# A SURVEY ON THE CAUCHY PROBLEM FOR THE KORTEWEG-DE VRIES EQUATION

by

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*To all the mathematicians before me,*

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## Abstract

In this thesis, we study the Cauchy problem for the classic Korteweg-de Vries equation

$$u_t + u_x + uu_x + u_{xxx} = 0 \quad \text{for } x \in \mathbb{R},\, t > 0$$
$$u(x, 0) = u_0(x) \quad \text{for } x \in \mathbb{R}$$

describing the propagation of long waves in shallow waters. We first use Bona and colleagues' approach of adding a regularizing term to the equation and show that the equation is well-posed for initial data $u_0 \in H^s$, $s \geq 3$, with solution lying in this space for each $t$ globally. We then use Kato's methods of semigroup theory in nonlinear study to lower the bound on $s$ to $s > 3/2$ for local solutions and to $s \geq 2$ for global solutions.

KORTEWEG-DE VRIES DENKLEMİ İÇİN CAUCHY PROBLEMİ ÜZERİNE BİR DERLEME

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## Özet

Bu tezde, sığ sulardaki uzun dalgaların davranışını ifade eden klasik Korteweg-de Vries denklemi için Cauchy problemi

$$u_t + u_x + uu_x + u_{xxx} = 0 \quad \text{for } x \in \mathbb{R},\, t > 0$$
$$u(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R}$$

incelenmektedir. İlk olarak Bona ve çalışma arkadaşlarının denkleme düzenleyici bir terim ekleme yaklaşımını kullanarak, başlangıç verisi $u_0 \in H^s$ $(s \geq 3)$ için problemin iyi konulmuş olduğunu ve çözümün global bir şekilde bu uzayda yer aldığını gösterdik. Ardından, $s$ üzerindeki sınırı yerel çözümler için $s > 3/2$, global çözümler içinse $s \geq 2$ olacak şekilde düşürmek için doğrusal olmayan durumlardaki yarı-grup teorisinde Kato'nun metodlarını kullandık.

# Table of Contents

# 1

# Preliminaries

This chapter is mainly devoted to the description of the problem we have at hand, and of tools we are going to use throughout the text.

We desire to grasp a basic comprehension of the Cauchy problem for the *Korteweg–de Vries equation*:

$$u_t() + u_x + u_{xxx} + uu_x = 0 \quad \text{for } x \in \mathbb{R}, t \geq 0 \tag{1.1a}$$

$$u(x,0) = f(x) \quad \text{for } x \in \mathbb{R} \tag{1.1b}$$

which was named after Dutch mathematicians D. J. Korteweg and G. de Vries, who tried to describe propagation of surface waves with long wavelengths in in shallow waters, that is, where depth of the water is comparable to the amplitude of the wave. This phenomenon is first observed by J. S. Russell, and problem he brought forth is then studied by Lord Rayleigh and J. V. Boussinesq. Though waves are traditionally studied via transport equation $u_t - cu_x = 0$ or wave equation $u_{tt} - c^2 \triangle_x u = 0$, there is the unmentioned assumption of amplitude being small compared to equilibrium depth. Lack of this assumption is what led mathematicians to look for different approaches regarding the problem. It is worth the time to note that Russell made his observations and experiments in 1834, Korteweg and de Vries published their paper in 1895, but equation became a topic of wide interest only in 1965, after a paper by M. Kruskal and N. Zabusky [10]. A more detailed investigation of history of the equation can be found in [7].

An important feature of the equation is nonlinearity added by the $uu_x$ term. Classically we would consider problem in $C^2(\mathbb{R} \times \mathbb{R}^+)$, but working in this space with aim to get a solution formula is quite hard. Turning our attention to showing whether Cauchy problem is well-posed or not, however, lets us use a variety of analytical tools, mainly functional analysis on various differential function spaces.

**Definition 1.1.** For domain $U \subset \mathbb{R}^n$, the function space $L^p(U)$ is defined as

$$L^p(U) = \left\{ f : U \mapsto \mathbb{R} \mid \int_U |f(x)|^p \, dx < \infty \right\}.$$

*Remark.* Note that, with this norm $\|f\|_p^p = \int_U |f(x)|^p\,dx$, $L^p(U)$ becomes a Banach space. Moreover, for $p = 2$, $L^2(U)$ is a Hilbert space with inner product

$$\langle f, g\rangle_{L^2(U)} = \int_U f(x)g(x)\,dx$$

Now, since what we have at hand is a differential equation, we need to be able to talk of derivatives of functions $f \in L^p(U)$. However, since functions in $L^p$ are only defined almost-everywhere, we need to step away from our usual calculus approach.

**Definition 1.2.** We say $f \in L^p(U)$ is *the $\alpha^{th}$ weak derivative* of $F \in L^p(U)$ if $f$ satisfies

$$\int F D^\alpha \phi\,dx = (-1)^{|\alpha|} \int f\phi\,dx \quad \forall \phi \in C_o^\infty(U)$$

where $C_o^\infty(U)$ is space of infinitely-differentiable functions that are identically 0 outside a compact subset of $U$. In this case, we denote $f$ by $F^{(\alpha)}$. Here, $\alpha$ is a multi-index $\alpha = (\alpha_1, \alpha_2, \dots \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \sum_{i=1}^n \alpha_i$ and $D^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\dots\partial_{x_n}^{\alpha_n}$.

Functions that do have weak derivatives up to an order $k$ are set in *Sobolev spaces*:

**Definition 1.3.** $W^{k,p}(U) = \{f \in L^p(U) \mid f^{(\alpha)} \in L^p(U), |\alpha| \le k\}$ is the Sobolev space of index $(k, p)$, where $f^{(\alpha)}$ denotes the $\alpha^{\text{th}}$ weak derivative of $f$. The norm

$$\|f\|_{W^{k,p}(U)}^p = \sum_{|\alpha|\le k} \|f^{(\alpha)}\|_{L^p(U)}^p$$

makes $W^{k,p}(U)$ a Banach space. We also define $W^{\infty,p}(U) = \bigcap_{k=0}^\infty W^{k,p}(U)$, however there is no norm structure on this space.

Proof of following theorem can be found in [5], Section 5.6.

**Theorem 1.4** (General Sobolev Inequalities)**.** *For $U \subset \mathbb{R}^n$ with sufficiently smooth boundary, and $u \in W^{k,p}(U)$, we have*

1. *If $kp < n$, then $u \in L^q(U)$ where $q = \frac{np}{n-kp}$. We also have the norm estimate*

   $$\|u\|_{L^q(U)} \le C\|u\|_{W^{k,p}(U)}$$

   *for some constant $C$ independent of $u$.*

2. *If $kp > n$, then $u$ is in Hölder space $C^{k-\lfloor n/p\rfloor-1,\gamma}(\bar{U})$ where $\gamma$ is*

   $$\gamma = \begin{cases} \lfloor n/p\rfloor + 1 - n/p, & \text{if } n/p \text{ is not an integer} \\ \text{any positive number} < 1, & \text{if } n/p \text{ is an integer} \end{cases}$$

   *We also have the norm estimate*

   $$\|u\|_{C^{k-\lfloor n/p\rfloor-1,\gamma}(\bar{U})} \le C\|u\|_{W^{k,p}(U)}$$

   *for constant $C$ independent of $u$.*

*Remark.* For formal definition of Hölder spaces see [5], Section 5.1. It is important to note that $C^{k+1}(\bar{U}) \subset C^{k,\gamma}(\bar{U}) \subset C^k(\bar{U})$ for every positive integer $k$, $0 < \gamma < 1$, and that these embeddings are continuous.

*Remark.* In our study, $n = 1$, $p = 2$ and $k \geq 1$ is an integer. With that perspective, theorem gives us that if $u \in W^{k,2}(U)$, then $u \in C^{k-1,1/2}(\bar{U})$.

*Remark.* From this point onwards, we will have $U = \mathbb{R}$, $p = 2$, and instead of $W^{k,2}(\mathbb{R})$ will write $H^k$, which is, like $L^2(\mathbb{R})$, a Hilbert space. Moreover, whenever it is clear from the context, $\|u\|$ and $\|u\|_k$ will be used instead of $\|u\|_{L^2}$ and $\|u\|_{H^k}$, respectively.

**Lemma 1.5.** *For $f, g \in H^k$ where $k \geq 1$ is an integer,*

1. *$f \in L^\infty(\mathbb{R})$ and $\|f\|_\infty \leq \|f\|_1$, where $\|\cdot\|_\infty$ denotes the $L^\infty$ norm of the function.*
2. *$fg \in H^k$ and $\|fg\|_k \leq C_K \|f\|_k \|g\|_k$ where $C_k$ is a constant depending only on $k$.*

*Proof.* 1. For all $x \in \mathbb{R}$,

$$(f(x))^2 = \int_{-\infty}^{x} f'(s)f(s)\,ds - \int_{x}^{\infty} f'(s)f(s)\,ds$$
$$\leq \int_{-\infty}^{\infty} 2|f(s)||f'(s)|\,ds \leq \int_{-\infty}^{\infty} |f(s)|^2 + |f'(s)|^2\,ds = \|f\|_1^2,$$

showing that $|f(x)| \leq \|f\|_1$ uniformly, and taking supremum over $\mathbb{R}$, we see that $\|f\|_\infty \leq \|f\|_1$.

2. Since $f, g \in H^k$, we have $f^{(i)}, g^{(i)} \in H^1$ for $0 \leq i \leq k-1$, and by first part of the lemma, $\|f^{(i)}\|_\infty \leq \|f^{(i)}\|_1 \leq \|f\|_k$, and similarly for $g$. Now, for $i \leq k-1$, $j \leq k$,

$$\|f^{(i)}g^{(j)}\|^2 = \int \left[(f^{(i)}(x))(g^{(j)}(x))\right]^2 dx \leq \|f^{(i)}\|_\infty^2 \int (g^{(j)}(x))^2\,dx \leq \|f\|_k^2 \|g\|_k^2.$$

Now we can use that $\|f^{(i)}g^{(j)}\| \leq \|f\|_k \|g\|_k$ to bound $\|(fg)^{(\alpha)}\|$.

$$\|(fg)^{(\alpha)}\| = \left\| \sum_{i=0}^{\alpha} C_i f^{(i)} g^{(\alpha-i)} \right\| \leq \sum_{i=0}^{\alpha} C_i \|f^{(i)} g^{(\alpha-i)}\| \leq \sum_{i=0}^{\alpha} C_i \|f\|_k \|g\|_k.$$

To finish the proof, look at $\|fg\|_k$.

$$\|fg\|_k^2 = \sum_{\alpha=0}^{k} \|(fg)^{(\alpha)}\|^2 \leq \sum_{\alpha=0}^{k} C_\alpha \|f\|_k^2 \|g\|_k^2 = C_k \|f\|_k^2 \|g\|_k^2.$$

In the end we got $\|fg\|_k \leq \sqrt{C_k} \|f\|_k \|g\|_k$, which was our aim.

□

**Corollary 1.6.** *The mapping $O(u) = u^2$ is locally Lipschitz on $H^k$.*

*Proof.* From Lemma 1.5 we have $\|u^2 - v^2\|_k \le C_k \|u+v\|_k \|u-v\|_k$, and restricting $O$ to the ball $\{u \in H^k \mid \|u\|_k \le R\}$, we get $\|u^2 - v^2\|_k \le 2C_k R \|u-v\|_k$, ending the proof. $\square$

The special attention $L^2(\mathbb{R})$ and $H^k$ receive is a consequence of the ever-useful operator Fourier transform. With this newfound perspective, let us play around with the norm of $f \in H^k$.

$$\begin{aligned}
\|f\|_{H^k}^2 &= \sum_{\alpha=0}^{k} \|f^{(\alpha)}\|_{L^2(\mathbb{R}_x)}^2 = \sum_{\alpha=0}^{k} \|\widehat{f^{(\alpha)}}\|_{L^2(\mathbb{R}_\xi)}^2 = \sum_{\alpha=0}^{k} \|(-i\xi)^\alpha \widehat{f}\|_{L^2(\mathbb{R}_\xi)}^2 \\
&= \sum_{\alpha=0}^{k} \int_{\mathbb{R}} |(-i\xi)^\alpha \widehat{f}(\xi)|^2 \, d\xi = \int_{\mathbb{R}} \sum_{\alpha=0}^{k} |(-i\xi)^\alpha \widehat{f}(\xi)|^2 \, d\xi \\
&= \int_{\mathbb{R}} \left( \sum_{\alpha=0}^{k} |\xi^{2\alpha}| \right) |\widehat{f}(\xi)|^2 \, d\xi \approx \int_{\mathbb{R}} (1+\xi^2)^k |\widehat{f}(\xi)|^2 \, d\xi \\
&= \|(1+\xi^2)^{k/2} \widehat{f}\|_{L^2(\mathbb{R}_\xi)}^2
\end{aligned}$$

Thus we obtain an alternative norm on $H^k$ by defining $\|f\|_{H^k}$ as $\|(1+\xi^2)^{k/2}\widehat{f}\|_{L^2(\mathbb{R}_\xi)}$. Note that in the calculations above, we claimed the last two integrals to be equivalent. This is an immediate result of $\lim\limits_{\xi \to \infty} \frac{\sum_{\alpha=0}^{k} \xi^{2\alpha}}{(1+\xi^2)^k} = 1$ for all $k$. With this alternative point of view on characterization of $H^k$, we can now more easily grasp a concept not so easy to conceptualize otherwise:

**Definition 1.7.** For a non-integer $s \ge 0$, $H^s = \{f \in L^2 \mid \|\,(1+\xi^2)^{s/2}\,\widehat{f}\|_{L^2} < \infty\}$.

