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Double Negation Semantics for Generalisations of Heyting Algebras

Abstract. This paper presents an algebraic framework for investigating proposed translations of classical logic into intuitionistic logic, such as the four negative translations introduced by Kolmogorov, Gödel, Gentzen and Glivenko. We view these as *variant semantics* and present a semantic formulation of Troelstra's syntactic criteria for a satisfactory negative translation. We consider how each of the above-mentioned translation schemes behaves on two generalisations of Heyting algebras: bounded pocrims and bounded hoops. When a translation fails for a particular class of algebras, we demonstrate that failure via specific finite examples. Using these, we prove that the syntactic version of these translations will fail to satisfy Troelstra's criteria in the corresponding substructural logical setting.

Keywords: pocrims, hoops, algebraic semantics, negative translation, double negation translation, involutive core, involutive replica.

1. Introduction

Schemes for translating classical logic into intuitionistic logic have been studied since the 1920s and are important for understanding the computational content of classical logic. These so-called *negative translations* or *double negation translations* such as those proposed by Kolmogorov, Gödel, Gentzen and Glivenko are generally presented as syntactic translations and are studied by mainly syntactic methods (e.g., see [9, 11]). In this paper we use an algebraic framework for investigating proposed double negation translations.

The arguments justifying the syntactic Kolgomorov and Gödel translations do not need the rule of contraction and hence we develop our framework in the context of two generalisations of Heyting algebras: bounded pocrims and bounded hoops. In logical terms these correspond to the conjunction-implication fragment of *intuitionistic affine logic* and what we call *intuitionistic Łukasiewicz logic*, respectively. We view a translation as a variant semantics for the logical language and we give a semantic formulation of Troelstra's criteria for a satisfactory translation.

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The algebras that correspond to classical logic are called involutive (i.e., they satisfy $\neg\neg x = x$). We associate with each bounded pocrim $\mathbf{A}$ two involutive pocrims:

- a bounded pocrim $\mathbf{A}^C$ called the *involutive core* of $\mathbf{A}$, whose universe is a subset of the universe of $\mathbf{A}$, and
- a bounded pocrim $\mathbf{A}^R$ called the *involutive replica* of $\mathbf{A}$, whose universe is a quotient of the universe of $\mathbf{A}$.

A generalisation of the first construction (involutive core) was studied in [20], where it is called a c-retraction. The injection $\iota : \mathbf{A}^C \to \mathbf{A}$ and the projection $\pi : \mathbf{A} \to \mathbf{A}^R$ are not necessarily homomorphisms when $\mathbf{A}$ is a general bounded pocrim, but they are homomorphisms when $\mathbf{A}$ is a bounded hoop. The involutive core and the involutive replica turn out to be naturally isomorphic via the composite $\pi \circ \iota$. The two constructions give complementary ways of viewing the double negation operation $\delta(x) = \neg\neg x$.

Using the involutive core and the involutive replica, we show that the Kolmogorov and Gödel translations satisfy our algebraic formulation of Troelstra's criteria for a satisfactory negative translation in any reasonable class of bounded pocrims. We also show by explicit finite examples, that the Gentzen and Glivenko translations fail to satisfy our algebraic formulation of Troelstra's criteria in general. The proofs that the Gentzen and Glivenko translations fail are based on specific finite classes of finite bounded pocrims. Using these counter-examples we can prove that the syntactic versions of these translations fail to satisfy Troelstra's formulation of his criteria.

For bounded hoops, the situation is much simpler. The double negation operation is a homomorphism implying that all reasonable double negation translation schemes are equivalent and hence satisfy our formulation of Troelstra's formulation. The results for bounded hoops is dependent on certain algebraic identities, some of which are not easy to derive from the axioms for this class of algebra. We use an indirect semantic method to verify the harder identities (see Section 4.2).

1.1. Related work

Cignoli and Torrell [8] investigate Glivenko's negative translation scheme in the setting of bounded BCK algebras, the algebraic models of the implicative fragment of intuitionistic affine logic. They study an analogue for BCK algebras of what we call the involutive core of a bounded pocrim, and discuss extensions of their results on the Glivenko translation to bounded pocrims and bounded hoops. In the present paper, we are interested in negative translation schemes in general and give a framework for comparing different translations.

Galatos and Ono [14] look at the Glivenko and Kolmogorov translations for substructural logics over the full Lambek calculus, taking again an algebraic approach studying involutive sub-structures of residuated lattices. In particular, they show that every involutive sub-structural logic has a *minimal* substructural logic that contains the first via a double negation interpretation. Commutativity is not assumed, so the paper has to deal with two forms of negation. A proof-theoretic presentation of the results in [14] for the Glivenko translation are then presented by Ono [20], looking at the weakest extension of full Lambek calculus needed to derive the Glivenko theorem for classical logic.

The work that is perhaps closest to ours is that of Farahani and Ono [10], where they also study various negative translations, analysing the role of the *double negation shift* principle in the treatment of the quantifiers in predicate logic. In their final section on "algebras" they discuss a construction (c-retraction), which can be viewed as a generalisation of our involutive core construction. In the present paper our goal is to create a general framework for negative translations, enabling us to identify situations where particular translation schemes fail to have the required algebraic properties for a negative translation. In our study we also an alternative to the c-retraction/involutive core construction, the involutive replica, which turns out to fit more naturally in some cases.

1.2. Syntactic Negative Translations

As mentioned above, we are studying here classes of algebras that capture the semantics of some well-known logics. A formula is provable in the conjunction-implication fragment of *intuitionistic affine logic* iff it is valid in all bounded pocrims. Similarly, provability in the conjunction-implication fragment of GBL (the fragment that we call *intuitionistic Łukasiewicz logic*) is captured by validity in the algebraic class of hoops. The classical counterparts of these logics, i.e. the extension of these logics with the *double negation elimination* (DNE) principle $A^{\perp\perp} \to A$, can be also captured by the sub-class of *involutive* pocrims/hoops, i.e. bounded pocrims/hoops which satisfy $x^{\perp\perp} = x$.

Negative translations provide a way to eliminate DNE from classical proofs of a formula A, turning these into intuitionistic proofs of the translation of A. Although various negative translations have been proposed in the literature [15, 16, 17, 19], it is well known that all negative translations which satisfy Troelstra's criteria [22, Section 1.10] are intuitionistically equivalent. Formally, Troelstra calls a formula translation $A \mapsto A^N$ a *negative translation* if

(i) A and A^N are classically equivalent;

(ii) If A is provable classically then A^N is provable intuitionistically;

(iii) A^N is equivalent to a formula in the negative fragment (negated atomic formulas, implication and conjunction).

The point behind (iii) is that, for this negative fragment, classical and intuitionistic provability coincide, and in particular $(A^{N_1})^{N_2}$ is intuitionistically equivalent to A^{N_1}. Assume then that two translations A^{N_1} and A^{N_2} satisfy the above. By (DNS1), we have that $A^{N_1} \to A$ holds classically. Hence, by (DNS2), $(A^{N_1} \to A)^{N_2}$ is intuitionistically valid. With a further assumption that these translations are modular (see [11]), we also have $(A^{N_1})^{N_2} \to A^{N_2}$ and hence $A^{N_1} \to A^{N_2}$.

2. Pocrims

The most general class of algebras we consider is the class of pocrims: partially ordered, commutative, residuated, integral monoids [3]. Pocrims provide the natural algebraic models for the fragment of intuitionistic logic known as *minimal affine logic*, whose connectives are implication ($\phi \Rightarrow \psi$) and a form of conjunction ($\phi \otimes \psi$) that is not required to be idempotent (so that the law of contraction need not hold). The underlying ordered set of a pocrim is bounded above but not necessarily below; bounded pocrims, i.e., those in which the order is bounded below provide the context for our study of negation.

Definition 2.1 (Pocrim). *A* pocrim *is a structure for the signature* $(\top, \cdot, \to)$ *of type* $(0, 2, 2)$ *satisfying the following laws, in which* $x \le y$ *is an abbreviation for* $x \to y = \top$*:*

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \qquad [\mathsf{m}_1]$$

$$x \cdot y = y \cdot x \qquad [\mathsf{m}_2]$$

$$x \cdot \top = x \qquad [\mathsf{m}_3]$$

$$x \le x \qquad [\mathsf{o}_1]$$

$$\textit{if } x \le y \textit{ and } y \le z, \textit{ then } x \le z \qquad [\mathsf{o}_2]$$

$$\textit{if } x \le y \textit{ and } y \le x, \textit{ then } x = y \qquad [\mathsf{o}_3]$$

$$\textit{if } x \le y, \textit{ then } x \cdot z \le y \cdot z \qquad [\mathsf{o}_4]$$

$$x \le \top \qquad [\mathsf{t}]$$

$$x \cdot y \le z \textit{ iff } x \le y \to z. \qquad [\mathsf{r}]$$

We will refer to the operations $\cdot$ and $\to$ as conjunction and residuation respectively. We adopt the convention that residuation associates to the right and has lower precedence than conjunction. So the brackets in $x \cdot ((x \to y) \to y)$ are all necessary while those in $(x \cdot z) \to (y \to z)$ may all be omitted.

Throughout this paper, we adopt the convention that if $\mathbf{P}$ is a structure then P is its universe. If $\mathbf{P}$ is a pocrim, the laws $[\mathsf{m}_i]$, $[\mathsf{o}_j]$ and $[\mathsf{t}]$ say that $(P; \top, \cdot; \le)$ is a partially ordered commutative monoid with the identity $\top$ as top element. Law $[\mathsf{r}]$, the *residuation property*, says that for any y and z the set $\{x \mid x \cdot y \le z\}$ is non-empty and has supremum $y \to z$. It is an easy exercise in the use of the axioms to show that $x \to y$ is monotonic in y and antimonotonic in x.

