Existence and Stability of Traveling Waves for a Class of Nonlocal Nonlinear Equations

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Abstract

In this article we are concerned with the existence and orbital stability of traveling wave solutions of a general class of nonlocal wave equations: $u_{tt} - Lu_{xx} = B(\pm|u|^{p-1}u)_{xx}$, $p > 1$. The main characteristic of this class of equations is the existence of two sources of dispersion, characterized by two coercive pseudo-differential operators L and B. Members of the class arise as mathematical models for the propagation of dispersive waves in a wide variety of situations. For instance, all Boussinesq-type equations and the so-called double-dispersion equation are members of the class. We first establish the existence of traveling wave solutions to the nonlocal wave equations considered. We then obtain results on the orbital stability or instability of traveling waves. For the case $L = I$, corresponding to a class of Klein-Gordon-type equations, we give an almost complete characterization of the values of the wave velocity for which the traveling waves are orbitally stable or unstable by blow-up.

Keywords: Solitary waves, Orbital stability, Boussinesq equation, Double dispersion equation, Concentration-compactness, Instability by blow-up, Klein-Gordon equation.
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1. Introduction

The present paper is concerned with the existence and stability of traveling wave solutions $u(x,t) = \phi_c(x-ct)$ of a general class of nonlocal nonlinear equations of the form

$$u_{tt} - Lu_{xx} = B(g(u))_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \tag{1.1}$$

where $c \in \mathbb{R}$ is the wave velocity, $u = u(x,t)$ is a real-valued function, $g(u) = \pm|u|^{p-1}u$ with $p > 1$, and L and B are linear pseudo-differential operators with smooth symbols $l(\xi)$ and $b(\xi)$, respectively. The orders of L and B will be denoted by ρ and $-r$, respectively. Here, and throughout this paper, we assume that *(i)* $r \geq 0$, *(ii)* for all k the symbols $l(\xi)$ and $b(\xi)$ satisfy the decay properties

$$\frac{d^k}{d\xi^k} l(\xi) = \mathcal{O}(|\xi|^{\rho-k}), \qquad \frac{d^k}{d\xi^k} b(\xi) = \mathcal{O}(|\xi|^{-r-k}) \quad \text{as} \quad |\xi| \to \infty, \tag{1.2}$$

and *(iii)* the pseudo-differential operators L and B are coercive elliptic operators; namely there exist positive constants c_1, c_2, c_3 and c_4 such that

$$c_1^2(1+\xi^2)^{\rho/2} \leq l(\xi) \leq c_2^2(1+\xi^2)^{\rho/2}, \tag{1.3}$$

$$c_3^2(1+\xi^2)^{-r/2} \leq b(\xi) \leq c_4^2(1+\xi^2)^{-r/2}, \tag{1.4}$$

for all $\xi \in \mathbb{R}$. Throughout the study we assume that the above constants c_i are chosen as the best constants. The aim of the present study is twofold: first to show the existence of traveling wave solutions $u(x,t) = \phi_c(x-ct)$ of (1.1) for the above-defined class of pseudo-differential operators L and B, and then to investigate the orbital stability and instability of those traveling wave solutions.

Equation (1.1) was first proposed in [1] as a general equation governing the propagation of doubly dispersive nonlinear waves. To illustrate the double nature of dispersion we rewrite (1.1) in the form $B^{-1}u_{tt} - LB^{-1}u_{xx} = (g(u))_{xx}$, where the first and second terms on the left-hand side represent two sources of dispersive effect. Clearly, for suitable choices of L and B, (1.1) will reduce to the well-known Boussinesq-type equations, including the Boussinesq equation [2], the improved Boussinesq equation [3] and the double dispersion equation [4] (see Section 3 of the present study and [1] for further details). An interesting reduction of (1.1) is established considering the operator B as a convolution integral

$$(Bv)(x) = (\beta * v)(x) = \int_{\mathbb{R}} \beta(x-y)v(y)dy \tag{1.5}$$

with the kernel function $\beta(x)$ and taking $L = B$. The resulting nonlocal nonlinear wave equation

$$u_{tt} = [\beta * (u + g(u))]_{xx} \tag{1.6}$$

describes the propagation of nonlinear strain waves in a one-dimensional, nonlocally elastic medium [5] (We refer the reader to [6, 7] for two different extensions of the model). The local existence, global existence and blow-up results for solutions of the Cauchy problem of (1.1) with initial data in suitable Sobolev spaces were provided in [1]. In a recent study [8], thresholds for global existence versus blow-up were established for (1.1) with power-type nonlinearities.

Existence and stability of traveling wave solutions of nonlinear wave equations are well studied in the literature starting from [9, 10] (see [11] for a recent overview of previous work). There have been a number of reliable existence, stability and instability results on the topic of solitary wave solutions of Boussinesq-type equations: [12, 13, 14, 15]. There are some studies addressing similar issues for unidirectional nonlocal wave equations involving pseudo-differential operators, see e.g., [16, 17, 18, 19, 20, 21, 22, 23]. With specific forms of L and B, the same questions for the nonlocal bidirectional wave equation (1.1) were studied in [24]. The purpose of the present study is to investigate existence and stability properties of traveling waves for the general class (1.1). We emphasize that the present study does not require any homogeneity and similar assumptions on the symbols $l(\xi)$ and $b(\xi)$.

It is well known that wave velocity ranges of the solitary waves are different for the Boussinesq equation (3.1) and the improved Boussinesq equation (3.4) (for details, see the examples in Section 3). To summarize, the Boussinesq equation has solitary waves for small values of c^2 when $g(u) = -|u|^{p-1}u$, while the improved Boussinesq equation has solitary waves for large values of c^2 when $g(u) = |u|^{p-1}u$. In the present study, we first observe that this is a general phenomena; traveling wave solutions of the class (1.1) with power nonlinearities exist for two different regimes. In the first regime, c^2 is small and $g(u) = -|u|^{p-1}u$ while in the second regime c^2 is large and $g(u) = |u|^{p-1}u$. Clearly, the Boussinesq equation and the improved Boussinesq equation are the most representative and studied examples of these two regimes, respectively. In the case of power nonlinearities, $g(u) = \pm|u|^{p-1}u$, the traveling wave solutions $u = \phi_c(x - ct)$ of

(1.1) satisfy the equation

$$(L - c^2 I)B^{-1}\phi_c \pm |\phi_c|^{p-1}\phi_c = 0, \tag{1.7}$$

where I is the identity operator. Then the order of L, i.e. ρ, is the determining parameter in this distinction regarding (1.1): for $\rho > 0$ the first regime occurs and for $\rho < 0$ the second regime occurs. The case $\rho = 0$ is of particular interest because both regimes occur. That is, when $\rho = 0$, traveling waves exist either for small c^2 and $g(u) = -|u|^{p-1}u$ or for large c^2 and $g(u) = |u|^{p-1}u$, as is observed for the double dispersion equation (3.7). In short, ρ determines the sign of $g(u)$ for which the traveling waves exist as well as the allowed values of c. Therefore, in the sequel, we consider the two regimes separately, which we will refer to shortly as the cases $\rho \geq 0$ and $\rho \leq 0$.

We first prove the existence of traveling wave solutions of (1.1) for both $\rho \geq 0$ and $\rho \leq 0$, separately. In both cases, the proof is based on a constrained variational problem, where traveling wave solutions appear as the critical points. We note that, in order to compensate for the non-homogeneity of the symbols, we use functionals that are not conserved integrals of (1.1). The concentration-compactness lemma of Lions [25, 26] is the main tool in establishing the existence of a minimizer of the constrained variational problem. In the case of $\rho \geq 0$ the traveling wave solution is also a minimizer of a certain conserved quantity allowing us to go further. On the other hand, for $\rho \leq 0$ the traveling wave solution turns out to be a saddle point and hence, as in the case of the improved Boussinesq equation, it does not allow us to get a stability result.