We could similarly define $H^s$ for negative $s$, but since by doing so we lose our condition $f \in L^2$ and instead obtain distributions, it will not be done in this text.

One benefit of working with time-dependent functions is our ability to mainly focus on space-variable, in this case $x \in \mathbb{R}$. In other words, even though graphs of any function of two variables are two-dimensional, we can "slice" it into functions of space-variable only at time $t = t_0$. There are some ways to take this approach which we will use them in Chapters 2 and 3, but the main idea is that we will not be using Sobolev spaces $H^k$ on $\mathbb{R} \times (0,T)$ or $\mathbb{R}_x \times \mathbb{R}_t$, but only on $\mathbb{R}_x$ at fixed times $t$.

Best way to solidify this point of view is using the following space with the respective norm:

**Definition 1.8.**

$$C([0,T]; H^k) = \{f : [0,T] \mapsto H^k \mid f(t) \text{ is continuous in } H^k \text{ norm}\},$$

that is $\|f(t) - f(t_0)\|_k \to 0$ whenever $t \to t_0$. In this space, as we would normally have in $C(\mathbb{R})$, we use the sup-norm to obtain

$$\|f\|_{C([0,T];H^k(\mathbb{R}))} = \sup\{\|f(t)\|_k \mid t \in [0,T]\}.$$

With this norm, $C([0,T];H^k)$ becomes a Banach space. From this point onwards, instead of $C([0,T];H^k(\mathbb{R}))$, we will write $\mathcal{H}^k_T$, or $\mathcal{H}^k$ when $T$ is understood from the context.

Also, as a final tool, there are methods called *energy estimates* and *conserved quantities.* With these methods, we are basically taking the time-slices of $u(x,t)$, in other words $u_t(x)$, and trying to determine what relations are present for this family of functions, by considering various integrals of them.

**Example 1.9.** Throughout the text, we will try to exemplify concepts we introduced with the classic example of one-dimensional heat equation:

$$u_t - \kappa u_{xx} = 0$$

$$u(x,0) = u_0(x)$$

where $\kappa > 0$. Multiplying the differential equation by $u$ and integrating by parts, we obtain $\int_{\mathbb{R}} (u^2/2)_t \, dx + \kappa \int_{\mathbb{R}} (u_x)^2 \, dx = 0$, which is equivalent to

$$\frac{d}{dt} \int_{\mathbb{R}} u^2 \, dx = -2\kappa \int_{\mathbb{R}} u_x^2 \, dx \leq 0.$$

Thus we conclude $\int_{\mathbb{R}} u(x,t)^2 \, dx$ is a decreasing function of $t$, and from that,

$$\int_{\mathbb{R}} u(x,t)^2 \, dx \leq \int_{\mathbb{R}} u(x,0)^2 \, dx = \int_{\mathbb{R}} u_0(x)^2 \, dx$$

for all $t \geq 0$.

**Example 1.10.** Turning our attention to KdV equation, we see that multiplying (1.1a) with $u$ and integrating by parts gives us

$$\int_{\mathbb{R}} u^2(x,t) \, dx = \int_{\mathbb{R}} u^2(x,0) \, dx = \int_{\mathbb{R}} u_0^2(x) \, dx \quad \text{for all } t$$

since, assuming $u$ and its derivatives vanish at infinity, $\int_{\mathbb{R}} u u_x \, dx = \int_{\mathbb{R}} u u_{xxx} \, dx = \int_{\mathbb{R}} u^2 u_x \, dx = 0$. Therefore we get

$$\frac{d}{dt} \int_{\mathbb{R}} \frac{u^2}{2} \, dx = 0,$$

which results in what is written above. Note that this implies, for initial data $u_0 \in L^2$, solution $u(t) \in L^2$ for all $t$. Similarly, multiplying the equation by $u^2 + 2u_{xx}$ and integrating by parts, we get

$$\frac{d}{dt} \int_{\mathbb{R}} \left( u_x^2(x,t) - \frac{u^3(x,t)}{3} \right) dx = 0,$$

in other words

$$\int_{-\infty}^{\infty} \left( u_x^2(x) - \frac{u^3(x)}{3} \right) dx = \int_{\mathbb{R}} \left( (u_0')^2(x,t) - \frac{u_0^3(x,t)}{3} \right) dx \text{ for all } t.$$

It is shown in [6] that there are infinitely many such conserved quantities for the KdV equation.

**2**

# Approaching the KdV Equation Through a Regularized Equation

This chapter heavily relies the on work of J. L. Bona and colleagues (see [1–3]). Tools and the methodology used are more or less simple, yet the motivation behind it is impressive, to say the least. We will first introduce a "regularizing" term into the equation, and obtain a new equation easier to deal with. From this new equation, we will derive bounds depending on the regularizing term and study limit behaviour of bounds as regularizing term vanishes. With these methods, we will prove well-posedness of KdV equation for initial data $u_0 \in H^k$, where k is an integer $k \geq 3$. Finally, using an interpolation theorem, we will extend the result to non-integer $s \geq 3$.

## 2.1 Introducing the BBM Equation

Starting with the KdV equation, (1.1a), change of variables $\tilde{x} = x, \tilde{t} = t - x$ gives us the equation $u_{\tilde{t}} + uu_{\tilde{x}} + u_{\tilde{x}\tilde{x}\tilde{x}} = 0$, which accepts regularizing term we will introduce below better than the original equation.

**Lemma 2.1.** *For a function u satisfying*

$$u_t + uu_x + u_{xxx} - \epsilon u_{xxt} = 0 \tag{2.1a}$$

$$u(x, 0) = u_0(x) \tag{2.1b}$$

*the function* $v(x, t) = \epsilon u\left(\sqrt{\epsilon}(x - t), \sqrt{\epsilon^3}t\right)$ *satisfies*

$$v_t + v_x + vv_x - v_{xxt} = 0 \tag{2.2a}$$

$$v_0(x) = v(x, 0) = \epsilon u_0(\sqrt{\epsilon}x). \tag{2.2b}$$

Equation (2.2) is called the *BBM equation*, and is first brought forth by Benjamin, Bona and Mahony in [1] as an alternative approach to problem of long waves in shallow water. Leaving fluid mechanics aside, it has, at least mathematically , the advantage of being easier to show well-posedness.

Taking Fourier transform of the BBM equation with respect to $x$, we obtain

$$\hat{v}_t + (i\xi)\hat{v} + (i\xi)\frac{\widehat{v^2}}{2} - (i\xi)^2\hat{v}_t = 0$$

which is a linear ordinary differential equation of $\hat{v}$ if we are to consider $\widehat{v^2}$ as an inhomogeneous term $F(\hat{v})$. Indeed, solution $\hat{v}$ of

$$(1+\xi^2)\hat{v}_t + (i\xi)\hat{v} = (-i\xi)\frac{\widehat{v^2}}{2} = F(\hat{v})$$

is expected to be

$$\hat{v}(\xi,t) = \left(\int_0^t e^{\frac{i\xi}{1+\xi^2}(\tau-t)}\frac{F(\hat{v})}{1+\xi^2}\,d\tau\right) + \widehat{v_0}(\xi)e^{-\frac{i\xi}{1+\xi^2}t}$$

and taking inverse Fourier transform of it, we obtain

$$v(x,t) = \frac{-1}{2}\left(\int_0^t \frac{i\xi}{1+\xi^2}e^{\frac{i\xi}{1+\xi^2}(\tau-t)}\widehat{(v^2)}\,d\tau\right)^{\vee} + \left(\widehat{v_0}(\xi)e^{-\frac{i\xi}{1+\xi^2}t}\right)^{\vee}. \tag{2.3}$$

Question then becomes showing that this mapping $O : \mathcal{H}_T^k \mapsto \mathcal{H}_T^k$ where function $w \in \mathcal{H}_T^k$ is mapped to the term (2.3) above, with $v^2$ inside the integral replaced by $w^2$, has a unique fixed-point, at least for a small time $T$. As it is often the case, showing that this mapping is a contraction will be sufficient and the result will follow from Banach's fixed-point theorem.

For $w_1, w_2 \in \mathcal{H}_T^k$, fixing arbitrary $t \leq T$, we get

$$\begin{aligned}
\|O(w_1)(t) - O(w_2)(t)\|_k &= \frac{1}{2}\left\|\left(\int_0^t \frac{i\xi}{1+\xi^2}e^{\frac{i\xi}{1+\xi^2}(\tau-t)}(w_1^2-w_2^2)\hat{}\,d\tau\right)^{\vee}\right\|_k \\
&= \frac{1}{2}\left\|(1+\xi^2)^{k/2}\int_0^t \frac{i\xi}{1+\xi^2}e^{\frac{i\xi}{1+\xi^2}(\tau-t)}(w_1^2-w_2^2)\hat{}\,d\tau\right\|_{L^2(\mathbb{R}_\xi)} \\
&\leq \frac{1}{2}\int_0^t \left\|(1+\xi^2)^{k/2}\frac{i\xi}{1+\xi^2}e^{\frac{i\xi}{1+\xi^2}(\tau-t)}(w_1^2-w_2^2)\hat{}\right\|_{L^2(\mathbb{R}_\xi)} d\tau \\
&= \frac{1}{2}\int_0^t \left(\int_{-\infty}^{\infty}\left|(1+\xi^2)^{k/2}\frac{i\xi}{1+\xi^2}e^{\frac{i\xi}{1+\xi^2}(\tau-t)}(w_1^2-w_2^2)\hat{}\right|^2 d\xi\right)^{\frac{1}{2}} d\tau
\end{aligned}$$

since $\left|e^{\frac{i\xi}{1+\xi^2}(\tau-t)}\right| = 1$, we continue as

$$\begin{aligned}
&= \frac{1}{2}\int_0^t \left(\int_{-\infty}^{\infty} |1+\xi^2|^k \left|\frac{\xi}{1+\xi^2}\right|^2 \left|(w_1^2-w_2^2)\hat{}\right|^2 d\xi\right)^{\frac{1}{2}} d\tau \\
&\leq \frac{1}{2}\int_0^t \left(\int_{-\infty}^{\infty} |1+\xi^2|^{k-1} \left|(w_1^2-w_2^2)\hat{}\right|^2 d\xi\right)^{\frac{1}{2}} d\tau
\end{aligned}$$

$$= \frac{1}{2}\int_0^t \|w_1^2 - w_2^2\|_{k-1}\, d\tau \le \frac{1}{2}\, t \sup_{\tau\in[0,t]} \|w_1^2(\tau) - w_2^2(\tau)\|_{k-1}$$

$$\le \tfrac{1}{2}\, t \sup_{\tau\in[0,t]} \|w_1^2(\tau) - w_2^2(\tau)\|_k \le \tfrac{1}{2}\, T\|w_1^2 - w_2^2\|_{\mathcal{H}_T^k}$$

and by Corollary 1.6 we have $\|fg\|_k \le C_k\|f\|_k\|g\|_k$. Therefore,

$$\le \frac{1}{2} TC_k\|w_1 + w_2\|_{\mathcal{H}_T^k}\|w_1 - w_2\|_{\mathcal{H}_T^k}.$$

Calculations up to this point were for fixed $t \in [0, T]$, however we found a bound independent of $t$. We can now take supremum over $t$ and get the desired result,

$$\|O(w_1) - O(w_2)\|_{\mathcal{H}_T^k} < TC_kR\|w_1 - w_2\|_{\mathcal{H}_T^k}, \tag{2.4}$$

where $R$ is the radius of ball in $\mathcal{H}_T^k$, containing both $w_1$ and $w_2$.

**Proposition 2.2.** *There exists $R, T > 0$ such that map $O : \bar{B}(R) \subset \mathcal{H}_T^k \mapsto \bar{B}(R)$ defined above becomes a contraction.*

*Proof.* First find $R$ such that mapping is well-defined, that is, $O(w) \in \bar{B}(R)$. We have

$$\|O(w)\|_k \le \left\| \frac{1}{2}\left(\int_0^t \frac{i\xi}{1+\xi^2} e^{\frac{i\xi}{1+\xi^2}(\tau - t)}\widehat{(w^2)}\, d\tau\right)^{\vee}\right\|_k + \left\|\left(\widehat{v_0}(\xi) e^{-\frac{i\xi}{1+\xi^2}t}\right)^{\vee}\right\|_k.$$

First term will be called $I_1$ and the other $I_2$. Using the calculations above, we see that

$$I_1 \le \frac{1}{2} T\|w^2\|_{\mathcal{H}_T^k} \le \frac{1}{2} TC_kR^2$$

and since $|e^{-(it\xi)/(1+\xi^2)}| = 1$,

$$I_2 = \left\|(1+\xi^2)^{k/2}\widehat{v_0}(\xi)e^{-\frac{i\xi}{1+\xi^2}t}\right\|_{L^2} = \left\|(1+\xi^2)^{k/2}\widehat{v_0}(\xi)\right\|_{L^2} = \|v_0\|_k.$$

Now, we want $I_1 + I_2 \le R$, in other words $(TC_kR^2)/2 + \|v_0\|_k \le R$. However, combining the calculations above and with aim to get $TC_kR < 1$ from (2.4), we see that $\|v_0\|_k \le R/2$ suffices. Now with $R$ satisfying this condition, we can find $T < (C_kR)^{-1}$ so that $TC_kR < 1$, completing the proof. □

Discussion so far can be summed up in the theorem below.

