RIESZ BASES OF EIGENFUNCTIONS OF 1D DIRAC OPERATOR WITH STRICTLY REGULAR BOUNDARY CONDITIONS

by

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ABSTRACT

One dimensional Dirac operators

$$L_{bc}(v)y = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{dy}{dx} + v(x)y, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad x \in [0, \pi]$$

considered with L^2-potentials

$$v(x) = \begin{pmatrix} 0 & P(x) \\ Q(x) & 0 \end{pmatrix}, \quad P, Q \in L^2\left([0, \pi]\right),$$

and subject to regular boundary conditions bc have discrete spectrum. In this thesis, we study basic properties of Riesz bases, prove existence of Riesz bases consisting of root functions of Dirac operators L_{bc} subject to strictly regular bc, find adjoint operator $(L_{bc})^*$, find all self-adjoint bc, and calculate some special self-adjoint extensions.

ÖZET

$$L_{bc}(v)y = i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{dy}{dx} + v(x)y, \quad y = \begin{pmatrix}y_1\\ y_2\end{pmatrix}, \quad x \in [0,\pi]$$

denklemiyle verilen,

$$\begin{pmatrix}0 & P(x)\\ Q(x) & 0\end{pmatrix}, \quad P, Q \in L^2\left([0,\pi]\right)$$

L^2 potansiyeli ve regüler sınır şartlarıyla düşünülen, tek boyutlu Dirac operatörünün ayrık spektrumu vardır. Bu tezde, Riesz tabanının genel özelliklerini inceliyoruz, güçlü regüler sınır şartlarıyla düşünülen Dirac operatörü L_{bc}'nin özvektörlerinden oluşan Riesz tabanının varlığını ispatlıyoruz, eşlenik operatörü $(L_{bc})^*$'ı buluyoruz, özeşlenik sınır şartlarını belirliyoruz ve bazı özel özeşlenik genişlemeleri hesaplıyoruz.

TABLE OF CONTENTS

1 Introduction

The differential expression

$$L(v)y = i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{dy}{dx} + v(x)y, \quad y = \begin{pmatrix}y_1\\ y_2\end{pmatrix},$$

is known as one dimensionsal Dirac operator. The matrix

$$v(x) = \begin{pmatrix}0 & P(x)\\ Q(x) & 0\end{pmatrix}$$

is called Dirac potential. In this thesis, we consider Dirac operators L_{bc} on $[0,\pi]$ with L^2 potentials, that is $P, Q \in L^2([0,\pi])$, and with domain

$$\begin{aligned} Dom\,(L_{bc}(v)) \quad = \quad & \left\{ y = \begin{pmatrix}y_1\\ y_2\end{pmatrix} : y_1 \text{ and } y_2 \text{ are absolutely continuous}, y \text{ satisfies} \right. \\ & \left. \text{the boundary conditions } bc, \text{ and } y_1', y_2' \in L^2([0,\pi]) \right\}. \end{aligned}$$

If $v \equiv 0$, then $L_{bc}(0)$ is denoted by L^0_{bc} and called the free Dirac operator. A regular boundary condition bc is given by the linear system of equations

$$y_1(0) + by_1(\pi) + ay_2(0) = 0,$$

$$dy_1(\pi) + cy_2(0) + y_2(\pi) = 0,$$

where $bc - ad \neq 0$. Moreover, bc is called strictly regular if $(b-c)^2 + 4ad \neq 0$.

In the second section, we study the basic properties of Riesz bases. If $(e_\gamma,\ \gamma \in \Gamma)$ is an orthonormal basis in a Hilbert space H and $A : H \to H$ is an automorphism, then the system $(f_\gamma,\ \gamma \in \Gamma)$, $f_\gamma = Ae_\gamma$, is called a Riesz basis. Riesz bases are unconditional bases. Moreover, Bari-Markus theorem is proven which states that if $(e_n,\ n \in \mathbb{N})$ is a Riesz basis in a Hilbert space H and $(f_n,\ n \in \mathbb{N})$ is a minimal system of vectors such that

$$\sum_{n=1}^{\infty} \|f_n - e_n\|^2 < \infty,$$

then $(f_n,\ n \in \mathbb{N})$ is also a Riesz basis. Bari-Markus theorem will be used to show the existence of a Riesz basis consisting of root functions of the Dirac operator L_{bc}.

In the third section, we study eigenvalues and eigenfunctions of Dirac operators. Dirac operators subject to regular boundary conditions bc have discrete spectrum. It is shown that for strictly regular bc, every eigenvalue of the free Dirac operator is simple and has the form $\lambda^0_{k,\alpha} = \tau_\alpha + k$, where $\alpha = 1, 2$ and $k \in 2\mathbb{Z}$, and spectrum consists only of eigenvalues. For each strictly regular bc, there is an $N \in 2\mathbb{N}$ such that

$$Sp(L_{bc}) \subset R_N \cup \bigcup_{|n|>N} \left(D^1_n \cup D^2_n\right),$$

where R_N is a rectangle containing $2N$ eigenvalues of L_{bc} and each of the discs $D^\alpha_n = \left\{z : \left|z - \lambda^0_{n,\alpha}\right| < \rho = \rho(bc)\right\}$, $\alpha = 1, 2$ and $|n| > N$, contains exactly one simple eigenvalue of L_{bc}. Using this spectra localization of the operators L_{bc} and Bari-Markus theorem, it is shown that there is a Riesz basis which consists of eigenfunctions and (at most finitely many) associated functions.

In the fourth section, we show that the adjoint operator of Dirac operator $L_{bc}(v)$ subject to regular boundary conditions is $L_{bc^*}(v^*)$, where boundary conditions bc^* given by the system

$$\begin{aligned} \overline{b}g_1(0) + g_1(\pi) + \overline{d}g_2(\pi) = 0 \\ \overline{a}g_1(0) + g_2(0) + \overline{c}g_2(\pi) = 0, \end{aligned}$$

and

$$v^* = \begin{pmatrix} 0 & \overline{Q} \\ \overline{P} & 0 \end{pmatrix}.$$

In the last two sections, we find the form of self-adjoint boundary conditions bc and self-adjoint Dirac operators. Furthermore, we give a characterization of self-adjoint extensions of an unbounded operator and we calculate some special self-adjoint extensions.

2 Riesz bases

In this section, we give basic facts about bases. We define Riesz bases and give some basic properties of Riesz bases. We consider only separable Hilbert spaces.

Definition 1. *Let H be a Hilbert space. A system $(e_n,\ n \in \mathbb{N})$ is called a basis in H if*

$$x = \sum_{n=1}^{\infty} c_n e_n, \quad \forall x \in H \tag{2.1}$$

where c_n's are uniquely determined and the series converges in norm. If the series converges unconditionally, then $(e_n,\ n \in \mathbb{N})$ is called an unconditional basis. Moreover a basis is called orthonormal if it is an orthonormal system in H which means

$$\langle e_n, e_m \rangle = \delta_{n,m}, \quad \forall n, m \in \mathbb{N}.$$

It is a general fact that there are orthonormal bases in every Hilbert space H.

Definition 2. *Two systems $(e_n,\ n \in \mathbb{N})$ and $(f_m, m \in \mathbb{N})$ in H are called biorthogonal if*

$$\langle e_n, f_m \rangle = \delta_{n,m}, \quad \forall n, m \in \mathbb{N}.$$

Theorem 3. *Assume that $(f_n,\ n \in \mathbb{N})$ is a basis in a Hilbert space H. Then there exists a unique family $\left(\widetilde{f_n},\ n \in \mathbb{N}\right)$ in H such that*

$$x = \sum_n \left\langle x, \widetilde{f_n} \right\rangle f_n, \quad \forall x \in H,$$

where $(f_n,\ n \in \mathbb{N})$ *and* $\left(\widetilde{f}_n,\ n \in \mathbb{N}\right)$ *are biorthogonal.*

Proof. Suppose $(f_n,\ n \in \mathbb{N})$ is a basis in a Hilbert space H. Then

$$x = \sum_{n=1}^{\infty} c_n(x) f_n,$$

where $c_n(.)$ are linear functions due to the uniqueness of expansion (**??**). Let

$$P_n(x) = c_n(x) f_n$$

and let

$$S_N(x) = \sum_{k=1}^{N} c_k(x) f_k.$$

First we show that the projections S_N, $N = 1, 2, \cdots$, are uniformly bounded. Now define

$$\|x\|_1 := \sup_N \left\| \sum_{k=1}^{N} c_k(x) f_k \right\| < \infty.$$

This norm is well-defined since $\sum_n c_n(x) f_n$ converges. The fact that $\|.\|_1$ is indeed a norm in H easily follows from the fact that $\|.\|$ is a norm in H. We show that $(H, \|.\|_1)$ is a Banach space. Let $(x_n,\ n \in \mathbb{N})$ be a Cauchy sequence in $(H, \|.\|_1)$. Fix $\varepsilon > 0$. Then there is μ such that

$$\|x_n - x_m\|_1 = \sup_N \left\| \sum_{k=1}^{N} \left[c_k(x_n) - c_k(x_m)\right] f_k \right\| < \varepsilon, \quad \text{for } n, m \geq \mu,$$

which implies

$$\left\| \sum_{k=1}^{N} \left[c_k(x_n) - c_k(x_m)\right] f_k \right\| < \varepsilon, \quad \forall N,\ \forall n, m \geq \mu.$$

Then for every $N \in \mathbb{N}$ and for every $n, m \geq \mu$

$$\|\left[c_N(x_n) - c_N(x_m)\right] f_N\| = \left\| \sum_{k=1}^{N} \left[c_k(x_n) - c_k(x_m)\right] f_k - \sum_{k=1}^{N-1} \left[c_k(x_n) - c_k(x_m)\right] f_k \right\|$$

$$\leq \left\| \sum_{k=1}^{N} \left[c_k(x_n) - c_k(x_m)\right] f_k \right\| + \left\| \sum_{k=1}^{N-1} \left[c_k(x_n) - c_k(x_m)\right] f_k \right\| < 2\varepsilon.$$

Therefore

$$|c_N(x_n - x_m)| < 2\varepsilon / \|f_N\|, \quad \text{for } n, m \geq \mu.$$

So, for every $N \in \mathbb{N}$, $(c_N(x_n))$ is a Cauchy sequence of numbers and converges, say to c_N^*. By definition of $\|.\|_1$,

$$\|x_n - x_m\| \leq \|x_n - x_m\|_1 < \varepsilon, \quad \forall n, m \geq \mu.$$

Therefore x_n is a Cauchy sequence in $(H, \|.\|)$ and $\|x_n - x\| \to 0$ for some x in H. Let $m \to \infty$. Then by the last inequality, we obtain

$$\|x_n - x\| \leq \varepsilon, \quad \forall n \geq \mu. \tag{2.2}$$

We know that

$$\left\| \sum_{k=1}^{N} \left[c_k(x_n) - c_k(x_m) \right] f_k \right\| < \varepsilon, \quad \forall N, \ \forall n, m \geq \mu.$$

Let $m \to \infty$. Then we obtain

$$\left\| \sum_{k=1}^{N} \left[c_k(x_n) - c_k^* \right] f_k \right\| \leq \varepsilon, \quad \forall N, \ \forall n \geq \mu. \tag{2.3}$$

Fix $n > \mu$. Then we have for every $N \in \mathbb{N}$,

$$\begin{aligned} \left\| \sum_{k=1}^{N} c_k(x) f_k - \sum_{k=1}^{N} c_k^* f_k \right\| \leq \left\| \sum_{k=1}^{N} c_k(x) f_k - x \right\| + \|x - x_n\| + \left\| x_n - \sum_{k=1}^{N} c_k(x_n) f_k \right\| \\ + \left\| \sum_{k=1}^{N} c_k(x_n) f_k - \sum_{k=1}^{N} c_k^* f_k \right\|. \end{aligned}$$

By (**??**) and (**??**), the second and the fourth term on the right hand side are less then ε. Since (f_k) is a basis, for large enough N, the first and the third terms are also less than ε. Thus it follows that for large enough N

$$\left\| \sum_{k=1}^{N} c_k(x) f_k - \sum_{k=1}^{N} c_k^* f_k \right\| < 4\varepsilon.$$

