mathcal{A}^{\mathfrak{b}} F^{*\mathfrak{c}}{}_i + 2C^{\mathfrak{a}}{}_{\mathfrak{bc}} \epsilon_{ijk} \mathcal{A}^{\mathfrak{b}j} F^{\mathfrak{c}k}, \tag{6.8d}$$

$$D_i \mathcal{A}^{\mathfrak{a}}{}_j - \not{D}_j \mathcal{A}^{\mathfrak{a}}{}_i = \epsilon^m{}_{kl} \ell_i{}^k \ell_j{}^l F^{*\mathfrak{a}}{}_m - C^{\mathfrak{a}}{}_{\mathfrak{bc}} \mathcal{A}^{\mathfrak{b}}{}_i \mathcal{A}^{\mathfrak{c}}{}_j, \tag{6.8e}$$

$$D\mathcal{A}^{\mathfrak{a}}{}_i - D_i \mathcal{A}^{\mathfrak{a}} = F^{\mathfrak{a}}{}_i - C^{\mathfrak{a}}{}_{\mathfrak{bc}} \mathcal{A}^{\mathfrak{b}} \mathcal{A}^{\mathfrak{b}}{}_i. \tag{6.8f}$$

This system is supplemented with the corresponding decomposition of equation (3.62), namely

$$\not{D}^i \mathcal{A}^{\mathfrak{a}}{}_i + \not{D} \mathcal{A}^{\mathfrak{a}} = f^{\mathfrak{a}}(x). \tag{6.9}$$

A set of fields $F^{\mathfrak{a}}{}_i$, $F^{*\mathfrak{a}}{}_i$, $\mathcal{A}^{\mathfrak{a}}{}_i$ and $\mathcal{A}^{\mathfrak{a}}$ which are solution to equations (6.8a)-(6.8b) and (6.8e) constitute the initial data set for a given set of gauge source functions $f^{\mathfrak{a}}(x)$; the corresponding time derivatives can be obtained from the remaining expressions. Next, we discuss the boundary data by following a similar approach to the one in the Maxwell case. Additional to the components $f^{\mathfrak{a}}{}_i \equiv s^i{}_j F^{\mathfrak{a}}{}_j$, we also prescribe the components $\mathcal{A}$ and $\nu^i A^{\mathfrak{a}}{}_i$ of the gauge potential. Using this information, and the initial data set, equation (6.8f) allows us to compute the components $s_i{}^j \mathcal{A}^{\mathfrak{a}}_j$. On the other hand, data for the component $f \equiv \not{n}^i F^{*\mathfrak{a}}{}_i$ can be extracted from (6.8a). Relation (6.8d) yields an evolution equation from where the component $f^{\mathfrak{a}} \equiv \not{n}^i F^{\mathfrak{a}}{}_i$ can be obtained. Finally, a direct calculation shows that expression (6.8c) establishes boundary data for $f^{*\mathfrak{a}}{}_i \equiv s_i{}^j F^{*\mathfrak{a}}$. In similarity to the Maxwell field, one can also prescribe the field $f^{*\mathfrak{a}}{}_i$ instead of $f^{\mathfrak{a}}{}_i$.

6.3.3.1 Data for the subsidiary fields

The final piece in the analysis of this matter field corresponds to proving that the basic data on $\mathscr{I}$ implies vanishing data for the relevant subsidiary fields. The system (6.8a)-(6.8b) and (6.8e) represents the relations

$$\not{n}^a M^{\mathfrak{a}}{}_a \simeq 0, \quad \ell_a{}^d \ell_b{}^e \ell_c{}^f M^{\mathfrak{a}}{}_{def} \simeq 0, \quad \ell_a{}^c \ell_b{}^d M^{\mathfrak{a}}{}_{cd} \simeq 0.$$

Analogous to the Maxwell field, the construction of the data for $F^{\mathfrak{a}}{}_{abc}$ implies that $Q^{\mathfrak{a}}{}_{abc} \simeq 0$. The parallelism between the Yang-Mills and Maxwell fields makes clear that the components $\ell_a{}^b M^{\mathfrak{a}}{}_b$ and $\not{n}^c \ell_a{}^d \ell_b{}^e M^{\mathfrak{a}}{}_{cde}$ vanish on the conformal boundary. Concerning $M^{\mathfrak{a}}{}_{ab}$, combining equations (3.69a) and (3.69b), we obtain the identity

$$\tfrac{1}{2} C^{\mathfrak{a}}{}_{\mathfrak{bc}} F^{\mathfrak{b}ab} M^{\mathfrak{c}}{}_{ab} - C^{\mathfrak{a}}{}_{\mathfrak{bc}} A^{\mathfrak{b}a} M^{\mathfrak{c}}{}_a = 0.$$

As it has been shown that $M^{\mathfrak{a}}{}_a \simeq 0$, it then follows that for an arbitrary field $F^{\mathfrak{a}}{}_{ab}$ the condition $M^{\mathfrak{a}}{}_{ab} \simeq 0$ must be satisfied. To conclude, recall that the Yang-Mills coupling has required the introduction of the field $P^{\mathfrak{a}}$ — see (3.77). Trivially, equation (6.9) implies that $P^{\mathfrak{a}} \simeq 0$.