**Theorem 2.3.** *For initial data $v_0 \in H^k$, $k \ge 1$, there is some $T > 0$ so that the BBM equation (2.2) has a unique solution in $\mathcal{H}_T^k$.*

In the end, we get that the BBM equation has a solution, at least for small $t$, and in turn the regularized KdV, (2.1), has a solution. Note that, in this proof, $R$ is bounded from below, so it can be arbitrarily large. Having that, though, requires $t$ going to 0 as $R \to \infty$. Similarly, the lower bound on $R$ means that this proof cannot give global solutions. However, extending results to global solution is possible by using conserved quantity $\int_{\mathbb{R}} u^2 + u'^2 = \|u\|_1 = \|u_0\|_1$ obtained by multiplying the BBM equation by $u$ and integrating with respect to spatial and time variables. Now, having $\|u(t)\|_1$ fixed means, with help from the explanation made in Section 3.3.2, that we can extend the solution for every $s$ globally.

## 2.2 Smooth and Regularized Approximation

We would like to show that the solutions corresponding to regularized KdV equation form a Cauchy sequence as $\epsilon$ approaches 0. However, since we also want to benefit from the bounds obtained in [3] for the case with smooth initial data, we cannot have $u$ as solution to (2.1) with initial data $u_0$. Instead, we will take convolution with a smooth function with parameter $\epsilon$ and use that smoothed function as our initial data. For $g \in L^2$, smoothing $g_\epsilon$ is given as

$$g_\epsilon(x) = \left((\phi(\epsilon^{1/6}\xi))^\vee * g\right)(x) \tag{2.5}$$

for a smooth function $\phi$ such that $0 < \phi(x) < 1$ for all $x$, $\phi(0) = 1$ and $\phi^{(n)}(0) = 0$ for $n = 1, 2, \ldots$, and tends to 0 at $\pm\infty$ exponentially.

In the end, the solution to equation

$$u_t + uu_x + u_{xxx} - \epsilon u_{xxt} = 0, \tag{2.6a}$$

$$u(x, 0) = u_{0\epsilon}(x) \tag{2.6b}$$

will be called $u_\epsilon$, where $u_0 \in H^k$ is the initial data of the original equation. We have the following lemmas from Section 4 and Section 5 of [3]:

**Lemma 2.4** (Norm estimates on smooth approximation)**.** *Let $g \in H^k$ where $k \geq 3$ and $g_\epsilon$ be the smooth function obtained from $g$ as in (2.5). Then, as $\epsilon \to 0$, we have the following estimates:*

1. $\|g_\epsilon\|_{k+j} = O(\epsilon^{-\frac{1}{6}j})$ *for* $j = 1, 2, \ldots$
2. $\|g - g_\epsilon\|_{k-j} = o(\epsilon^{\frac{1}{6}j})$ *for* $j = 0, 1, 2 \ldots$

*Moreover, on bounded subsets of $H^k$ the first estimate, on compact subsets of $H^k$ the second estimate are uniform.*

**Lemma 2.5** (Bounds for smooth initial data)**.** *Let $u$ be the solution of the KdV equation regularized with $-\epsilon u_{xxt}$, with initial data $u_0 \in H^\infty$. Fixing arbitrary $T$ where solutions are considered for $t \in [0, T]$,*

- $\|u\|_1 \leq a(\|u_0\|_1)$*, independent of* $\epsilon$ *and* $t \in [0, T]$*,*
- $\|u\|_2 \leq b(\|u_0\|_3)$ *for* $0 < \epsilon \leq \epsilon_0$ *where* $\epsilon_0$ *depends on* $T$ *and* $\|u_0\|_3$*, independent of* $t \in [0, T]$*,*
- *for* $\epsilon_0$ *as above and* $\epsilon \leq \epsilon_0$*, solution* $u$ *is bounded in* $\mathcal{H}_T^k$*, with a bound depending only on* $T, \epsilon_0, \|u_0\|_k$ *and* $\sqrt{\epsilon}\|u_0\|_{k+1}$*,*

*where $a, b : \mathbb{R}^+ \mapsto \mathbb{R}^+$ are monotone increasing continuous functions such that $a(0) = b(0) = 0$.*

*A Sketch of the Proof.* The first results in theorem, namely the ones regarding $\|u\|_1$ and $\|u\|_2$ are based upon manipulations of the KdV equation. The general case is proven by induction. Multiplying the Equation (2.6) by $\partial_x^{2m} u$ (shown as $u^{(2m)}$ from here on for simplicity) and integrating by parts gives

$$\frac{d}{dt}\int_{-\infty}^{\infty} (u^{(m)})^2 + \epsilon(u^{(m+1)})^2\, dx = -\int_{-\infty}^{\infty} (u^2)^{(m+1)} u^{(m)}\, dx.$$

This method will be used almost the same way for the next theorem. The induction hypothesis and various manipulations on the second term will result in

$$\frac{d}{dt}\int_{-\infty}^{\infty} (u^{(m)})^2 + \epsilon(u^{(m+1)})^2\, dx \leq C\Big(\int_{-\infty}^{\infty} (u^{(m)})^2\, dx + 1\Big),$$

which, by naming $\int_{-\infty}^{\infty} (u^{(m)})^2 + \epsilon(u^{(m+1)})^2\, dx = E_m(t)$, turns into $E_m'(t) \leq C(E_m + 1)$. Now we have

$$E_m(t) \leq E_m(0)e^{Ct} + e^{Ct} - 1$$

and the observation $(E_m(0))^{1/2} \leq \|u_0\|_m + \epsilon^{1/2}\|u_0\|_{m+1}$ completes the sketch of proof. □

Combining the two lemmas, we conclude that:

**Corollary 2.6.** *Letting $k \geq 3$, $u_\epsilon$ is bounded in $\mathcal{H}_T^k$ for any finite $T$, independently of (sufficiently small) $\epsilon$. Moreover, $\epsilon^{\frac{1}{6}m} u_\epsilon$ is bounded in $\mathcal{H}_T^{k+m}$ for each $m \geq 1$, finite $T$, independently of $\epsilon$. Lastly, by taking the inverse of operator $(1 - \partial_x^2)$, it can be shown that $\partial_t u_\epsilon$ is bounded in $\mathcal{H}_T^{k-3}$ and $\epsilon^{\frac{1}{6}m}\partial_x^{k+m-3}\partial_t u_\epsilon$ is bounded in $\mathcal{H}_T$, independently of $\epsilon$, for $m \leq 5$ and for any $T$.*

**Theorem 2.7.** *For $u_0 \in H^k, k \geq 3$ and $u_\epsilon$ the solution to Cauchy problem (2.6), $\{u_\epsilon\}$ is Cauchy in $\mathcal{H}^k$ as $\epsilon \to 0$.*

*Proof.* Our primary aim is to show that $\|u_\epsilon - u_\delta\|_k \to 0$ independent of $t$ as $|\epsilon - \delta| \to 0$ and $\epsilon, \delta \to 0$. An easy way to have this construction is to have $0 < \delta < \epsilon$ and let $\epsilon \to 0$. Then we will call $u_\epsilon = u, u_\delta = v, u - v = w$, which turns our problem into showing $\|w\|_k \to 0$. It should be noted that while it is not explicitly written, $w$ is indeed a sequence in $\mathcal{H}^k$, $\{w_\epsilon\}$. It is easy to see that $w$ satisfies

$$w_t + (uw - \tfrac{1}{2}w^2)_x + w_{xxx} - \delta w_{xxt} = (\epsilon - \delta)u_{xxt}, \tag{2.7}$$

which, by multiplying by $w^{(2j)}$ for $j \leq k$, and integrating over $x \in \mathbb{R}$ and over $t \in [0, T]$, gives the integral equation

$$\begin{aligned}\int_{-\infty}^{\infty} \big[\big(w^{(j)}\big)^2 + \delta\big(w^{(j+1)}\big)^2\big]\, dx &= \int_{-\infty}^{\infty} \big[\big(w_0^{(j)}\big)^2 + \delta\big(w_0^{(j+1)}\big)^2\big]\, dx \\ &\quad - 2\int_0^t \int_{-\infty}^{\infty} \big[(uw - \tfrac{1}{2}w^2)^{(j+1)} - (\epsilon - \delta)u_t^{(j+2)}\big] w^{(j)}\, dx\, d\tau. \end{aligned} \tag{2.8}$$

We are interested in $\|w\|_k$ at each $t$, which leads us to investigate $\|w^{(j)}\|$ for $j \leq k$. In the integral equation above, we have applicable terms, so we will call

$$V_j(t) = \left[\int_{-\infty}^{\infty} [\big(w^{(j)}\big)^2 + \delta\big(w^{(j+1)}\big)^2]\, dx\right]^{1/2}$$

and we will have $\|w^{(j)}(t)\| \leq V_j(t)$. If we can, using these tools, show that $\|w\|_k \to 0$ as $\epsilon \to 0$ for $k = 3$, then we can use induction to show that it also holds for $k > 3$, completing the proof. One thing to note is $w \in H^\infty$ since initial data $w_0 \in H^\infty$, therefore $w^{(j)} \in L^2$ for all $j$ (see [1]).

[**The case** $k = 3$] Starting with $j = 0$, equation (2.8) turns into

$$V_0^2(t) = V_0^2(0) - 2\int_0^t \int_{-\infty}^{\infty} \left[(\tfrac{1}{2}u_x - w_x)w^2\right] dx\, d\tau + 2(\epsilon - \delta)\int_0^t \int_{-\infty}^{\infty} u_{xxt} w\, dx\, d\tau$$

and bounding $|\frac{1}{2}u_x - w_x|$ by constant $C_1$ , $\epsilon^{1/3}\|u_{xxt}\|$ by $C_2$ using Corollary (2.6), we reach

$$V_0^2(t) \leq V_0^2(0) + 2C_1 \int_0^t V_0^2(\tau)\, d\tau + 2\epsilon^{\frac{2}{3}} C_2 \int_0^t V_0(\tau)\, d\tau$$

and through simple calculations, we obtain

$$\|w\| \leq V_0(t) \leq V_0(0)e^{C_1 T} + \epsilon^{2/3} C_1^{-1} C_2 (e^{C_1 T} - 1). \tag{2.9}$$

It is easy to see that second term goes to 0 as $\epsilon \to 0$. We shall now bound the first term, $V_0(0)$, by $\epsilon$ by Lemma (2.4),

$$\begin{aligned} V_0(0) &= \left[\int_{-\infty}^{\infty} [\big(u_{0\epsilon} - u_{0\delta}\big)^2 + \delta\big(u'_{0\epsilon} - u'_{0\delta}\big)^2]\, dx\right]^{1/2} \\ &= \big(\|u_{0\epsilon} - u_{0\delta}\|^2 + \delta\|u'_{0\epsilon} - u'_{0\delta}\|^2\big)^{1/2} \leq \|u_{0\epsilon} - u_{0\delta}\|_1 \\ &\leq \|u_{0\epsilon} - u_0\|_1 + \|u_0 - u_{0\delta}\|_1 \leq C\epsilon^{1/3} + C\delta^{1/3} \leq C\epsilon^{1/3}, \end{aligned}$$

where $C$, here and for the rest of the proof, is a constant, depending on $T$ and norms of $g$ up to order $k$, but independent of $\epsilon$. Combining (2.9) and $V_0(0) \leq C\epsilon^{1/3}$, we see that $\|w(t)\| \to 0$ as $\epsilon \to 0$, uniform in $t \in [0, T]$.

Study of $V_1, V_2, V_3$ is essentially the same. $V_j(t)$ is bounded by terms in (2.8), $V_j(0)$ and an integral. To bound the integral, Corollary (2.6) is used to derive bounds on either $L^\infty$ or $L^2$ norm of various functions. Using the bounds, integral is simplified into another integral of $V_j(\tau)$ and $V_j^2(\tau)$ with a term of $\epsilon$ in it. To bound $V_j(0)$, triangle inequality is used. One thing to note is that, rather than using

$$V_j(0) \leq \|u_{0\epsilon} - u_{0\delta}\|_{j+1} \leq \|u_{0\epsilon} - u_0\|_{j+1} + \|u_0 - u_{0\delta}\|_{j+1}$$

as above,

$$V_j(0) \le \|u_{0\epsilon}^{(j)} - u_{0\delta}^{(j)}\| + \sqrt{\delta}\|u_{0\epsilon}^{(j+1)} - u_{0\delta}^{(j+1)}\| \le \|u_{0\epsilon} - u_{0\delta}\|_j + \sqrt{\delta}\|u_{0\epsilon} - u_{0\delta}\|_{j+1} \tag{2.10}$$

$$\le \|u_{0\epsilon} - u_0\|_j + \|u_0 - u_{0\delta}\|_j + \sqrt{\delta}\|u_{0\epsilon} - u_0\|_{j+1} + \sqrt{\delta}\|u_0 - u_{0\delta}\|_{j+1} \tag{2.11}$$

or

$$\le \|u_{0\epsilon} - u_0\|_j + \|u_0 - u_{0\delta}\|_j + \sqrt{\delta}\|u_{0\epsilon}\|_{j+1} + \sqrt{\delta}\|u_{0\delta}\|_{j+1} \tag{2.12}$$

are used instead. It is done so, because $u_0 \notin H^4$, therefore when $j = 3$, $\|u_{0\epsilon} - u_0\|_4$ and $\|u_0 - u_{0\delta}\|_4$ are undefined, or when $j = 2$, by Lemma 2.4, we lose $\epsilon$ dependency on norms $\|u_0 - u_{0\epsilon}\|_3$ and $\|u_0 - u_{0\delta}\|_3$. Though it still approaches 0, it is better to have a control over the bound, in the form of $\sqrt{\delta}$ in this case. In the end, however, all goes to 0 as $\epsilon \to 0$, showing that the sequence $\{u_\epsilon\}$ is indeed Cauchy when $k = 3$. Detailed calculations of the proof can be found in [3], Section 5.