A pocrim is said to be *bounded* if it has a (necessarily unique) *annihilator*, i.e., an element $\bot$ such that for every x we have:

$$\bot = x \cdot \bot. \qquad [\mathsf{ann}]$$

Note that any finite pocrim $\mathbf{P}$ is bounded, the annihilator being given by $\prod_{x \in P} x$. In a bounded pocrim $\mathbf{P}$, we have that $\bot = x \cdot \bot \le x \cdot \top = x$ for any x, so that $(M; \le)$ is indeed a bounded ordered set. We write $\neg x$ for $x \to \bot$ (and give $\neg$ higher precedence than the binary operators).

LEMMA 2.2. *The following are valid in all bounded pocrims:*

1. $\neg\neg\neg x = \neg x$.
2. $x \to y \le \neg y \to \neg x$.
3. $\neg(x \cdot y) = x \to \neg y = y \to \neg x$.

PROOF. The proofs are easy exercises in the use of the bounded pocrim axioms. ■

An element x of a bounded pocrim is said to be *regular* if it satisfies the double-negation identity:

$$\neg\neg x = x. \qquad [\mathsf{dne}]$$

For example, $\top$ and $\bot$ are regular in any bounded pocrim. A bounded pocrim is said to be *involutive* if all its elements are regular. This class of algebras corresponds to the $(\top, \bot, \Rightarrow, \wedge)$-fragment of *classical affine logic*. See [21] for further information about pocrims in general and involutive pocrims in particular.

We will often write $\delta(x)$ for $\neg\neg x$.

LEMMA 2.3. *The following are valid in all bounded pocrims:*

1. $x \le \delta(x)$.
2. $\delta^2(x) = \delta(x)$.
3. δ *is monotonic: if* $x \le y$ *then* $\delta(x) \le \delta(y)$.
4. $x \to \delta(y) = \delta(x) \to \delta(y)$.
5. $x \cdot \delta(y) \le \delta(x \cdot y)$.
6. $\delta(x \to y) \le x \to \delta(y)$.

PROOF. Let us prove part 6: using $[\mathsf{r}]$ several times, we have that $(*)$ $x \cdot \neg y \le \neg(x \to y)$, whence:

$$\begin{aligned} \delta(x \to y) \cdot x \cdot \neg y &\le \delta(x \to y) \cdot \neg(x \to y) && (*),[\mathsf{o}_4] \\ &\le \bot && [\mathsf{r}] \\ \delta(x \to y) \cdot x &\le \neg y \to \bot = \delta(y) && [\mathsf{r}] \\ \delta(x \to y) &\le x \to \delta(y) && [\mathsf{r}] \end{aligned}$$

The proofs of the other parts are similar exercises in the use of the bounded pocrim axioms together with Lemma 2.2 and the monotonicity properties of $\cdot$ and $\to$ as necessary. ∎

In any bounded pocrim, the set $\{\bot, \top\}$ is closed under $\cdot$ and $\to$ and so, as $\neg\bot = \top$ and $\neg\top = \bot$, $\{\bot, \top\}$ is the universe of an involutive subpocrim.

EXAMPLE 2.4. *There is a unique pocrim* $\mathbb{B}$ *with two elements. It is involutive and provides the standard model for classical Boolean logic.*

DEFINITION 2.5 (Ordinal sum). *If* $\mathbf{C}$ *and* $\mathbf{D}$ *are pocrims, the* ordinal sum*,* $\mathbf{C} \oplus \mathbf{D}$*, is the pocrim* $((C \setminus \{\top\}) \sqcup D, \top, \cdot, \to)$ *where* $\cdot$ *and* $\to$ *extend the given operations on* C *and* D *to the disjoint union* $(C \setminus \{\top\}) \sqcup D$ *in such a way that whenever* $\top \ne c \in C$ *and* $d \in D$*,* $c \cdot d = c$ *(implying that* $d \to c = c$ *and* $c \to d = \top$*).*

Thus the order type of $\mathbf{C} \oplus \mathbf{D}$ is the concatenation of the partial orders $(C \setminus \{\top\}; \le)$ and $(D; \le)$. If $C \ne \{\top\}$, $\mathbf{C} \oplus \mathbf{D}$ is bounded iff $\mathbf{C}$ is bounded and can only be involutive if $D = \{\top\}$ (in which case $\mathbf{C} \oplus \mathbf{D}$ is isomorphic to $\mathbf{C}$), since if $\top \ne d \in D$, then, in $\mathbf{C} \oplus \mathbf{D}$, we have $\neg d = \bot$, so that $\neg\neg d = \top \ne d$.

REMARK 2.6. *As alluded to in Section 1.2 the equational theory of pocrims can be viewed as a logical theory, where a term t is viewed as a formula that holds in a pocrim $\mathbf{P}$ iff $t = \top$ under all assignments of variables in t to values in P. Conversely, as $x = y$ in a pocrim iff $(x \to y) \cdot (y \to x) = \top$, the equational theory can be recovered from the logical theory. In the sequel, we concentrate on the case of bounded pocrims. If $\mathcal{C}$ is a class of bounded pocrims, we write $\mathsf{Th}(\mathcal{C})$ for the logical theory of $\mathcal{C}$, i.e., the set of all terms t over the signature $(\top, \bot, \cdot, \to)$ of a bounded pocrim with variables drawn from the set $\mathsf{Var} = \{v_1, v_2, \ldots\}$, such that $t = \top$ under any assignment $\mathsf{Var} \to P$ taking values in a member $\mathbf{P}$ of $\mathcal{C}$. It can be shown that a deductive system called intuitionistic affine logic, which we will refer to as $\mathbf{AL_i}$ is sound and complete for the logical theory of all bounded pocrims. $\mathbf{AL_i}$ is essentially the usual intuitionistic propositional logic $\mathbf{IL}$ without the rule of contraction.*

2.1. Involutive pocrims

If $\mathbf{P}$ is a bounded pocrim, let $N = \mathsf{im}(\delta) = \{\delta(x) \mid x \in P\}$. Since $\delta(\neg x) = \neg x$, $N = \mathsf{im}(\neg)$. In general, N is not closed under conjunction and hence is not a subpocrim and δ does not respect either $\cdot$ or $\to$:

EXAMPLE 2.7. *There is a bounded pocrim $\mathbf{U}$ with elements $\top > a > b > c > \bot$ and with $\cdot$, $\to$ and δ as follows:*

$\cdot$	$\top$	a	b	c	$\bot$
$\top$	$\top$	a	b	c	$\bot$
a	a	b	b	$\bot$	$\bot$
b	b	b	b	$\bot$	$\bot$
c	c	$\bot$	$\bot$	$\bot$	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

$\to$	$\top$	a	b	c	$\bot$
$\top$	$\top$	a	b	c	$\bot$
a	$\top$	$\top$	a	c	c
b	$\top$	$\top$	$\top$	c	c
c	$\top$	$\top$	$\top$	$\top$	a
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$

	δ
$\top$	$\top$
a	a
b	a
c	c
$\bot$	$\bot$

So, in $\mathbf{U}$, $\delta(a \to b) = a \neq \top = \delta(a) \to \delta(b)$, $\delta(a \cdot a) = a \neq b = \delta(a) \cdot \delta(a)$ and $\delta(\delta(a) \cdot \delta(a)) \neq \delta(a) \cdot \delta(a)$. The image of negation is $N = \{\top, a, c, \bot\}$, which is not closed under conjunction, since $a \cdot a = b$.

However, in the above example, if we define $x \mathbin{\hat{\cdot}} y = \delta(x \cdot y)$, we find that $\mathbf{N} = (N; \top, \hat{\cdot}, \to, \bot)$ is an involutive pocrim whose residuation agrees with that of $\mathbf{U}$. Dually, we find that the equivalence relation whose equivalence classes form the partition $Q = \{\{\top\}, \{a, b\}, \{c\}, \{\bot\}\}$ of U is a congruence on the monoid $(U, \top, \cdot)$ and if we define $[x] \mathbin{\check{\to}} [y] = [x \to \delta(y)]$ (or equivalently $[\delta(x) \to \delta(y)]$, by Lemma 2.3, part 4), then $\mathbf{Q} = (Q; [\top], \check{\cdot}, \check{\to}, [\bot])$ is an involutive pocrim where $\check{\cdot}$ is induced from $\cdot$ by the monoid congruence. Using the following lemma, we will see that these constructions generalise to all bounded pocrims.

LEMMA 2.8. *Let* $\mathbf{P}$ *be a bounded pocrim.*

1. *The set* $N = \mathsf{im}(\delta)$ *is closed under* $\rightarrow$.
2. *Let the relation* θ *be defined on* P *by* $x \;\theta\; y$ *iff* $\delta(x) = \delta(y)$. *Then* θ *is a congruence on the monoid* $(P, \top, \cdot)$.