For orbital stability, in the case $\rho \geq 0$, we adopt a well-known general criteria in terms of convexity of a certain function $d(c)$ related to conserved quantities. In particular cases of (1.1), one can compute $d(c)$ explicitly, and obtain stability intervals for the wave velocity c. In our general case, this is not possible unless one makes further assumptions on the pseudo-differential operators L and B. Nevertheless, we are able to show that for general L and B the function $d(c)$ is not convex when c^2 is sufficiently small. Moreover, for $c = 0$ we further show the instability by blow-up using the blow-up threshold obtained in [8]. One case where we can compute $d(c)$ explicitly is when $L = I$ and general B, which gives rise to a class of Klein-Gordon-type equations. We thus obtain the orbital stability interval. Moreover, in this particular case, we are able to improve the blow-up result mentioned above for $c = 0$ and obtain an interval of c for instability by blow-up. It turns out that these two intervals complement one another. Hence, for this class of Klein-Gordon-type equations, we have an almost complete characterization of stability/instability regions in terms of c.

The structure of the paper is as follows. Section 2 reviews some previously known results, including the local existence theorem and the conserved quantities for (1.1). In Section 3, we start with some well-known examples that lead us to two regimes: $\rho \geq 0$ and $\rho \leq 0$. We then establish the existence of traveling wave solutions of (1.1) in both regimes by introducing constrained variational problems in a Sobolev space setting and using the concentration-compactness lemma of Lions [25, 26]. In Section 4, for the case $\rho \geq 0$, we prove some orbital stability and instability by blow-up results for the traveling wave solutions of (1.1). In Section 5, for the case $L = I$, we provide an almost complete characterization of stability/instability regions.

The remaining part of this section is devoted to the notation that is used in the rest of the paper. Throughout the paper, the symbol $\widehat{u}$ represents the Fourier transform of u, defined by $\widehat{u}(\xi) = \int_{\mathbb{R}} u(x)e^{-i\xi x}dx$. The L^p, $1 \leq p < \infty$ and L^∞ norms of u on $\mathbb{R}$ are denoted by $\|u\|_{L^p}$ and $\|u\|_{L^\infty}$, respectively. The inner product of u and v in $L^2(\mathbb{R})$ is represented by $\langle u, v\rangle$. The L^2 Sobolev space of order s on $\mathbb{R}$ is denoted by $H^s = H^s(\mathbb{R})$ with the norm $\|u\|^2_{H^s} = \int_{\mathbb{R}}(1+\xi^2)^s|\widehat{u}(\xi)|^2 d\xi$. The symbol $\mathbb{R}$ in $\int_{\mathbb{R}}$ will be mostly suppressed to simplify exposition. C is a generic positive constant. D_x is the partial derivative with respective to x.

2. Preliminaries: Local Existence and Conserved Quantities

In the study of existence and stability of traveling wave solutions of nonlinear dispersive equations both the local well-posedness theory of the inital-value problem and the conservation laws of energy and momentum play a key role. For the convenience of the reader, this section contains background material on these issues that will be used in later sections.

To make our exposition self-contained we start with the statement of the local existence theorem proved in [1] for the Cauchy problem

$$u_{tt} - Lu_{xx} = B(g(u))_{xx}, \quad x \in \mathbb{R}, \quad t > 0 \tag{2.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R} \tag{2.2}$$

with a general nonlinear function $g(u)$.

Theorem 2.1. *[1] Let $s > \frac{1}{2}$, $u_0 \in H^s$, $u_1 \in H^{s-1-\frac{\rho}{2}}$ and $g \in C^{[s]+1}$. Assume that L and B satisfy (1.3)-(1.4) with $\rho \geq 0$ and $r + \frac{\rho}{2} \geq 1$. Then, there exists some $T > 0$ so that the Cauchy problem (2.1)-(2.2) is locally well-posed with solution $u \in C([0,T), H^s) \cap C^1([0,T), H^{s-1-\frac{\rho}{2}})$.*

Before giving the conserved quantities, we make two remarks regarding Theorem 2.1. First, even though it was proved for $\rho \geq 0$ in [1], here we remark that the proof also works when $\rho > -2$. This is due to the acting semigroup

$$\mathcal{S}(t)v = \mathcal{F}^{-1}\left(\frac{\sin(\xi\sqrt{l(\xi)t})}{\xi\sqrt{l(\xi)}}\right)\mathcal{F}v,$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier and inverse Fourier transform operators. We note that $\rho + 2$ is in fact the order of the operator $\partial_x^2 L$. Observing that one may prove this new assertion in the same fashion as Theorem 2.1 was proved, we leave the details to the reader. Secondly, when $\rho \leq -2$, (2.1) becomes an H^s-valued ordinary differential equation and then the local well-posedness proof of [5] applies. Below we state these two observations as a theorem:

Theorem 2.2. *Let $s > \frac{1}{2}$, and $g \in C^{[s]+1}$.*

(i) If L and B satisfy (1.3)-(1.4) with $\rho > -2$ and $r + \frac{\rho}{2} \geq 1$, then there exists some $T > 0$ so that the Cauchy problem (2.1)-(2.2) is locally well-posed with solution $u \in C([0,T), H^s) \cap C^1([0,T), H^{s-1-\frac{\rho}{2}})$ for initial data $u_0 \in H^s$ and $u_1 \in H^{s-1-\frac{\rho}{2}}$.

(ii) If L and B satisfy (1.3)-(1.4) with $\rho \leq -2$ and $r \geq 2$, then there exists some $T > 0$ so that the Cauchy problem (2.1)-(2.2) is locally well-posed with solution $u \in C^1([0,T), H^s)$ for initial data $u_0 \in H^s$ and $u_1 \in H^s$.

As it was done in [8], for convenience we rewrite (2.1) as a system of equations and consider the Cauchy problem

$$u_t = w_x, \quad x \in \mathbb{R}, \quad t > 0 \tag{2.3}$$

$$w_t = Lu_x + B(g(u))_x, \quad x \in \mathbb{R}, \quad t > 0 \tag{2.4}$$

$$u(x,0) = u_0(x), \quad w(x,0) = w_0(x), \quad x \in \mathbb{R}. \tag{2.5}$$

Below we state the local well-posedness theorem of the Cauchy problem (2.3)-(2.5) in terms of the pair (u, w).

Theorem 2.3. *Let $s > \frac{1}{2}$, and $g \in C^{[s]+1}$.*

(i) If L and B satisfy (1.3)-(1.4) with $\rho > -2$ and $r + \frac{\rho}{2} \geq 1$, then there exists some $T > 0$ so that the Cauchy problem (2.3)-(2.5) is locally well-posed with solution $(u, w) \in C([0,T), H^s) \times C([0,T), H^{s-\frac{\rho}{2}})$ for initial data $(u_0, w_0) \in H^s \times H^{s-\frac{\rho}{2}}$.

(ii) If L and B satisfy (1.3)-(1.4) with $\rho \leq -2$ and $r \geq 2$, then there exists some $T > 0$ so that the Cauchy problem (2.3)-(2.5) is locally well-posed with solution $(u, w) \in C([0,T), H^s) \times C([0,T), H^{s+1})$ for initial data $(u_0, w_0) \in H^s \times H^{s+1}$.