[**Inductive step:** $n-1 \Rightarrow n$] Now, we assume that the theorem is true for a certain $k - 1 \ge 3$, that is, for $u_0 \in H^{k-1}$, $\{u_\epsilon\}$ is Cauchy in $\mathcal{H}^{k-1}$ as $\epsilon \to 0$. Now our aim is to show that it also holds for $k$, in other words that $v_0 \in H^k$ implies $\{v_\epsilon\}$ is Cauchy in $\mathcal{H}^k$ as $\epsilon \to 0$. However, an important note is that $v_0 \in H^k$ gives us $v_0 \in H^{k-1}$, therefore we already have $\{v_\epsilon\}$ is Cauchy in $\mathcal{H}^{k-1}$. This observation is telling us that our primary concern for inductive step will be to show $\|w^{(k)}(t)\| \to 0$ uniformly as $\epsilon \to 0$. Here, $w$ and $V_j(t)$ are the same as above. Now, recalling (2.8), we have

$$\|w^{(k)}\|^2 \le V_k^2(t) = V_k^2(0) + 2I_k$$

where $I_k = -\int_0^t \int_{-\infty}^{\infty} \left[(uw - \frac{w^2}{2})^{(k+1)} - (\epsilon - \delta)u_t^{(k+2)}\right] w^{(k)}\, dx\, d\tau$. Using Leibnitz' rule,

$$I_k = -\int_0^t \int_{-\infty}^{\infty} \Big(\sum_{n=0}^{k+1} c_n w^{(k+1-n)} u^{(n)} w^{(k)} + c_n w^{(k+1-n)} w^{(n)} w^{(k)}\Big) - (\epsilon - \delta) u_t^{(k+2)} w^{(k)}\, dx\, d\tau$$

and separating the first and last terms of the sums,

$$I_k \le C \int_0^t \int_{-\infty}^{\infty} \Bigg(\sum_{n=1}^{k+1} |w^{(k+1-n)} u^{(n)} w^{(k)}| + \sum_{n=1}^{k} |w^{(k+1-n)} w^{(n)} w^{(k)}| + \epsilon|u_t^{(k+2)} w^{(k)}| - \big(uw^{(k+1)}w^{(k)} + 2ww^{(k)}w^{(k+1)}\big)\Bigg)\, dx\, d\tau \tag{2.13}$$

is obtained. We now want to somehow bound the terms in the integral, with a factor containing $\epsilon$ if possible. Primary motivation is, as we did in (2.9), to send $V_k(t)$ to 0 as $\epsilon \to 0$. To this end, let us examine what we already have at hand:

1. Initial data $v_0 \in H^k$, thus $u, v, w$ are bounded in $H^k$, let us say by $C_k$, by Corollary 2.6. We thus have $|u^{(i)}|, |v^{(i)}|, |w^{(i)}|, \|u^{(j)}\|, \|v^{(j)}\|, \|w^{(j)}\| \le C_k$ for integers $i < k$,

$j \leq k$. Moreover, $\{w_\epsilon\}$ is Cauchy in $H^{k-1}$, meaning that $|w^{(i)}|, \|w^{(j)}\| \to 0$ for integers $i < k-1$ , $j \leq k-1$. To be precise, it is shown in [3] that $\|w\|_{k-1} \leq \bar{C}\epsilon^{1/6}$ as $\epsilon \to 0$.

2. By second part of Corollary 2.6, $\epsilon^{5/6}\|u_t^{(k+2)}\|$ is also bounded, say by constant $\tilde{C}$.

3. By integrating by parts, last term in the integral transforms into

$$\int_0^t \int_{-\infty}^{\infty} -\big(uw^{(k)}w^{(k+1)} + 2ww^{(k)}w^{(k+1)}\big)\,dx\,d\tau = \int_0^t \int_{-\infty}^{\infty} \big(\tfrac{u_x}{2} + w_x\big)\big(w^{(k)}\big)^2\,dx\,d\tau \leq \tfrac{3C_k}{2}\int_0^t \|w^{(k)}\|^2\,d\tau$$

since $|u_x/2 + w_x|$ is bounded, as seen in the first item.

Combining all of the facts above, (2.13) and $I_k$ are bounded from above:

$$I_k \leq C\int\limits_0^t \Bigg(\Big(C_kV_k^2(\tau) + \big(\sum_{n=2}^{k} C_k\bar{C}\epsilon^{1/6}V_k(\tau)\big) + |w|\|u^{(k+1)}\|V_k(\tau)\Big) + \Big(2C_kV_k^2(\tau) + \big(\sum_{n=2}^{k-1} C_k\bar{C}\epsilon^{1/6}V_k(\tau)\big)\Big) + \Big(\epsilon^{1/6}\tilde{C}V_k(\tau)\Big) + \Big(\tfrac{3C_k}{2}V_k^2(\tau)\Big)\Bigg)\,d\tau. \quad (2.14)$$

Getting rid of all the various constants and simply calling them $\mathbf{C}$ is possible. The main question, however, is whether we can bound $|w|\|u^{(k+1)}\|V_k(\tau)$. It is known from Corollary 2.6 that $\epsilon^{1/6}u_\epsilon$ ($= \epsilon^{1/6}u$ in this case) is bounded in $\mathcal{H}^{k+1}$ independently of $\epsilon$. Then, if we can show that $|w|$ has a power of $\epsilon$ greater than 1/6, we will be able express $I_k$ as an integral of $V_k$ only. Fortunately, it is shown by Bona and Smith in [3], when proving induction step in the proof. While it is skipped now, main idea is that since initial data $v_0 \in H^k$, $k > 3$, Lemma 2.4 gives $\|v_0 - v_{0\epsilon}\|_1 \leq C\epsilon^{1/3}$, concluding $\|w\|_1 \leq C\epsilon^{1/3}$. From there, it is our conclusion in Chapter 1 telling us $|w| \leq C\epsilon^{1/3}$. The final estimate for $I_k$ turns out to be

$$I_k \leq \mathbf{C}\int\limits_0^t V_k^2(\tau) + \epsilon^{1/6}V_k(\tau)\,d\tau.$$

We can now, once again, turn our attention to $V_k(t)$. As we did in (2.9), from

$$V_k^2(t) \leq V_k^2(0) + 2\mathbf{C}\int\limits_0^t V_k^2(\tau) + \epsilon^{1/6}V_k(\tau)\,d\tau$$

we get

$$\|w^{(k)}\| \leq V_k(t) \leq V_k(0)e^{\mathbf{C}T} + \epsilon^{1/6}(e^{\mathbf{C}T} - 1).$$

Now the final step is showing $V_k(0)$ going to 0 with $\epsilon$, which is easy by using estimate (2.12).

$$V_k(0) \leq \|v_0 - v_{0\epsilon}\|_k + \|v_0 - v_{0\delta}\|_k + C\epsilon^{1/3}.$$

From this, we get the result $\|w^{(k)}\| \to 0$ as $\epsilon \to 0$, concluding that $\|w\|_k \to 0$ with $\epsilon \to 0$. Thus the proof is complete with this inductive step. $\square$

**Corollary 2.8.** *For $u_\epsilon$ as above, $\{\partial_t u_\epsilon\}$ is Cauchy in $\mathcal{H}_T^{k-3}$ as $\epsilon \to 0$.*

*Proof.* For $w$ as in the previous theorem, leaving $w_t$ term by itself in (2.7) gives us

$$w_t = -(uw - \tfrac{1}{2}w^2)_x - w_{xxx} + \delta w_{xxt} + (\epsilon - \delta)u_{xxt}.$$

Last two terms go to 0 in norm since $\epsilon \to 0$, and other two terms go to 0 in norm since $w$ is Cauchy in $\mathcal{H}_T^k$ and $w_{xxx}$ is Cauchy in $\mathcal{H}_T^{k-3}$ by previous theorem. From this, it immediately follows that $w_t$ is Cauchy in $H^{k-3}$. $\square$

It should be emphasized that great lengths Bona and Smith go in [3] to prove that $\|w\|_{k-1} \le \bar{C}\epsilon^{1/6}$ is essential in the proof. Without that kind of control over norms $\|w^{(j)}\|$ for $j = 0, 1, \dots k$, we have no apparent way to control terms $\int_{-\infty}^{\infty} |w^{(k+1-n)}u^{(n)}w^{(k)}|\, dx$ and $\int_{-\infty}^{\infty} |w^{(k+1-n)}w^{(n)}w^{(k)}|\, dx$ appearing in (2.13). Without such bounds, combining $\|w^{(k)}\| \le V_k(t)$ and $\|w^{(j)}\| \le C_k$ would turn out to be insufficient as it only gives us $\int_{-\infty}^{\infty} |w^{(k+1-n)}u^{(n)}w^{(k)}|\, dx \le C_k^2 V_k(\tau)$ which does not vanish. In the end,

We can now state our primary theorem for the section.

**Theorem 2.9** (Existence - Uniqueness)**.** *For initial data $u_\epsilon \in H^k$ with $k \ge 3$, Cauchy problem shown in (1.1) has a unique solution $u \in \mathcal{H}_T^k$ for all finite $T$.*

*Proof.* Uniqueness is immediate with the following argument: Calling different solutions to the same initial data $u_1, u_2$, $u_1 - u_2 = w$ satisfies

$$w_t + w_x + w_{xxx} + [(u_1 + u_2)w]_x = 0.$$

Multiplying by $w$ and integrating with respect to $x$, we get the integral equation

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{\infty} w^2\, dx &= -\int_{-\infty}^{\infty} 2[(u_1+u_2)w]_x w\, dx = \int_{-\infty}^{\infty} 2(u_1+u_2)ww_x\, dx \\ &= \int_{-\infty}^{\infty} (u_1+u_2)_x w^2\, dx \le C\int_{-\infty}^{\infty} w^2\, dx\end{aligned}$$

meaning that $\int_{-\infty}^{\infty} w^2\, dx = 0$ for all $t$, and since we consider the continuous version of $w$, $w \equiv 0$ identically.

For existence, we will use tools from Theorem 2.7 and Corollary 2.8. From machinery built in the respective proofs, we know that $u_\epsilon$ is convergent in $\mathcal{H}_T^k$, $\partial_t u_\epsilon$ is convergent in $\mathcal{H}_T^{k-3}$ to some functions $u, v$ respectively. It is also trivial that $\partial_x^3 u_\epsilon \to \partial_x^3 u$ in $\mathcal{H}_T^{k-3}$, and so $\partial_x(u_\epsilon^2) \to \partial_x(u^2)$ in $\mathcal{H}_T^{k-1}$ once we recall our result Corollary 1.6. Moreover, since $u_\epsilon \to u$ in distribution sense, we can say $\partial_t u_\epsilon \to \partial_t u$ in distribution sense, but that just means $v = u_t$. Finally, note that $\partial_t u_\epsilon$ is bounded in $\mathcal{H}_T^{k-3}$ by Corollary 2.6, therefore $\partial_x^2 \partial_t u_\epsilon$ is bounded in $\mathcal{H}_T^{k-5}$. From this, we get that $\epsilon \partial_x^2 \partial_t u_\epsilon \to 0$ as $\epsilon \to 0$, at least in the distribution sense. Combining all of these, solutions to the regularized KdV equation converges to a solution of the KdV equation. $\square$

It should be noted that we never mentioned space $\mathcal{H}_\infty^k$. We have arbitrarily large, but finite $T$. This distinction is not essential when defining the function, since we can have the following definition:

$$f : \mathbb{R}_x \times \mathbb{R}_t^+ \mapsto \mathbb{R}$$

$$f(x,t) = f_T(x,t)$$

for some $T > t$ and $f_T$ solution to KdV equation on $\mathcal{H}_T^k$ we found above. By uniqueness property, this $f$ is well-defined. However, this definition gets important when we attempt to conclude continuous dependence on initial data. It turns out that we cannot have continuous dependence on space $\mathcal{H}_\infty^k$. A simple counterexample is *solitary waves*, the phenomenon first observed by S. Russell. It primarily says that certain kind of waves of different speeds of propagation diverge, therefore $\|u_C - u_D\|_k \to \|u_C\|_k + \|u_D\|_k$ even when initial data converge. However, even when we cannot have continuous dependence for space $\mathcal{H}_\infty^k$, we can have it for $\mathcal{H}_T^k$ for arbitrary $T > 0$.

**Theorem 2.10.** *The mapping $U : H^k \mapsto \mathcal{H}_T^k$, where $u = U(g)$ is the unique solution of the KdV equation with initial data $g \in H^k$, $k \geq 3$ in time interval $[0,T]$, is continuous.*

*Proof.* We will use sequential characterization of continuity. That is, for $g_n \to g$ in $H^k$, we want to get $\|u^n - u\|_{\mathcal{H}_T^k} \to 0$, in other words $\|u^n - u\|_k \to 0$ uniformly in $[0,T]$, where $u^n$ and $u$ are solutions to problems with initial data $g_n$ and $g$, respectively. To be precise, fixing $\delta > 0$, we want to find $N$ such that $n \geq N$ implies $\|u^n - u\|_k \leq \delta$ uniformly. Proof is relatively straightforward.