PROOF. 1. As remarked above, $N = \mathsf{im}(\neg)$, which is closed under $\rightarrow$ since $\neg x \rightarrow \neg y = \neg(\neg x \cdot y)$.
2. Clearly θ is an equivalence relation and we have only to show that if $x \;\theta\; y$ then $x \cdot z \;\theta\; y \cdot z$. Now, if $x \;\theta\; y$, then $(*)$ $\neg y = \neg(\delta(y)) = \neg(\delta(x)) = \neg x$ by Lemma 2.2, part 1 and we have:

$$\begin{aligned}
\delta(x \cdot z) \cdot \neg(y \cdot z) &= \delta(x \cdot z) \cdot (z \rightarrow \neg y) && \text{Lemma 2.2, part 3} \\
&= \delta(x \cdot z) \cdot (z \rightarrow \neg x) && (*) \\
&= \delta(x \cdot z) \cdot \neg(x \cdot z) && \text{Lemma 2.2, part 3} \\
&= \bot && [\mathsf{r}]
\end{aligned}$$

So using $[\mathsf{r}]$ again we have $\delta(x \cdot z) \leq \neg\neg(y \cdot z) = \delta(y \cdot z)$. By symmetry we also have $\delta(y \cdot z) \leq \delta(x \cdot z)$, whence $\delta(x \cdot z) = \delta(y \cdot z)$, i.e., $x \cdot z \;\theta\; y \cdot z$, as required. ■

Lemma 2.8 justifies the following definition:

DEFINITION 2.9. *Given a bounded pocrim* $\mathbf{P}$ *we define the following structures over the signature of a bounded pocrim:*

- $\mathbf{P}^C$, *the* involutive core of $\mathbf{P}$, *is* $(P^C, \top, \bot, \hat{\cdot}, \hat{\rightarrow})$ *where* $P^C = \mathsf{im}(\delta) \subseteq P$, *where* $\top$ *and* $\bot$ *are as in* P *and where* $\hat{\cdot}$ *and* $\hat{\rightarrow}$ *are defined as follows:*

$$\begin{aligned}
x \mathbin{\hat{\cdot}} y &:= \delta(x \cdot y) \\
x \mathbin{\hat{\rightarrow}} y &:= x \rightarrow y
\end{aligned}$$

  We write $\iota \colon P^C \rightarrow P$ *for the inclusion.*

- $\mathbf{P}^R$, *the* involutive replica of $\mathbf{P}$ *is* $(P^R, [\top], [\bot], \check{\cdot}, \check{\rightarrow})$ *where* P^R *is the quotient* P/θ *of* P *by the equivalence relation defined by* $x \;\theta\; y$ *iff* $\delta(x) = \delta(y)$ *and where, writing* $[x]$ *for the equivalence class in* P^R *of* $x \in P$, *we define* $\check{\cdot}$ *and* $\check{\rightarrow}$ *as follows:*

$$\begin{aligned}
[x] \mathbin{\check{\cdot}} [y] &:= [x \cdot y] \\
[x] \mathbin{\check{\rightarrow}} [y] &:= [x \rightarrow \delta(y)]
\end{aligned}$$

  We write $\pi \colon P \rightarrow P^R$ *for the projection.*

We will write $\hat{\leq}$ and $\check{\leq}$ for the order relation on $\mathbf{P}^C$ and $\mathbf{P}^R$ respectively.

THEOREM 2.10. *Let* $\mathbf{P}$ *be a bounded pocrim. Then:*

1. $\mathbf{P}^C$ *is an involutive pocrim and the inclusion of* $(P^C, \hat{\leq})$ *in* $(P, \leq)$ *is strictly monotonic* ($x \mathbin{\hat{\leq}} y$ *iff* $x \leq y$)*.*
2. $\mathbf{P}^R$ *is an involutive pocrim and the projection of* $(P, \leq)$ *onto* $(P^R, \hat{\leq})$ *is weakly monotonic* ($x \leq y$ *implies* $[x] \mathbin{\check{\leq}} [y]$)*.*
3. $\mathbf{P}^C$ *and* $\mathbf{P}^R$ *are isomorphic bounded pocrims via the composition of the inclusion* $\iota \colon P^C \to P$ *and the projection* $\pi \colon P \to P^R$*.*

PROOF. 1. Noting that $\hat{\to}$ is the restriction to P^C of $\to$, the claim about strong monotonicity is clear and we can write $\leq$ for $\hat{\leq}$. The bounded pocrim axioms are then easily proved with the exception of $[\mathsf{m}_1]$ (associativity of $\hat{\cdot}$) and $[\mathsf{r}]$ (residuation). For associativity, we have:

$$\begin{aligned} x \mathbin{\hat{\cdot}} (y \mathbin{\hat{\cdot}} z) &= \delta(x \cdot \delta(y \cdot z)) && \text{Definition} \\ &\leq \delta(x \cdot y \cdot z) && \text{Lemma 2.3, part 5} \\ &\leq \delta(x \cdot \delta(y \cdot z)) && [\mathsf{o}_4]\text{, Lemma 2.3, parts 1, 3} \end{aligned}$$

So $x \mathbin{\hat{\cdot}} (y \mathbin{\hat{\cdot}} z) = \delta(x \cdot y \cdot z)$ and similarly $(x \mathbin{\hat{\cdot}} y) \mathbin{\hat{\cdot}} z = \delta(x \cdot y \cdot z)$, giving us the associativity of $\hat{\cdot}$.
For residuation, the right-to-left direction is clear: if $x \leq y \mathbin{\hat{\to}} z = y \to z$, then $x \cdot y \leq z$ and then $x \mathbin{\hat{\cdot}} y = \delta(x \cdot y) \leq \delta(z)$ by Lemma 2.3, part 3. But $z = \delta(z)$ since $z \in P^C = \mathsf{im}(\delta)$. Hence, $x \mathbin{\hat{\cdot}} y \leq z$. For the converse, assume $x \mathbin{\hat{\cdot}} y \leq z$, i.e. $\delta(x \cdot y) \leq z$. By Lemma 2.3, part 5, we have that $x \cdot \delta(y) \leq \delta(x \cdot y)$, and hence $x \cdot \delta(y) \leq z$. But $y = \delta(y)$ since $y \in P^C = \mathsf{im}(\delta)$, so we have $x \cdot y \leq z$, which by residuation in $\mathbf{P}$ gives $x \leq y \to z$. To conclude the proof of part 1, we must show that $\mathbf{P}^C$ is involutive, but this is clear since negation in $\mathbf{P}^C$ is the restriction to $P^C = \mathsf{im}(\delta)$ of the negation in $\mathbf{P}$ and all the elements of $\mathsf{im}(\delta)$ are regular by Lemma 2.2, part 1.
2. That $(P^R, [\top], \check{\cdot})$ is a monoid follows immediately from Lemma 2.8, part 2. By definition, $[x] \mathbin{\check{\leq}} [y]$ iff $\delta(x \to \delta(y)) = \delta(\top) = \top$. We have

$$\begin{aligned} x \to \delta(y) &\leq \delta(x \to \delta(y)) && \text{Lemma 2.3, part 1} \\ &\leq x \to \delta(y) && \text{Lemma 2.3, part 6} \end{aligned}$$

So $\delta(x \to \delta(y)) = x \to \delta(y)$. Hence $[x] \mathbin{\hat{\leq}} [y]$ iff $x \leq \delta(y)$. Using this, weak monotonicity and axioms $[\mathsf{o}_1]$–$[\mathsf{o}_4]$ are easily checked. For residuation, we have that $[x] \mathbin{\check{\cdot}} [y] \mathbin{\check{\leq}} [z]$ iff $x \cdot y \leq \delta(z)$ iff $x \leq y \to \delta(z) = \delta(y \to \delta(z))$ iff $[x] \mathbin{\check{\leq}} [y] \mathbin{\check{\to}} [\delta(z)] = [y] \mathbin{\check{\to}} [z]$.

Finally, we must show that $\mathbf{P}^R$ is involutive. Now $[x] \mathbin{\check{\rightarrow}} [\bot] = [x \rightarrow \delta(\bot)] = [x \rightarrow \bot]$, so negation and hence, also, double negation commute with the projection of P onto P^R. As, by construction $[\delta(x)] = [x]$, $\mathbf{P}^R$ is indeed involutive.
3. We must show that $\pi \circ \iota$ is one-to-one, onto and respects the pocrim operations. To see that $\pi \circ \iota$ is one-to-one, let $x, y \in P^C$, so that $x = \delta(x)$ and $y = \delta(y)$, and assume $[x] = [y]$ in P^R. By definition $\delta(x) = \delta(y)$, hence $x = y$. To see that $\pi \circ \iota$ is onto, observe that $[x] = [\delta(x)]$ and $\delta(x) \in P^C$, for any $x \in P$. Clearly, $(\pi \circ \iota)(\top) = [\top]$ and $(\pi \circ \iota)(\bot) = [\bot]$. To see that $\pi \circ \iota$ respects conjunction, we must show $[x \mathbin{\hat{\cdot}} y] = [x] \mathbin{\check{\cdot}} [y]$ for $x, y \in P^C$. By definition $[x] \mathbin{\check{\cdot}} [y] = [x \cdot y]$, and $[x \mathbin{\hat{\cdot}} y] = [x \cdot y]$ iff $\delta(x \mathbin{\hat{\cdot}} y) = \delta(x \cdot y)$, but, by definition, $\delta(x \mathbin{\hat{\cdot}} y) = \delta(\delta(x \cdot y)) = \delta(x \cdot y)$. Finally, to see that $\pi \circ \iota$ respects residuation, we must show $[x \mathbin{\hat{\rightarrow}} y] = [x] \mathbin{\check{\rightarrow}} [y]$ for $x, y \in P^C$. By definition $[x] \mathbin{\check{\rightarrow}} [y] = [x \rightarrow \delta(y)] = [x \rightarrow y]$, since $y \in P^C$. Moreover, $[x \mathbin{\hat{\rightarrow}} y] = [x \rightarrow y]$ iff $\delta(x \mathbin{\hat{\rightarrow}} y) = \delta(x \rightarrow y)$, which holds by definition. ■

REMARK 2.11. *For any bounded pocrim* $\mathbf{P}$, $\iota \colon P^C \rightarrow P$ *is a homomorphism of the* $(\top, \bot, \rightarrow)$*-reduct of* $\mathbf{P}$*, and* $\pi \colon P \rightarrow P^R$ *is a homomorphism of the* $(\top, \bot, \cdot)$*-reduct of* $\mathbf{P}$*. In general, however, neither map is a pocrim homomorphism (see the discussion of the bounded pocrim* $\mathbf{U}$ *in Example 2.7).*

As $\mathbf{P}^C$ and $\mathbf{P}^R$ are isomorphic pocrims, one could focus attention on one of the two constructions, and several authors work solely with their analogue of $\mathbf{P}^C$. We prefer to have both constructions available, since, in some contexts it is convenient for the $(\top, \bot, \rightarrow)$-structure to be respected, while in other contexts it is more convenient for the $(\top, \bot, \cdot)$-structure to be respected (cf. the proofs of Theorems 3.5 and 3.6).