Remark 2.4. *Clearly, the solution predicted by Theorem 2.3 can be extended to the maximal time interval $[0, T_{\max})$ where $T_{\max}$, if finite, is characterized by the blow-up conditions*

$$\limsup_{t \to T_{\max}^-} \left(\|u(t)\|_s + \|w(t)\|_{s-\frac{\rho}{2}} \right) = \infty \qquad \text{in case (i)}$$

and

$$\limsup_{t \to T_{\max}^-} \left(\|u(t)\|_s + \|w(t)\|_{s+1} \right) = \infty \qquad \text{in case (ii).}$$

The laws of conservation of energy and momentum for the system (2.3)-(2.5) with $g(u) = \pm |u|^{p-1} u$ are

$$\begin{aligned} \mathcal{E}(u(t), w(t)) &= \frac{1}{2} \left\| B^{-1/2} w(t) \right\|_{L^2}^2 + \frac{1}{2} \left\| L^{1/2} B^{-1/2} u(t) \right\|_{L^2}^2 \pm \frac{1}{p+1} \|u(t)\|_{L^{p+1}}^{p+1} \\ &= \mathcal{E}(u_0, w_0) \qquad (2.6) \\ \mathcal{M}(u(t), w(t)) &= \int \left(B^{-1/2} u(t) \right) \left(B^{-1/2} w(t) \right) dx = \mathcal{M}(u_0, w_0), \qquad (2.7) \end{aligned}$$

respectively. For the details of deriving these conservation laws we refer the reader to [8].

3. Existence of traveling waves

In this section we prove that (1.1) with $g(u) = \pm |u|^{p-1} u$, $p > 1$ has traveling wave solutions of the form $u(x,t) = \phi_c(x - ct)$ for suitable values of wave velocity c and the appropriate choice of the sign $\pm$. Assuming that ϕ_c, $LB^{-1}\phi_c$, $B^{-1}\phi_c$ and their first-order derivatives decay sufficiently rapidly at infinity, it is readily seen that $u(x,t)=\phi_c(x-ct)$ satisfies (1.1) if ϕ_c solves (1.7). We will prove the existence of solutions of (1.7) through a constrained variational problem.

To motivate our investigation we first consider the following three classical examples.

Example 1. (The Boussinesq Equation) If we take $L=I-\partial_x^2$, $B=I$ (for which $\rho=2$ and $r=0$, respectively) and $g(u)=-|u|^{p-1}u$, then (1.1) reduces to the (generalized) Boussinesq equation [2]

$$u_{tt}-u_{xx}+u_{xxxx}=-(|u|^{p-1}u)_{xx}. \tag{3.1}$$

Solitary wave solutions to the Boussinesq equation satisfy

$$\phi_c''-(1-c^2)\phi_c+|\phi_c|^{p-1}\phi_c=0, \tag{3.2}$$

where the prime represents the derivative with respect to $\zeta=x-ct$. When $c^2<1$, the explicit solution is given by

$$\phi_c(\zeta)=\left[\frac{1}{2}(p+1)(1-c^2)\right]^{\frac{1}{p-1}}\operatorname{sech}^{\frac{2}{p-1}}\left[\frac{1}{2}(p-1)(1-c^2)^{\frac{1}{2}}\zeta\right]. \tag{3.3}$$

Example 2. (The Improved Boussinesq Equation) If we take $L=B=(I-\partial_x^2)^{-1}$ (for which $\rho=-2$ and $r=2$) and $g(u)=|u|^{p-1}u$, then (1.1) reduces to the improved Boussinesq equation [3]

$$u_{tt}-u_{xx}-u_{xxtt}=(|u|^{p-1}u)_{xx}. \tag{3.4}$$

Solitary wave solutions to the improved Boussinesq equation satisfy

$$c^2\phi_c''-(c^2-1)\phi_c+|\phi_c|^{p-1}\phi_c=0. \tag{3.5}$$

When $c^2>1$, the explicit solution is given by

$$\phi_c(\zeta)=\left[\frac{1}{2}(p+1)(c^2-1)\right]^{\frac{1}{p-1}}\operatorname{sech}^{\frac{2}{p-1}}\left[\frac{1}{2}(p-1)(1-\frac{1}{c^2})^{\frac{1}{2}}\zeta\right]. \tag{3.6}$$

Example 3. (The Double Dispersion Equation) Let $L=(I-a_1\partial_x^2)^{-1}(I-a_2\partial_x^2)$ and $B=(I-a_1\partial_x^2)^{-1}$ for two positive constants a_1 and a_2 in which $\rho=0$ and $r=2$. Then (1.1) reduces to the double dispersion equation [4]

$$u_{tt}-u_{xx}-a_1u_{xxtt}+a_2u_{xxxx}=(g(u))_{xx}. \tag{3.7}$$

Solitary wave solutions to the double dispersion equation satisfy

$$(a_2 - a_1 c^2)\phi_c'' - (1 - c^2)\phi_c = g(\phi_c). \tag{3.8}$$

It is worth noting that sech-type solitary wave solutions to (3.8) may be obtained in two regimes. The first regime is identified by the equations

$$c^2 < \min\{1, \frac{a_2}{a_1}\}, \qquad g(\phi_c) = -|\phi_c|^{p-1}\phi_c \tag{3.9}$$

and with the solitary wave solutions

$$\phi_c(\zeta) = \left[\frac{1}{2}(p+1)(1-c^2)\right]^{\frac{1}{p-1}} \operatorname{sech}^{\frac{2}{p-1}} \left[\frac{1}{2}(p-1)\left(\frac{1-c^2}{a_2 - a_1 c^2}\right)^{\frac{1}{2}} \zeta\right], \tag{3.10}$$

whereas the second regime is described by

$$c^2 > \max\{1, \frac{a_2}{a_1}\}, \qquad g(\phi_c) = +|\phi_c|^{p-1}\phi_c \tag{3.11}$$

and with the solitary wave solutions

$$\phi_c(\zeta) = \left[\frac{1}{2}(p+1)(c^2-1)\right]^{\frac{1}{p-1}} \operatorname{sech}^{\frac{2}{p-1}} \left[\frac{1}{2}(p-1)\left(\frac{c^2-1}{a_1 c^2 - a_2}\right)^{\frac{1}{2}} \zeta\right]. \tag{3.12}$$

We note that the coercivity constants of L in this particular case are

$$c_1^2 = \min\{1, \frac{a_2}{a_1}\}, \qquad c_2^2 = \max\{1, \frac{a_2}{a_1}\},$$

hence the inequalities of (3.9) and (3.11) can be expressed as $c^2 < c_1^2$ and $c^2 > c_2^2$, respectively. Note that, in the limiting cases $(a_1, a_2) = (0, 1)$ or $(a_1, a_2) = (1, 0)$, (3.7) reduces to the Boussinesq equation or the improved Boussinesq equation, respectively. Indeed, in those limiting cases, one of the two regimes disappears.

As the above examples show, the sign of the order of the operator L and the sign of the nonlinear term determine together the range of c for which a traveling wave solution exists. The general case of (1.1) can be handled in much the same way by identifying two regimes. We describe the two regimes characterized by the equations

$$\rho \geq 0, \qquad g(\phi_c) = -|\phi_c|^{p-1}\phi_c, \tag{3.13}$$

and by

$$\rho \leq 0, \qquad g(\phi_c) = +|\phi_c|^{p-1}\phi_c, \tag{3.14}$$

respectively. While the Boussinesq equation serves as a prototype equation for the case defined in (3.13), the improved Boussinesq equation provides a prototype equation for the case (3.14). In the same manner, we see that the double dispersion equation for which $\rho = 0$ belongs to both of the two regimes. In the next two subsections we will prove the existence of traveling wave solutions of (1.1) for the regimes defined by (3.13) and (3.14), respectively.