We will first, using triangle inequality and bounds $\|u - u_\epsilon\|_k \leq C\epsilon^{1/6} + C\|g - g_\epsilon\|_k$, $\|u^n - u_\epsilon^n\|_k \leq C\epsilon^{1/6} + C\|g_n - g_{n\epsilon}\|_k$ obtained in [3], make the observation that

$$\begin{aligned}
\|u^n - u\|_k &\leq \|u^n - u_\epsilon^n\|_k + \|u_\epsilon^n - u_\epsilon\|_k + \|u_\epsilon - u\|_k \\
&\leq (C\epsilon^{1/6} + C\|g_n - g_{n\epsilon}\|_k) + \|u_\epsilon^n - u_\epsilon\|_k + (C\epsilon^{1/6} + C\|g - g_\epsilon\|_k) \\
&\leq \tfrac{\delta}{3} + \|u_\epsilon^n - u_\epsilon\|_k + \tfrac{\delta}{3}.
\end{aligned}$$

Last step is a result of Lemma (2.4). Since $g_n \to g$, all are bounded in $H^k$, thus the bound is uniform. Therefore we may find some $\epsilon$ small enough that we may bound the terms in this manner. Now we have variable $n$ and fixed $\epsilon > 0$.

Now, since $\epsilon$ is fixed, we can benefit from transformation done in (2.5), and consider $v^n$ and $v$ instead, with initial data $h_n(x) = \epsilon g_{n\epsilon}(\sqrt{\epsilon}x)$ and $h(x) = \epsilon g_\epsilon(\sqrt{\epsilon}x)$. Now, if we can have $v^n \to v$ in $\mathcal{H}_R^k$ for fixed $R > 0$, we can then invert the transformation and get our desired result, $u_\epsilon^n \to u_\epsilon$ in $\mathcal{H}_T^k$. This inversion of transformation is where $\epsilon$ being fixed helps.

Recalling how we defined transformation that smooths the initial data, we can see that $g_n \to g$ in $L^2$ gives us $h_n \to h$ in $H^r$ for all $r = 0, 1, \dots$ Getting closer to desired result, we will name $v^n - v = w^n$ and once again construct the equation that $w^n$ satisfies:

$$w_t^n + w_x^n + w^n w_x^n + (vw^n)_x - w_{xxt}^n = 0$$

$$w^n(x, 0) = h_n(x) - h(x) = f_n(x).$$

For the rest of the proof, we drop superscript $n$ and simply write $w$ instead of $w^n$. Multiplying the above equation by $w^{(2j)}$ and integrating by parts as we done before, and calling $W_j(t) = \int_{-\infty}^{\infty} \left[(w^{(j)})^2 + (w^{(j+1)})^2\right] dx$, we get the integral equation

$$W_j(t) = W_j(0) - 2\int_0^t \int_{-\infty}^{\infty} \left[(\tfrac{w^2}{2})^{(j+1)} + (vw)^{(j+1)}\right] w^{(j)}\, dx\, d\tau.$$

Using Leibniz' rule, we get the bound

$$W_j(t) \le W_j(0) + C\left|\int_0^t \int_{-\infty}^{\infty} \left[\sum_{n=0}^{j+1} w^{(j+1-n)} w^{(n)} w^{(j)} + \sum_{n=0}^{j+1} v^{(j+1-n)} w^{(n)} w^{(j)}\right] dx\, d\tau\right| \tag{2.15}$$

and letting $j = 0$, it turns into

$$\begin{aligned} W_0(t) &\le W_0(0) + C\left|\int_0^t \int_{-\infty}^{\infty} v_x w^2\, dx\, d\tau\right| \le W_0(0) + C\int_0^t \int_{-\infty}^{\infty} w^2\, dx\, d\tau \\ &\le W_0(0) + C\int_0^t W_0(\tau)\, d\tau \end{aligned}$$

where $C$ are all various constants. Now the last inequality is implying $W_0(t) \le W_0(0)e^{Ct}$, which lets us to conclude $\|w^n\|_1 \le \|f_n\|_1 e^{CR}$, giving us $w_n \to 0 \in \mathcal{H}_R^1$ since $h_n \to h$ in $H^1$. Now assume inductively $\|w^n\|_j \to 0$ as $n \to \infty$ uniformly on $[0, R]$. Then (2.15) turns into

$$W_j(t) \le W_j(0) + C\int_0^t W_j(\tau) + a_n W_j^{1/2}(\tau)\, d\tau$$

where $a_n$ is a combination of various norms $\|w^n\|_i$, $i \le j$. Then we have $a_n \to 0$ as $n \to \infty$. Integral equation above turns into

$$W_j^{1/2}(t) \le W_j^{1/2}(0)e^{CR} + a_n(e^{CR} - 1),$$

and since $a_n \to 0$ by discussion above, $W_j(0) \to 0$ by the fact that $W_j(0) \le \|f_n\|_{j+1} \downarrow 0$, we conclude $W_j \to 0$ uniformly.

We just showed that $v^n \to v$ in $\mathcal{H}_R^k$, but that means $u_\epsilon^n \to u_\epsilon$ in $\mathcal{H}_T^k$. Combining this with the triangle inequality gives $\|u_n - u\|_k \le \delta$, completing the proof. $\square$

We have answered the most important question of the chapter: Cauchy problem for the KdV equation is well-posed if initial data $u_0 \in H^k$, $k = 3, 4, \dots$. It is not the whole work of Bona and Smith [3]. They also show more smoothness properties for time derivatives, and in the appendix, weaken the condition for initial data $u_0 \in H^2$ by the use of distribution techniques. However, we will weaken this condition in the next chapter, following the footsteps of Tosio Kato, using semigroup theory. Yet we still have one more issue to take care of: extension of the results for non-integer $s$, that is, $u_0 \in H^s$, $s \in [3, \infty)$. This will be done in the following section.

## 2.3 Non-integer Interpolation

The extension of our results to non-integer values of $s$ in $H^s$ will use an interpolation method, applicable to nonlinear situations as well. The following results are from Bona and Scott [2]. Throughout the section, we will first state the theorems, then define the concepts introduced in the theorem.

**Theorem 2.11.** *Let $B_0, B_1, C_0, C_1$ be Banach spaces such that $B_0 \supset B_1$, $C_0 \supset C_1$ with both inclusions continuous, and $(\lambda, q)$ pair in the range $0 < \lambda < 1$, $1 \leq q \leq \infty$. Let $A$ be a mapping satisfying the conditions*

1. *$A : B_{\lambda,q} \mapsto C_0$ and for $f, g \in B_{\lambda,q}$, we have*

$$\|Af - Ag\|_{C_0} \leq c_0(\|f\|_{B_{\lambda,q}} + \|g\|_{B_{\lambda,q}})\|f - g\|_{B_0},$$

2. *$A : B_1 \mapsto C_1$ and for $h \in B_1$, we have*

$$\|Ah\|_{C_1} \leq c_1(\|h\|_{B_{\lambda,q}})\|h\|_{B_1}$$

*where $c_0, c_1 : \mathbb{R}^+ \mapsto \mathbb{R}^+$ are continuous non-decreasing functions.*

*Then, for $B_{\lambda,q} \supseteq B_{\theta,p}$, $A$ maps $B_{\theta,p}$ into $C_{\theta,p}$ with norm estimate*

$$\|Af\|_{c_{\theta,p}} \leq c(\|f\|_{B_{\lambda,q}})\|f\|_{B_{\theta,p}}$$

*for $f \in B_{\theta,p}$, $c(x) = 4c_0(4x)^{1-\theta}c_1(3x)^{\theta}$.*

Theorem 2.11 will be given without proof. Still, for it to make sense, some definitions and properties need to be presented.

**Definition 2.12.** For Banach spaces $B_0 \supset B_1$, continuously included,

- $B_{\theta,p}$ (or $[B_0, B_1]_{\theta,p}$ to avoid ambiguity) with $0 < \theta < 1$, $1 \leq p \leq \infty$ is an intermediate Banach space such that $B_0 \supset B_{\theta,p} \supset B_1$, with inclusion mappings continuous. Parameter $p$, as usual, expresses norm power, while $\theta$ shows the "weight" relative to original spaces $B_0$ and $B_1$.

- $f \in B_0$ is in $B_{\theta,p}$, by definition, if and only if the norm below is finite:

$$\|f\|_{B_{\theta,p}}^p = \int_0^\infty \frac{K(f,\epsilon)^p}{\epsilon^{\theta p+1}}\, d\epsilon$$

where $K(f,\epsilon) = \inf_{g \in B_1} \{\|f-g\|_{B_0} + \epsilon \|g\|_{B_1}\}$.

- This consturction bestows a linear ordering on family $B_{\theta,p}$. Intermediate spaces corresponding to pairs $(\theta_1, p_1)$ and $(\theta_2, p_2)$ are related as follows:

$$B_{\theta_1,p_1} \supset B_{\theta_2,p_2} \iff \text{ either } \theta_1 = \theta_2 \textit{ and } p_1 > p_2 \text{ or } \theta_1 < \theta_2.$$

Now we see that the theorem above is about the extension of mappings satisfying certain conditions to intermediate spaces. The next theorem will allow the extension to be continuous as well. Required definitions will be given later.

**Theorem 2.13.** *Let us take $B_0, B_1, C_0, C_1, \lambda, q$ and $A$ as in Theorem 2.11 and also assume that pair $B_0, B_1$ has a $(\theta, p)$ approximate identity $\{S_\epsilon\}$. Then, if $A$ satisfies the additional condition*

*3. $A : B_1 \mapsto C_1$ is a continuous mapping,*

*then $A : B_{\theta,p} \mapsto C_{\theta,p}$ is also a continuous mapping.*

**Definition 2.14.** For Banach spaces $B_0 \supset B_1$, continuously included, and $\theta$ and $p$ as constructed above, we say that $B_0, B_1$ pair has a $(\theta, p)$ *approximate identity* if there is a family of continuous mappings $\{S_\epsilon\}$ for $0 < \epsilon \leq 1$ with $S_\epsilon : B_{\theta,p} \mapsto B_1$ satisfying

1. $\|S_\epsilon f\|_{B_{\theta,p}} + \epsilon^{1-\theta}\|S_\epsilon f\|_{B_1} \leq c\|f\|_{B_{\theta,p}}$ for all $f \in B_{\theta,p}$ and $\epsilon \in (0,1]$.
2. $\|S_\epsilon f - f\|_{B_{\theta,p}} + \epsilon^{-\theta}\|S_\epsilon f - f\|_{B_0} \to 0$ as $\epsilon \to 0$ for all $f \in B_{\theta,p}$, uniformly on compact subsets of $B_{\theta,p}$.

Applying these results according to our goals is natural at most points. Obviously we have $A = U : H^k \mapsto \mathcal{H}_T^k$, sending initial data to the solution of the KdV equation, a continuous mapping by virtue of Theorem 2.10. Therefore, it is safe to call $B_1 = H^k$, $C_1 = \mathcal{H}_T^k$ for $k > 3$. However, to be able to truly use the theorems, we need to determine intermediate spaces. There is no harm in calling $H^0\ (= L^2) = B_0$ and $\mathcal{H}_T^0 = C_0$. Also, since our Banach spaces are in fact Hilbert spaces, and norm power is 2, we will have $p = q = 2$. This much is sensible enough, but for $B_0 = L^2$, $B_1 = H^k$, question of what $B_{\lambda,2}$ is remains. However, it turns out that in this case $B_{\lambda,2} \cong H^{\lambda k}$, so choosing $\lambda = \frac{k-1}{k}$, we can identify $B_{\lambda,2}$ as $H^{k-1}$. It follows that, for $\theta > \lambda$, $B_{\theta,2} = H^s$ for non-integer $s$. We might have expected that $[\mathcal{H}_T^0, \mathcal{H}_T^k]_{\theta,2} \cong C([0,T]; H^s) = \mathcal{H}_T^s$, but it does not hold. Fortunately we have $C_{\theta,2} \subset \mathcal{H}_T^s$, shown in [2], so extension of mapping $A$ will be useful. Now the final conditions before checking condition 1 and 2 from Theorem 2.11 is whether $L^2 = H^0$ and $H^k$ has a $(\theta, 2)$ approximate identity or not. This is non-trivial, and will be explained below.

**Example 2.15** (From Bona and Scott, [2])**.** We want to have a family of operators $S_\epsilon : H^s \mapsto H^k$ where $s = \theta k$, $0 < \epsilon \le 1$. Knowing that $B_0 = L^2, B_{\theta,2} = H^s, B_1 = H^k$, conditions in Definition 2.14 turn into

1. $\|S_\epsilon f\|_s + \epsilon^{1-s/k}\|S_\epsilon f\|_k \le c\|f\|_s$ for all $f \in H^s$, $\epsilon \in (0,1]$.
2. $\|S_\epsilon f - f\|_s + \epsilon^{-s/k}\|S_\epsilon f - f\| \to 0$ as $\epsilon \to 0$ for all $f \in H^s$, uniformly on compact subsets of $H^s$.