Since $\sum_{k=1}^{N} c_k(x) f_k \to x$, it follows that $\sum_{k=1}^{N} c_k^* f_k \to x$, that is $x = \sum_{k=1}^{N} c_k^* f_k$. Hence $c_k^* = c_k(x)$ by uniqueness of c_k. Since

$$\left\| \sum_{k=1}^{N} \left[c_k(x_n) - c_k^* \right] f_k \right\| \leq \varepsilon, \quad \forall N, \ \forall n \geq \mu,$$

we obtain that for every $n \geq \mu$,

$$\|x_n - x\|_1 = \sup_N \left\| \sum_{k=1}^{N} \left[c_k(x_n) - c_k(x) \right] f_k \right\| = \sup_N \left\| \sum_{k=1}^{N} \left[c_k(x_n) - c_k^* \right] f_k \right\| \leq \varepsilon.$$

So $\|x_n - x\|_1 \to 0$ as $n \to \infty$. Thus $(H, \|.\|_1)$ is a Banach space.
By definition of $\|.\|_1$, we have

$$\|x\| \leq \|x\|_1, \quad \forall x \in H,$$

and this means that the identity operator

$$I : (H, \|.\|_1) \to (H, \|.\|)$$

is a continuous one to one mapping. So by Open Mapping Theorem, it has a continuous inverse. So there is a constant $c > 0$ such that

$$\|x\|_1 \leq c\,\|x\|, \quad \forall x \in H.$$

Therefore

$$S_N(x) = \left\| \sum_{k=1}^{N} c_k(x) f_k \right\| \leq c\,\|x\|, \quad \forall N.$$

By this way, we have shown that S_N's are uniformly bounded. Then

$$\|P_n(x)\| = \|S_n(x) - S_{n-1}(x)\| \leq \|S_n(x)\| + \|S_{n-1}(x)\| \leq 2c\,\|x\|.$$

So

$$\|P_n(x)\| = |c_n(x)|\,\|f_n\| \leq 2c\,\|x\|,$$

which implies

$$|c_n(x)| \leq \frac{2c}{\|f_n\|}\,\|x\|.$$

Hence $c_n(x)$ is continuous for $n = 1, 2, \cdots$. By Riesz Representation Theorem, each $c_n(x)$ can be written in the form

$$c_n(x) = \left\langle x, \widetilde{f_n} \right\rangle,$$

where $\widetilde{f_n}$ is a uniquely determined element in H. So every $x \in H$ can be written as

$$x = \sum_{n=1}^{\infty} \left\langle x, \widetilde{f_n} \right\rangle f_n.$$

Since $c_n(f_m) = \delta_{n,m}$ for $n, m = 1, 2, \cdots$, c_n's and f_n's are biorthogonal systems. And $c_n(f_m) = \delta_{n,m}$ means that

$$c_n(f_m) = \left\langle f_m, \widetilde{f_n} \right\rangle = \delta_{n,m}.$$

Thus f_n's and $\widetilde{f_n}$'s are biorthogonal. Hence we have shown that for any basis $(f_n,\ n \in \mathbb{N})$ in H, there is a unique biorthogonal system $\left(\widetilde{f_n},\ n \in \mathbb{N}\right)$ in H such that

$$x = \sum_{n=1}^{\infty} \left\langle x, \widetilde{f_n} \right\rangle f_n, \quad \forall x \in H.$$

□

The bases that we consider are unconditional bases, so we don't have convergence problems related to order of the elements. Thus for practical uses we use countable bases of the form $(e_\gamma,\ \gamma \in \Gamma)$ where Γ is a countable set of indices, instead of bases of the form $(e_n,\ n \in \mathbb{N})$.

Let H be a Hilbert space and let $(e_\gamma,\ \gamma \in \Gamma)$ be an orthonormal basis in H. If $A : H \to H$ is an automorphism, then the system $(f_\gamma,\ \gamma \in \Gamma)$ given by

$$f_\gamma = Ae_\gamma, \quad \gamma \in \Gamma \tag{2.4}$$

is also a basis in H. For every $x \in H$, we have

$$x = A\left(A^{-1}x\right) = A\left(\sum_\gamma \left\langle A^{-1}x, e_\gamma \right\rangle e_\gamma\right) = \sum_\gamma \left\langle x, \left(A^{-1}\right)^* e_\gamma \right\rangle f_\gamma = \sum_\gamma \left\langle x, \widetilde{f_\gamma} \right\rangle f_\gamma.$$

Definition 4. *A basis of the form* (**??**) *is called a Riesz basis.*

So we have also showed that (f_γ) is a basis with its biorthogonal system

$$\widetilde{f_\gamma} = \left(A^{-1}\right)^* e_\gamma, \quad \gamma \in \Gamma. \tag{2.5}$$

Riesz bases are unconditional bases since orthonormal bases are unconditional bases.

Lemma 5. *Let H_1 and H_2 be two Hilbert spaces and let $A : H_1 \to H_2$ be an isomorphism. If $(e_\gamma,\ \gamma \in \Gamma)$ is an orthonormal basis in H_1, then the system*

$$f_\gamma = Ae_\gamma, \quad \gamma \in \Gamma$$

is a Riesz basis in H_2.

Proof. Assume H_1 and H_2 are two Hilbert spaces and $A : H_1 \to H_2$ is an isomorphism. Let $(e_\gamma,\ \gamma \in \Gamma)$ be an orthonormal basis in H_1. Take any orthonormal basis in H_2, say $(\phi_\gamma,\ \gamma \in \Gamma)$. Then the operator $B : H_2 \to H_1$ defined by

$$B\phi_\gamma = e_\gamma, \quad \forall \gamma \in \Gamma.$$

Then B is clearly an isomorphism. Now take $C : H_2 \to H_2$ given by $C = A \circ B$. Then C is an isomorphism and

$$C\phi_\gamma = f_\gamma, \quad \forall \gamma \in \Gamma.$$

Thus $(f_\gamma,\ \gamma \in \Gamma)$ is a Riesz basis in H_2. □

Recall that $\ell^2(\Gamma)$ is the space consisting of the generalized sequences $(x_\gamma,\ \gamma \in \Gamma)$ such that

$$\sum_\gamma |x_\gamma|^2 < \infty.$$

We consider $\ell^2(\Gamma)$ equipped with the inner product

$$\langle x, y\rangle = \sum_\gamma x_\gamma \overline{y_\gamma},$$

where $x = (x_\gamma,\ \gamma \in \Gamma)$ and $y = (y_\gamma,\ \gamma \in \Gamma)$.

Now we give a characterization of Riesz bases by the following theorem.

Theorem 6. *Suppose that $(f_\gamma,\ \gamma \in \Gamma)$ is a basis in H and $\left(\widetilde{f_\gamma},\ \gamma \in \Gamma\right)$ is its biorthogonal system. Then $(f_\gamma,\ \gamma \in \Gamma)$ is a Riesz basis if and only if*

$$c \le \|f_\gamma\| \le C, \quad \forall \gamma \in \Gamma \tag{2.6}$$

and

$$m\|x\|^2 \le \sum_\gamma \left|\left\langle x, \widetilde{f_\gamma}\right\rangle\right|^2 \|f_\gamma\|^2 \le M\|x\|^2, \tag{2.7}$$

for some positive constants c, C, m and M.

Proof. First let $(f_\gamma,\ \gamma \in \Gamma)$ be a Riesz basis in H with its biorthogonal system $\left(\widetilde{f_\gamma},\ \gamma \in \Gamma\right)$. Then there is an orthonormal basis $(e_\gamma,\ \gamma \in \Gamma)$ in H and an automorphism A such that

$$A(e_\gamma) = f_\gamma.$$

Then we have

$$\|f_\gamma\| = \|Ae_\gamma\| \le \|A\| \quad \text{and} \quad 1 = \|e_\gamma\| = \left\|A^{-1}f_\gamma\right\| \le \left\|A^{-1}\right\| \|f_\gamma\|,$$

which gives us

$$\frac{1}{\|A^{-1}\|} \le \|f_\gamma\| \le \|A\|. \tag{2.8}$$

So we get (**??**) with $c = \frac{1}{\|A^{-1}\|}$ and $C = \|A\|$. Also we have that

$$\begin{aligned}
\sum_\gamma \left|\left\langle x, \left(A^{-1}\right)^* e_\gamma\right\rangle\right|^2 \|f_\gamma\|^2 &= \sum_\gamma \left|\langle A^{-1}x, e_\gamma\rangle\right|^2 \|f_\gamma\|^2 \\
&\le \|A\|^2 \sum_\gamma \left|\langle A^{-1}x, e_\gamma\rangle\right|^2 \quad \text{by } (\mathbf{??}) \\
&= \|A\|^2 \left\|A^{-1}x\right\|^2 \\
&\le \|A\|^2 \left\|A^{-1}\right\|^2 \|x\|^2.
\end{aligned}$$

and

$$\begin{aligned}
\sum_\gamma \left|\left\langle x, \left(A^{-1}\right)^* e_\gamma\right\rangle\right|^2 \|f_\gamma\|^2 &= \sum_\gamma \left|\left\langle A^{-1}x, e_\gamma\right\rangle\right|^2 \|f_\gamma\|^2 \\
&\geq \tfrac{1}{\|A^{-1}\|^2} \sum_\gamma \left|\left\langle A^{-1}x, e_\gamma\right\rangle\right|^2 \quad \text{by (??)} \\
&= \tfrac{1}{\|A^{-1}\|^2} \left\|A^{-1}x\right\|^2 \\
&\geq \tfrac{1}{\|A^{-1}\|^2} \tfrac{1}{\|A\|^2} \|x\|^2 .
\end{aligned}$$

since

$$\|x\| = \left\|A^{-1}Ax\right\| \leq \|A\| \left\|A^{-1}x\right\|$$

which means

$$\left\|A^{-1}x\right\| \geq \frac{\|x\|}{\|A\|}.$$

Combining these results, we get

$$\frac{1}{\|A^{-1}\|^2} \frac{1}{\|A\|^2} \|x\|^2 \leq \sum_\gamma \left|\left\langle x, \widetilde{f_\gamma}\right\rangle\right|^2 \|f_\gamma\|^2 \leq \|A\|^2 \left\|A^{-1}\right\|^2 \|x\|^2 , \tag{2.9}$$

which proves (??) with $m = \frac{1}{\|A^{-1}\|^2 \|A\|^2}$ and $M = \|A\|^2 \left\|A^{-1}\right\|^2$.

Now let $(f_\gamma,\ \gamma \in \Gamma)$ be a basis in H and $\left(\widetilde{f_\gamma},\ \gamma \in \Gamma\right)$ be its biorthogonal system such that (??) and (??) holds. Since Γ is a countable set, we may think that $\Gamma = \{\gamma_i,\ i = 1, 2, \cdots\}$. With this enumaration, consider the operator $B : \ell^2(\Gamma) \to H$ given by

$$B\left((x_\gamma)\right) = \sum_{i=1}^{\infty} x_{\gamma_i} f_{\gamma_i}.$$

Let

$$S_N = \sum_{i=1}^{N} x_{\gamma_i} f_{\gamma_i}.$$

Then

$$\left\langle S_N, \widetilde{f_{\gamma_i}}\right\rangle = x_{\gamma_i}, \quad i = 1, 2, \cdots, N.$$

So by using (??) and (??), we get that

$$k \|S_N\|^2 \leq \sum_\gamma |x_{\gamma_i}|^2 \leq K \|S_N\|^2 ,$$

where $k = m/C^2$ and $K = M/c^2$. By the same argument, we get that

$$k \|S_{N+M} - S_N\|^2 \leq \sum_{i=N}^{N+M} |x_{\gamma_i}|^2 \to 0 \quad \text{as } N \to \infty$$

since $(x_\gamma) \in \ell^2$. So (S_N) is a Cauchy sequence in H. Since H is complete, (S_N) is convergent to some s in H. Thus the series $\sum_{i=1}^{\infty} x_{\gamma_i} f_{\gamma_i}$ converges which shows that B is well-defined.