3. Generalised and Double Negation Semantics

Beginning with Kolmogorov [19], logicians have studied *double negation translations* (or *negative translations*) that represent classical logic in intuitionistic logic. Kolmogorov's translation inductively replaces every subformula of a formula by its double negation. Other authors have devised more economical translations: Gödel's translation [17] applies double negation to the right-hand operands of implications and at the outermost level; Gentzen's translation [15] applies double negation to atomic formulas only; and Glivenko's translation [16] is the most economical off all and just applies double negation once at the outermost level. In this section we undertake an algebraic study of these translations.

3.1. Generalised semantics

We wish to undertake an algebraic analysis of translations such as the various double negation translations. We will view the translations as variant semantics and so we need a framework to compare different semantics.

Typically, these translations are defined by recursion over the syntactic structure of a term, sometimes composed with an additional top-level transformation. See, for example, [11] where top-level transformations are handled by redefining the provability relation. Here, rather than working with syntax, we prefer to think of a syntactic term t as its denotation viewed as a family of maps $\alpha \mapsto x$, where x ranges over the universe of a bounded pocrim $\mathbf{P}$ and α is an assignment of values in P to the free variables of t. The modularity properties of a translation scheme which are needed for our proofs (see, for example, Theorem 3.11) are then captured by the following definition:

DEFINITION 3.1. *Let* $\mathsf{Poc}_\bot$ *be the category of bounded pocrims and homomorphisms and let* Set *be the category of sets. Given any set* X*, let* $H_X : \mathsf{Poc}_\bot \to \mathsf{Set}$ *be the functor that maps a pocrim* $\mathbf{P}$ *to* $\mathsf{Hom}_{\mathsf{Set}}(X, P)$*, i.e., the set of all functions from* X *to* P*, and maps a homomorphism* $h : \mathbf{P} \to \mathbf{Q}$ *to* $f \mapsto h \circ f : \mathsf{Hom}_{\mathsf{Set}}(X, P) \to \mathsf{Hom}_{\mathsf{Set}}(X, Q)$*. Now let* $\mathsf{Ass} = H_{\mathsf{Var}}$ *and* $\mathsf{Sem} = H_{\mathcal{L}}$ *where* $\mathcal{L}$ *is the set of all terms over the signature* $(\top, \bot, \cdot, \to)$ *of a bounded pocrim with variables drawn from the set* $\mathsf{Var} = \{v_1, v_2, \ldots\}$*. We define a* semantics *to be a natural transformation* $\mu : \mathsf{Ass} \to \mathsf{Sem}$*.*

So given a bounded pocrim $\mathbf{P}$, $\mathsf{Ass}(\mathbf{P})$ denotes the set of assignments $\alpha : \mathsf{Var} \to P$, while $\mathsf{Sem}(\mathbf{P})$ denotes the set of all possible functions $s : \mathcal{L} \to P$. A semantics μ is a family of functions $\mu_{\mathbf{P}}$ indexed by bounded pocrims $\mathbf{P}$ such that $\mu_{\mathbf{P}} : \mathsf{Ass}(\mathbf{P}) \to \mathsf{Sem}(\mathbf{P})$ and such that for any homomorphism $f : \mathbf{P} \to \mathbf{Q}$ the following diagram commutes.

$$\begin{array}{ccc} \mathsf{Ass}(\mathbf{P}) & \xrightarrow{\mathsf{Ass}(f)} & \mathsf{Ass}(\mathbf{Q}) \\ \downarrow{\scriptstyle \mu_{\mathbf{P}}} & & \downarrow{\scriptstyle \mu_{\mathbf{Q}}} \\ \mathsf{Sem}(\mathbf{P}) & \xrightarrow{\mathsf{Sem}(f)} & \mathsf{Sem}(\mathbf{Q}) \end{array}$$

The standard semantics μ^{S} is the one that simply uses the given assignment $\alpha : \mathsf{Var} \to P$ to give values to the variables in a term in $\mathcal{L}$ and then calculates its value interpreting the operations in the obvious way:

$$
\begin{aligned}
\mu^{\mathsf{S}}_{\mathbf{P}}(\alpha)(v_i) &= \alpha(v_i)\\
\mu^{\mathsf{S}}_{\mathbf{P}}(\alpha)(\top) &= \top\\
\mu^{\mathsf{S}}_{\mathbf{P}}(\alpha)(\bot) &= \bot\\
\mu^{\mathsf{S}}_{\mathbf{P}}(\alpha)(s \cdot t) &= \mu^{\mathsf{S}}_{\mathbf{P}}(\alpha)(s) \cdot \mu^{\mathsf{S}}_{\mathbf{P}}(\alpha)(t)\\
\mu^{\mathsf{S}}_{\mathbf{P}}(\alpha)(s \to t) &= \mu^{\mathsf{S}}_{\mathbf{P}}(\alpha)(s) \to \mu^{\mathsf{S}}_{\mathbf{P}}(\alpha)(t)
\end{aligned}
$$

The Kolmogorov translation corresponds to a semantics μ^{Kol} defined like μ^{S}, but applying double negation to everything in sight:

$$
\begin{aligned}
\mu^{\mathsf{Kol}}_{\mathbf{P}}(\alpha)(v_i) &= \delta(\alpha(v_i))\\
\mu^{\mathsf{Kol}}_{\mathbf{P}}(\alpha)(\top) &= \top\\
\mu^{\mathsf{Kol}}_{\mathbf{P}}(\alpha)(\bot) &= \bot\\
\mu^{\mathsf{Kol}}_{\mathbf{P}}(\alpha)(s \cdot t)) &= \delta(\mu^{\mathsf{Kol}}_{\mathbf{P}}(\alpha)(s) \cdot \mu^{\mathsf{Kol}}_{\mathbf{P}}(\alpha)(t))\\
\mu^{\mathsf{Kol}}_{\mathbf{P}}(\alpha)(s \to t)) &= \delta(\mu^{\mathsf{Kol}}_{\mathbf{P}}(\alpha)(s) \to \mu^{\mathsf{Kol}}_{\mathbf{P}}(\alpha)(t))
\end{aligned}
$$

The Gödel translation[1] corresponds to a semantics that applies double negation to the right operands of residuation and at the outermost level. We define it using an auxiliary semantics μ^*.

$$
\begin{aligned}
\mu^{*}_{\mathbf{P}}(\alpha)(v_i) &= \alpha(v_i)\\
\mu^{*}_{\mathbf{P}}(\alpha)(\top) &= \top\\
\mu^{*}_{\mathbf{P}}(\alpha)(\bot) &= \bot\\
\mu^{*}_{\mathbf{P}}(\alpha)(s \cdot t)) &= \mu^{*}_{\mathbf{P}}(\alpha)(s) \cdot \mu^{*}_{\mathbf{P}}(\alpha)(t)\\
\mu^{*}_{\mathbf{P}}(\alpha)(s \to t)) &= \mu^{*}_{\mathbf{P}}(\alpha)(s) \to \delta(\mu^{*}_{\mathbf{P}}(\alpha)(t))\\
\mu^{\mathsf{G\ddot{o}d}}_{\mathbf{P}}(\alpha)(s) &= \delta(\mu^{*}_{\mathbf{P}}(\alpha)(s))
\end{aligned}
$$

It is easily verified that μ^{S}, μ^{Kol} and $\mu^{\mathsf{G\ddot{o}d}}$ are indeed natural transformations $\mathsf{Ass} \to \mathsf{Sem}$.

The Gentzen and Glivenko translations correspond to semantics obtained by composing the standard semantics with double negation:

$$
\begin{aligned}
\mu^{\mathsf{Gen}} &= \mu^{\mathsf{S}} \circ \delta^{\mathsf{Var}}\\
\mu^{\mathsf{Gli}} &= \delta^{\mathcal{L}} \circ \mu^{\mathsf{S}}
\end{aligned}
$$

[1] Gödel's translation of implication is originally stated as $(A \to B)^N = \neg(A^N \wedge \neg B^N)$, since Gödel's motivation was to interpret all connectives in terms of conjunction and negation. But since $\neg(A^N \wedge \neg B^N)$ is equivalent to $A^N \to \neg\neg B^N$ (even in affine logic), nowadays this more intuitive definition is taken as the translation of implication.

where δ^X denotes the natural transformation from $H_X = \mathsf{Hom}_{\mathsf{Set}}(X, \cdot)$ to itself with $\delta^X_{\mathbf{P}} = f \mapsto \delta \circ f$.

3.2. Double negation semantics

DEFINITION 3.2 (Double negation semantics). *Let $\mathcal{C}$ be a class of bounded pocrims, we say that a semantics μ is a* double negation semantics *for $\mathcal{C}$ if the following conditions hold:*

(DNS1) *If $\mathbf{P} \in \mathcal{C}$ is involutive, then $\mu_{\mathbf{P}} = \mu^{\mathsf{S}}_{\mathbf{P}}$.*

(DNS2) *Given a term t, if, for every involutive $\mathbf{Q} \in \mathcal{C}$ and every $\beta : \mathsf{Var} \to Q$, we have:*

$$\mu^{\mathsf{S}}_{\mathbf{Q}}(\beta)(t) = \top,$$

then, for every $\mathbf{P} \in \mathcal{C}$ and every $\alpha : \mathsf{Var} \to P$, we have:

$$\mu_{\mathbf{P}}(\alpha)(t) = \top.$$

(DNS3) *$\delta^{\mathcal{L}} \circ \mu_{\mathbf{P}} = \mu_{\mathbf{P}}$, for every $\mathbf{P} \in \mathcal{C}$.*

Note that these condition are trivially true if $\mathcal{C}$ is empty. If $\mathcal{C}$ is non-empty but does not contain any involutive pocrim, the conditions only hold if $\mu_{\mathbf{P}}(\alpha)(t) = \top$ for every $\mathbf{P} \in \mathcal{C}$, assignment $\alpha : \mathsf{Var} \to P$ and term t.