Now, for fixed $\phi \in C^\infty$ such that $\phi \equiv 1$ on $[-1,1]$, $\phi \equiv 0$ on $\mathbb{R} \setminus (-2,2)$ and $0 \le \phi \le 1$, $S_\epsilon$ defined as $\widehat{S_\epsilon u}(\xi) = \phi(\epsilon^{1/k}\xi)\hat{u}(\xi)$ satisfies the following conditions

$$\|S_\epsilon f\|_s = \|(1+\xi^2)^{s/2}\widehat{S_\epsilon f}\| = \|(1+\xi^2)^{s/2}\phi(\epsilon^{1/k}\xi)\hat{f}\| \le \|(1+\xi^2)^{s/2}\hat{f}\| = \|f\|_s, \quad (2.16)$$

and

$$\begin{aligned}\|S_\epsilon f\|_k &= \left(\int_{-\infty}^{\infty}(1+\xi^2)^k\phi^2(\epsilon^{1/k}\xi)|\hat{f}(\xi)|^2\,d\xi\right)^{1/2}\\ &\le \left(\sup_{|\xi|<\frac{2}{\epsilon^{1/k}}}\left|(1+\xi^2)^{k-s}\right|\int_{-\infty}^{\infty}(1+\xi^2)^s|\hat{f}(\xi)|^2\,d\xi\right)^{1/2} \le \left(C_1(4\epsilon^{-2/k})^{k-s}\|f\|_s^2\right)^{1/2}\\ &\le \left(C_2(\epsilon^{-(1-s/k)})^2\|f\|_s^2\right)^{1/2} = C\epsilon^{-(1-s/k)}\|f\|_s. \quad (2.17)\end{aligned}$$

Therefore condition 1 is satisfied. For condition 2, note that for $r \le s$, the following holds:

$$\begin{aligned}\|f - S_\epsilon f\|_r^2 &= \int_{-\infty}^{\infty}(1+\xi^2)^r(1-\phi(\epsilon^{1/k}\xi))^2|\hat{f}(\xi)|^2\,d\xi\\ &\le \int_{|\xi|>\epsilon^{-1/k}}(1+\xi^2)^r|\hat{f}(\xi)|^2\,d\xi\\ &\le \sup_{|\xi|>\epsilon^{-1/k}}\left|(1+\xi^2)^{r-s}\right|\int_{|\xi|>\epsilon^{-1/k}}(1+\xi^2)^s|\hat{f}(\xi)|^2\,d\xi\\ &\le C_1\epsilon^{(-2/k)(r-s)}\int_{|\xi|>\epsilon^{-1/k}}(1+\xi^2)^s|\hat{f}(\xi)|^2\,d\xi,\end{aligned}$$

and as $\epsilon \to 0$, $\int_{|\xi|>\epsilon^{-1/k}}(1+\xi^2)^s|\hat{f}(\xi)|^2\,d\xi \to 0$ uniformly on compact subsets of $H^k$. Therefore we can say $\|f - S_\epsilon f\|_r = o(\epsilon^{(s-r)/k})$ when $\epsilon \to 0$. Now plugging in extremes $r = 0$, $r = s$, we get $\|f - S_\epsilon f\| = o(\epsilon^{s/k})$ and $\|f - S_\epsilon f\|_s = o(1)$, which is precisely what condition 2 is. Therefore we showed that this particular family of mappings satisfies conditions, thus $L^2$ and $H^k$ have a $(\theta, 2)$ approximate identity for all $\theta \in (0,1)$.

Final step that is missing is simply conditions 1 and 2 of Theorem 2.11 for our mapping $U$, which translates as $\|Uf - Ug\|_{\mathcal{H}_T^0} \le c_0(\|f\|_{k-1} + \|g\|_{k-1})\|f - g\|$ and

$\|Uh\|_{\mathcal{H}_T^k} \leq c_1(\|h\|_{k-1})\|h\|_k$ in our case. Second one requires the use of conserved quantities for the KdV equation, which is shown in [2] after lengthy calculations. First one, however, is easier to tackle, especially with us accepting Condition 2 of Theorem 2.11. Calling $Uf = u, Ug = vUf - Ug = w$, we see that $w$ satisfies the partial differential equation $w_t + [(u+v)w]_x/2 + w_{xxx} = 0$ with initial data $w(x,0) = (f-g)(x)$. Multiplying the equation by $w$ and integrating over $\mathbb{R}$ and $t$ as before, we get

$$\frac{d}{dt}\int_{-\infty}^{\infty} w^2\,dx \leq \frac{1}{2}(\|u\|_k + \|v\|_k)\int_{-\infty}^{\infty} w^2\,dx.$$

Using second condition, we see that $\|u\|_k \leq \|f\|_k c_1(\|f\|_{k-1})$, $\|v\|_k \leq \|g\|_k c_1(\|g\|_{k-1})$, so we can have $\|u\|_k + \|v\|_k \leq (\|f\|_k + \|g\|_k)c_1(\|f\|_k + \|g\|_k)$. Then we have

$$\frac{d}{dt}\|w(t)\|^2 \leq c(\|f\|_k + \|g\|_k)\|w\|^2$$

where $c(x) = xc_1(x)$ for $c_1$ above. Using Gronwall's inequality, in the end we get

$$\|w(t)\|^2 \leq \|w(0)\|^2 e^{ct}.$$

Here, $c$ is a constant depending on $\|f\|_k, \|g\|_k, c_1$, but not on $t$. On bounded time interval $[0,T]$ taking the square root, for $c_T(x) = e^{Tc(x)/2}$, we have

$$\|Uf - Ug\|_{\mathcal{H}_T^0} \leq c_T(\|f\|_k + \|g\|_k)\|f-g\|,$$

giving us the first condition. Note that we did not get $c(\|f\|_{k-1} + \|g\|_{k-1})$, but there is no problem in using a greater bound. With this, this chapter comes to its conclusion, summarized by the theorem below.

**Theorem 2.16.** *Let us be given $u_0 \in H^s$, $s \geq 3$, not necessarily an integer. Then the Cauchy problem for the Korteweg-de Vries equation*

$$u_t u_x + +uu_x + u_{xxx} = 0, \quad x \in \mathbb{R}, t > 0$$

$$u(x,0) = u_0(x) \quad x \in \mathbb{R}$$

*has a solution $u \in \mathcal{H}_T^s$ for all $T > 0$. The solution is unique, and depends continuously on the initial data.*

3

# Semigroup Approach to the Problem

Following the theorem from last chapter with initial data $u_0 \in H^s$, $s \geq 3$, our primary goal for this chapter is lowering the bound for $s$ to $s > 3/2$ for local solutions, and to $s \geq 2$ for global solutions. We will heavily follow Tosio Kato's work [8,9]. To familiarize ourselves with the methods a brief study of linear situation will be presented, then the result will be stated and exposed, followed by application of the general theorem to our specific case, that is, the KdV equation.

## 3.1 Studying the Linear Situation

Throughout the chapter we will, in a sense, be interested in ordinary differential equations with Banach space values. Linear situation, in general, is expressed as

$$\frac{d}{dt}\mathbf{u}(t) + A(t)\mathbf{u}(t) = f(t) \quad \text{for } t \in [0, T] \tag{3.1a}$$

$$\mathbf{u}(0) = u. \tag{3.1b}$$

Here, $\mathbf{u} : [0, T] \mapsto X$ is a function of time $t$ with values in a Banach space $X$, which, in our case, will be $H^s$ for some $s$. Moreover, $u, f(t)$ are elements in $X$ and $A(t)$ is a family of operators on $X$.

Now that we have mentioned operators, some definitions need to presented.

**Definition 3.1.** 1. Given two real Banach spaces, $X$, $Y$, and a linear operator $O : X \to Y$, operator norm of $O$, $\|O\|_{X,Y}$, is defined as

$$\|O\|_{X,Y} = \sup \left\{ \|O(x)\|_Y \mid \|x\|_X = 1 \right\}.$$

2. Set of all bounded linear operators from $X$ to $Y$ is shown as

$$\mathcal{B}(X, Y) = \{O : X \to Y \mid \|O\|_{X,Y} < \infty\}.$$

*Remark.* When $X = Y$ we use $\|O\|_X$, $\mathcal{B}(X)$ instead of $\|O\|_{X,Y}$, $\mathcal{B}(X,Y)$, respectively.

Before going further, a new operator must be derived from the equation to express how $\mathbf{u}(t)$ and $u$ are related.

**Definition 3.2.** For equation (3.1a) with $f \equiv 0$, operator family, called *evolution operator*, $\{U(t,s)\} \subset \mathcal{B}(X)$ on triangle $0 \le s \le t \le T$ is defined as

$$U(t,s)\mathbf{u}(s) = \mathbf{u}(t),$$

taking solution at time $s$ to solution at time $t$.

Obviously, $\mathbf{u}(t) = U(t,0)\mathbf{u}(0) = U(t,0)u$. The inhomogeneous case has solution

$$\mathbf{u}(t) = U(t,0)u + \int_0^t U(t,s)f(s)\,ds.$$

We established a relation between $\mathbf{u}(t)$ and $u$ via evolution operators. We now want to relate $U(t,s)$ to $A(t)$. If instead of a family of operators $A(t)$ we had a constant bounded operator $A$, $U(t,s)$ would be $e^{-(t-s)A}$ where $e^A$ is defined in form of Taylor sums, $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$. Though the exact methodology is still obscure, let us simply follow the following terminology.

**Definition 3.3.** For Banach space $X$ and $A$ as in (3.1a):

1. We say evolution operators $U(t,s)$ form a $C_0$*-semigroup* if the following conditions are satisfied:

   (a) $\forall s \in [0,T]$, $U(s,s) = I_x$, identity operator on $X$.

   (b) $U(t,s)U(s,r) = U(t,r)$, independent of choice of $t, s$ and $r$.

   (c) The mappings $t \to U(t,s)$ and $s \to U(t,s)$ are continuous in the operator norm. In other words, for $t_n \to t_0$, we have $\|U(t_n,s)u - U(t_0,s)u\|_X \to 0$ for all $u \in X$ and similarly for $s_n \to s$.

2. We say $A$ *generates a* $C_0$*-semigroup* if the evolution operator $U(t,s)$ associated with the differential equation is a $C_0$ semigroup. Semigroup generated by $A$ is shown as $\{e^{-tA}\}$.

3. The set of all negative generators of $C_0$-semigroups is shown as $G(X)$. We also define $G(X,M,\beta) = \{A \in G(X) \mid \|e^{-tA}\|_X \le Me^{t\beta}\}$.

*Remark.* We are interested in *negative* generators because taking $A(t)\mathbf{u}(t)$ to the other side of the equation 3.1a we get $\mathbf{u}'(t) = -A(t)\mathbf{u}(t) + f(t)$. In the end, it is just a convention to word the definition as it is.

**Example 3.4.** To solidify the concepts introduced so far, let us go back to the heat equation once again. Rewriting the equation in both old and new notation, we have

$$\begin{aligned} &u_t - \kappa u_{xx} = 0, && \mathbf{u}(t) + A\mathbf{u} = 0, \\ &u(x,0) = u_0(x) \text{ for } x \in \mathbb{R}, && \mathbf{u}(0) = u_0. \end{aligned}$$

Classical methods give the solution for this problem as

$$u(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\kappa t}} u_0(y)\, dy.$$

Here, our operator $A$ is $-\kappa D_x^2$, our evolution operator $U(t,0)u_0$ is $\frac{1}{\sqrt{4\pi\kappa t}} (f_t * u_0)(x)$ with $f_t(x) = e^{-x^2/4\kappa t}$. Also, from Example (1.9), we know that $\|\mathbf{u}(t)\| \leq \|u_0\|$ for all $t$, so we can say $-\kappa D_x^2 \in G(L^2, 1, 0)$.

We will now cite some conditions from [9] which will allow us to successfully conclude that evolution operator $U(t,s)$ can be constructed for $A$ satisfying the conditions. Then some remarks regarding the conditions will be presented, followed by the main theorem, stating properties and estimates of $U(t,s)$. We will thus conclude this section and expand our case to nonlinear one.

1. $\{A(t)\}_{0\leq t\leq T}$ is a stable family of operators in $G(X)$ with stability index $M, \beta$.
2. There is a Banach space $Y \subset X$, embedded densely and continuously in $X$, and an isomorphism $S : Y \hookrightarrow X$ such that, for all $t \in [0,T]$,

   $$SA(t)S^{-1} = A(t) + B(t), \quad \text{where } B(t) \in \mathcal{B}(X) \text{ satisfies the conditions}$$

   (a) $t \to B(t)x$ is a measurable $X$-valued function for every $x \in X$.

   (b) $t \to \|B(t)\|_X$ is upper integrable on $[0,T]$.
3. $Y \subset D(A(t))$, the domain which $A$ is defined in $X$, for all $0 \leq t \leq T$, so that $A(t) \in \mathcal{B}(Y, X)$.

*Remark.* 1. In our study, instead of "stability index", having $A(t) \in G(X, M, \beta)$ will suffice. General case may be demanding the distinction, but our case does not.

2. Note, first of all, that $SA(t)S^{-1} \in \mathcal{B}(X)$. Moreover, for the definition to make sense, we need to have $A(t) : Y \to Y$ for all $t$.

3. The condition $Y \subset D(A(t))$, combined with the remark above, gives us $A(t) \in \mathcal{B}(Y)$. Clearly, $t \to A(t) \in \mathcal{B}(Y,X)$ is continuous in norm, since $t \to \mathcal{B}(X)$ is, and $Y \subset X$ is continuously embedded.