3.1. The case $\rho \geq 0$ and $g(u) = -|u|^{p-1}u$

Throughout this subsection we assume that we are in the regime described by (3.13). To satisfy the requirements imposed by Theorem 2.1 we also assume that L and B satisfy (1.2)-(1.4) with $r + \frac{\rho}{2} \geq 1$ in addition to $\rho \geq 0$. Let

$$s_0 = \frac{r}{2} + \frac{\rho}{2}. \tag{3.15}$$

Note that the above inequalities imply $s_0 \geq \frac{1}{2}$. For $\psi \in H^{s_0}$, we now define the following functionals

$$\mathcal{I}_c(\psi) = \frac{1}{2}\int_{\mathbb{R}} (L^{1/2}B^{-1/2}\psi)^2 dx - \frac{c^2}{2}\int_{\mathbb{R}} (B^{-1/2}\psi)^2 dx \tag{3.16}$$

$$\mathcal{Q}(\psi) = \int_{\mathbb{R}} |\psi|^{p+1} dx. \tag{3.17}$$

It is worth pointing out that they are not conserved integrals of (1.1). By the Sobolev embedding theorem, we have $H^{s_0} \subset H^{1/2} \subset L^q$ for all $q \geq 2$. This insures that the functionals $\mathcal{I}_c(\psi)$ and $\mathcal{Q}(\psi)$ are well-defined on H^{s_0}. We also note that the space $H^{s_0} \times H^{s_0-\frac{\rho}{2}}$ is the natural space for the energy and momentum functionals in (2.6) and (2.7).

We begin by proving a coercivity estimate for $\mathcal{I}_c(\psi)$, which holds only for $c^2 < c_1^2$ where c_1 is the ellipticity constant of L.

Lemma 3.1. *Let $c^2 < c_1^2$. Then there are positive constants γ_1, γ_2 such that*

$$\gamma_1\|\psi\|_{H^{s_0}}^2 \leq \mathcal{I}_c(\psi) \leq \gamma_2\|\psi\|_{H^{s_0}}^2.$$

Proof. By (1.3) and (1.4) we have

$$(c_1^2 - c^2)(1+\xi^2)^{\rho/2} \leq c_1^2(1+\xi^2)^{\rho/2} - c^2 \leq l(\xi) - c^2 \leq c_2^2(1+\xi^2)^{\rho/2}, \tag{3.18}$$

and

$$\frac{1}{c_4^2}(1+\xi^2)^{r/2} \leq b^{-1}(\xi) \leq \frac{1}{c_3^2}(1+\xi^2)^{r/2}, \tag{3.19}$$

respectively. Using Parseval's theorem for (3.16) and combining

$$\mathcal{I}_c(\psi) = \frac{1}{2}\int \left(l(\xi) - c^2\right) b^{-1}(\xi) |\widehat{\psi}(\xi)|^2 d\xi$$

with (3.18) and (3.19) yields

$$\frac{c_1^2 - c^2}{2c_4^2}\|\psi\|_{H^{s_0}}^2 \leq \mathcal{I}_c(\psi) \leq \frac{c_2^2}{2c_3^2}\|\psi\|_{H^{s_0}}^2.$$

□

Remark 3.2. *The important point to note here is that the above proof works only under the assumption $\rho \geq 0$.*

Remark 3.3. *From Lemma 3.1 it follows that when $c^2 < c_1^2$, $\sqrt{\mathcal{I}_c(\psi)}$ defines a norm equivalent to the H^{s_0} norm.*

For $c^2 < c_1^2$ we now consider the variational problem

$$m_1(c) = \inf\left\{\mathcal{I}_c(\psi) : \psi \in H^{s_0}, \quad \mathcal{Q}(\psi) = 1\right\}. \tag{3.20}$$

A sequence $\{\psi_n\}$ in H^{s_0} is called a minimizing sequence for $m_1(c)$, if $\mathcal{Q}(\psi_n) = 1$ for all n and $\lim_{n\to\infty} \mathcal{I}_c(\psi_n) = m_1(c)$. Let $\{\tilde{\psi}_n\}$ be a sequence in H^{s_0} such that $\lim_{n\to\infty} \mathcal{I}_c(\tilde{\psi}_n) = m_1(c)$ and $\mathcal{Q}(\tilde{\psi}_n) = \lambda_n$ with $\lim_{n\to\infty} \lambda_n = 1$. Then $\psi_n = \lambda_n^{-1/(p+1)}\tilde{\psi}_n$ will be a minimizing sequence and the sequences $\{\psi_n\}$ and $\{\tilde{\psi}_n\}$ have the same limiting behavior. We will henceforth abuse the terminology and refer also to $\{\tilde{\psi}_n\}$ as a minimizing sequence.

We emphasize here two aspects of the variational problem. First, $m_1(c) > 0$. Since $\mathcal{Q}(\psi) = \|\psi\|_{L^{p+1}}^{p+1} = 1$, we have $1 = \|\psi\|_{L^{p+1}}^{p+1} \leq C\|\psi\|_{H^{s_0}}^{p+1}$ where C is the Sobolev embedding constant. By Lemma 3.1, $\mathcal{I}_c(\psi) \geq \gamma_1\|\psi\|_{H^{s_0}}^2 \geq \gamma_1 C^{-1} > 0$ so that $m_1(c) > 0$. Second, note that a minimizing sequence $\{\psi_n\}$ is always bounded in H^{s_0}. This is a direct consequence of $\|\psi_n\|_{H^{s_0}}^2 \leq \gamma_1^{-1}\mathcal{I}_c(\psi_n)$ together with the fact that $\mathcal{I}_c(\psi_n)$ is convergent.

The main results of this subsection are Theorem 3.11 establishing the existence of minimizers of (3.20) and Theorem 3.13 showing that the minimizers are in fact traveling wave solutions of (1.1). The rest of this section will be devoted mainly to the proof of Theorem 3.11, which is based on the Concentration Compactness Lemma of Lions [25, 26] given below.

Lemma 3.4. *(Concentration Compactness Lemma) Let $\{\rho_n\}$ be a sequence of nonnegative functions in L^1 satisfying $\int \rho_n(x)dx = \mu$ for all n and some $\mu > 0$. Then there is a subsequence ρ_{n_k} satisfying one of the following conditions:*

(i) (Compactness) There are real numbers y_k for $k = 1, 2, \cdots$, such that for any $\epsilon > 0$, there is a $R > 0$ large enough that

$$\int_{|x-y_k|\leq R} \rho_{n_k}(x)dx \geq \mu - \epsilon.$$

(ii) (Vanishing) For any $R > 0$, $\lim_{k\to\infty} \sup_{y\in\mathbb{R}} \int_{|x-y|\leq R} \rho_{n_k}(x)dx = 0$.