REMARK 3.3. *Subject to one proviso, the above definition can be seen to agree with the usual syntactic definition of a double negation translation due to Troelstra, as summarised in Section 1.2. The proviso is that we must have $\mathsf{Th}(\mathcal{I}) = \mathsf{Th}(\mathcal{C}) + [\mathsf{dne}]$, where $\mathcal{I}$ comprises the involutive pocrims in $\mathcal{C}$ and where $\mathsf{Th}(\mathcal{C}) + [\mathsf{dne}]$ denotes the smallest set of terms that contains $\mathsf{Th}(\mathcal{C})$ that is closed under rewriting with equations that either hold in every member of $\mathcal{C}$ or have one of the forms $\neg\neg x = x$ or $x = \neg\neg x$.*

DEFINITION 3.4. *We say a class $\mathcal{C}$ of bounded pocrims is* inv-closed *if whenever $\mathbf{P} \in \mathcal{C}$, then there is $\mathbf{Q} \in \mathcal{C}$ such that $\mathbf{Q}$ is isomorphic to the involutive core, or equivalently the involutive replica, of $\mathbf{P}$.*

THEOREM 3.5. *The Kolmogorov semantics, μ^{Kol}, is a double negation semantics for any inv-closed class $\mathcal{C}$ of bounded pocrims.*

PROOF. (DNS1) and (DNS3) are easy to verify. As for (DNS2), let $\mathbf{P} \in \mathcal{C}$ and let t be a term such such that $\mu^{\mathsf{S}}_{\mathbf{Q}}(\beta)(t) = \top$ for every assignment $\beta : \mathsf{Var} \to Q$ when $\mathbf{Q}$ is involutive. Then, if $\alpha : \mathsf{Var} \to P$, it is easy to see by induction on the structure of any term s that the Kolmogorov semantics of s in $\mathbf{P}$ under an assignment α agrees with the standard semantics of s on $\mathbf{P}^C$, the involutive core of $\mathbf{P}$, under the assignment $\delta \circ \alpha$:

$$\mu^{\mathsf{Kol}}_{\mathbf{P}}(\alpha)(s) = \mu^{\mathsf{S}}_{\mathbf{P}^C}(\delta \circ \alpha)(s)$$

(For the inductive step for residuation use the identity $\delta(\delta(x) \to \delta(y)) = \delta(x) \to \delta(x)$, which follows from Lemma 2.3 parts 3 and 6.) Now $\delta \circ \alpha$ is an assignment into the involutive pocrim $\mathbf{P}^C$, which by assumption is isomorphic to some $\mathbf{Q} \in \mathcal{C}$, via some isomorphism $\phi\colon \mathbf{P}^C \to \mathbf{Q}$. Hence, using our hypothesis on involutive members of $\mathcal{C}$, and the fact that μ^{S} is a natural transformation, we have:

$$\mu^{\mathsf{Kol}}_{\mathbf{P}}(\alpha)(t) = (\phi^{-1} \circ \mu^{\mathsf{S}}_{\mathbf{Q}}(\phi \circ \delta \circ \alpha))(t) = \top$$

completing the proof of (DNS2). ■

Theorem 3.6. *The Gödel semantics, $\mu^{\mathsf{Göd}}$ is a double negation semantics for any class $\mathcal{C}$ of inv-closed bounded pocrims.*

Proof. We follow a similar line to the proof of Theorem 3.5 using the involutive replica in place of the involutive core. Again (DNS1) and (DNS3) are easy. For (DNS2), given a bounded pocrim $\mathbf{P}$, we see by induction on the structure of a term s that for any assignment $\alpha : \mathsf{Var} \to P$, we have:

$$\pi(\mu^{\mathsf{Göd}}_{\mathbf{P}}(\alpha)(s)) = \mu^{\mathsf{S}}_{\mathbf{P}^R}(\pi \circ \alpha)(s)$$

where $\pi : P \to P^R$ is the natural projection onto the involutive replica. Hence if $\mu^{\mathsf{S}}_{\mathbf{Q}}(\beta)(t) = \top$ for every assignment $\beta : \mathsf{Var} \to Q$ where $\mathbf{Q}$ is involutive, then, using our hypothesis on involutive members of $\mathcal{C}$, and the fact that μ^{S} is a natural transformation, we have:

$$\pi(\mu^{\mathsf{Göd}}_{\mathbf{P}}(\alpha)(t)) = (\phi^{-1} \circ \mu^{\mathsf{S}}_{\mathbf{P}^R}(\phi \circ \pi \circ \alpha))(t) = \pi(\top)$$

where $\phi\colon \mathbf{P}^R \to \mathbf{Q}$ is an isomorphism of $\mathbf{P}^R$ with some involutive $\mathbf{Q} \in \mathcal{C}$. Now $\pi(x) = \pi(y)$ iff $\delta(x) = \delta(y)$, so $\delta(\mu^{\mathsf{Göd}}_{\mathbf{P}}(\alpha)(t)) = \delta(\top) = \top$, but clearly $\delta \circ \mu^{\mathsf{Göd}} = \mu^{\mathsf{Göd}}$ and we have proved (DNS2). ■

We will now exhibit classes of bounded pocrims where the Gentzen and Glivenko semantics fail to give double negation semantics. These classes involve the pocrims defined in the following examples.

EXAMPLE 3.7. *The pocrim* $\mathbf{P}_4$ *comprises the chain* $\top > p > q > \bot$. *The operation tables for* $\mathbf{P}_4$ *are as follows.*

$\cdot$	$\top$	p	q	$\bot$
$\top$	$\top$	p	q	$\bot$
p	p	$\bot$	$\bot$	$\bot$
q	q	$\bot$	$\bot$	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

$\rightarrow$	$\top$	p	q	$\bot$
$\top$	$\top$	p	q	$\bot$
p	$\top$	$\top$	p	p
q	$\top$	$\top$	$\top$	p
$\bot$	$\top$	$\top$	$\top$	$\top$

δ	
$\bot$	$\bot$
p	p
q	p
$\top$	$\top$

In $\mathbf{P}_4$, $\delta(q) = p$, *so* $\mathbf{P}_4$ *is not involutive. However, the involutive core of* $\mathbf{P}_4$ *is actually a subpocrim: namely the subpocrim with universe* $\{\bot, p, \top\}$ *which (in anticipation of Example 4.4), we will refer to as* $\mathbf{L}_3$.

EXAMPLE 3.8. *Consider the pocrim* $\mathbf{Q}_6$ *with six elements* $\top > p > q > r > s > \bot$ *and with* $\cdot$, $\rightarrow$ *and* δ *as shown in the following tables:*

$\cdot$	$\top$	p	q	r	s	$\bot$
$\top$	$\top$	p	q	r	s	$\bot$
p	p	p	r	r	s	$\bot$
q	q	r	r	r	$\bot$	$\bot$
r	r	r	r	r	$\bot$	$\bot$
s	s	s	$\bot$	$\bot$	$\bot$	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

$\rightarrow$	$\top$	p	q	r	s	$\bot$
$\top$	$\top$	p	q	r	s	$\bot$
p	$\top$	$\top$	q	q	s	$\bot$
q	$\top$	$\top$	$\top$	p	s	s
r	$\top$	$\top$	$\top$	$\top$	s	s
s	$\top$	$\top$	$\top$	$\top$	$\top$	q
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

δ	
$\top$	$\top$
p	$\top$
q	q
r	q
s	s
$\bot$	$\bot$

$\mathbf{Q}_6$ *is not involutive, as* $\delta(x) = x$ *fails for* $x \in \{p, r\}$. *In* $\mathbf{Q}_6$, *double negation is an implicative homomorphism:* $\neg\neg x \rightarrow \neg\neg y = \neg\neg(x \rightarrow y)$ *for all* x, y. *Double negation is not quite a conjunctive homomorphism in* $\mathbf{Q}_6$: $\neg\neg x \cdot \neg\neg y = \neg\neg(x \cdot y)$ *unless* $\{x, y\} \subseteq \{q, r\}$, *in which case* $\neg\neg x \cdot \neg\neg y = r < q = \neg\neg(x \cdot y)$.

The involutive replica of $\mathbf{Q}_6$ *turns out to be a quotient pocrim: as indicated by the block decomposition of the above operation tables, there is a homomorphism* $h : \mathbf{Q}_6 \rightarrow \mathbf{Q}_4$, *where* $\mathbf{Q}_4$ *is the involutive replica of* $\mathbf{Q}_6$ *and comprises the chain* $\top > u > v > \bot$ *with operation tables as follows:*

$\cdot$	$\top$	u	v	$\bot$
$\top$	$\top$	u	v	$\bot$
u	u	u	$\bot$	$\bot$
v	v	$\bot$	$\bot$	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

$\rightarrow$	$\top$	u	v	$\bot$
$\top$	$\top$	u	v	$\bot$
u	$\top$	$\top$	v	v
v	$\top$	$\top$	$\top$	u
$\bot$	$\top$	$\top$	$\top$	$\top$

δ	
$\top$	$\top$
u	u
v	v
$\bot$	$\bot$

The kernel congruence of h *has equivalence classes* $\{\top, p\}$, $\{q, r\}$, $\{s\}$ *and* $\{\bot\}$ *which are mapped by* h *to* $\top$, u, v, $\top$ *respectively in* $\mathbf{Q}_4$.