Next we prove that B is continuous. Set

$$x = B\left(\left(x_\gamma\right)\right) = \sum_{i=1}^{\infty} x_{\gamma_i} f_{\gamma_i}.$$

Then by (**??**) and (**??**)

$$k\left\|x\right\|^2 = k\left\|B\left(\left(x_\gamma\right)\right)\right\|^2 \leq \sum_{i=1}^{\infty}\left|x_{\gamma_i}\right|^2 = \left\|\left(x_\gamma\right)\right\|_{\ell^2}^2 \leq K\left\|B\left(\left(x_\gamma\right)\right)\right\|^2.$$

So B is continuous since

$$\left\|B\left(\left(x_\gamma\right)\right)\right\| \leq \frac{1}{\sqrt{k}}\left\|\left(x_\gamma\right)\right\|_{\ell^2},$$

and B^{-1} is continuous since

$$\left\|B\left(\left(x_\gamma\right)\right)\right\| \geq \frac{1}{\sqrt{K}}\left\|\left(x_\gamma\right)\right\|_{\ell^2}.$$

Thus B is an isomorphism.
Let $(e_\gamma,\ \gamma \in \Gamma)$ be the orthonormal basis in ℓ^2 given by

$$e_\gamma(\alpha) = \delta_{\gamma,\alpha}.$$

Then by definition of B,

$$Be_\gamma = f_\gamma, \quad \forall \gamma \in \Gamma.$$

Thus (f_γ) is a Riesz basis by the previous lemma. □

Definition 7. *A system $(f_n,\ n \in \mathbb{N})$ is called minimal if*

$$f_j \notin \overline{span}\left\{f_k\right\}_{k \neq j}, \quad \forall j \in \mathbb{N}.$$

Theorem 8. *(Bari-Markus Theorem) Let $(e_n,\ n \in \mathbb{N})$ be a Riesz basis in a Hilbert space H and let $(f_n,\ n \in \mathbb{N})$ be a minimal system of vectors such that*

$$\sum_{n=1}^{\infty}\left\|f_n - e_n\right\|^2 < \infty.$$

Then $(f_n,\ n \in \mathbb{N})$ is also a Riesz basis.

Proof. It sufficies to show that there is an isomorphism A such that $A\left(e_n\right) = f_n$. Since $(e_n,\ n \in \mathbb{N})$ is a Riesz basis, there is an isomorphism B and orthonormal basis $(\phi_n,\ n \in \mathbb{N})$ such that $B\left(\phi_n\right) = e_n$. So $A \circ B$ will be an isomorphism such that $B \circ A\left(\phi_n\right) = f_n$.

For $x = \sum\limits_{n=1}^{\infty} x_n e_n$, set

$$Tx := \sum_{n=1}^{\infty} x_n (e_n - f_n).$$

The operator T is bounded since

$$\begin{aligned} \|Tx\| &= \left\| \sum_{n=1}^{\infty} x_n (e_n - f_n) \right\| \leq \sum_{n=1}^{\infty} |x_n| \, \|e_n - f_n\| \\ &\leq \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} \|e_n - f_n\|^2 \right)^{1/2} \leq \frac{\sqrt{M}}{c^2} s \, \|x\|, \end{aligned}$$

where c and M are coming from the inequalities (**??**),(**??**) and $s^2 = \sum\limits_{n=1}^{\infty} \|f_n - e_n\|^2$. Let $(T_k,\ k \in \mathbb{N})$ be the sequence of finite rank operators given by

$$T_k x = \sum_{n=1}^{k} x_n(e_n - f_n), \quad \text{for } x = \sum_{n=1}^{\infty} x_n e_n.$$

We have $\|T_k - T\| \to 0$ as $k \to \infty$, since

$$\begin{aligned} \|T_k x - Tx\| &= \left\| \sum_{n=k+1}^{\infty} x_n(e_n - f_n) \right\| \leq \sum_{n=k+1}^{\infty} |x_n| \, \|e_n - f_n\| \\ &\leq \left(\sum_{n=k+1}^{\infty} |x_n|^2 \right)^{1/2} \left(\sum_{n=k+1}^{\infty} \|e_n - f_n\|^2 \right)^{1/2} \\ &\leq \|x\| \left(\sum_{n=k+1}^{\infty} \|e_n - f_n\|^2 \right)^{1/2}, \end{aligned}$$

and $\sum\limits_{n=1}^{\infty} \|e_n - f_n\|^2$ is a convergent series. Since T_k are finite-dimensional operators, it follows that T is a compact operator.

Now consider the operator $A = 1 - T$. A is invertible if 1 is not in the spectrum of T. Let $x \in \ker A$, that is

$$(1 - T)x = 0, \quad \text{for } x = \sum_{n=1}^{\infty} x_n e_n.$$

Then it follows

$$\sum_{n=1}^{\infty} x_n f_n = 0,$$

which implies (since $(f_n,\ n \in \mathbb{N})$ is a minimal system) that $x_n = 0$ for every n, hence $x = 0$. So 1 is not an eigenvalue of the operator T. Recall that spectrum of a compact operator contains only eigenvalues. Since T is a compact operator and 1 is not an eigenvalue of T, 1 is not in the spectrum of T. Then A is invertible, so it is an isomorphism. Hence $(f_n,\ n \in \mathbb{N})$ is also a Riesz basis. $\square$

3 Riesz Basis of Root Functions of Dirac Operator

In this section, we study the spectrum of free Dirac operator L^0_{bc} subject to strictly regular boundary conditions bc. We show that the eigenfunctions of L^0_{bc} form a Riesz basis in $L^2([0,\pi],\mathbb{C}^2)$. Moreover, we show the existence of Riesz basis of root functions of Dirac operator L_{bc} subject to strictly regular boundary conditions bc.

The differential expression

$$L(v)y = i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{dy}{dx} + \begin{pmatrix}0 & P(x)\\ Q(x) & 0\end{pmatrix}y. \tag{3.1}$$

is known as the one dimensional Dirac operator. The matrix $v = \begin{pmatrix}0 & P(x)\\ Q(x) & 0\end{pmatrix}$ is called the Dirac potential. In case $v \equiv 0$, we write $L^0 = L(0)$ and call L^0 free Dirac operator. A general boundary condition for the Dirac operator is given by a system of two linear equations

$$\begin{aligned} a_1y_1(0) + b_1y_1(\pi) + a_2y_2(0) + b_2y_2(\pi) = 0\\ c_1y_1(0) + d_1y_1(\pi) + c_2y_2(0) + d_2y_2(\pi) = 0. \end{aligned} \tag{3.2}$$

Of course, equivalent systems of the form (**??**) define one and the same bc. Each boundary condition is determined by the matrix of the coefficients of (**??**)

$$\begin{pmatrix}a_1 & b_1 & a_2 & b_2\\ c_1 & d_1 & c_2 & d_2\end{pmatrix}. \tag{3.3}$$

But if we multiply this matrix from the left by a 2x2 invertible matrix, we get another matrix that determines the same bc.

We may assign to every boundary condition bc of the form (**??**) a corresponding operator L^0_{bc} as follows. Let

$$\begin{aligned} Dom\,(L^0_{bc}) \;=\; & \Big\{ y = \begin{pmatrix}y_1\\ y_2\end{pmatrix} : y_1 \text{ and } y_2 \text{ are absolutely continuous}, y \text{ satisfies}\\ & \text{the boundary conditions } bc, \text{ and } y_1', y_2' \in L^2([0,\pi])\Big\}, \end{aligned}$$

and let

$$L^0_{bc}(y) = i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}y_1'\\ y_2'\end{pmatrix}.$$

Theorem 9. *L^0_{bc} is a closed densely defined operator.*

Proof. Let

$$\left(\begin{pmatrix}f_n\\ g_n\end{pmatrix}, L^0\begin{pmatrix}f_n\\ g_n\end{pmatrix}\right) \longrightarrow \left(\begin{pmatrix}f\\ g\end{pmatrix}, \begin{pmatrix}h_1\\ h_2\end{pmatrix}\right) \quad \text{in } L^2([0,\pi],\mathbb{C}^2).$$

This means that

$$\left(\begin{pmatrix}f_n\\ g_n\end{pmatrix}, \begin{pmatrix}if_n'\\ -ig_n'\end{pmatrix}\right) \longrightarrow \left(\begin{pmatrix}f\\ g\end{pmatrix}, \begin{pmatrix}h_1\\ h_2\end{pmatrix}\right) \quad \text{in } L^2([0,\pi],\mathbb{C}^2).$$

So

$$\left(f_n, if_n'\right) \xrightarrow{\|\cdot\|} (f, h_1) \text{ and } \left(g_n, -ig_n'\right) \xrightarrow{\|\cdot\|} (g, h_2).$$

Since f_n's are measurable functions and $f_n \xrightarrow{\|\cdot\|} f$, f_n converges to f in measure. Then there is a subsequence f_{nk} such that $f_{nk}(x) \xrightarrow{a.e.} f(x)$. Thus there exists $c \in [0, \pi]$ such that $f_n(c) \to f(c)$. Now define

$$H_1(x) := \frac{1}{i} \int_c^x h_1(t)dt.$$

Then

$$\begin{aligned}
|f_n(x) - f_n(c) - H_1(x)|^2 &= \left| \int_c^x \left(f_n'(t) - \tfrac{1}{i} h_1(t)\right) dt \right|^2 \\
&\leq \left(\int_0^\pi \left|f_n'(t) - \tfrac{1}{i} h_1(t)\right| dt \right)^2 \\
&\leq \left(\int_0^\pi \left|f_n'(t) - \tfrac{1}{i} h_1(t)\right|^2 dt \right) \left(\int_0^\pi 1^2 dt \right) \\
&= \left\| f_n' - \tfrac{1}{i} h_1 \right\|^2 \pi.
\end{aligned}$$

By this inequality, we get that

$$f_n(x) - f_n(c) \xrightarrow{unif} H_1(x),$$

since $f_n' \xrightarrow{\|\cdot\|} \frac{h_1}{i}$. And also $f_n(x) - f_n(c) \xrightarrow{a.e.} f(x) - f(c)$, so

$$H_1(x) = f(x) - f(c) \text{ a.e.}$$

Since we identify functions that are equal almost everywhere, we may think that the previous equality holds for every $x \in [0, \pi]$. So

$$h_1(x) = iH_1'(x) = if'(x) \text{ a.e.} \tag{3.4}$$

We can similarly define $H_2(x) := -\frac{1}{i} \int_c^x h_2(t)dt$ and get that

$$h_2(x) = -ig'(x). \tag{3.5}$$

Since we identify functions that are equal almost everywhere and f_n's are absolutely continuous functions, we may think that $f_n(x) \to f(x)$, $\forall x \in [0, \pi]$. So f satisfies bc. Similarly g also satisfies bc. So by (**??**) and (**??**)

$$\begin{pmatrix} f \\ g \end{pmatrix} \in dom(L_{bc}^0) \quad \text{and} \quad L^0 \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

Thus we have shown that the graph of L_{bc}^0 is closed, which means that L_{bc}^0 is a closed operator. □

Now we find the eigenvalues of the operator L^0 given by

$$L^0(y) = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{dy}{dx}$$

subject to a general boundary condition bc of the form (**??**) with associated matrix (**??**). Let A_{ij} denote the matrix given by i-th and j-th columns of the matrix (**??**). If

$$L^0 y = \lambda y, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

then

$$iy_1^{'} = \lambda y_1 \quad \text{and} \quad -iy_2^{'} = \lambda y_2.$$

So each solution of the equation has the form

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} k_1 e^{-i\lambda x} \\ k_2 e^{i\lambda x} \end{pmatrix} \tag{3.6}$$

for some constants k_1 and k_2. Now if we let $z = e^{i\lambda\pi}$, we get

$$y_1(0) = k_1, \quad y_2(0) = k_2, \quad y_1(\pi) = k_1 z^{-1} \quad \text{and} \quad y_2(\pi) = k_2 z.$$

So if we use these initial conditions of y, we can see that y satisfies the boundary conditions (**??**) iff (k_1, k_2) is a solution of the system of equations

$$\begin{aligned} a_1 k_1 + b_1 k_1 z^{-1} + a_2 k_2 + b_2 k_2 z = 0 \\ c_1 k_1 + d_1 k_1 z^{-1} + c_2 k_2 + d_2 k_2 z = 0 \end{aligned}$$

or equivalently

$$\begin{aligned} k_1 \left(a_1 + b_1 z^{-1}\right) + k_2 \left(a_2 + b_2 z\right) = 0 \\ k_2 \left(c_1 + d_1 z^{-1}\right) + k_2 \left(c_2 + d_2 z\right). \end{aligned} \tag{3.7}$$

So we have a non-zero solution y iff

$$\begin{vmatrix} a_1 + b_1 z^{-1} & a_2 + b_2 z \\ c_1 + d_1 z^{-1} & c_2 + d_2 z \end{vmatrix} = 0,$$

$$\Leftrightarrow a_1 c_2 + a_1 d_2 z + b_1 z^{-1} c_2 + b_1 z^{-1} d_2 z - c_1 a_2 - c_1 b_2 z - d_1 z^{-1} a_2 - d_1 z^{-1} b_2 z = 0,$$

$$\Leftrightarrow \left(a_1 d_2 - b_2 c_1\right) z + \left(b_1 d_2 - d_1 b_2 + a_1 c_2 - a_2 c_1\right) + \left(b_1 c_2 - d_1 a_2\right) z^{-1} = 0.$$

If we multiply both sides by z, it is equivalent to the quadratic equation

$$|A_{14}| z^2 + \left(|A_{13}| + |A_{24}|\right) z + |A_{23}| = 0. \tag{3.8}$$

The boundary condition (**??**) is called *strictly regular* if

$$|A_{14}| \neq 0, \quad |A_{23}| \neq 0, \quad \left(|A_{13}| + |A_{24}|\right)^2 \neq 4 |A_{14}| |A_{23}| \tag{3.9}$$

hold. So if we have strictly regular boundary conditions, then the quadratic equation (**??**) has two distinct roots, call z_1 and z_2.