**Theorem 3.5** (Kato, [9])**.** *Under conditions (1), (2) and (3), the evolution operator $U(t,s)$ corresponding to $A(t)$ uniquely exists on $\triangle = \{(t,s) \mid 0 \leq s \leq t \leq T\}$, with the following properties:*

1. *$U$ is continuous as a mapping $U : \triangle \mapsto \mathcal{B}(X)$ and $U(s,s) = I$ for all $s \leq T$.*
2. *$U(t,s)U(s,r) = U(t,r)$.*
3. *$U(t,s)Y \subset Y$ and mapping $U : \triangle \mapsto \mathcal{B}(Y)$ is also continuous.*
4. *$\frac{d}{dt}U(t,s) = -A(t)U(t,s)$ and $\frac{d}{ds}U(t,s) = U(t,s)A(s)$ which exists continuously on $\mathcal{B}(Y,X)$.*

*Moreover, we have the bounds*

$$\sup_{(t,s)\in\triangle} \|U(t,s)\|_X \leq Me^{\beta T} \tag{3.2}$$

$$\sup_{(t,s)\in\triangle} \|U(t,s)\|_Y \leq \|S\|_{Y,X}\|S^{-1}\|_{X,Y}Me^{\beta T+M\|\bar{B}\|_{X,1}} \tag{3.3}$$

*where $\|\bar{B}\|_{X,1}$ denotes the upper integral $\int_0^T \|B(t)\|_X \, dt$.*

## 3.2 Existence Theorem for Nonlinear Case

We now want to add nonlinearity to our equation. To do this, instead of (3.1a), we consider

$$\frac{d}{dt}\mathbf{u}(t) + A(t,\mathbf{v})\mathbf{u}(t) = f(t,\mathbf{v}) \quad \text{for } 0 \leq t \leq T, \quad \text{with } \mathbf{u}(0) = u. \tag{3.4}$$

The idea is, if this equation has a solution $\mathbf{u}(t)$, then we will have a mapping $\Phi(\mathbf{v}) = \mathbf{u}$ defined. If this $\Phi$ can be shown to be a contraction mapping, then the equation

$$\frac{d}{dt}\mathbf{u}(t) + A(t,\mathbf{u})\mathbf{u}(t) = f(t,\mathbf{u}) \quad \text{for } 0 \leq t \leq T, \quad \text{with } \mathbf{u}(0) = u \tag{3.5}$$

will have a unique solution. Conditions required for this approach will be, naturally, more strict.

**Theorem 3.6** (From [9], unique existence)**.** *For equation (3.5), let us assume the following conditions are satisfied.*

- *(X) $X$ is a reflexive Banach space and there is $Y \subset X$, continuously and densely embedded in $X$, also reflexive. There is also an isomorphism $S$ of $Y$ onto $X$.*
- *(A1) For the open ball $W \subset Y$, $A : [0,T] \times W \mapsto G(X, 1, \beta)$.*
- *(A2) $SA(t,y)S^{-1} = A(t,y) + B(t,y)$ where $B(t,y) \in \mathcal{B}(X)$ for all $y \in W$, $t \in [0,T]$ and $\|B(t,y)\|_X \leq \lambda_1$ uniformly.*

*(A3)* $A(t,y) \in \mathcal{B}(Y,X)$ *for all* $(t,y) \in [0,T] \times W$ *with the additional conditions*

- *The mapping* $t \to A(t,y)$ *is continuous in* $\mathcal{B}(Y,X)$ *for all fixed* $y \in W$.
- *The mapping* $y \to A(t,y)$ *is Lipschitz continuous in* $\mathcal{B}(Y,X)$ *uniformly for all fixed* $t \in [0,T]$, *that is,* $\|A(t,y) - A(t,z)\|_{Y,X} \leq \mu_1 \|y - z\|_X$.

*(A4)* *For* $y_0 \in W$, *center of the ball* $W$, *we have* $A(t,y)y_0 \in Y$, *uniformly bounded* $\|A(t,y)y_0\|_Y \leq \lambda_2$ *for all* $(t,y) \in [0,T] \times W$.

*(f1)* $f(t,y)$ *is uniformly bounded in* $Y$, $\|f(t,y)\|_Y \leq \lambda_3$ *for all* $(t,y) \in [0,T] \times W$ *with the additional conditions*

- *The mapping* $t \to f(t,y)$ *is continuous in* $X$ *for all fixed* $y \in W$.
- *The mapping* $y \to f(t,y)$ *is Lipschitz continuous in* $X$ *uniformly for all fixed* $t \in [0,T]$, *that is,* $\|f(t,y) - f(t,z)\|_X \leq \mu_2 \|y - z\|_X$.

*Then, for initial data* $u_0 \in W$, *(3.5) will have a unique solution*

$$\mathbf{u} \in C([0,T'];W) \cap C^1([0,T'];X) \quad \textit{with } \mathbf{u}(0) = u_0$$

*for some* $0 < T' \leq T$.

**Corollary 3.7.** *For the homogeneous equations with* $f \equiv 0$, *having*

$$\langle Aw(t), w(t) \rangle \geq -\beta \|w(t)\|^2$$

*suffices to obtain condition* (A1).

*Proof.* Having $A \in G(X,1,\beta)$ means $\|e^{-tA}\| \leq e^{t\beta}$, but this is in operator norm for an evolution operator, thus it is just having $\|w(t)\| = \|e^{-tA}w(0)\| \leq e^{t\beta}\|w(0)\|$. For a homogeneous equation, we have

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2 = \int_{-\infty}^{\infty} w_t(x,t)w(x,t)\,dx = \langle w_t(t), w(t) \rangle = -\langle Aw(t), w(t) \rangle \leq \beta \|w(t)\|^2,$$

which immediately gives $\|w(t)\|^2 \leq \|w(0)\|^2 e^{2t\beta}$, proving the result. □

### 3.2.1 Application to Korteweg-de Vries Equation

We will now apply Theorem 3.6 to the KdV equation,

$$u_t + u_{xxx} + uu_x = 0 \quad x \in \mathbb{R}, 0 \leq t \leq T, \tag{3.6}$$

$$u(x,0) = u_0(x) \quad x \in \mathbb{R}. \tag{3.7}$$

We first have to convert this equation into a Banach space valued differential equation. One immediate suggestion is having $X = L^2$, $Y = H^s$ for some possibly non-integer

$s$, $S = (1 - D^2)^{s/2}$, $A(t, y) = A(y) = D^3 + yD$, $f(t, y) \equiv 0$ where $D$ denotes spatial differentiation operator $d/dx$.

Condition *(X)* is almost trivially satisfied, possibly except $S$ being the desired isomorphism.

**Proposition 3.8.** *The mapping $S(f) = (1 - D^2)^{s/2} f$ is an isomorphism of $H^s$ into $L^2$.*

*Proof.* Taking the situation to Fourier transformed spaces, what we have is $\hat{S}(\hat{f}) = (1 + \xi^2)^{s/2} \hat{f}$. We have seen that $f \in H^s$ implies $\hat{S}(\hat{f}) \in L^2$ before, therefore mapping is well-defined. Moreover, obviously it preserves norm, since it is how we have defined $H^s$ norm for non-integer $s$. Lastly, this mapping is an isomorphism because it has an inverse, $\hat{S}^{-1}(\hat{g}) = (1 + \xi^2)^{-s/2} \hat{g}$. $\square$

**Definition 3.9.** Operator $(1 - D^2)^{1/2}$ is traditionally shown as $\Lambda$, so $S$ is $\Lambda^s$.

*Remark.* This method of considering Fourier transformed spaces also justifies the fact that $\Lambda, \Lambda^{-1}$ and $D$ all commute, since they all correspond to multiplication by functions of $\xi$ in Fourier variables, which do commute.

*Remark.* For conditions *(A1-4)*, we will take $W$ as the ball with center 0, radius $R$ in $H^s$. Moreover, for $A(y)$ to be defined on subsets of $H^s$, we need to have, at least, $s \geq 3$ because of the $D^3$ factor in $A$.

For condition *(A1)*, we easily obtain

$$\langle (D^3 + yD)v, v \rangle = \int_{\mathbb{R}} v^{(3)} v + yv'v \, dx = \int_{\mathbb{R}} -y' \left( v^2/2 \right) \, dx \geq -\frac{1}{2} \|y'\|_\infty \|v\|^2 \geq -\frac{R}{2} \|v\|^2$$

since $y \in W$, a ball in $H^s$ with radius $R$, $s \geq 3$, and this result, with Corollary 3.7, satisfies the condition.

Condition *(A2)* can be expressed as having $\Lambda^s A(y) \Lambda^{-s} - A(y) \in \mathcal{B}(X)$ and bounded uniformly. By the remark above $\Lambda$ and $D$ commute, therefore we quickly get that

$$\begin{aligned} \Lambda^s A(y) \Lambda^{-s} - A(y) &= \Lambda^s (D^3 + yD) \Lambda^{-s} - (D^3 + yD) = \Lambda^s y D \Lambda^{-s} - yD \\ &= (\Lambda^s y D - y D \Lambda^s) \Lambda^{-s} = (\Lambda^s y D - y \Lambda^s D) \Lambda^{-s} = (\Lambda^s y - y \Lambda^s) D \Lambda^{-s} \\ &= (\Lambda^s y - y \Lambda^s) \Lambda^{-s} D. \end{aligned}$$

Now, calling $\Lambda^s y - y \Lambda^s = [\Lambda^s, y]$, the *commutator* of $\Lambda^s$ and $y$, and $M_f$ the operator of multiplication by function $f$ and using the fact that $\|[\Lambda^s, M_f] \Lambda^{1-s}\| \leq c \|f\|_s$ with $s > 3/2$ from [9] and [4], we get

$$\begin{aligned} \left\| \left( \Lambda^s A(y) \Lambda^{-s} - A(y) \right) f \right\| &= \|[\Lambda^s, y] \Lambda^{-s} D f\| = \|[\Lambda^s, y] \Lambda^{1-s} \Lambda^{-1} D f\| \\ &\leq c \|y\|_s \|\Lambda^{-1} D f\| \leq cR \|f\|, \end{aligned}$$

last step following from $\|y\|_s \leq R$ and

$$\|\Lambda^{-1}Df\|^2 = \|(\Lambda^{-1}Df)^\frown\|^2 = \int_{\xi\in\mathbb{R}} \left|\frac{-i\xi}{(1+\xi^2)^{1/2}}\right|^2 |\hat{f}|^2\, d\xi \leq \int_{\xi\in\mathbb{R}} |\hat{f}|^2\, d\xi = \|f\|^2.$$

Thus the operator $\Lambda^s A(y)\Lambda^{-s} - A(y)$ is uniformly bounded in $\mathcal{B}(X)$ by $cR$, satisfying the condition.

For condition *(A3)*, $f \in H^s$, we have

$$\|A(y)f\| = \|f''' + yf'\| \leq \|f\|_s + \|y\|_\infty \|f'\| \leq (1+R)\|f\|_s.$$

We therefore have $A(y) \in \mathcal{B}(Y, X)$ for all $y \in W$. For continuity conditions on $A$, observe that mapping $t \to A(t, y)$ is constant, so first part of *(A3)* is automatically satisfied. For the second part, what we want is to have $\|A(y_1) - A(y_2)\|_{L^2, H^s} \leq C\|y_1 - y_2\|_s$, that is, $\|[A(y_1) - A(y_2)]f\| \leq C\|y_1 - y_2\|_s \|f\|_s$ for all $f \in H^s$, $\|y\|_s \leq R$. Since $s \geq 1$, we get the result:

$$\begin{aligned}\|[A(y_1) - A(y_2)]f\| &= \|[(D^3 + y_1 D) - (D^3 + y_2 D)]f\| = \|(y_1 - y_2)f'\| \\ &\leq \|y_1 - y_2\|_\infty \|f'\| \leq \|y_1 - y_2\|_1 \|f\|_1 \leq \|y_1 - y_2\|_s \|f\|_s.\end{aligned}$$

Condition *(A4)* is automatically satisfied, because we have center $y_0 = 0$, therefore $A(y)0 = (D^3 + yD)0 = 0 \in Y$ for all $y \in W$.

Similarly for condition *(f1)*, we have $f \equiv 0$, automatically satisfying the condition.

**Theorem 3.10.** *For initial data $u_0 \in H^s$, $s \geq 3$, the KdV equation (3.6) has a unique solution $u \in \mathcal{H}^s_T \cap C^1([0,T]; H^{s-3})$ for some $T < \infty$.*

*Proof.* Discussion so far gives us solution to the KdV equation, $u$, exists in $C([0,T]; W) \cap C^1([0,T]; L^2)$ with $W \subset H^s$. However, $u_t = -u_x - uu_x - u_{xxx}$ shows us that $u_t(t) \in \mathcal{H}^{s-3}_T$, in other words $u \in C^1([0,T]; H^{s-3})$, concluding the proof. □

This result, while nice, does not add anything to the discussion we had so far. However, we can indeed lower the bound on $s$ to $s > 3/2$ by considering a different operator than $A(y) = D^3 + yD$. The following discussion is from [8].