THEOREM 3.9. *(i) The Gentzen semantics μ^{Gen} is not a double negation semantics for any class of bounded pocrims that contains the pocrim $\mathbf{Q}_6$ of Example 3.8. (ii) The Glivenko semantics μ^{Gli} is not a double negation semantics for any class of bounded pocrims that contains the pocrim $\mathbf{P}_4$ of Example 3.7.*

PROOF. By the remarks after Definition 3.2 we can assume that the class of bounded pocrims contains at least one involutive pocrim in both cases.
(i): We show that (DNS2) does not hold for μ^{Gen} in $\mathbf{Q}_6$. Let $x, y \in \mathsf{Var}$ and let t be the formula $\delta(x \cdot y) \to x \cdot y$. Clearly, $\mu^{\mathsf{S}}_{\mathbf{P}}(\alpha)(t) = \top$, for any involutive pocrim $\mathbf{P}$ and any $\alpha : \mathsf{Var} \to P$. Thus (DNS2) requires $\mu^{\mathsf{Gen}}_{\mathbf{Q}_6}(\alpha)(t) = \top$ for any $\alpha : \mathsf{Var} \to Q_6$. However, if $\alpha(x) = \alpha(y) = r$, we have:

$$\begin{aligned}\mu^{\mathsf{Gen}}_{\mathbf{Q}_6}(\alpha)(t) &= \delta(\delta(r) \cdot \delta(r)) \to \delta(r) \cdot \delta(r) \\ &= \delta(q \cdot q) \to q \cdot q \\ &= \delta(r) \to r = q \to r = p \neq \top.\end{aligned}$$

(ii): we argue as in the proof of *(i)*, but taking t to be $\delta(x) \to x$. Then, if $\alpha(x) = q$, we have:

$$\begin{aligned}\mu^{\mathsf{Gli}}_{\mathbf{P}}(\alpha)(t) &= \delta(\delta(q) \to q) \\ &= \delta(p \to q) = \delta(p) = p \neq \top.\end{aligned}$$

■

THEOREM 3.10. *Let $\mathcal{C}_1$ comprise the two bounded pocrims $\mathbf{P}_4$ and $\mathbf{L}_3$ of Example 3.7 and let $\mathcal{C}_2$ comprise the two bounded pocrims $\mathbf{Q}_6$ and $\mathbf{Q}_4$ of Example 3.8. Then:*
(i) *The Gentzen semantics, μ^{Gen}, is a double negation semantics for $\mathcal{C}_1$, but the Glivenko semantics, μ^{Gli}, is not.*
(ii) *The Glivenko semantics, μ^{Gli}, is a double negation semantics for $\mathcal{C}_2$, but the Gentzen semantics, μ^{Gen}, is not.*

PROOF. *(i)*: By Theorem 3.9, μ^{Gli} is not a double negation semantics for $\mathcal{C}_1$. As for μ^{Gen}, (DNS1) is easily verified. For (DNS3) and (DNS2), note that for any $\alpha : \mathsf{Var} \to P_4$, we have:

$$\mu^{\mathsf{Gen}}_{\mathbf{P}_4}(\alpha) = (\mu^{\mathsf{S}}_{\mathbf{P}_4} \circ \delta^{\mathsf{Var}})(\alpha) = \mu^{\mathsf{S}}_{\mathbf{P}_4}(\delta \circ \alpha) = \mu^{\mathsf{S}}_{\mathbf{L}_3}(\delta \circ \alpha)$$

where in the last expression we have identified $\mathbf{L}_3$ with the bounded subpocrim of $\mathbf{P}_4$ whose universe is $\mathsf{im}(\delta)$. Thus evaluation under μ^{Gen} with an

assignment in any bounded pocrim in $\mathcal{C}_1$ is equivalent to evaluation under the standard semantics, μ^{S}, with an assignment in the involutive pocrim $\mathbf{L}_3$. (DNS3) and (DNS2) follow immediately from this.
(ii): By Theorem 3.9, μ^{Gen} is not a double negation semantics for $\mathcal{C}_2$. As for μ^{Gli}, (DNS1) and (DNS3) are immediate from the definition of μ^{Gli}. For (DNS2), let t be a formula, such that $\mu^{\mathsf{S}}_{\mathbf{Q}_4}(\alpha)(t) = \top$, for any assignment $\alpha : \mathsf{Var} \to \mathbf{Q}_4$. As $\mathbf{Q}_4$ is the only involutive pocrim in $\mathcal{C}_2$, we must show that $\mu^{\mathsf{Gli}}_{\mathbf{P}}(\alpha)(t) = \top$ for $\mathbf{P} \in \mathcal{C}_2$ under any assignment $\alpha : \mathsf{Var} \to P$. This is easy to see for $\mathbf{P} = \mathbf{Q}_4$, since the Glivenko semantics is the double negation of the standard semantics and $\mathbf{Q}_4$ is involutive. As for $\mathbf{P} = \mathbf{Q}_6$, let $\alpha : \mathsf{Var} \to \mathbf{Q}_6$ be given. As discussed in Example 3.8, there is a quotient projection $h : \mathbf{Q}_6 \to \mathbf{Q}_4$, so, as μ^{S} is a natural transformation, the following diagram commutes:

$$\begin{tikzcd}
\mathsf{Ass}(\mathbf{Q}_6) \arrow[r, "\mathsf{Ass}(h)"] \arrow[d, "\mu^{\mathsf{S}}_{\mathbf{Q}_6}"] & \mathsf{Ass}(\mathbf{Q}_4) \arrow[d, "\mu^{\mathsf{S}}_{\mathbf{Q}_4}"] \\
\mathsf{Sem}(\mathbf{Q}_6) \arrow[r, "\mathsf{Sem}(h)"] & \mathsf{Sem}(\mathbf{Q}_4)
\end{tikzcd}$$

Hence, by the assumption on t, we have:

$$(h \circ \mu^{\mathsf{S}}_{\mathbf{Q}_6}(\alpha))(t) = \mu^{\mathsf{S}}_{\mathbf{Q}_4}(h \circ \alpha)(t) = \top$$

So $\mu^{\mathsf{S}}_{\mathbf{Q}_6}(\alpha)(t) \in h^{-1}(\top) = \{\top, p\}$. As $\delta(\top) = \delta(p) = \top$, we can conclude:

$$\mu^{\mathsf{Gli}}_{\mathbf{Q}_6}(\alpha)(t) = \delta(\mu^{\mathsf{S}}_{\mathbf{Q}_6}(\alpha)(t)) = \top.$$

■

THEOREM 3.11. *There are extensions of intuitionistic affine logic* $\mathbf{AL_i}$ *in which the syntactic Gentzen translation meets Troelstra's criteria for a double negation translation but the syntactic Glivenko translation does not and vice versa.*

PROOF. By Remark 3.3 and Theorem 3.10 it is enough to prove that if $\mathbf{L}_3$, $\mathbf{P}_4$, $\mathbf{Q}_4$ and $\mathbf{Q}_6$ are as in Theorem 3.10, then:

$$\begin{aligned}
\mathsf{Th}(\mathbf{L}_3) &= \mathsf{Th}(\mathbf{P}_4) + [\mathsf{dne}] \\
\mathsf{Th}(\mathbf{Q}_4) &= \mathsf{Th}(\mathbf{Q}_6) + [\mathsf{dne}].
\end{aligned}$$

For the first equation, the right-to-left inclusion holds because identities are preserved in subalgebras. For left-to-right, let a term t be given and let us write $\mathbf{P} \models t$ to mean $\mu^{\mathsf{S}}_{\mathbf{P}}(\alpha)(t) = \top$ for every $\alpha : \mathsf{Var} \to P$. Assume $\mathbf{L}_3 \models t$ and let $w_1, \ldots, w_k$ be the variables occurring in t. Define s to be $(\delta(w_1) \to w_1) \cdot \ldots \cdot (\delta(w_k) \to w_k)$. Clearly $s \in \mathsf{Th}(\mathsf{Poc}_\bot) + [\mathsf{dne}] \subseteq \mathsf{Th}(\mathbf{P}_4) + [\mathsf{dne}]$ I claim that $\mu^{\mathsf{S}}_{\mathbf{P}_4}(\alpha)(s \cdot s \to t) = \top$ for every $\alpha : \mathsf{Var} \to P_4$, so that, as $s \in \mathsf{Th}(\mathbf{P}_4) + [\mathsf{dne}]$, $t \in \mathsf{Th}(\mathbf{P}_4) + [\mathsf{dne}]$. To see this, let an assignment $\alpha : \mathsf{Var} \to P_4$ be given. Then either *(i)* $\mathsf{im}(\alpha) \subseteq \{\top, p, 1\}$, in which case $\mu^{\mathsf{S}}_{\mathbf{P}_4}(\alpha)(s \cdot s \to t)/ge\mu^{\mathsf{S}}_{\mathbf{P}_4}(\alpha)(t) = \top$, since α is an assignment into the involutive subpocrim $\mathbf{L}_3$ and so $\mathbf{L}_3 \models t$ by assumption, or *(ii)* $\alpha(w_i) = q$ for some i, but then $\mu^{\mathsf{S}}_{\mathbf{P}_4}(\alpha)(\delta(w_i) \to w_i) = p \to q = p$ and so $\mu^{\mathsf{S}}_{\mathbf{P}_4}(\alpha)(s \cdot s) \leq p \cdot p = \bot$. In both cases, we have that $\mu^{\mathsf{S}}_{\mathbf{P}_4}(\alpha)(s \cdot s \to t) = \top$, proving the claim.

The proof of the second equation is similar using the facts that identities are preserved in quotient algebras and that, if $\mathbf{Q}_4 \models t$ and $\alpha : \mathsf{Var} \to Q_6$, then $\mu^{\mathsf{S}}_{\mathbf{Q}_6}(\alpha)(t) \in \{\top, p\}$, implying that $\mathbf{Q}_6 \models (\delta(t) \to t) \to t$. ∎

4. Hoops

If x and y are elements of a pocrim, $x \cdot (x \to y)$ is a lower bound for x and y as is $y \cdot (y \to x)$. Pocrims in which the two lower bounds coincide (and hence $x \cdot (x \to y)$ is the meet of x and y) turn out to have many pleasant properties, motivating the following definition.