(iii) (Dichotomy) There exists $\tilde{\mu} \in (0, \mu)$ such that for any $\epsilon > 0$, there exists $k_0 \geq 1$, and ρ_k^1, $\rho_k^2 \geq 0$ such that for $k \geq k_0$

$$\begin{aligned}
&\|\rho_{n_k} - (\rho_k^1 + \rho_k^2)\|_{L^1} \leq \epsilon, \\
&\left|\int \rho_k^1(x)dx - \tilde{\mu}\right| \leq \epsilon, \qquad \left|\int \rho_k^2(x)dx - (\mu - \tilde{\mu})\right| \leq \epsilon, \\
&\textit{supp } \rho_k^1 \cap \textit{supp } \rho_k^2 = \emptyset, \qquad \textit{dist}\{\textit{supp } \rho_k^1, \textit{supp } \rho_k^2\} \to \infty \textit{ as } k \to \infty.
\end{aligned}$$

Remark 3.5. *Lemma 3.4 also holds under the weaker condition $\lim_{n\to\infty} \int \rho_n(x)dx = \mu$ for some $\mu > 0$.*

For later analysis, it will be convenient to express the functional $\mathcal{I}_c$ in the form

$$\mathcal{I}_c(\psi) = \frac{1}{2}\|K_c\psi\|_{L^2}^2 + \frac{1}{2}\gamma_c\|\psi\|_{L^2}^2$$

where K_c is a suitable coercive operator with the symbol $k_c(\xi)$ and γ_c is a positive constant. This is equivalent to saying that

$$(L - c^2 I)B^{-1} = K_c^2 + \gamma_c I$$

or, in terms of the symbols $\left(l(\xi) - c^2\right) b^{-1}(\xi) = k_c^2(\xi) + \gamma_c$. By (3.18) and (3.19) it is obvious that $(l(\xi) - c^2)b^{-1}(\xi) \geq (c_1^2 - c^2)c_4^{-2}$. So taking $\gamma_c = (c_1^2 - c^2)/(2c_4^2)$ we get

$$k_c^2(\xi) = (l(\xi) - c^2)b^{-1}(\xi) - \frac{c_1^2 - c^2}{2c_4^2} \geq \frac{c_1^2 - c^2}{2c_4^2}.$$

Clearly K_c is a pseudo-differential operator of order s_0, exhibiting decay properties similar to those in (1.2).

Let the sequence $\{\rho_n(x)\}$ be defined by

$$\rho_n(x) = \frac{1}{2}|K_c\psi_n(x)|^2 + \frac{1}{2}\gamma_c|\psi_n(x)|^2$$

for a minimizing sequence $\{\psi_n\}$. By the definition of a minimizing sequence we have $\lim_{n\to\infty}\int \rho_n dx = m_1(c) > 0$. In what follows, we will apply the concentration-compactness principle of Lions to the above-defined sequence ρ_n. We follow the classical approach and show that neither vanishing nor dichotomy holds. To this end, we have divided our task into a sequence of lemmas. To rule out vanishing we will use the following lemma [11] (pp 125), which is a variant of Lemma I.1 in [26]:

Lemma 3.6. *Suppose $\alpha > 0$ and $\delta > 0$ are given. Then there exists $\eta = \eta(\alpha, \delta) > 0$ such that if $f_n \in H^{1/2}$ with $\|f_n\|_{H^{1/2}} \leq \alpha$ and $\|f_n\|_{L^{p+1}} \geq \delta$, then*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{y-2}^{y+2} | f_n(x) |^{p+1} dx \geq \eta.$$

We can now state and prove the following.

Lemma 3.7. *Vanishing does not occur.*

Proof. We proceed by contradiction and assume that vanishing occurs. Then

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}} \int_{y-2}^{y+2} | \psi_{n_k}(x) |^2 dx = 0.$$

Since ψ_{n_k} is bounded in $H^{s_0} \subset H^{1/2}$, we have $\|\psi_{n_k}\|_{H^{1/2}} \leq \alpha$ and $\|\psi_{n_k}\|_{L^{p+1}} = 1$. It follows from Lemma 3.6 that there is some $\eta > 0$ for which

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}} \int_{y-2}^{y+2} | \psi_{n_k}(x) |^{p+1} dx \geq \eta.$$

On the other hand,

$$\begin{aligned}
\left(\int_{y-2}^{y+2} | \psi_{n_k}(x) |^{p+1} dx\right)^2 &\leq \left(\int_{y-2}^{y+2} | \psi_{n_k}(x) |^{2p} dx\right)\left(\int_{y-2}^{y+2} | \psi_{n_k}(x) |^2 dx\right) \\
&\leq \|\psi_{n_k}\|_{L^{2p}}^{2p} \int_{y-2}^{y+2} | \psi_{n_k}(x) |^2 dx \\
&\leq C\|\psi_{n_k}\|_{H^{1/2}}^{2p} \int_{y-2}^{y+2} | \psi_{n_k}(x) |^2 dx \\
&\leq C\alpha^{2p} \int_{y-2}^{y+2} | \psi_{n_k}(x) |^2 dx,
\end{aligned}$$

which implies

$$\eta^2 \leq \lim_{k\to\infty} \sup_{y\in\mathbb{R}} \left(\int_{y-2}^{y+2} |\, \psi_{n_k}(x)\, |^{p+1}\, dx \right)^2 \leq C\alpha^{2p} \lim_{k\to\infty} \sup_{y\in\mathbb{R}} \int_{y-2}^{y+2} |\, \psi_{n_k}(x)\, |^2\, dx.$$

This contradicts our assumption. □

To prove that dichotomy does not occur, it is convenient to define the family of variational problems

$$m_\lambda(c) = \inf \{\mathcal{I}_c(\phi) : \phi \in H^{s_0}, \quad \mathcal{Q}(\phi) = \lambda\} \tag{3.21}$$

where $\lambda > 0$. Note that as $\mathcal{I}_c$ and $\mathcal{Q}$ are homogeneous of degrees 2 and $p+1$, respectively, we have the scaling $m_\lambda(c) = \lambda^{\frac{2}{p+1}} m_1(c)$. Moreover, since $g(\eta) = \eta^{\frac{2}{p+1}} + (1-\eta)^{\frac{2}{p+1}} > 1$ for all $\eta \in (0,1)$, we obtain the strict subadditivity condition of $m_\lambda(c)$ described in the following lemma:

Lemma 3.8. *For any* $\lambda \in (0,1)$,

$$m_\lambda(c) + m_{1-\lambda}(c) > m_1(c).$$

We need commutator estimates for pseudo-differential operators to control nonlocal terms. The following lemma is due to [22] (Lemma 2.12). Below we give an alternative proof relying, as in [22], on the commutator estimate of Coifman and Meyer (Theorem 35 of [31]). We note that for $N = s_0 = 0$ the assertion of Lemma 3.9 reduces to Coifman and Meyer's estimate.