In the following, we consider only strictly regular boundary conditions. Now if we multiply A_{14}^{-1} with (**??**), we get that

$$A_{14}^{-1}\begin{pmatrix} a_1 & b_1 & a_2 & b_2 \\ c_1 & d_1 & c_2 & d_2 \end{pmatrix} = \frac{1}{a_1d_2 - b_2c_1}\begin{pmatrix} b_2 & -b_2 \\ -c_1 & a_1 \end{pmatrix}\begin{pmatrix} a_1 & b_1 & a_2 & b_2 \\ c_1 & d_1 & c_2 & d_2 \end{pmatrix} = \begin{pmatrix} 1 & b & a & 0 \\ 0 & d & c & 1 \end{pmatrix} \tag{3.10}$$

where $\begin{pmatrix} b & a \\ d & c \end{pmatrix} = A_{14}^{-1}A_{23}$.

So we found an equivalent system to the boundary condition (**??**) given by

$$\begin{aligned} y_1(0) + by_1(\pi) + ay_2(0) = 0 \\ dy_1(\pi) + cy_2(0) + y_2(\pi) = 0. \end{aligned} \tag{3.11}$$

This system is associated with the matrix (**??**). From now on we consider the boundary conditions in the form (**??**) with matrices (**??**). So the conditions (**??**) adjusted to this new form of the boundary conditions means that

$$|A_{23}| = bc - ad \neq 0 \tag{3.12}$$

and

$$\begin{aligned} (|A_{13} + A_{24}|)^2 - 4\,|A_{14}|\,|A_{23}| &= (c+b)^2 - 4(bc - ad) \\ &= c^2 + 2bc + b^2 - 4bc + 4ad \\ &= (b-c)^2 + 4ad \neq 0. \end{aligned} \tag{3.13}$$

Now the system (**??**) becomes

$$\begin{aligned} k_1\left(1 + bz^{-1}\right) + k_2a = 0 \\ k_1dz^{-1} + k_2\,(c+z) = 0 \end{aligned} \tag{3.14}$$

which means

$$\begin{pmatrix} 1 + b/z & a \\ d/z & c+z \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} z+b & a \\ d & c+z \end{pmatrix}\begin{pmatrix} k_1/z \\ k_2 \end{pmatrix} = 0. \tag{3.15}$$

And the quadratic equation (**??**) becomes

$$z^2 + (b+c)\,z + bc - ad = 0. \tag{3.16}$$

So from these equations, we get the following lemma.

Lemma 10. *The complex number $-z$ is an eigenvalue of the matrix $A_{23} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$ if and only if z is a root of the quadratic equation* (**??**). *And also, $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ is a non-zero solution of* (**??**) *if and only if $\begin{pmatrix} k_1/z \\ k_2 \end{pmatrix}$ is an eigenvector of A_{23} corresponding to the eigenvalue $-z$.*

The quadratic equation (**??**) has two distinct nonzero roots z_1 and z_2 corresponding to the given strictly regular boundary condition. So the matrix A_{23} has two distinct nonzero eigenvalues $-z_1$ and $-z_2$ by the previous lemma. Let τ_1 and τ_2 be chosen such that

$$z_1 = e^{i\pi\tau_1}, \quad z_2 = e^{i\pi\tau_2}.$$

and

$$|Re\,\tau_1 - Re\,\tau_2| \leq 1, \quad |Re\,\tau_1| \leq 1.$$

Then

$$z_1 = e^{i\pi\lambda} \quad \Leftrightarrow \quad \lambda = \tau_1 + k, \ k \in 2\mathbb{Z}$$

and

$$z_2 = e^{i\pi\lambda} \quad \Leftrightarrow \quad \lambda = \tau_2 + k, \ k \in 2\mathbb{Z}.$$

So the set

$$E = \{\tau_1 + k, \tau_2 + k; \ k \in 2\mathbb{Z}\} \tag{3.17}$$

gives us all eigenvalues of L^0.

Now let us fix eigenvectors $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ and $\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ corresponding to eigenvalues $-z_1$ and $-z_2$. Then these eigenvectors are linearly independent. Let us define

$$\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}^{-1} := \begin{pmatrix} \alpha_1^{'} & \alpha_2^{'} \\ \beta_1^{'} & \beta_2^{'} \end{pmatrix}. \tag{3.18}$$

By the previous lemma, for each eigenvalue there is an eigenvector of L^0 of the form (**??**) with

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 z_1 \\ \alpha_2 \end{pmatrix} \quad \text{if } \lambda = \tau_1 + k, \quad \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} \beta_1 z_1 \\ \beta_2 \end{pmatrix} \quad \text{if } \lambda = \tau_2 + k.$$

So all eigenfunctions of L^0_{bc} with boundary conditions (**??**) are

$$\Phi^1 = \left\{\varphi_k^1, \ k \in 2\mathbb{Z}\right\}, \quad \varphi_k^1 := \begin{pmatrix} z_1 \alpha_1 e^{-i(\tau_1 + k)x} \\ \alpha_2 e^{i(\tau_1 + k)x} \end{pmatrix} = \begin{pmatrix} \alpha_1 e^{i\tau_1(\pi - x)} e^{-ikx} \\ \alpha_2 e^{i\tau_1 x} e^{ikx} \end{pmatrix} \tag{3.19}$$

and

$$\Phi^2 = \left\{\varphi_k^2, \ k \in 2\mathbb{Z}\right\}, \quad \varphi_k^2 := \begin{pmatrix} z_2 \beta_1 e^{-i(\tau_2 + k)x} \\ \beta_2 e^{i(\tau_2 + k)x} \end{pmatrix} = \begin{pmatrix} \beta_1 e^{i\tau_2(\pi - x)} e^{-ikx} \\ \beta_2 e^{i\tau_2 x} e^{ikx} \end{pmatrix}. \tag{3.20}$$

Theorem 11. *The set* $\Phi = \Phi^1 \cup \Phi^2$ *is a Riesz basis in the space* $L^2([0,\pi], \mathbb{C}^2)$ *with its biorthogonal system* $\tilde{\Phi} = \tilde{\varphi}^1 \cup \tilde{\varphi}^2$, *where*

$$\tilde{\Phi}^1 = \left\{\tilde{\varphi}_k^1, \ k \in 2\mathbb{Z}\right\}, \quad \tilde{\varphi}_k^1 := \begin{pmatrix} \overline{\alpha_1^{'}} e^{i\overline{\tau_1}(\pi - x)} e^{-ikx} \\ \overline{\alpha_2^{'}} e^{i\overline{\tau_1} x} e^{ikx} \end{pmatrix}, \tag{3.21}$$

and

$$\tilde{\Phi}^2 = \left\{\tilde{\varphi}_k^2, \ k \in 2\mathbb{Z}\right\}, \quad \tilde{\varphi}_k^2 := \begin{pmatrix} \overline{\beta_1^{'}} e^{i\overline{\tau_2}(\pi - x)} e^{-ikx} \\ \overline{\beta_2^{'}} e^{i\overline{\tau_2} x} e^{ikx} \end{pmatrix}, \tag{3.22}$$

$\alpha_1^{'}, \alpha_2^{'}, \beta_1^{'}, \beta_2^{'}$ *are coming from* (**??**).

Proof. The system consisting of

$$e_k^1 := \begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix}, \quad e_k^2 := \begin{pmatrix} 0 \\ e^{ikx} \end{pmatrix}, \quad k \in 2\mathbb{Z} \tag{3.23}$$

forms an orthonormal basis in $L^2([0,\pi], \mathbb{C}^2)$. Now we construct an automorphism on $L^2([0,\pi], \mathbb{C}^2)$ which maps this system to Φ. Consider the operator given by

$$A : L^2\left([0,\pi], \mathbb{C}^2\right) \to L^2\left([0,\pi], \mathbb{C}^2\right) \tag{3.24}$$

$$A \begin{pmatrix} f \\ g \end{pmatrix} := \begin{pmatrix} \alpha_1 e^{i\tau_1(\pi-x)} f(\pi - x) \\ \alpha_2 e^{i\tau_1 x} f(x) \end{pmatrix} + \begin{pmatrix} \beta_1 e^{i\tau_2(\pi-x)} g(\pi - x) \\ \beta_2 e^{i\tau_2 x} g(x) \end{pmatrix}.$$

It is obvious that A maps the system in (**??**) to Φ. Now we show that A is bounded, A^{-1} exists and is also bounded.

Observe that for any a and b, we have

$$(a-b)^2 = a^2 - 2ab + b^2 \geq 0 \quad \text{equivalently} \quad a^2 + b^2 \geq 2ab$$

which gives

$$(a+b)^2 = a^2 + 2ab + b^2 \leq 2a^2 + 2b^2.$$

So if we use this inequality, we get

$$\begin{aligned}
\left\| A \begin{pmatrix} f \\ g \end{pmatrix} \right\|^2 &= \left\| \begin{pmatrix} \alpha_1 e^{i\tau_1(\pi-x)} f(\pi - x) \\ \alpha_2 e^{i\tau_1 x} f(x) \end{pmatrix} + \begin{pmatrix} \beta_1 e^{i\tau_2(\pi-x)} g(\pi - x) \\ \beta_2 e^{i\tau_2 x} g(x) \end{pmatrix} \right\|^2 \\
&\leq 2 \left(\left\| \begin{pmatrix} \alpha_1 e^{i\tau_1(\pi-x)} f(\pi - x) \\ \alpha_2 e^{i\tau_1 x} f(x) \end{pmatrix} \right\|^2 + \left\| \begin{pmatrix} \beta_1 e^{i\tau_2(\pi-x)} g(\pi - x) \\ \beta_2 e^{i\tau_2 x} g(x) \end{pmatrix} \right\|^2 \right). \\
&\leq \tfrac{2}{\pi} \left(\int_0^\pi \left[|\alpha_1|^2 \left| e^{i\tau_1(\pi-x)} \right|^2 |f(\pi - x)|^2 + |\alpha_2|^2 \left| e^{i\tau_1 x} \right|^2 |f(x)|^2 \right] dx \right) \\
&\quad + \tfrac{2}{\pi} \left(\int_0^\pi \left[|\beta_1|^2 \left| e^{i\tau_2(\pi-x)} \right|^2 |g(\pi - x)|^2 + |\beta_2|^2 \left| e^{i\tau_2 x} \right|^2 |g(x)|^2 \right] dx \right).
\end{aligned}$$

Now let

$$c_1 := \max_{x \in [0,\pi]} \left\{ \left| e^{i\tau_1 x} \right|, \left| e^{i\tau_2 x} \right| \right\}, \ c_2 := \max \left\{ |\alpha_1|, |\alpha_2|, |\beta_1|, |\beta_2| \right\} \quad \text{and} \quad \tilde{c} = 2c_1c_2.$$

Then we have

$$\begin{aligned}
\left\| A \begin{pmatrix} f \\ g \end{pmatrix} \right\| &\leq 2c_1c_2 \left(\tfrac{1}{\pi} \int_0^\pi |f(\pi - x)|^2 dx + \int_0^\pi |g(x)|^2 dx \right)^{1/2} \\
&= \tilde{c} \left(\tfrac{1}{\pi} \int_0^\pi \left(|f(x)|^2 + |g(x)|^2 \right) dx \right)^{1/2} \\
&= \tilde{c} \left\| \begin{pmatrix} f \\ g \end{pmatrix} \right\|.
\end{aligned}$$

Thus we have shown that A is bounded. Let us find its inverse. Let

$$A\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}.$$

Then by definition of A, solving this equation is equivalent to solve the system

$$\begin{aligned} \alpha_1 e^{i\tau_1 x} f(x) + \beta_1 e^{i\tau_2 x} g(x) &= F(\pi - x) \\ \alpha_2 e^{i\tau_1 x} f(x) + \beta_2 e^{i\tau_2 x} g(x) &= G(x), \end{aligned}$$

which can be written in the form

$$\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix} \begin{pmatrix} e^{i\tau_1 x} f(x) \\ e^{i\tau_2 x} g(x) \end{pmatrix} = \begin{pmatrix} F(\pi - x) \\ G(x) \end{pmatrix}.$$

So by (**??**) we get that

$$\begin{pmatrix} e^{i\tau_1 x} f(x) \\ e^{i\tau_2 x} g(x) \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}^{-1} \begin{pmatrix} F(\pi - x) \\ G(x) \end{pmatrix} = \begin{pmatrix} \alpha_1' F(\pi - x) + \alpha_2' G(x) \\ \beta_1' F(\pi - x) + \beta_2' G(x) \end{pmatrix},$$

which gives us

$$A^{-1}\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} e^{-i\tau_1 x}\left[\alpha_1' F(\pi - x) + \alpha_2' G(x)\right] \\ e^{-i\tau_2 x}\left[\beta_1' F(\pi - x) + \beta_2' G(x)\right] \end{pmatrix}.$$

So we found A^{-1} and similar with the operator A, the inverse of A is also bounded.