DEFINITION 4.1 (Hoop, [5], see also [1, 2, 12]). *A* hoop[2] *is a pocrim that satisfies* commutativity of weak conjunction*:*

$$x \cdot (x \to y) = y \cdot (y \to x). \qquad [\mathsf{cwc}]$$

Logically, we can view $\phi \otimes (\phi \Rightarrow \psi)$ or $\psi \otimes (\psi \Rightarrow \phi)$ as a weak form of conjunction of ϕ and ψ. If we define $\phi \wedge \psi \equiv \phi \otimes (\phi \Rightarrow \psi)$, then [cwc] says that $\wedge$ is commutative, i.e. the two forms of weak conjunction coincide. This is known in substructural logic as the *axiom of divisibility*.

The following lemma provides some useful characterizations of hoops.

LEMMA 4.2. *If* $\mathbf{P}$ *is a pocrim, the following are equivalent:*

1. $\mathbf{P}$ *is a hoop. I.e.,* $\mathbf{P}$ *satisfies* $x \cdot (x \to y) = y \cdot (y \to x)$.
2. $\mathbf{P}$ *is* naturally ordered. *I.e., for every* $x, y \in P$ *such that* $x \leq y$*, there is* $z \in P$ *such that* $x = y \cdot z$.

[2] Büchi and Owens [5] write of hoops that "their importance ... merits recognition with a more euphonious name than the merely descriptive "commutative complemented monoid"". Presumably they chose "hoop" as a euphonious companion to "group" and "loop".

3. For every $x, y \in P$ such that $x \leq y$, $x = y \cdot (y \to x)$.

4. **P** *satisfies $x \cdot (x \to y) \leq y \cdot (y \to x)$*

PROOF. *1* $\Rightarrow$ *2*: Assume that **P** satisfies $x \cdot (x \to y) = y \cdot (y \to x)$ and that $x, y \in P$ satisfy $x \leq y$, i.e., $x \to y = 1$. Taking $z = y \to x$, we have:

$$x = x \cdot 1 = x \cdot (x \to y) = y \cdot (y \to x) = y \cdot z.$$

2 $\Rightarrow$ *3*: Assume that **P** is naturally ordered and that $x, y \in P$ satisfy $x \leq y$. Then $x = y \cdot z$ for some z. By the residuation property, we have $z \leq y \to x$, hence $x = y \cdot z \leq y \cdot (y \to x) \leq x$ and so $x = y \cdot (y \to x)$.
3 $\Rightarrow$ *4*: assume that **P** satisfies $x = y \cdot (y \to x)$ whenever $x, y \in P$ and $x \leq y$. Given any $x, y \in P$, we have $x \cdot (x \to y) \leq y$, whence:

$$x \cdot (x \to y) = y \cdot (y \to x \cdot (x \to y)) \leq y \cdot (y \to x).$$

4 $\Rightarrow$ *1*: exchange x and y and use the fact that $\leq$ is antisymmetric. ■

The axiom [cwc] is often referred to as the axiom of divisibility in the literature, for reasons which become clear if one uses the alternative notation x/y for $y \to x$, so that the formula of part 3 of Lemma 4.2 reads $x = y \cdot (x/y)$.

4.1. Involutive hoops

EXAMPLE 4.3. *We write* **I** *for the involutive hoop whose universe is the unit interval* $[0, 1]$ *and whose operations are defined by*

$$\begin{aligned} \top &= 1 \\ x \cdot y &= \mathsf{max}(x + y - 1, 0) \\ x \to y &= \mathsf{min}(1 - x + y, 1) \end{aligned}$$

I *provides an infinite model of classical Łukasiewicz logic, (which we refer to as* $\mathbf{ŁL_C}$*).*

EXAMPLE 4.4. *For $n \geq 2$, let $\mathbf{L}_n$ be the subhoop of* **I** *generated by $\frac{1}{n-1}$. It is easy to see that the universe of $\mathbf{L}_n$ is $L_n = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$. The hoops $\mathbf{L}_n$ are involutive and provide natural finite models of classical Łukasiewicz logic* $\mathbf{ŁL_C}$.

A hoop **H** is said to be *Wajsberg*, see [1, 13], if it satisfies

$$(x \to y) \to y = (y \to x) \to x.$$

LEMMA 4.5. *A bounded hoop is Wajsberg iff it is involutive.*

PROOF. In a bounded Wajsberg hoop **H** we have

$$\delta(x) = (x \to \bot) \to \bot = (\bot \to x) \to x = \top \to x = x$$

therefore **H** is involutive. For the other direction, assume **H** is an involutive hoop and let $x, y \in H$. Since **H** is involutive, it is enough to show that $\neg((x \to y) \to y)$ is symmetric in x and y which one may prove as follows:

$$\begin{aligned}
\neg((x \to y) \to y) &= \delta(x \to y) \cdot \neg y && \text{Theorem 4.11} \\
&= (x \to y) \cdot \neg y && \text{involutivity of } \mathbf{H} \\
&= (x \to y) \cdot (y \to x) \cdot ((y \to x) \to \neg y) && \text{Lemma 4.2, part 3} \\
&= (x \to y) \cdot (y \to x) \cdot \neg(y \cdot (y \to x)) && \text{Lemma 2.2, part 3}
\end{aligned}$$

where the application of Lemma 4.2 uses that $\neg y \leq y \to x$. By [cwc], the last expression is symmetric in x and y. ■

There are, however, unbounded Wajsberg hoops, for instance:

EXAMPLE 4.6. *Let* **O** *be the unbounded hoop whose universe is the half-open interval* $(0, 1]$ *and whose operations are:*

$$\begin{aligned}
\top &= 1 \\
x \cdot y &= xy \\
x \to y &= \mathsf{min}(\frac{y}{x}, 1)
\end{aligned}$$

O *is easily seen to be a Wajsberg hoop because* $(x \to y) \to y = \mathsf{max}(x, y)$.

EXAMPLE 4.7. *Apart from* $\mathbf{L}_3$ *there is one other pocrim with 3 elements, namely* $\mathbf{G}_3 = \mathbb{B} \oplus \mathbb{B}$. $\mathbf{G}_3$ *is the first non-Boolean example in the sequence of idempotent pocrims defined by the equations* $\mathbf{G}_2 = \mathbb{B}$ *and* $\mathbf{G}_{n+1} = \mathbf{G}_n \oplus \mathbb{B}$. G_n *can be taken to be a set of* n *real numbers* $\{1, x_1, x_2, \ldots, x_{n-2}, 0\}$ *with* $\top = 1 > x_1 > x_2 \ldots > x_{n-2} > 0 = \bot$ *and with operations defined by*

$$x \cdot y = \mathsf{min}\{x, y\} \qquad x \to y = \begin{cases} y & \text{if } y < x \\ \top & \text{otherwise} \end{cases}$$

The $\mathbf{G}_n$ *are finite Heyting algebras. They were used by Gödel to prove that intuitionistic propositional logic requires infinitely many truth values [18]. In* $\mathbf{G}_n$, $\neg x = \bot$ *unless* $x = \bot$, *so for* $n > 2$, $\mathbf{G}_n$ *is not involutive.*

It is easy to check from the definitions that $\mathbf{C} \oplus \mathbf{D}$ is a hoop iff both $\mathbf{C}$ and $\mathbf{D}$ are hoops.

Example 4.8. *It can be shown that there are 7 pocrims with 4 elements:* $\mathbb{B} \times \mathbb{B}$, $\mathbf{L}_4$, $\mathbf{G}_4$, $\mathbb{B} \oplus \mathbf{L}_3$, $\mathbf{L}_3 \oplus \mathbb{B}$, $\mathbf{P}_4$ *and* $\mathbf{Q}_4$, *where* $\mathbf{P}_4$ *and* $\mathbf{Q}_4$ *are as described in Examples 3.7 and 3.8 respectively.* $\mathbf{P}_4$ *and* $\mathbf{Q}_4$ *are the smallest pocrims that are not hoops:* $\mathbf{P}_4$ *is not a hoop since it is not naturally ordered: there is no* z *with* $p \cdot z = q$. *Likewise* $\mathbf{Q}_4$ *is not a hoop, because there is no* z *with* $u \cdot z = v$.

4.2. De Morgan identities in hoops

In this section we prove two De Morgan identities for conjunction and residuation in bounded hoops. The proof of the identity for conjunction is elementary. The identity for residuation is proved using an indirect method captured in the following lemma.

Lemma 4.9. *Let* $\phi(x_1, \ldots, x_n)$ *be an identity in the language of a bounded hoop, then* $\phi(x_1, \ldots, x_n)$ *holds in all hoops iff it holds under every interpretation of the* x_i *in a bounded hoop* $\mathbf{H}$ *that falls under one of the following three cases:*
case (i)*:* $\mathbf{H} \cong \mathbf{F} \oplus \mathbf{S}$ *with* $F = \{\top\}$*;*
case (ii)*:* $\mathbf{H} \cong \mathbf{F} \oplus \mathbf{S}$ *with* $F \neq \{\top\}$. *There is a subcase for each choice of* $I = \{i \mid x_i \in S\} \neq \emptyset$ *and* $J = \{j \mid x_j \in F\} \neq \emptyset$, *with* $\mathbf{F}$ *generated by the* x_j *with* $j \in J$*;*
case (iii)*:* $\mathbf{H} \cong \mathbb{B} \oplus \mathbf{S}$, *with all* $x_i \in S$.
Here in each case $\mathbf{S}$ *is subdirectly irreducible, Wajsberg and generated by the* $x_i \in S$. $\mathbf{S}$ *is not necessarily bounded in cases* (i) *and* (ii).

Proof. The proof uses Birkhoff's theorem (e.g., see [6, Theorem II.8.6]) to show that $\mathbf{H}$ is isomorphic to a subdirect product of subdirectly irreducible hoops and then uses the characterization of subdirectly irreducible hoops due to Blok and Ferreirim [1, Thorem 2.9]. Details are left to the reader. ∎

Note that in case (i) of the lemma $\mathbf{H}$ is isomorphic to $\mathbf{S}$ and so is a bounded Wajsberg hoop and hence involutive.