If we make the transformation

$$u(t) = P(t)v(t) \quad \text{with } P(t) = e^{-tD^3},$$

we get $u_t(t) = -D^3 u(t) + P(t)v_t(t)$, so the KdV equation becomes

$$u_t + u_{xxx} + uu_x = P(t)v_t(t) + \big(P(t)v(t)\big)\big(DP(t)v(t)\big) = 0.$$

This equation can be rewritten as

$$\frac{d}{dt}v + A(t, v)v = 0 \tag{3.8}$$

with $A(t,y) = P(-t)M_{P(t)y}DP(t)$, $M_{P(t)y}$ being multiplication by function $P(t)y$. Now we want to apply Theorem 3.6 to this new equation. Showing the existence of such a $v$ will give us solution $u$ to the KdV equation.

We will again pick $X = L^2$, $Y = H^s$ (this time with $s > 3/2$), $W$ a ball in $H^s$ with center 0 and radius $R$ for this problem, $S = \Lambda^s$, $f \equiv 0$. Following the discussion above regarding the case with $A(y) = D^3 + yD$, conditions *(X, A4, f1)* are seen to be automatically satisfied in this case as well.

For condition *(A1)*, when checking this new condition introduced in Corollary 3.7 for our new operator $A(t,y) = P(-t)M_{P(t)y}DP(t)$, it should be noted that $P(t) = e^{-tD^3}$ is a continuous family of unitary operators, seen by Fourier transform of the operator, $e^{-ti\xi^3}$. Therefore, showing that the inequality holds for

$$\langle M_{P(t)y}Dw, w\rangle = \int_{\mathbb{R}} (P(t)y)(Dw)w\,dx = \int_{\mathbb{R}} P(t)y(w^2/2)'\,dx$$

suffices. Since $D$ and $P(t)$ commute, integrating by parts gives us

$$\int_{\mathbb{R}} P(t)y(w^2/2)'\,dx = \int_{\mathbb{R}} -P(t)y'(w^2/2)\,dx \geq -\frac{1}{2}\|P(t)y'\|_\infty\|w\| \geq -\frac{R}{2}\|w\|.$$

This holds because $y \in H^s$, therefore $y' \in H^{s-1} \subset L^\infty$, and $P(t)$ is unitary. Hence, condition *(A1)* is satisfied.

For condition *(A2)*, we are interested in $[\Lambda^s, A(t,y)]\Lambda^{-s}$. Direct manipulations give

$$\begin{aligned}
[\Lambda^s, A(t,y)]\Lambda^{-s} &= (\Lambda^s P(-t)M_{P(t)y}DP(t) - P(-t)M_{P(t)y}DP(t)\Lambda^s)\Lambda^{-s} \\
&= (P(-t)\Lambda^s M_{P(t)y}DP(t) - P(-t)M_{P(t)y}\Lambda^s DP(t))\Lambda^{-s} \\
&= P(-t)(\Lambda^s M_{P(t)y} - M_{P(t)y}\Lambda^s)DP(t)\Lambda^{-s} \\
&= P(-t)[\Lambda^s, M_{P(t)y}]\Lambda^{-s}DP(t)
\end{aligned}$$

since $\Lambda, D$ and $P(t)$ are all differential operators and commute. Now we were looking for $[\Lambda^s, A(t,y)]\Lambda^{-s} \in \mathcal{B}(L^2)$, but calculation above says it is same as aiming to get $P(-t)[\Lambda^s, M_{P(t)y}]\Lambda^{-s}DP(t) \in \mathcal{B}(L^2)$, and $P(t), P(-t)$ being unitary, it suffices to have $[\Lambda^s, M_{P(t)y}]\Lambda^{-s}D \in \mathcal{B}(L^2)$. In the end, as before, we get

$$\begin{aligned}
\|\big(\Lambda^s A(t,y)\Lambda^{-s} - A(t,y)\big)f\| &= \|[\Lambda^s, M_{P(t)y}]\Lambda^{-s}Df\| \leq \|[\Lambda^s, M_{P(t)y}]\Lambda^{1-s}\Lambda^{-1}Df\| \\
&\leq c\|P(t)y\|_s\|\Lambda^{-1}Df\| = c\|y\|_s\|\Lambda^{-1}Df\| \leq cR\|f\|,
\end{aligned}$$

showing that *(A2)* is satisfied.

Finally, condition *(A3)* is studied. First, we have $\|A(t,y)\|_{H^s,L^2} \leq \lambda_3$ for all $y \in W$, $0 \leq t \leq T$, seen from

$$\begin{aligned}
\|A(t,y)f\|^2 = \int_{\mathbb{R}} |(P(t)y(x))f'(x)|^2\,dx &\leq \|P(t)y\|_\infty^2\|f'\|^2 \\
&\leq \|P(t)y\|_1^2\|f\|_s^2 = \|y\|_1^2\|f\|_s^2 \leq R^2\|f\|_s^2. \quad (3.9)
\end{aligned}$$

For Lipschitz continuity of $y \to A(t,y)$, we look at

$$\|(A(t,y) - A(t,z))f\| = \|(M_{P(t)y} - M_{P(t)z})Df\|$$
$$= \|(M_{P(t)(y-z)})Df\| = \|A(t,y-z)f\| \leq \|y-z\|_s\|f\|_s \quad (3.10)$$

giving us the desired result. For continuity of $t \to A(t,y)$, note that continuity of $P(t)$, combined with the fact that $M_{P(t)y}$ is continuous in $\mathcal{B}(X)$, gives us the result.

This concludes the discussion, showing that there is a solution $v$. In turn, this is equivalent to showing $u$, solution to KdV equation. We can now finally state the main theorem for this section.

**Theorem 3.11.** *For initial data $u_0 \in H^s$, $s > 3/2$, the KdV equation (3.6) has a unique solution $u \in \mathcal{H}^s_T \cap C([0,T]; L^2)$ for some $T < \infty$ depending on $\|u_0\|_s$.*

## 3.3 Continuous Dependence and Global Extension

### 3.3.1 Continuous Dependence on Initial Data

As before, we will state the theorem from Kato [9] and apply it to the KdV equation with $A(t,y) = P(-t)M_{P(t)y}DP(t)$.

**Theorem 3.12.** *Assume that we have a sequence of equations*

$$\frac{d}{dt}u^n(t) + A^n(t,u^n)u^n = f^n(t,u^n) \quad 0 \leq t \leq T \quad (3.11a)$$

$$u^n(0) = u_{n0} \quad (3.11b)$$

*such that each term satisfies conditions* (X), (A1-4), (f1) *set in Theorem 3.6, in addition to the two conditions* (A5), (f2) *below, with bounds on estimates, $\lambda$, and Lipschitz constants, $\mu$, being uniform, that is, independent of $n$.*

*(A5) Mapping $y \to B(t,y)$ is Lipschitz continuous in $\mathcal{B}(X)$ uniformly for all $t \in [0,T]$,*

$$\|B(t,y) - B(t,z)\|_X \leq \mu_3\|y-z\|_Y.$$

*(f2) Mapping $y \to f(t,y)$ is Lipschitz continuous in $\mathcal{B}(Y)$ uniformly for all $t \in [0,T]$,*

$$\|f(t,y) - f(t,z)\|_Y \leq \mu_4\|y-z\|_Y.$$

*Then, if each term $A^n(t,y)$, $B^n(t,y)$ and $f^n(t,y)$ are convergent in their respective spaces $\mathcal{B}(Y,X)$, $\mathcal{B}(X)$ and $Y$ for each $(t,y) \in [0,T] \times W$ and $u_{n0}$ converges in $W$, then there is $T'' \leq T$ such that sequence of unique solutions $\{u^n\}$ corresponding to sequence of equations above converge to $u \in C([0,T'']; W) \cap C^1([0,T]; X)$, unique solution to problem*

$$\frac{d}{dt}u(t) + A(t,u)u = f(t,u) \quad 0 \leq t \leq T'',\ u(0) = u_0.$$

*Moreover, we have $u^n(t) \to u(t)$ in $Y$, uniformly in $t$.*

We want to deal with the same KdV equation, so only initial data $u_{n0}$ is changing. Moreover, in our study in the last section we had $f \equiv 0$, so condition *(f2)* is automatically satisfied. Therefore, checking Lipschitz continuity of $y \to B(t,y)$ suffices. For $f \in L^2$, $y, z \in W \subset H^s$,

$$\begin{aligned}
\big\|\big(B(t,y) - B(t,z)\big)f\big\| &= \big\|\big([\Lambda^s, A(t,y)]\Lambda^{-s} - [\Lambda^s, A(t,z)]\Lambda^{-s}\big)f\big\| \\
&= \Big\|\Big(P(-t)\big([\Lambda^s, M_{P(t)y}] - [\Lambda^s, M_{P(t)z}]\big)\Lambda^{-s}DP(t)\Big)f\Big\| \\
&= \big\|[\Lambda^s, M_{P(t)(y-z)}]\Lambda^{-s}Df\big\| = \big\|[\Lambda^s, M_{P(t)(y-z)}]\Lambda^{1-s}\Lambda^{-1}Df\big\| \\
&\leq c\|P(t)(y-z)\|_s\|\Lambda^{-1}Df\| \leq c\|y-z\|_s\|f\|,
\end{aligned}$$

last step following from $P(t)$ being unitary and $\|\Lambda^{-1}Df\| \leq \|f\|$ as we have done before. This concludes the discussion, showing that $y \to B(t,y)$ is indeed Lipschitz continuous, independent of $t$. We can now state our strongest result so far.

**Corollary 3.13.** *For $u_0 \in H^s$, $s > 3/2$, Theorem 3.11 holds with the mapping $u_0 \to u(t)$ continuous in $H^s$. Moreover, if $u_{0n} \to u_0$ in $H^s$, we have $u_n(t) \to u(t)$ in $H^s$, uniformly for $t \leq T' < T$.*

### 3.3.2 Global Extension to $T = \infty$

It should be noted that discussion in this section is not specific to the KdV equation. Basically any equation satisfying the conditions set forth can have a global extension. Methodology depends on combination of various ideas, which in turn can be applied to BBM equation introduced before in Section 2.1.

First, let us take $u(t) \in H^s$, solution to the KdV equation for some $s \geq 4$, $0 \leq t \leq T < \infty$. Then, after multiplying the equation by $u^3 + 3\big((uu_x)_x + uu_{xx}\big) + \frac{18}{5}u_{xxxx}$ and lengthy integrations by parts, we get

$$\frac{d}{dt}\int_{-\infty}^{\infty} \frac{9}{5}u_{xx}^2 - 3uu_x^2 + \frac{1}{4}u^4\,dx = 0$$

implying that a monotone increasing function $q : [0,\infty) \mapsto [0,\infty)$ can be found so that we have $\|u(t)\|_2 \leq q(\|u(0)\|_2)$ for all $u$, solution to the KdV equation in $H^s$ for some $s > 2$. A result of Kato [8] states that solutions of KdV equation have intervals of existence independent of $s$ in $H^s$, that is, if a solution to equation exists in both $\mathcal{H}_{T_1}^{s_1}$ and $\mathcal{H}_{T_2}^{s_2}$, then $T_1 = T_2$. With that, we see that $\|u(t)\|_2 \leq q(\|u(0)\|_2)$ is true for all solutions to the KdV equation in $\mathcal{H}_T^2$, not only for those in $\mathcal{H}_T^2$. This is simply because we can approximate $\phi \in H^2$ with a sequence $\phi_n \in H^s$, and solutions corresponding to $\phi_n$ will converge to solution corresponding to $\phi$ in $\mathcal{H}_T^2$.

With that, for any solution to the KdV equation in $\mathcal{H}_T^s$, we consider it in $\mathcal{H}_T^2$, extend it globally to get a result in $\mathcal{H}_\infty^2$, which, in turn implies $u \in \mathcal{H}_\infty^s$. This concludes this section, showing that global solutions to KdV equation exists for all initial data $u_0 \in H^s$, $s \geq 2$.

4

# Conclusion

Finally we have showed that the Cauchy problem for K-dV equation is well-posed for initial data $u_0 \in H^s$ with $s$ having lower bound $s > 3/2$. Tools we have used for this purpose are varied, from being specifically tailored to fit our needs to being simply an application of much more general theory.

We first used analysis with the aim of obtaining a Cauchy sequence in a Banach space. A well-posedness result was obtained, but it was not as strong as we would like. We then used non-integer interpolation, which had the benefit of being applicable in nonlinear situations such as ours. Though the conditions required by the interpolation is strict, it may be used in a wide range of problems.

In the last chapter, we peeked into semigroup theory and saw how operator theory has applications in the study of both partial and ordinary differential equations. After all, we have simply considered Banach space valued ordinary differential equations. This theory, easily and surely, can be applied to ordinary differential equations as well. However, main result regarding semigroup theory to be obtained from this thesis should be thus: this theory is not a trivial matter and indeed deserves serious study from those interested in analysis. Regardless, we managed to lower the bound on $s$ to $3/2$ for local existence, and this bound originates mainly from restrictions on Kato's theorem regarding norm estimate of commutator of $\Lambda^s$ and $M_f$. Similarly, bound $s \geq 2$ arises from relation $\|u(t)\|_2 \leq q(\|u_0\|_2)$. If these bounds could be played with, and lowered, we would have less and less smoothness requirements on initial data $u_0$.

Indeed, it is my knowledge that $s$ has been lowered as low as $-1$, but once again, we lose our familiar notion of *functions* once we are in the unfamiliar setting of $H^s$ for negative $s$, and obtain distributions instead, a discussion we have tried our best to not stray into. It should be stated that this study of the well-posedness for the KdV equation covers the results presented in mid-to-late 1970's, and not the more recent results regarding the problem.

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