Lemma 3.9. *Let* $u \in H^{s_0}$ *and* $\theta \in C^\infty(\mathbb{R})$ *with bounded derivatives of all orders. Then, for the commutator* $[K_c, \theta]\, u = K_c(\theta u) - \theta K_c u$ *we have the estimate*

$$\|\, [K_c, \theta]\, u\|_{L^2} \leq C \left(\sum_{n=1}^{N+1} \|\theta^{(n)}\|_{L^\infty} \right) \|u\|_{H^{s_0}},$$

where $N = [s_0]$ *and* C *is a positive constant.*

Proof. Before embarking on the proof, let us write down $k_c(\xi)$ in the form:

$$k_c(\xi) = k_c(0) + \sum_{j=1}^{N} \frac{k_c^{(j)}(0)}{j!} \xi^j + \xi^{N+1} r(\xi)$$

where a superscript in parenthesis indicates order of the derivative. We thus get $K_c = k_c(0)I + P(D_x) + D_x^{N+1}R$, where $P(D_x)$ is the differential operator of order N with vanishing constant term and R is the operator with symbol $r(\xi)$ of nonpositive order. Also we have the decay estimates

$$|D_\xi^n r(\xi)| = \mathcal{O}(|\xi|^{-n}) \quad \text{as} \quad |\xi| \to \infty \quad \text{for every} \quad n \in \mathbb{N}.$$

Hence R satisfies the hypotheses of Theorem 35 in [31] and thus there exists a constant C such that

$$\|[R,\theta]f'\|_{L^2} \leq C\|\theta'\|_{L^\infty}\|f\|_{L^2}.$$

An easy computation shows that the commutator satisfies

$$[K_c,\theta] = [P(D_x),\theta] + [RD_x^{N+1},\theta]. \tag{3.22}$$

Note that D_x commutes with R. By the Leibniz rule we have

$$[P(D_x),\theta]u = P(D_x)(\theta u) - \theta P(D_x)u = \sum_{n=1}^{N} \theta^{(n)} P_{N-n}(D_x)u,$$

where $P_{N-n}(D_x)$ is a differential operator of order $N-n$. We thus get

$$\begin{aligned} \|[P(D_x),\theta]u\|_{L^2} &\leq \sum_{n=1}^{N} \|\theta^{(n)} P_{N-n}(D_x)u\|_{L^2} \leq C\left(\sum_{n=1}^{N} \|\theta^{(n)}\|_{L^\infty}\|D_x^{N-n}u\|_{L^2}\right) \\ &\leq C\left(\sum_{n=1}^{N} \|\theta^{(n)}\|_{L^\infty}\right)\|u\|_{H^N}. \end{aligned} \tag{3.23}$$

Using the Leibniz rule again we obtain

$$\begin{aligned} [RD_x^{N+1},\theta]u &= RD_x^{N+1}(\theta u) - \theta(RD_x^{N+1}u) = R\left(\sum_{n=0}^{N+1} C_{N+1}^n \theta^{(n)} D_x^{N+1-n}u\right) - \theta(RD_x^{N+1}u) \\ &= \sum_{n=1}^{N+1} C_{N+1}^n R(\theta^{(n)} D_x^{N+1-n}u) + R(\theta D_x^{N+1}u) - \theta(RD_x^{N+1}u) \\ &= \sum_{n=1}^{N+1} C_{N+1}^n R(\theta^{(n)} D_x^{N+1-n}u) + [R,\theta]D_x^{N+1}u, \end{aligned}$$

where the C_{N+1}^n's are constants. We proceed to show that

$$\begin{aligned}\left\|\sum_{n=1}^{N+1} C_{N+1}^n R\left(\theta^{(n)} D_x^{N+1-n} u\right)\right\|_{L^2} &\leq \sum_{n=1}^{N+1} C_{N+1}^n \|R(\theta^{(n)} D_x^{N+1-n} u)\|_{L^2} \\ &\leq C \sum_{n=1}^{N+1} \|\theta^{(n)}\|_{L^\infty} \|D_x^{N+1-n} u\|_{L^2} \\ &\leq C \left(\sum_{n=1}^{N+1} \|\theta^{(n)}\|_{L^\infty}\right) \|u\|_{H^N}. \end{aligned} \tag{3.24}$$

By Coifman and Meyer's theorem [31] it follows that

$$\|[R,\theta] D_x^{N+1} u\|_{L^2} = \|[R,\theta](D_x^N u)'\|_{L^2} \leq C\|\theta'\|_{L^\infty} \|D_x^N u\|_{L^2} \leq C\|\theta'\|_{L^\infty} \|u\|_{H^N}. \tag{3.25}$$

Finally, combining (3.22), (3.23), (3.24) and (3.25) yields the result. □

Next, we rule out dichotomy through the following lemma.

Lemma 3.10. *Dichotomy does not occur.*

Proof. Suppose dichotomy occurs. Then, by Lemma 3.4 there is $\tilde{\mu} \in (0,\mu)$ such that for any $\epsilon > 0$, there exists $k_0 \geq 1$, and $\rho_k^1, \rho_k^2 \geq 0$ such that for $k \geq k_0$

$$\begin{aligned}&\|\rho_{n_k} - (\rho_k^1 + \rho_k^2)\|_{L^1} \leq \epsilon, \\ &|\int \rho_k^1 dx - \tilde{\mu}| \leq \epsilon, \quad |\int \rho_k^2 dx - (\mu - \tilde{\mu})| \leq \epsilon, \\ &\text{supp } \rho_k^1 \cap \text{supp } \rho_k^2 = \emptyset, \quad \text{dist}\{\text{supp } \rho_k^1, \text{supp } \rho_k^2\} \to \infty, \text{ as } k \to \infty.\end{aligned}$$

As in Lions [25], assume that the supports of ρ_k^1 and ρ_k^2 are of the form:

$$\text{supp } \rho_k^1 \subset (y_k - R_k, y_k + R_k), \qquad \text{supp } \rho_k^2 \subset (-\infty, y_k - 2R_k) \cup (y_k + 2R_k, \infty)$$

for some $R_k \to \infty$. Thus we have for $k \geq k_0$

$$\int_{R_k \leq |x-y_k| \leq 2R_k} \rho_{n_k} dx \leq \|\rho_{n_k} - (\rho_k^1 + \rho_k^2)\|_{L^1} \leq \epsilon.$$

We now choose a function $\theta^1(x) \in C^\infty(\mathbb{R})$ so that $0 \leq \theta^1 \leq 1$. Let $\theta^1(x) = 1$ for $|x| \leq 1$ and $\theta^1(x) = 0$ for $|x| \geq 2$. Let $\theta^2(x)$ be defined by $\theta^2(x) = 1 - \theta^1(x)$. Define $\theta_k^i(x) = \theta^i(\frac{x-y_k}{R_k})$ and $\psi_k^i(x) = \theta_k^i(x)\psi_{n_k}(x)$ for $i = 1, 2$. Hence we have $\psi_{n_k}(x) = \psi_k^1(x) + \psi_k^2(x)$ and

$$\mathcal{I}_c(\psi_{n_k}) = \mathcal{I}_c(\psi_k^1) + \mathcal{I}_c(\psi_k^2) + \int (K_c\psi_k^1)\left(K_c\psi_k^2\right) dx + \gamma_c \int \psi_k^1 \psi_k^2 dx. \tag{3.26}$$

We first rewrite the first integral term as follows:

$$\begin{aligned}\int (K_c\psi_k^1)\left(K_c\psi_k^2\right)dx &= \int (K_c\theta_k^1\psi_{n_k})\left(K_c\theta_k^2\psi_{n_k}\right)dx \\ &= \int \left\{\theta_k^1 K_c\psi_{n_k} + [K_c,\theta_k^1]\psi_{n_k}\right\}\left\{\theta_k^2 K_c\psi_{n_k} + [K_c,\theta_k^2]\psi_{n_k}\right\}dx \\ &= \int \Big\{\theta_k^1\theta_k^2\left(K_c\psi_{n_k}\right)^2 + \left([K_c,\theta_k^1]\psi_{n_k}\right)\left([K_c,\theta_k^2]\psi_{n_k}\right) \\ &\qquad + \left(\theta_k^1[K_c,\theta_k^2]\psi_{n_k} + \theta_k^2[K_c,\theta_k^1]\psi_{n_k}\right)K_c\psi_{n_k}\Big\}\,dx.\end{aligned}$$