Example 4.10. *If* $0 < k \in \mathbb{N}$, *the identity* $\neg x^k \rightarrow \delta(x) \rightarrow x = \top$ *clearly holds in any involutive hoop. It also holds in any hoop of the form* $\mathbb{B} \oplus \mathbf{S}$ *(since in such a hoop, either* $x = \bot$ *or* $\neg x^k = \bot$*). This covers cases* (i) *and* (iii) *in Lemma 4.9. As the identity has only one variable, there is nothing to prove in case* (ii). *Hence,* $\neg x^k \rightarrow \delta(x) \rightarrow x = \bot$ *holds in any bounded hoop.*

In any bounded pocrim, we have $\neg(x \cdot y) = x \cdot y \to \bot = x \to y \to \bot = x \to \neg y$, so the first identity in the following theorem is easily proved. In a bounded hoop, we have a kind of dual identity: $\neg(x \to y) = \neg\neg x \cdot \neg y$.

THEOREM 4.11. *The following identities are satisfied in any bounded hoop:*

$$\neg(x \cdot y) = x \to \neg y \qquad \neg(x \to y) = \neg\neg x \cdot \neg y$$

PROOF. See the above remarks for the first identity. For the second we use Lemma 4.9, which reduces the problem to the following cases for a hoop $\mathbf{H}$ and its elements x and y.
Case (i): Our assumptions imply that $\mathbf{H}$ is involutive. Using Lemma 2.2, in an involutive hope we have $x \to y \leq \neg y \to \neg x \leq \delta(x) \to \delta(y) = x \to y$, whence $x \to y = \neg y \to \neg x = \neg(x \cdot \neg y)$ and we have:

$$\neg(x \to y) = \neg\neg(x \cdot \neg y) = x \cdot \neg y = \neg\neg x \cdot \neg y.$$

Case (ii): $\mathbf{H} = \mathbf{F} \oplus \mathbf{S}$, $\{x, y\} \cap S \neq \emptyset$, $\{x, y\} \cap F \setminus \{\top\} \neq \emptyset$: this leads to two subcases that are proved using elementary properties of $\mathbf{F} \oplus \mathbf{S}$, as follows.
Subcase (ii)(a): $x \in S$, $y \in F \setminus \{\top\}$:

$$\neg(x \to y) = \neg y = \top \cdot \neg y = \neg\neg x \cdot \neg y$$

Subcase (ii)(b): $x \in F \setminus \{\top\}$, $y \in S$:

$$\neg(x \to y) = \neg\top = \bot = \neg\neg x \cdot \bot = \neg\neg x \cdot \neg y.$$

Case (iii): $\mathbf{H} = \mathbb{B} \oplus \mathbf{S}$ where $x, y \in S$: for $u \in S$, $\neg u = \bot$, so as $x \to y \in S$, we have $\neg(x \to y) = \bot = \top \cdot \bot = \neg\neg x \cdot \neg y$. ∎

4.3. Double negation semantics for hoops

THEOREM 4.12. *Let $\mathbf{H}$ be a bounded hoop.*

1. *The double negation mapping, δ, is a homomorphism $\mathbf{H} \to \mathbf{H}$.*
2. *For $x, y \in H^C$, $x \mathbin{\hat{\cdot}} y = x \cdot y$, hence $\mathbf{H}^C$ is a bounded subhoop of $\mathbf{H}$.*
3. *For $x, y \in H$, $[x] \mathbin{\check{\to}} [y] = [x \to y]$, hence $\mathbf{H}^R$ is a quotient bounded hoop of $\mathbf{H}$ via the the projection $\pi \colon H \to H^R$.*
4. *$\mathbf{H}^C$ and $\mathbf{H}^R$ are isomorphic hoops via the composition $\pi \circ \iota \colon H^C \to H^R$.*

PROOF. 1. By Theorem 4.11, we have[3]:

$$\delta(x) \cdot \delta(y) = \neg(x \to \neg y) = \delta(x \cdot y)$$
$$\delta(x) \to \delta(y) = \neg(\delta(x) \cdot \neg y) = \delta(x \to y)$$

2. For $x, y \in H^C$, we have

$$\begin{aligned} x \hat{\cdot} y &= \delta(x \cdot y) && \text{by definition} \\ &= \delta(x) \cdot \delta(y) && \text{by part 1} \\ &= x \cdot y && \text{since } x, y \in H^C \end{aligned}$$

I.e. $\iota \colon H^C \to H$ respects conjunction. Hence, by Remark 2.11, ι is a hoop homomorphism.
3. We have

$$\begin{aligned} [x] \check{\to} [y] &= [x \to \delta(y)] && \text{by definition} \\ &= [\delta(x) \to \delta(y)] && \text{by Lemma 2.3, part 4} \\ &= [\delta(x \to y)] && \text{by part 1} \\ &= [x \to y] && \text{since } \delta(\delta(x \to y)) = \delta(x \to y) \end{aligned}$$

I.e. $\pi \colon H \to H^R$ respects residuation. Hence, by Remark 2.11, π is a hoop homomorphism.
4. Immediate from Theorem 2.10 (part 3). ■

REMARK 4.13. *Cignoli and Torrens ([8, Theorem 4.8]) show that the set of regular elements in a bounded hoop is a subhoop if and only if the double negation map respects conjunction. Theorem 4.12 shows that this is always the case.*

THEOREM 4.14. *The Kolmogorov semantics, μ^{Kol}, the Gödel semantics, $\mu^{\mathsf{G\ddot{o}d}}$, the Gentzen semantics, μ^{Gen}, and the Glivenko semantics, μ^{Gli} are double negation semantics for any class $\mathcal{C}$ of hoops that is closed under taking involutive cores (or equivalently involutive replicas).*

PROOF. It is clear from Theorem 4.12 (part 1) that the Kolmogorov, Gödel, Gentzen and Glivenko semantics all agree when restricted to hoops. The result is therefore immediate from either Theorem 3.5 or Theorem 3.6. ■

[3] This is a strengthening of Lemma 1.3 of [7], which shows that these equations hold in the more restricted setting of BL algebras.

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References

[1] Blok, W. J., and I. M. A. Ferreirim, 'On the structure of hoops', *Algebra Universalis*, 43 (2000), 2-3, 233–257.

[2] Blok, W. J., and D. Pigozzi, 'On the structure of varieties with equationally definable principal congruences. III.', *Algebra Univers.*, 32 (1994), 4, 545–608.

[3] Blok, Willem J., and James G. Raftery, 'Varieties of commutative residuated integral pomonoids and their residuation subreducts', *Journal of Algebra*, 190 (1997), 280–328.

[4] Bova, Simone, and Franco Montagna, 'The consequence relation in the logic of commutative GBL-algebras is PSPACE-complete', *Theor. Comput. Sci.*, 410 (2009), 12-13, 1143–1158.

[5] Büchi, J. R., and T. M. Owens, 'Complemented monoids and hoops', , c. 1974. Unpublished manuscript.

[6] Burris, Stanley, and H.P. Sankappanavar, *A Course in Universal Algebra.*, Springer-Verlag, 1981. An updated version is available online via http://www.math.uwaterloo.ca/ snburris/.

[7] Cignoli, Roberto, and Antoni Torrens, 'Free algebras in varieties of BL-algebras with a Boolean retract', *Algebra Universalis*, 48 (2002), 55–79.

[8] Cignoli, Roberto, and Antoni Torrens, 'Glivenko like theorems in natural expansions of bck-logic', *Math. Log. Q.*, 50 (2004), 111–125.

[9] Escardó, M. H., and P. Oliva, 'The Peirce translation and the double negation shift', in F. Ferreira, B. Löwe, E. Mayordomo, and L. M. Gomes, (eds.), *Programs, Proofs, Processes - CiE 2010, LNCS 6158*, Springer, 2010, pp. 151–161.

[10] Farahani, Hadi, and Hiroakira Ono, 'Glivenko theorems and negative translations in substructural predicate logics', *Archive for Mathematical Logic*, 51 (2012).

[11] Ferreira, G., and P. Oliva, 'On the relation between various negative translations', *Logic, Construction, Computation, Ontos-Verlag Mathematical Logic Series*, 3 (2012), 227–258.

[12] Ferreirim, Isabel M. A., *On Varieties and Quasivarieties of Hoops and their Reducts*, Ph. D. thesis, University of Illinois at Chicago, 1992.

[13] Font, Josep M., Antonio J. Rodríguez, and Antoni Torrens, 'Wajsberg algebras.', *Stochastica*, 8 (1984), 5–31.

[14] Galatos, Nikolaos, and Hiroakira Ono, 'Glivenko theorems for substructural logics over FL', *Journal of Symbolic Logic*, 71 (2006), 4, 1353–1384.

[15] GENTZEN, G., 'Über das Verhältnis zwischen intuitionistischer und klassischer Arithmetik, galley proof (received in 1933)', *Mathematische Annalen*, (1933).

[16] GLIVENKO, V. I., 'Sur quelques points de la logique de M. Brouwer', *Bulletin de la Société Mathématique de Belgique*, 15 (1929).

[17] GÖDEL, K., 'Zur intuitionistischen Arithmetik und Zahlentheorie', *Ergebnisse eines Mathematischen Kolloquiums*, 4 (1933), 34–38.

[18] GÖDEL, KURT, 'Zum intuitionistischen Aussagenkalkül', *Anzeiger der Akademie der Wissenschaften in Wien*, 69 (1932), 65–66.

[19] KOLMOGOROV, A. N., 'On the principle of the excluded middle (Russian)', *Mat. Sb.*, 32 (1925), 646–667.

[20] ONO, HIROAKIRA, 'Glivenko theorems revisited', *Annals of Pure and Applied Logic*, 161 (2009), 2, 246 – 250. Festschrift on the occasion of Franco Montagna's 60th birthday.

[21] RAFTERY, JAMES G., 'On the variety generated by involutive pocrims', *Rep. Math. Logic*, 42 (2007), 71–86.

[22] TROELSTRA, A. S., *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*, vol. 344 of *Lecture Notes in Mathematics*, Springer, Berlin, 1973.

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