Remark 51. Vanishing data for the subsidiary variables on $\mathcal{S}_\star$ follows from an argument similar to the one in Remark 50.

6.3.3.2 Summary

Next, we summarise the above discussion:

Lemma 16. *Let $F^{\mathfrak{a}}{}_{ab}$ and $A^{\mathfrak{a}}{}_a$ be the fields satisfying the Yang-Mills equations* (2.28a)-(2.28c) *with energy-momentum tensor given by* (2.30)*, and a set of gauge source functions given by* (3.62)*. Then, the fields $f^{\mathfrak{a}}{}_i$, $\mathcal{A}^{\mathfrak{a}}$ and $\nu^i A^{\mathfrak{a}}{}_i$ defined on $\mathscr{I}$ represent the basic boundary data required for the system* (3.25a)-(3.25e)*.*

6.4 The local existence result

After having analysed each matter field, we can now state the main result regarding the construction of anti-de Sitter-like spacetimes coupled to one of the tracefree matter models introduced in Section 2.4.

Theorem 4. *Let $\mathcal{S}_\star$ be a 3-dimensional spacelike hypersurface with boundary $\partial\mathcal{S}_\star$ and smooth tracefree anti-de Sitter-like initial data defined on it. Let ℓ_{ij} be a smooth Lorentzian 3-metric defined on $\mathscr{I}$. Assume that the data on $\mathcal{S}_\star$ and on $\mathscr{I}$ satisfy, up to some order, compatibility conditions at $\partial\mathcal{S}_\star$. Consider suitable initial and boundary data for either the conformally* *invariant scalar field, the Maxwell field or the Yang-Mills field. Then, there exists a smooth solution to the Einstein field equations with* $\lambda < 0$ *coupled to one of the aforementioned matter models in a neighbourhood of* $\partial\mathcal{S}_\star$.

Proof. The result is obtained following a reasoning similar to the one in Theorem 3, so here we just emphasise the differences. The determination of $\not{D}\not{d}_{ij}$ in the tracefree matter case needs, additional to ℓ_{ij}, the prescription of the field T_{ij} on $\mathscr{I}$. These basic data allow us to establish a well-posed problem for the system (3.25a)-(3.25e) coupled to the corresponding wave equations for each matter model. While the vanishing of the geometric zero-quantities has been established in Section 5.4.1, it has also been proven that the basic data for the matter fields imply trivial data for subsidiary variables associated to them on $\mathscr{I}$. Thus, this permits us to link solutions to the system of wave equations for the conformal fields to solutions to the MTCEFE which, in turn, provide a solution to the EFE with tracefree matter. □

Chapter 7

Final remarks

In this work, the tools from conformal geometry have been exploited to analyse several properties of the EFE and, in particular, of anti-de Sitter-like spacetimes. The relation between the MTCEFE and the system of wave equations they imply has been established via a system of homogeneous wave equations for the corresponding zero-quantities; these, remarkably, arise from a set of integrability conditions along with a number of relations showing the manner these fields are intertwined. As the existence of these relations is non-trivial, nor the homogeneous character of the resulting evolution system is, this seems to point towards some more fundamental properties of the EFE related to their symmetry group. Ideally, a further analysis might provide a practical criterion for the existence of integrability conditions based on the structure of the equations.

Regarding the Killing boundary data problem in anti-de Sitter-like spacetimes, the analysis has shown how the problem on the "bulk" can be reduced to a simplified version contained on the conformal boundary. Furthermore, the obstruction tensor emerges as the main object determining the existence of a continuous symmetry on this type of spacetimes, as its vanishing enables the obtention of a homogeneous system for the corresponding zero-quantities. It was pointed out that a conformally flat hypersurface trivially yields a vanishing obstruction tensor; in this sense, the next natural step is to find larger families of 3-dimensional timelike hypersurfaces with this property.

The construction of vacuum and tracefree matter anti-de Sitter-like spacetimes heavily relied on the second order evolution system implied by the MTCEFE, along with the number of formulae relating the subsidiary variables to each other. The other main ingredient allowing us to establish a well-posed problem has been the systematic construction of initial and boundary data. In particular, it has been found that the boundary data set can be fully determined by prescribing the metric on the conformal boundary. However, a complete characterisation of suitable boundary data for this class of spacetimes is still an open question. A successful solution to this problem might, potentially, shed some light on the issue of the conjectured instability of the anti-de Sitter spacetime, as previous studies only assume a restricted class of boundary conditions.

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