For large k we estimate

$$\int \theta_k^1\theta_k^2(K_c\psi_{n_k})^2dx \le \int_{R_k\le|x-y_k|\le 2R_k}(K_c\psi_{n_k})^2dx \le \int_{R_k\le|x-y_k|\le 2R_k}\rho_{n_k}dx \le \epsilon.$$

Note that we have

$$\begin{aligned}\int \left([K_c,\theta_k^1]\psi_{n_k}\right)\left([K_c,\theta_k^2]\psi_{n_k}\right)dx &\le \|[K_c,\theta_k^1]\psi_{n_k}\|_{L^2}\|[K_c,\theta_k^2]\psi_{n_k}\|_{L^2}, \\ \int \left(\theta_k^1[K_c,\theta_k^2]\psi_{n_k} + \theta_k^2[K_c,\theta_k^1]\psi_{n_k}\right)K_c\psi_{n_k}dx &\le \|K_c\psi_{n_k}\|_{L^2}\left(\|[K_c,\theta_k^1]\psi_{n_k}\|_{L^2} + \|[K_c,\theta_k^2]\psi_{n_k}\|_{L^2}\right),\end{aligned}$$

By the commutator estimate of Lemma 3.9, we get

$$\|[K_c,\theta_k^i]\psi_{n_k}\|_{L^2} \le C\left(\sum_{n=1}^{N+1}\|\theta_k^{i(n)}\|_{L^\infty}\right)\|\psi_{n_k}\|_{H^{s_0}} \le \frac{C^i}{R_k}$$

for $i = 1, 2$. Having disposed of the above results, we now return to the first integral term in (3.26). Thus, for large k we have

$$\int (K_c\psi_k^1)\left(K_c\psi_k^2\right)dx = \mathcal{O}(\epsilon).$$

The last integral term in (3.26) can be handled similarly. From what has already been proved, we deduce that

$$\mathcal{I}_c(\psi_{n_k}) = \mathcal{I}_c(\psi_k^1) + \mathcal{I}_c(\psi_k^2) + \mathcal{O}(\epsilon).$$

Since $\epsilon > 0$ is arbitrary, it follows from (3.20) that

$$m_1(c) = \lim_{k\to\infty}\mathcal{I}_c(\psi_{n_k}) \ge \lim_{k\to\infty}\inf\mathcal{I}_c(\psi_k^1) + \lim_{k\to\infty}\inf\mathcal{I}_c(\psi_k^2). \tag{3.27}$$

Since $\|\psi_{n_k}\|_{H^{s_0}}$ and $\|\psi_{n_k}\|_{L^{2p}}$ are uniformly bounded, we see that

$$\begin{aligned}\int(|\psi_{n_k}|^{p+1}-|\psi_k^1|^{p+1}-|\psi_k^2|^{p+1})dx &= \int_{R_k\le|x-y_k|\le 2R_k}|\psi_{n_k}|^{p+1}|1-(\theta_k^1)^{p+1}-(\theta_k^2)^{p+1}|dx\\ &\le \sup_k\|\psi_{n_k}\|_{L^{2p}}^p\left(\int_{R_k\le|x-y_k|\le 2R_k}|\psi_{n_k}|^2dx\right)^{1/2}\\ &\le \sup_k\|\psi_{n_k}\|_{L^{2p}}^p\left(\int_{R_k\le|x-y_k|\le 2R_k}\rho_{n_k}dx\right)^{1/2}=\mathcal{O}(\epsilon).\end{aligned}$$

Combining this with (3.17) yields

$$1=\mathcal{Q}(\psi_{n_k})=\mathcal{Q}(\psi_k^1)+\mathcal{Q}(\psi_k^2)+\mathcal{O}(\epsilon).$$

By passing to a subsequence if necessary, we can assume that, for $i=1,2$, $\lim_{k\to\infty}\mathcal{Q}(\psi_k^i)=\lambda_i$ with $\lambda_1+\lambda_2=1$. Note that

$$\lim_{k\to\infty}\inf\mathcal{I}_c(\psi_k^i)\ge m_{\lambda_i}(c)\quad\text{for}\quad i=1,2.$$

We now show that λ_1 (and similarly λ_2) is non-zero. To this end, suppose $\lambda_1=0$. This gives $\lambda_2=1$ and $\lim_{k\to\infty}\inf\mathcal{I}_c(\psi_k^2)\ge m_1(c)$. On the other hand, by the commutator estimates we have

$$\begin{aligned}\mathcal{I}_c(\psi_k^1) &= \frac{1}{2}\|K_c\psi_k^1\|_{L^2}^2+\frac{1}{2}\gamma_c\|\psi_k^1\|_{L^2}^2\\ &\ge \frac{1}{2}\|\theta_k^1K_c\psi_{n_k}\|_{L^2}^2+\frac{1}{2}\gamma_c\|\theta_k^1\psi_{n_k}\|_{L^2}^2-\|[K_c,\theta_k^1]\psi_{n_k}\|_{L^2}\|K_c\psi_{n_k}\|_{L^2}\\ &\ge \int\theta_k^1\rho_{n_k}dx-\mathcal{O}(\epsilon)\\ &\ge \int_{|x-y_k|\le R_k}\rho_{n_k}dx-\mathcal{O}(\epsilon)\\ &\ge \int_{|x-y_k|\le R_k}\rho_k^1dx-\|\rho_{n_k}-(\rho_k^1+\rho_k^2)\|_{L^1}-\mathcal{O}(\epsilon),\end{aligned}$$

where we have used the fact that ρ_k^1 has support in $|x-y_k|\le R_k$ and ρ_k^2 vanishes there. As $k\to\infty$ this yields

$$\lim_{k\to\infty}\inf\mathcal{I}_c(\psi_k^1)\ge\tilde{\mu},$$

and by (3.27), we obtain $m_1(c)\ge\tilde{\mu}+m_1(c)$, contradicting $\tilde{\mu}>0$. Then it follows that $\lambda_i\neq 0$ for $i=1,2$. We thus get

$$m_1(c)\ge m_{\lambda_1}(c)+m_{1-\lambda_1}(c)$$

which contradicts the subadditivity property of Lemma 3.8. This completes the proof that the dichotomy does not occur. ☐

So far, with Lemmas 3.7 and 3.10 we have ruled out the possibility of both vanishing and dichotomy. The Concentration-Compactness Lemma implies that "compactness" occurs. We are then in a position to prove the following theorem establishing the existence of global minimizers.

Theorem 3.11. *Assume that $\rho \geq 0$, $r + \frac{\varrho}{2} \geq 1$ and $c^2 < c_1^2$. Let $\{\psi_n\}$ be a minimizing sequence for (3.20). Then there exists a subsequence $\{\psi_{n_k}\}$ and a sequence $\{y_{n_k}\}$ of real numbers such that $\psi_{n_k}(. + y_{n_k})$ converges to some $\psi \in H^{s_0}$ and ψ is a minimizer for (3.20).*

Proof. Let $\{\psi_n\}$ be a minimizing sequence for (3.20). Since vanishing and dichotomy are ruled out, the concentration-compactness lemma implies that there is a subsequence $\{\psi_{n_k}\}$ such that for any $\epsilon > 0$ there are $R > 0$ and real numbers y_k satisfying

$$\int_{|x| \geq R} \mid \psi_{n_k}(x + y_{n_k}) \mid^2 dx < \epsilon.$$

Since the sequence $\{\psi_n(. + y_{n_k})\}$ is bounded in H^{s_0}, replacing it by a subsequence if necessary, we can assume that it converges weakly to some $\psi \in H^{s_0}$. The tails of the functions $\psi_n(. + y_{n_k})$ are uniformly bounded by ϵ outside some interval $[-R, R]$ in the L^2 norm. $H^{s_0}([-R, R])$ is compactly embedded in $L^2([-R, R])$ so that $\psi_{n_k}(. + y_{n_k})$ restricted to $[-R, R]$ converges strongly to ψ restricted to $[-R, R]$, in $L^2([-R, R])$. But then we have

$$\|\psi_{n_k}(. + y_{n_k}) - \psi\|_{L^2} \leq \|\psi_{n_k}(. + y_{n_k}) - \psi\|_{L^2([-R,R])} + 2\epsilon. \tag{3.28}$$