Now only thing left is to calculate the biorthogonal system of Φ. First we calculate the adjoint operator of A^{-1}. Only by using definitions, we get

$$\begin{aligned} \left\langle A^{-1}\begin{pmatrix} F \\ G \end{pmatrix}, \begin{pmatrix} f \\ 0 \end{pmatrix} \right\rangle &= \tfrac{1}{\pi}\int_0^\pi e^{-i\tau_1 x}\left[\alpha_1' F(\pi - x) + \alpha_2' G(x)\right] \overline{f(x)}dx \\ &= \tfrac{1}{\pi}\int_0^\pi \left(F(x)\, \overline{\overline{\alpha_1'} f(\pi - x)\, e^{i\overline{\tau_1}(\pi - x)}} + G(x)\, \overline{\overline{\alpha_2'} f(x)\, e^{i\overline{\tau_1} x}} \right) dx \\ &= \left\langle \begin{pmatrix} F \\ G \end{pmatrix}, \begin{pmatrix} \overline{\alpha_1'} f(\pi - x)\, e^{i\overline{\tau_1}(\pi - x)} \\ \overline{\alpha_2'} f(x)\, e^{i\overline{\tau_1} x} \end{pmatrix} \right\rangle, \end{aligned}$$

which means

$$\left(A^{-1}\right)^* \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} \overline{\alpha_1'} f(\pi - x)\, e^{i\overline{\tau_1}(\pi - x)} \\ \overline{\alpha_2'} f(x)\, e^{i\overline{\tau_1} x} \end{pmatrix}.$$

Similarly the following equation holds

$$\left(A^{-1}\right)^* \begin{pmatrix} 0 \\ g \end{pmatrix} = \begin{pmatrix} \overline{\beta_1'} g(\pi - x)\, e^{i\overline{\tau_2}(\pi - x)} \\ \overline{\beta_2'} f(x)\, e^{i\overline{\tau_2} x} \end{pmatrix}.$$

Since $(A^{-1})^*$ is linear

$$\left(A^{-1}\right)^* \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \overline{\alpha_1'} f(\pi - x) e^{i\overline{\tau_1}(\pi-x)} \\ \overline{\alpha_2'} f(x) e^{i\overline{\tau_1} x} \end{pmatrix} + \begin{pmatrix} \overline{\beta_1'} g(\pi - x) e^{i\overline{\tau_2}(\pi-x)} \\ \overline{\beta_2'} f(x) e^{i\overline{\tau_2} x} \end{pmatrix}.$$

In view of (**??**), (**??**) and (**??**) really gives the biorthogonal system of Φ and Φ is a Riesz basis for $L^2([0,\pi], \mathbb{C}^2)$. We are done. □

Theorem 12. *The spectrum of L^0_{bc}, considered with strictly regular boundary conditions bc of the form* (**??**)*, consists only of its eigenvalues.*

Proof. Assume λ is not an eigenvalue of L^0_{bc}. Since

$$\left(\lambda - L^0_{bc}\right) \varphi^1_k = \left[\lambda - (\tau_1 + k)\right] \varphi^1_k,$$

we have

$$\left(\lambda - L^0_{bc}\right)^{-1} \varphi^1_k := \frac{1}{\lambda - (\tau_1 + k)} \varphi^1_k,$$

where φ^1_k is an eigenvector of the form (**??**) and $(\tau_1 + k)$ is the corresponding eigenvalue. Similarly

$$\left(\lambda - L^0_{bc}\right)^{-1} \varphi^2_k := \frac{1}{\lambda - (\tau_2 + k)} \varphi^2_k,$$

for an eigenvector φ^2_k of the form (**??**) with corresponding eigenvalue $(\tau_2 + k)$.
Let $f \in L^2([0,\pi], \mathbb{C}^2)$. By using the previous theorem, we can write f as

$$f = \sum_{k \in 2\mathbb{Z}} \left(f^{k,1} \varphi^1_k + f^{k,2} \varphi^2_k\right),$$

where $f^{k,i} = \langle f, \tilde{\varphi}^i_k \rangle$ for $i = 1, 2$.

Now we can define the inverse by

$$\begin{aligned} \left(\lambda - L^0_{bc}\right)^{-1}(f) &= \left(\lambda - L^0_{bc}\right)^{-1} \left(\sum_{k \in 2\mathbb{Z}} \left(f^{k,1} \varphi^1_k + f^{k,2} \varphi^2_k\right)\right) \\ &:= \sum_{k \in 2\mathbb{Z}} \left(f^{k,1} \tfrac{1}{\lambda - (\tau_1 + k)} \varphi^1_k + f^{k,2} \tfrac{1}{\lambda - (\tau_2 + k)} \varphi^2_k\right). \end{aligned}$$

Since this formula gives the algebraic inverse, it remains to show that this inverse operator is bounded. But we have

$$\begin{aligned} \left\| \left(\lambda - L^0_{bc}\right)^{-1} f \right\| &= \left\| \sum_{k \in 2\mathbb{Z}} \left(f^{k,1} \tfrac{1}{\lambda - (\tau_1 + k)} \varphi^1_k + f^{k,2} \tfrac{1}{\lambda - (\tau_2 + k)} \varphi^2_k\right) \right\| \\ &\leq \sum_{k \in 2\mathbb{Z}} \left\| f^{k,1} \tfrac{1}{\lambda - (\tau_1 + k)} \varphi^1_k \right\| + \sum_{k \in 2\mathbb{Z}} \left\| f^{k,2} \tfrac{1}{\lambda - (\tau_2 + k)} \varphi^2_k \right\| \\ &\leq \left(\sum_{k \in 2\mathbb{Z}} \left\|f^{k,1}\right\|^2 \left\|\varphi^1_k\right\|^2\right)^{1/2} \left(\sum_{k \in 2\mathbb{Z}} \tfrac{1}{|\lambda - (\tau_1 + k)|^2}\right)^{1/2} \\ &\quad + \left(\sum_{k \in 2\mathbb{Z}} \left\|f^{k,2}\right\|^2 \left\|\varphi^2_k\right\|^2\right)^{1/2} \left(\sum_{k \in 2\mathbb{Z}} \tfrac{1}{|\lambda - (\tau_2 + k)|^2}\right)^{1/2} \\ &\leq \left[\left(\sum_{k \in 2\mathbb{Z}} \tfrac{1}{|\lambda - (\tau_1 + k)|^2}\right)^{1/2} + \left(\sum_{k \in 2\mathbb{Z}} \tfrac{1}{|\lambda - (\tau_2 + k)|^2}\right)^{1/2}\right] \|A\| \left\|A^{-1}\right\| \|f\| \end{aligned}$$

where A is the operator defined in (??), and we get the last inequality by using (??). So it is only left to show the convergence of the series in the last part of the equality.

For this fixed λ, there is $n \in 2\mathbb{Z}$ such that

$$|Re\,(\lambda - (\tau_1 + n))| \leq 1,$$

since $|Re\,\tau_1| \leq 1$. This implies, for $k \neq n$

$$|\lambda - (\tau_1 + k)| \geq |Re\,(\lambda - (\tau_1 + n) + n - k)| \geq |n - k| - |Re\,(\lambda - (\tau_1 + n))|$$
$$\geq |n - k| - 1 \geq \frac{1}{2}\,|n - k|\,,$$

since n, k are even numbers. Now for the first series, we get

$$\begin{aligned} \sum_{k\in 2\mathbb{Z}} \tfrac{1}{|\lambda-(\tau_1+k)|^2} &= \tfrac{1}{|\lambda-\tau_1-n|^2} + \sum_{k\neq n} \tfrac{1}{|\lambda-\tau_1-k|^2} \\ &\leq \tfrac{1}{|\lambda-\tau_1-n|^2} + \sum_{k\neq n} \tfrac{1}{(|n-k|-1)^2} \\ &\leq \tfrac{1}{|\lambda-\tau_1-n|^2} + \sum_{k\neq n} \tfrac{2^2}{|n-k|^2}. \end{aligned}$$

So we have shown that the first series is convergent. Similar argument proves that the second series converges.

This proves the operator $(\lambda - L_{bc}^0)^{-1}$ is bounded if λ is not an eigenvalue of L_{bc}^0. This means that spectrum of L_{bc}^0 only contains its eigenvalues. So the proof is completed. $\square$

Now we consider the spectra localization of the operators $L_{bc} = L_{bc}^0 + V$, where V denotes the operator of multiplication by the matrix $v(x) = \begin{pmatrix} 0 & P(x) \\ Q(x) & 0 \end{pmatrix}$. We subdivide the complex plane $\mathbb{C}$ into the strips

$$H_m = \left\{ z \in \mathbb{C} : -1 \leq Re\left(z - m - \frac{\tau_1 + \tau_2}{2} \right) \leq 1 \right\}, \quad m \in 2\mathbb{Z},$$

and set

$$H^N = \bigcup_{|m|\leq N} H_m,$$

$$R_N = \left\{ z = x + it : \left| x - Re\frac{\tau_1 + \tau_2}{2} \right| < N + 1, |t| < N \right\},$$

where $N \in 2\mathbb{N}$. Let

$$\rho := \min\left(1 - |Re\,(\tau_1 - \tau_2)|\,/2, |\tau_1 - \tau_2|\,/2\right),$$

and

$$D_m^\mu = \{ z \in \mathbb{C} : |z - \tau_m - m| < \rho \}\,, \quad m \in 2\mathbb{Z}.$$

It is known that (see [?], Theorem 12.) for each strictly regular bc, there is an $N \in 2\mathbb{N}$ such that

$$Sp(L_{bc}) \subset R_N \cup \bigcup_{|n|>N} \left(D_n^1 \cup D_n^2\right).$$

Moreover, each disc D_n^α, $\alpha = 1, 2$, $|n| > N$ contains exactly one simple eigenvalue of L_{bc}, while R_N contains $2N$ eigenvalues of L_{bc}. Let us consider the Riesz projections associated with L_{bc}

$$S_N = \frac{1}{2\pi i} \int_{\partial R_N} (\lambda - L)^{-1} \, d\lambda, \quad P_{n,\alpha} = \frac{1}{2\pi i} \int_{\partial D_n^\alpha} (\lambda - L)^{-1} \, d\lambda, \quad \alpha = 1, 2, \quad (3.25)$$

and let S_N^0 and $P_{n,\alpha}^0$ be the Riesz projections associated with the free operator L_{bc}^0. Next we use the following theorem (see [?], Theorem 15).

Theorem 13. *Suppose L_{bc} and L_{bc}^0 are, respectively, the Dirac operator with an L^2 potential and the corresponding free Dirac operator, subject to the same strictly regular boundary conditions bc. Then, there is an $N \in 2\mathbb{N}$ such that the Riesz projections $S_N, P_{n,\alpha}$ and $S_N^0, P_{n,\alpha}^0, n \in 2\mathbb{Z}, |n| > N, \alpha = 1, 2,$ associated with L and L^0 are well defined by (??), and we have*

$$\dim P_{n,\alpha} = \dim P_{n,\alpha}^0 = 1, \quad \dim S_N = \dim S_N^0 = 2N; \quad (3.26)$$

$$\sum_{|n|>N} \left\| P_{n,\alpha} - P_{n,\alpha}^0 \right\|^2 < \infty, \quad \alpha = 1, 2, \quad (3.27)$$

$$\textit{If } S_N(x) = 0, \ P_{n,\alpha} = 0 \quad \forall n, \alpha \Rightarrow x = 0. \quad (3.28)$$

Let φ_n^α, $\alpha = 1, 2$ are unit eigenfunctions of the free Dirac operator L_{bc}^0 such that

$$L_{bc}^0 \varphi_n^\alpha = \lambda_{n,0}^\alpha \varphi_n^\alpha,$$

where $\lambda_{n,0}^\alpha = \tau_\alpha + n$. For $|n| > N$, set

$$\Psi_n^\alpha = P_{n,\alpha}(\varphi_n^\alpha).$$

Then Ψ_n^α are eigenvectors of the L_{bc} such that

$$L_{bc} \Psi_n^\alpha = \lambda_n^\alpha \Psi_n^\alpha,$$

where $\lambda_n^\alpha \in D_n^\alpha$. By using the previous theorem, we obtain that

$$\sum_{|n|>N} \|\Psi_n^\alpha - \varphi_n^\alpha\|^2 = \sum_{|n|>N} \left\| P_{n,\alpha}(\varphi_n^\alpha) - P_{n,\alpha}^0(\varphi_n^\alpha) \right\|^2 \le \sum_{|n|>N} \left\| P_{n,\alpha} - P_{n,\alpha}^0 \right\|^2 < \infty$$

since $\|\varphi_n^\alpha\| = 1$. Thus $(\Psi_n^\alpha, \ |n| > N, \ \alpha = 1, 2)$ forms a Riesz bases in its closed linear span.

We can write $H = L^2([0,\pi],\mathbb{C}^2)$ as direct sum of the spaces (not orthogonal) H_1 and H_0

$$H = H_0 \oplus H_1$$

where

$$H_0 = Ran(S_N), \quad H_1 = Ran(1 - S_N).$$

By (**??**), H_1 is the closed linear span of $(\Psi_n^\alpha,\ |n| > N,\ \alpha = 1,2)$, so $(\Psi_n^\alpha,\ |n| > N,\ \alpha = 1,2)$ forms a Riesz basis in H_1. H_0 is a finite dimensional invariant subspace . So we can choose a basis for H_0 consisting of root functions of L_{bc} corresponding to eigenvalues in R_N. Then the union of this chosen basis and $(\Psi_n^\alpha,\ |n| > N,\ \alpha = 1,2)$ forms a Riesz basis in $L^2([0,\pi],\mathbb{C}^2)$. Hence we have shown the existence of Riesz basis consisting of root functions of L_{bc}.

4 Adjoint of the Dirac operator

In this section, we find the adjoint operator of $L_{bc}(v)$ subject to regular boundary conditions.

We may assign to every boundary condition bc of the form (**??**) a corresponding operator $L_{bc}(v)$ as follows. Let

$$\begin{aligned} Dom\,(L_{bc}(v)) \quad = \quad & \Big\{ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} : y_1 \text{ and } y_2 \text{ are absolutely continuous}, y \text{ satisfies} \\ & \text{the boundary conditions } bc \text{ and } y_1', y_2' \in L^2([0,\pi]) \Big\}. \end{aligned}$$

and let

$$L_{bc}(v)y = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} + \begin{pmatrix} 0 & P(x) \\ Q(x) & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

In the following we assume that $P, Q \in L^2([0,\pi])$. By C_0^∞, we denote the set of all infinitely differentiable functions φ such that $supp(\varphi) \subset (0,\pi)$.

Lemma 14. *If $f \in L^2([a,b])$ and $\int_a^b \phi' \overline{f(x)} dx = 0$, for every $\phi \in C_0^\infty([a,b])$, then f is constant.*

Proof. Fix $\phi_0 \in C_0^\infty$ such that $\int_a^b \phi_0(t)dt = 1$. Let $\psi \in C_0^\infty$ and let

$$c = \int_a^b \psi(t)dt.$$

Then

$$\psi(x) - c\phi_0(x) = \phi'(x),$$

where

$$\phi(x) = \int_a^x (\psi(t) - c\phi_0(t))dt.$$

Observe that $\phi \in C_0^\infty$, so

$$\int_a^b \phi^{'}(t)\overline{f(t)}dt = 0.$$

Since $\psi(x) - c\phi_0(x) = \phi^{'}(x)$, we get

$$\int_a^b [\psi(t) - c\phi_0(t)]\overline{f(t)}dt = 0,$$

which gives

$$\int_a^b \psi(t)\overline{f(t)}dt = c\int_a^b \phi_0(t)\overline{f(t)}dt.$$

Now let

$$d = \int_a^b \phi_0(t)\overline{f(t)}dt.$$

So last equation means that

$$\int_a^b \psi(t)\overline{f(t)}dt = d\int_a^b \psi(t)dt,$$

which also means

$$\int_a^b \psi(t)[\overline{f(t)} - d]dt = 0, \quad \forall \psi \in C_0^\infty.$$

Since C_0^∞ is dense in $L^2([0, \pi])$, we have

$$\overline{f(t)} - d = 0.$$

Thus f is constant. $\square$

Theorem 15. *Let $L_{bc}(v)$ be the Dirac operator with boundary conditions bc given by (**??**). Then its adjoint operator $(L_{bc}(v))^*$ is $L_{bc^*}(v^*)$ where boundary conditions bc^* given by the system*

$$\overline{b}g_1(0) + g_1(\pi) + \overline{d}g_2(\pi) = 0$$
$$\overline{a}g_1(0) + g_2(0) + \overline{c}g_2(\pi) = 0,$$

and

$$v^* = \begin{pmatrix} 0 & \overline{Q} \\ \overline{P} & 0 \end{pmatrix}.$$

Proof. Let $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in Dom\left((L_{bc}(v))^*\right)$. Then there exists $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in L^2\left([0,\pi], \mathbb{C}^2\right)$ such that

$$\left\langle L\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\rangle, \quad \forall f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in Dom\left(L_{bc}(v)\right).$$

Since

$$L\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} f_1' \\ f_2' \end{pmatrix} + \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} if_1' + Pf_2 \\ -if_2' + Qf_1 \end{pmatrix},$$

we have

$$\left\langle \begin{pmatrix} if_1' + Pf_2 \\ -if_2' + Qf_1 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\rangle. \tag{4.1}$$

Therefore

$$\frac{1}{\pi}\int_0^\pi \left(\left[if_1'(x) + P(x)f_2(x)\right]\overline{g_1(x)} + \left[-if_2'(x) + Q(x)f_1(x)\right]\overline{g_2(x)}\right)dx$$

$$= \frac{1}{\pi}\int_0^\pi (f_1(x)\overline{h_1(x)} + f_2(x)\overline{h_2(x)})dx.$$

Let us define

$$H_1(x) = \int_0^x h_1(t)dt \quad \text{and} \quad H_2(x) = \int_0^x h_2(t)dt,$$

$$I_1(x) = \int_0^x \overline{P(t)}g_1(t)dt \quad \text{and} \quad I_2(x) = \int_0^x \overline{Q(t)}g_2(t)dt.$$

So if we plug in these functions in the last equation and do integration by parts, we get that

$$\int_0^\pi (if_1'(x)\overline{g_1(x)})dx - \int_0^\pi (if_2'(x)\overline{g_2(x)})dx + f_2(\pi)\overline{I_1(\pi)} - f_2(0)\overline{I_1(0)} - \int_0^\pi f_2'(x)\overline{I_1(x)}dx$$

$$+f_1(\pi)\overline{I_2(\pi)} - f_1(0)\overline{I_2(0)} - \int_0^\pi f_1'(x)\overline{I_2(x)}dx$$

$$= f_1(\pi)\overline{H_1(\pi)} - f_1(0)\overline{H_1(0)} - \int_0^\pi f_1'(x)\overline{H_1(x)}dx + f_2(\pi)\overline{H_2(\pi)} - f_2(0)\overline{H_2(0)} - \int_0^\pi f_2'(x)\overline{H_2(x)}dx.$$

This equality holds for every $f \in Dom\,(L_{bc}(v))$. Since $C_0^\infty \subset Dom\,(L_{bc}(v))$, we can take $f \in C_0^\infty$. Then $f_1(\pi) = f_2(\pi) = f_1(0) = f_2(0) = 0$. So we have

$$\int_0^\pi [f_1'(x)(i\overline{g_1(x)} - \overline{I_2(x)} + \overline{H_1(x)}) + f_2'(x)(-i\overline{g_2(x)} - \overline{I_1(x)} + \overline{H_2(x)})]dx = 0.$$

If we take $f_2(x) = 0$ and use the previous lemma, we get

$$-ig_1 - I_2 + H_1 = \text{ constant}.$$

And similarly if we take $f_1(x) = 0$, we get

$$ig_2 - I_1 + H_2 = \text{ constant}.$$

By taking derivatives of the last two equations, it follows that

$$h_1 = ig_1' + \overline{Q}g_2 \quad \text{ and } \quad h_2 = -ig_2' + \overline{P}g_1.$$

Thus we have found that

$$(L_{bc}(v))^* \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} ig_1' + \overline{Q}g_2 \\ -ig_2' + \overline{P}g_1 \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} g_1' \\ g_2' \end{pmatrix} + \begin{pmatrix} 0 & \overline{Q} \\ \overline{P} & 0 \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.$$

And also g_1 and g_2 are absolutely continuous functions, since H_1 and H_2 are absolutely continuous functions by their construction. Now, (**??**) becomes

$$\left\langle \begin{pmatrix} if_1' + Pf_2 \\ -if_2' + Qf_1 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} ig_1' + \overline{Q}g_2 \\ -ig_2' + \overline{P}g_1 \end{pmatrix} \right\rangle, \quad \forall f \in Dom(L_{bc}(v)).$$

Therefore

$$\int_0^\pi \left(\left[if_1'(x) + P(x)f_2(x) \right] \overline{g_1(x)} + \left[-if_2'(x) + Q(x)f_1(x) \right] \overline{g_2(x)} \right) dx$$

$$= \int_0^\pi \left(f_1(x)\overline{\left[ig_1'(x) + \overline{Q(x)}g_2(x) \right]} + f_2(x)\overline{\left[-ig_2'(x) + \overline{P(x)}g_1(x) \right]} \right) dx,$$

which gives us

$$\int_0^\pi \left[i f_1'(x)\overline{g_1(x)} + P(x) f_2(x)\overline{g_1(x)} - i f_2'(x)\overline{g_2(x)} + Q(x) f_1(x)\overline{g_2(x)} \right] dx$$

$$= \int_0^\pi \left[-i f_1(x)\overline{g_1'(x)} + f_1(x)Q(x)\overline{g_2(x)} + i f_2(x)\overline{g_2'(x)} + f_2(x)P(x)\overline{g_1(x)} \right] dx.$$

By canceling the terms which appear on both sides, we get

$$\int_0^\pi \left[\left(f_1'(x)\overline{g_1(x)} + f_1(x)\overline{g_1'(x)} \right) - \left(f_2'(x)\overline{g_2(x)} + f_2(x)\overline{g_2'(x)} \right) \right] dx = 0,$$

which also can be written as

$$\int_0^\pi \left[\frac{d}{dx}\left(f_1(x)\overline{g_1(x)} \right) - \frac{d}{dx}\left(f_2(x)\overline{g_2(x)} \right) \right] dx = 0.$$

Finally by evaluating the integral, we get the equation

$$f_1(\pi)\overline{g_1(\pi)} - f_1(0)\overline{g_1(0)} - f_2(\pi)\overline{g_2(\pi)} + f_2(0)\overline{g_2(0)} = 0. \tag{4.2}$$

We use boundary conditions bc of the form (**??**). First we write $f_1(0)$ and $f_2(\pi)$ in terms of $f_1(\pi)$ and $f_2(0)$, that is

$$f_1(0) = -b f_1(\pi) - a f_2(0)$$

$$f_2(\pi) = -d f_1(\pi) - c f_2(0).$$

If we plug in these two equations in (**??**), we get that

$$f_1(\pi)\overline{g_1(\pi)} - \left(-b f_1(\pi) - a f_2(0)\right)\overline{g_1(0)} - \left(-d f_1(\pi) - c f_2(0)\right)\overline{g_2(\pi)} + f_2(0)\overline{g_2(0)} = 0.$$

Therefore

$$f_1(\pi)\left[\overline{g_1(\pi)} + \overline{b g_1(0)} + \overline{d g_2 \pi)}\right] + f_2(0)\left[\overline{a g_1(0)} + \overline{c g_2(\pi)} + \overline{g_2(0)}\right] = 0.$$

And this identity holds for every $f \in Dom(L^0_{bc}(v))$. We can find an f such that $f_1(\pi) = 1$ and $f_2(0) = 0$. Similarly we can find an f such that $f_1(\pi) = 0$ and $f_2(0) = 1$. So boundary conditions of the adjoint operator $(L_{bc}(v))^*$ are given by the equations

$$\overline{b} g_1(0) + g_1(\pi) + \overline{d} g_2(\pi) = 0$$

$$\overline{a} g_1(0) + g_2(0) + \overline{c} g_2(\pi) = 0. \tag{4.3}$$

□

Let bc^* be the boundary conditions defined by (**??**). It is associated with the matrix

$$\begin{pmatrix} \overline{b} & 1 & 0 & \overline{d} \\ \overline{a} & 0 & 1 & \overline{c} \end{pmatrix},$$

so bc^* is not in the canonical form (**??**). In order to get that form, we multiply this matrix from the left by

$$\begin{pmatrix} \overline{b} & \overline{d} \\ \overline{a} & \overline{c} \end{pmatrix}^{-1} = \frac{1}{\overline{bc} - \overline{da}} \begin{pmatrix} \overline{c} & -\overline{d} \\ -\overline{a} & \overline{b} \end{pmatrix},$$

and we get

$$\begin{pmatrix} 1 & \frac{\overline{c}}{\overline{bc}-\overline{da}} & -\frac{\overline{d}}{\overline{bc}-\overline{da}} & 0 \\ 0 & -\frac{\overline{a}}{\overline{bc}-\overline{da}} & \frac{\overline{b}}{\overline{bc}-\overline{da}} & 1 \end{pmatrix}. \tag{4.4}$$

The system associated with this matrix gives us an equivalent system to (**??**).

So throughout this proof, we have also shown that

$$\begin{aligned} Dom((L_{bc}(v))^*) \quad = \quad & \left\{ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} : y_1 \text{ and } y_2 \text{ are absolutely continuous}, y \text{ satisfies} \right. \\ & \left. \text{the boundary conditions } bc^* \text{ and } y_1', y_2' \in L^2([0,\pi]) \right\}. \end{aligned}$$

where bc^* is given by (**??**) when bc is of the form (**??**). And also bc^* is equivalent to the boundary condition associated with the matrix (**??**).

Corollary 16. *The operator L_{bc} is closed.*

Proof. By the previous theorem, we have that

$$(L_{bc}(v))^* = L_{bc^*}(v^*),$$

and

$$(L_{bc^*}(v^*))^* = L_{bc^{**}}(v).$$

But we also have

$$((L_{bc}(v))^*)^* = \overline{L_{bc}(v)}.$$

Recall that we consider bc given by the matrix (**??**) and corresponding bc^* is given by the matrix (**??**). We have

$$\det \begin{pmatrix} \frac{\overline{c}}{\overline{bc}-\overline{da}} & -\frac{\overline{d}}{\overline{bc}-\overline{da}} \\ -\frac{\overline{a}}{\overline{bc}-\overline{da}} & \frac{\overline{b}}{\overline{bc}-\overline{da}} \end{pmatrix} = \frac{\overline{cb} - \overline{da}}{\left(\overline{bc} - \overline{da}\right)^2} = \frac{1}{\overline{bc} - \overline{da}}.$$

So bc^{**} is given by the matrix

$$\begin{pmatrix} 1 & b & a & 0 \\ 0 & d & c & 1 \end{pmatrix}.$$

Then bc^{**} and bc are given by the same matrix. Thus

$$\overline{L_{bc}(v)} = ((L_{bc}(v))^*)^* = L_{bc^{**}}(v) = L_{bc}(v).$$

Hence $L_{bc}(v)$ is a closed operator. $\square$

Let L^0_{00} be the free Dirac operator with domain

$$Dom(L^0_{00}) = \left\{ f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : \ f_1, f_2 \text{ are absolutely continuous functions,} \right.$$
$$\left. f_1^{'}, f_2^{'} \in L^2\left([0,\pi]\right), f_1(0) = f_1(\pi) = 0, f_2(0) = f_2(\pi) = 0 \right\}.$$

The same argument that we use in the proof of Theorem 9 shows that L^0_{00} is closed. L^0_{00} is obviously a densely defined operator with this domain.

Now we find the adjoint of L^0_{00}. Exactly with the same calculations done for finding adjoint of L_{bc}, we can show that L^0_{00} is a symmetric operator.

Let $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in Dom((L^0_{00})^*)$. Then for all $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in Dom(L^0_{00})$, again by he calculations done before, we obtain

$$f_1(\pi)\overline{g_1(\pi)} - f_1(0)\overline{g_1(0)} - f_2(\pi)\overline{g_2(\pi)} + f_2(0)\overline{g_2(0)} = 0.$$

Since $f \in Dom(L^0_{00})$, we have $f_1(0), f_1(\pi), f_2(0), f_2(\pi) = 0$. So the last equality holds for any $g \in Dom\left((L^0_{00})^*\right)$. Thus adjoint of L^0_{00} is the free Dirac operator with domain

$$Dom\left((L^0_{00})^*\right) = \left\{ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} : y_1 \text{ and } y_2 \text{ are absolutely continuous}, y_1', y_2' \in L^2([0,\pi]) \right\}.$$

5 Self-adjoint Dirac Operators and Self-adjoint bc

In this section, we study self-adjoint boundary conditions and self-adjoint Dirac operators.

Recall that a densely defined unbounded operator A is called self-adjoint if $A = A^*$, that is $domA = domA^*$ and $A^*f = Af$ for every $f \in DomA$. So if A satisfies some boundary conditions bc, then A^* must also satisfy the same bc.

We have seen that if the boundary conditions of $L_{bc}(v)$ is given by the matrix

$$\begin{pmatrix} 1 & b & a & 0 \\ 0 & d & c & 1 \end{pmatrix},$$

then the boundary conditions bc^* of $(L_{bc}(v))^*$ is given by the matrix

$$\begin{pmatrix} 1 & \frac{\bar{c}}{\overline{bc - da}} & -\frac{\bar{d}}{\overline{bc - da}} & 0 \\ 0 & -\frac{\bar{a}}{\overline{bc - da}} & \frac{\bar{b}}{\overline{bc - da}} & 1 \end{pmatrix}.$$

So if the operator $L_{bc}(v)$ is self-adjoint, then

$$\begin{pmatrix} b & a \\ d & c \end{pmatrix} = \begin{pmatrix} \frac{\overline{c}}{\overline{bc-da}} & -\frac{\overline{d}}{\overline{bc-da}} \\ -\frac{\overline{a}}{\overline{bc-da}} & \frac{\overline{b}}{\overline{bc-da}} \end{pmatrix}.$$

The determinants of these two matrices must be equal, that is

$$\Delta = bc - ad = \frac{\overline{cb - da}}{\left(\overline{bc - da}\right)^2},$$

which gives us

$$|\Delta| = |bc - ad| = 1.$$

Therefore

$$|b| = \left|\frac{\overline{c}}{\overline{bc - da}}\right| = |c| \qquad \text{and} \qquad |a| = \left|-\frac{\overline{d}}{\overline{bc - da}}\right| = |d|\,.$$

Since $|\Delta| = 1$, let

$$\Delta = e^{i\theta}, \quad a = \rho e^{i\alpha} \quad \text{and} \quad b = |b|\, e^{i\beta},$$

for some $\alpha, \beta, \theta \in [0, 2\pi)$ and $\rho \in \mathbb{R}^+$. Then

$$|\Delta| = |bc - ad| = \left|\frac{b\overline{b} + a\overline{a}}{\overline{\Delta}}\right| = \left||b|^2 + |\rho|^2\right| = 1,$$

so

$$|b| = \sqrt{1 - \rho^2}.$$

Then

$$\Delta = e^{i\theta}, \quad a = \rho e^{i\alpha} \quad \text{and} \quad b = \sqrt{1-\rho^2}\, e^{i\beta}, \tag{5.1}$$

for some $\alpha, \beta, \theta \in [0, 2\pi)$ and $\rho \in (0, 1)$. Therefore

$$c = \frac{\overline{b}}{\overline{\Delta}} = \overline{b}\Delta = \sqrt{1-\rho^2}\, e^{i(\theta-\beta)}, \tag{5.2}$$

$$d = -\frac{\overline{a}}{\overline{\Delta}} = -\overline{a}\Delta = -\rho e^{i(\theta-\alpha)}. \tag{5.3}$$

Now assume that bc is given by the matrix (**??**) and a, b, c and d satisfies (**??**), (**??**) and (**??**). Then the terms of the matrix (**??**) becomes

$$\frac{\overline{c}}{\overline{\Delta}} = \overline{c}\Delta = \sqrt{1-\rho^2}\, e^{i(\beta-\theta)}e^{i\theta} = \sqrt{1-\rho^2}\, e^{i\beta} = b,$$

$$-\frac{\overline{d}}{\overline{\Delta}} = -\overline{d}\Delta = \rho e^{i(\alpha-\theta)}e^{i\theta} = \rho e^{i\alpha} = a,$$

$$-\frac{\overline{a}}{\overline{\Delta}} = -\overline{a}\Delta = -\frac{(-d\overline{\Delta})}{\overline{\Delta}} = d, \quad \text{since } a = -\overline{d}\Delta,$$

$$\frac{\overline{b}}{\overline{\Delta}} = \overline{b}\Delta = \frac{(c\overline{\Delta})}{\overline{\Delta}} = c, \quad \text{since } b = \overline{c}\Delta.$$

Thus bc and bc^* are equal which means bc is self-adjoint. By this argument the following proposition holds.

Proposition 17. *If bc is a self-adjoint boundary condition given by the matrix* (**??**), *then there are uniquely determined numbers* $\alpha, \beta, \theta \in [0, 2\pi]$ *and* $\rho \in (0, 1)$ *such that* (**??**), (**??**) *and* (**??**) *hold. And conversely, if* a, b, c, d *are given by* (**??**), (**??**) *and* (**??**), *then the matrix* (**??**) *determines self-adjoint boundary condition.*

Thus if $L_{bc}(v)$ subject to boundary conditions bc given by the matrix (**??**) is self-adjoint, then $\overline{Q} = P$ and there are uniquely determined numbers $\alpha, \beta, \theta \in [0, 2\pi]$ and $\rho \in (0, 1)$ such that (**??**),(**??**) and (**??**) hold. And conversely, if $L_{bc}(v)$ is subject to boundary conditions bc given by the matrix (**??**) such that a, b, c, d are given by (**??**),(**??**),(**??**) and $\overline{Q} = P$, then $L_{bc}(v)$ is self-adjoint.

6 Self-adjoint Extensions

In this section, we give a characterization of self-adjoint extensions of an unbounded operator. Then we find all self-adjoint extensions of L^0_{00}, corresponding to partial isometries which can be represented by real-valued matrices.

Definition 18. *Let A be a closed symmetric operator. The deficiency subspaces of A are the spaces*

$$L_+ = \ker(A^* - i) = [ran\,(A + i)]^\perp\,,$$

$$L_- = \ker(A^* + i) = [ran\,(A - i)]^\perp\,.$$

The deficiency indices of A are the numbers $n_+ = \dim L_+$ *and* $n_- = \dim L_-$.

Definition 19. *A partial isometry is an operator W such that for h in* $(\ker W)^\perp$, $\|Wh\| = \|h\|$. *The space* $(\ker W)^\perp$ *is called the initial space of W and the space* $ran\,W$ *is called the final space of W.*

The following theorem is well-known(see [**?**], Theorem 2.17 or [**?**], Theorem X.2).

Theorem 20. *Let A be a closed symmetric operator. If W is a partial isometry with initial space in* L_+ *and final subspace in* L_-, *let*

$$D_W = \{f + g + Wg :\ f \in dom\,(A)\,, g \in initial\ W\} \tag{6.1}$$

and define A_W *on* D_W *by*

$$A_W\,(f + g + Wg) = Af + ig - iWg. \tag{6.2}$$

Then A_W *is a closed symmetric extension of A. Conversely, if B is any closed symmetric extension of A, then there is a unique partial isometry W such that* $B = A_W$ *as in* (**??**).

So this theorem gives one to one correspondence between closed symmetric extensions of a closed symmetric operator A and partial isometries with initial space in L_+ and final space in L_-. Moreover, it is known that if $n_+ = n_-$, then the set of self-adjoint extensions is in natural correspondence with the set of isomorphisms of L_+ and L_-, respectively(see [?], Theorem 2.20).

Now we find self-adjoint extensions of L_{00}^0. First we find the deficiency subspaces of L_{00}^0. Let $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \ker((L_{00}^0)^* + i)$. Then

$$((L_{00}^0)^* + i)f = \begin{pmatrix} if_1' + if_1 \\ -if_2' + if_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Solving these two differential equations, we get that for some constants c_1 and c_2

$$f_1(x) = c_1 e^{\pi - x} \quad \text{and} \quad f_2(x) = c_2 e^x.$$

We choose $f_1(x) = c_1 e^{\pi - x}$ instead of $f_1(x) = c_1 e^{-x}$, since

$$\left\| e^{\pi - x} \right\| = \frac{1}{\pi} \int_0^\pi e^{2(\pi - x)} dx = \frac{1}{\pi} \int_0^\pi e^{2x} dx = \|e^x\|.$$

So $n_- = 2$ since

$$\begin{aligned} L_- = \ker((L_{00}^0)^* + i) &= \left\{ \begin{pmatrix} c_1 e^{\pi - x} \\ c_2 e^x \end{pmatrix}, c_1 \text{ and } c_2 \text{ are constants} \right\} \\ &= \left\{ c_1 \begin{pmatrix} e^{\pi - x} \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e^x \end{pmatrix}, c_1 \text{ and } c_2 \text{ are constants} \right\}. \end{aligned}$$

Now let $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \ker((L_{00}^0)^* - i)$. Then

$$((L_{00}^0)^* - i)f = \begin{pmatrix} if_1' - if_1 \\ -if_2' - if_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Similarly solving these two differential equations, we get that for some constants c_1 and c_2

$$f_1(x) = c_1 e^x \quad \text{and} \quad f_2(x) = c_2 e^{\pi - x}.$$

So $n_+ = 2$ since

$$\begin{aligned} L_+ = \ker((L_{00}^0)^* - i) &= \left\{ \begin{pmatrix} c_1 e^x \\ c_2 e^{\pi - x} \end{pmatrix}, c_1 \text{ and } c_2 \text{ are constants} \right\} \\ &= \left\{ c_1 \begin{pmatrix} e^x \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e^{\pi - x} \end{pmatrix}, c_1 \text{ and } c_2 \text{ are constants} \right\}. \end{aligned}$$

Consider the isometries between L_+ and L_-. Since $n_- = n_+ = 2$, they can be represented by 2x2 matrices. Let

$$e^1 = \begin{pmatrix} e^x \\ 0 \end{pmatrix}, e^2 = \begin{pmatrix} 0 \\ e^{\pi - x} \end{pmatrix}, \phi^1 = \begin{pmatrix} e^{\pi - x} \\ 0 \end{pmatrix} \text{ and } \phi^2 = \begin{pmatrix} 0 \\ e^x \end{pmatrix}.$$

Then $\|e^1\| = \|e^2\| = \|\phi^1\| = \|\phi^2\|$. Let $W : L_+ \to L_-$ be an isometry such that

$$We^1 = w_{11}\phi^1 + w_{21}\phi^2,$$
$$We^2 = w_{12}\phi^1 + w_{22}\phi^2.$$

Further we identify W by its matrix representation

$$W = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}.$$

Since $We^1 \perp We^2$, we have

$$w_{11}\overline{w_{12}} + w_{21}\overline{w_{22}} = 0.$$

W is an isometry, so $\|We^1\| = \|e^1\|$ and $\|We^2\| = \|e^2\|$ which gives

$$|w_{11}|^2 + |w_{21}|^2 = 1,$$
$$|w_{12}|^2 + |w_{22}|^2 = 1.$$

By the equations above, about the entries of W,

$$\begin{pmatrix} w_{11} & w_{21} \\ w_{12} & w_{22} \end{pmatrix}^{-1} = \begin{pmatrix} \overline{w_{11}} & \overline{w_{12}} \\ \overline{w_{21}} & \overline{w_{22}} \end{pmatrix}.$$

So

$$\det \begin{pmatrix} w_{11} & w_{21} \\ w_{12} & w_{22} \end{pmatrix} \det \begin{pmatrix} \overline{w_{11}} & \overline{w_{12}} \\ \overline{w_{21}} & \overline{w_{22}} \end{pmatrix} = 1$$

which means

$$\det W \cdot \overline{detW} = 1 \quad \text{and} \quad |\det W| = 1.$$

By the previous theorem, we have a self-adjoint extension B_W of L_{00}^0 corresponding to each isometry W and

$$Dom(B_W) = \left\{ f + g + Wg : \ f \in Dom(L_{00}^0), g \in L_+ \right\}.$$

Next we show that the functions in $Dom(B_W)$ satisfy certain $bc = bc(W)$ that are uniquely determined if given by the matrix of the form (**??**). So next we look for a matrix

$$\begin{pmatrix} 1 & b & a & 0 \\ 0 & d & c & 1 \end{pmatrix}$$

such that every $h \in Dom(B_W)$ satisfies the bc defined by that matrix.
Let $f \in Dom(L_{00}^0)$ and $g \in L_+$ such that

$$g = c_1 e^1 + c_2 e^2 = \begin{pmatrix} c_1 e^x \\ c_2 e^{\pi - x} \end{pmatrix}.$$

Then

$$Wg = c_1 We^1 + c_2 We^2$$

and

$$f + g + Wg = f + c_1(e^1 + We^1) + c_2(e^2 + We^2).$$

Let

$$k := e^1 + We^1 = \begin{pmatrix} e^x + w_{11}e^{\pi - x} \\ w_{21}e^x \end{pmatrix},$$

$$l := e^2 + We^2 = \begin{pmatrix} w_{12}e^{\pi - x} \\ e^{\pi - x} + w_{22}e^x \end{pmatrix}.$$

Since $(f + g + Wg) \in Dom(B_W)$, it must satisfy the boundary conditions. Since $f \in Dom(L^0_{bc})$ and boundary conditions are given by linear equations, if we let $c_1 = 1$ and $c_2 = 0$, then k must satisfy the boundary conditions of B_W. So

$$k_1(0) + bk_1(\pi) + ak_2(0) = 0$$

$$dk_1(\pi) + ck_2(0) + k_2(\pi) = 0$$

which means

$$1 + w_{11}e^{\pi} + b(e^{\pi} + w_{11}) + aw_{21} = 0$$

$$d(e^{\pi} + w_{11}) + cw_{21} + w_{21}e^{\pi} = 0.$$

Similarly if we take $c_2 = 1$ and $c_1 = 0$, then l must also satisfy the boundary conditions of B_W. So

$$l_1(0) + bl_1(\pi) + al_2(0) = 0$$

$$dl_1(\pi) + cl_2(0) + l_2(\pi) = 0$$

which means

$$w_{12}e^{\pi} + bw_{21} + a(e^{\pi} + w_{22}) = 0$$

$$dw_{21} + c(e^{\pi} + w_{22}) + 1 + w_{22}e^{\pi} = 0.$$

By solving these equations for a, b, c, d, we get that

$$a = \frac{w_{12}(e^{2\pi} - 1)}{\Delta},$$

$$b = -\frac{w_{21}w_{12}e^{\pi} - (e^{\pi} + w_{22})(1 + w_{11}e^{\pi})}{\Delta},$$

$$c = \frac{w_{12}w_{21}e^{\pi} - (e^{\pi} + w_{11})(1 + w_{22}e^{\pi})}{\Delta},$$

$$d = \frac{w_{21}(e^{2\pi} - 1)}{\Delta},$$

where $\Delta = w_{21}w_{12} - (e^{\pi} + w_{11})(e^{\pi} + w_{22})$.

So for each isometry W, we found corresponding boundary conditions bc given by the matrix

$$\begin{pmatrix} 1 & b & a & 0 \\ 0 & d & c & 1 \end{pmatrix}$$

where a, b, c, d are uniquely determined by the equalities above and every $h \in Dom(B_W)$ satisfy bc.

Now we consider the partial case where the entries of the matrix W are real numbers. Then W can be written in the form

$$W = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

for some $\theta \in [0, 2\pi)$, where $\det W = \cos^2\theta + \sin^2\theta = 1$. Now if boundary conditions of B_W are given by the matrix (**??**), then the numbers a, b, c, d becomes

$$a = \frac{\sin\theta(e^{2\pi} - 1)}{\Delta},$$

$$b = \frac{2e^{\pi} + \cos\theta + \cos\theta e^{2\pi}}{\Delta},$$

$$c = b \quad \text{and} \quad d = -a$$

where $\Delta = -1 - e^{2\pi} - 2e^{\pi}\cos\theta$.

The boundary conditions given by the matrix (**??**) where a, b, c, d are given by the above equalities is self adjoint if $bc - ad = 1$, by the proposition about self-adjoint boundary conditions. But

$$bc - ad = b^2 + a^2 = \frac{\left(2e^{\pi} + \cos\theta(e^{2\pi} + 1)\right)^2 + \sin^2\theta\left(e^{2\pi} - 1\right)^2}{\left(1 + e^{2\pi} + 2e^{\pi}\cos\theta\right)^2}$$

$$= \frac{4e^{2\pi} + 4e^{\pi}\cos\theta(e^{2\pi} + 1) + \cos^2\theta(e^{2\pi} + 1)^2 + \sin^2\theta(e^{2\pi} - 1)^2}{(e^{2\pi} + 1)^2 + 4e^{\pi}\cos\theta(e^{2\pi} + 1) + 4e^{2\pi}\cos^2\theta}.$$

Since

$$\cos^2\theta(e^{2\pi}+1)^2+\sin^2\theta(e^{2\pi}-1)^2 = e^{4\pi}\cos^2\theta+2e^{2\pi}\cos^2\theta+\cos^2\theta+e^{4\pi}\sin^2\theta-2e^{2\pi}\sin^2\theta+\sin^2\theta$$

$$= e^{4\pi} + 1 + 2e^{2\pi}(2\cos^2\theta - 1),$$

we get

$$bc - ad = \frac{4e^{2\pi} + 4e^{\pi}\cos\theta(e^{2\pi} + 1) + 1 + e^{4\pi} + 4e^{2\pi}\cos^2\theta - 2e^{2\pi}}{e^{4\pi} + 2e^{2\pi} + 1 + 4e^{\pi}\cos\theta(e^{2\pi} + 1) + 4e^{2\pi}\cos^2\theta} = 1.$$

Thus the boundary conditions bc corresponding to the isometry W is self-adjoint.

Recall that the boundary condition given by the matrix (**??**) is strictly regular if

$$(b - c)^2 + 4ad \neq 0.$$

We have found the form of the boundary conditions bc of B_W, the self-adjoint extension of L_{00}^0 which corresponds to an isometry W defined by a real-valued matrix. If these boundary conditions are not strictly regular then

$$(b - c)^2 + 4ad = -4a^2 = -4\frac{\sin^2\theta\left(e^{2\pi-1}\right)^2}{\Delta^2} = 0,$$

which implies

$$\sin\theta = 0.$$

So there are two cases, either $\theta = 0$ or $\theta = \pi$. First case is that $\theta = 0$. Then the matrix which gives the boundary conditions becomes

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

This boundary condition is called periodic since we have

$$y_1(0) = y_1(\pi),$$

$$y_2(0) = y_2(\pi).$$

In the second case $\theta = \pi$. Then the matrix which gives the boundary conditions becomes

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

This boundary condition is called anti-periodic since we have

$$y_1(0) = -y_1(\pi),$$

$$y_2(0) = -y_2(\pi).$$

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