This shows that $\psi_{n_k}(. + y_{n_k})$ converges strongly to ψ in L^2. Moreover, it follows from the embedding $H^{s_0} \subset L^{2p}$ that there is some $C > 0$ so that $\|\psi_{n_k}(. + y_{n_k})\|_{L^{2p}} \leq C$ for all n_k. Then we have

$$\begin{aligned} \|\psi_{n_k}(. + y_{n_k}) - \psi\|_{L^{p+1}}^{p+1} &\leq \|\psi_{n_k}(. + y_{n_k}) - \psi\|_{L^{2p}}^{p} \|\psi_{n_k}(. + y_{n_k}) - \psi\|_{L^2} \\ &\leq (2C)^p \|\psi_{n_k}(. + y_{n_k}) - \psi\|_{L^2}. \end{aligned}$$

Hence $\psi_{n_k}(. + y_{n_k})$ also converges to $\psi \in L^{p+1}$ strongly and hence $Q(\psi) = 1$. By the definition of $m_1(c)$, we get $\mathcal{I}_c(\psi) \geq m_1(c)$. As it has already been stated in Remark 3.3,

$\sqrt{\mathcal{I}_c(\psi)}$ defines a Hilbertian norm on H^{s_0} equivalent to the standard norm. Denoting the corresponding inner product by $\langle .,. \rangle_c$ and recalling that $\psi_{n_k}(.+y_{n_k})$ is also a minimizing sequence, we get

$$\begin{aligned}\mathcal{I}_c(\psi) = \langle \psi, \psi \rangle_c = \lim_{k\to\infty} \langle \psi, \psi_{n_k}(.+y_{n_k}) \rangle_c \quad &\leq \quad \lim_{k\to\infty} \sup \sqrt{\mathcal{I}_c(\psi)}\sqrt{\mathcal{I}_c(\psi_{n_k}(.+y_{n_k}))} \\ &= \quad \sqrt{\mathcal{I}_c(\psi)}\sqrt{m_1(c)}\end{aligned}$$

so that $\mathcal{I}_c(\psi) \leq m_1(c)$. Combining with the reverse inequality above we obtain $\mathcal{I}_c(\psi) = m_1(c)$, so ψ is the minimizer. This completes the proof. □

Remark 3.12. *Note that in the above proof we have*

$$\mathcal{I}_c(\psi) = \lim_{k\to\infty} \mathcal{I}_c(\psi_{n_k}(.+y_{n_k})),$$

so the weak limit preserves the norm. Then it follows that it is a strong limit; in other words $\psi_{n_k}(.+y_{n_k})$ converges strongly to $\psi \in H^{s_0}$.

With Theorem 3.11 in hand, we can now prove the following main result, namely, the existence of traveling wave solutions:

Theorem 3.13. *Assume that $\rho \geq 0$ and $r + \frac{\rho}{2} \geq 1$. Let $c^2 < c_1^2$ and $g(u) = -|u|^{p-1}u$. Then the traveling wave solutions of (1.1) exist.*

Proof. The proof consists of two steps, first we show that a proper scaling of the minimizer is a weak solution of (1.7). Then applying a regularity argument, we deduce that this weak solution is actually strong and exhibits the necessary decay properties. A minimizer $\psi \in H^{s_0}$ of the variational problem (3.20) is a weak solution of the Euler-Lagrange equation

$$(L - c^2 I)B^{-1}\psi - \theta(p+1)|\psi|^{p-1}\psi = 0, \tag{3.29}$$

where θ denotes a Lagrange multiplier. Multiplying (3.29) by ψ and integrating gives $2m_1(c) = \theta(p+1)$. Then

$$\phi_c = [2m_1(c)]^{1/(p-1)}\psi \in H^{s_0}$$

is a weak solution of (1.7):

$$(L - c^2 I)B^{-1}\phi_c - |\phi_c|^{p-1}\phi_c = 0. \tag{3.30}$$

As $s_0 \geq \frac{1}{2}$ and $p > 1$, we have $|\phi_c|^{p-1}\phi_c \in L^2$. Then, $(L - c^2I)^{-1}B$ is an operator of order $-(\rho + r)$, we get

$$\phi_c = (L - c^2I)^{-1}B(|\phi_c|^{p-1}\phi_c) \in H^{\rho+r} = H^{2s_0}.$$

Thus ϕ_c is a strong solution of (1.7). We note that the regularity of ϕ_c may be improved: since $2s_0 \geq 1$ so $\phi_c \in L^\infty$ and $D_x\phi_c \in L^2$. This in turn shows that $D_x(|\phi_c|^{p-1}\phi_c) = p|\phi_c|^{p-1}D_x\phi_c \in L^2$, implying that $|\phi_c|^{p-1}\phi_c \in H^1$. But then $\phi_c = (L - c^2I)^{-1}B(|\phi_c|^{p-1}\phi_c) \in H^{2s_0+1} \subset H^2$. This bootstrap argument can be repeated for larger p. In fact, when p is odd, $\phi_c \in C^\infty$. □

3.2. The case $\rho \leq 0$ and $g(u) = |u|^{p-1}u$

Throughout this subsection we assume that we are in the regime described by (3.14). In addition to $\rho \leq 0$ we also assume that either $\rho \leq -2$ and $r \geq 2$ or $\rho > -2$ and $\frac{\rho}{2} + r \geq 1$. Under the assumption that L and B satisfy (1.2)-(1.4) the requirements of Theorem 2.3 are satisfied. In what follows we take

$$s_0 = \frac{r}{2}.$$

The important point to note here is that $s_0 \geq \frac{1}{2}$ for both sets of parameter values. An immediate consequence of this fact is that the Sobolev embeddings of in the previous subsection also apply to the present case.

The crucial fact about $\mathcal{I}_c(\psi)$ for the present case is that, when $\rho < 0$, or when $\rho = 0$ and c^2 is large, the term $\|B^{-1/2}\psi\|_{L^2}^2$ in (3.16) dominates the others in $\mathcal{I}_c(\psi)$. Hence $\mathcal{I}_c(\psi)$ is no longer bounded from below. Nevertheless, we note that it is bounded from above for large values of c^2. This is due to the change in the sign of the nonlinear term.

Given the form of the nonlinear term, we look for a solution of the equation

$$(L - c^2I)B^{-1}\phi_c + |\phi_c|^{p-1}\phi_c = 0. \tag{3.31}$$

We now define a new functional, $\mathcal{J}_c(\psi)$, as the negative of what we have considered above:

$$\mathcal{J}_c(\psi) = -\mathcal{I}_c(\psi).$$

As a result, a new range of wave velocities is established to be able to prove a coercivity estimate for $\mathcal{J}_c(\psi)$. The range is provided by the following lemma; the proof is very similar to that of Lemma 3.1.

Lemma 3.14. *Let $c^2 > c_2^2$. Then there are positive constants γ_1, γ_2 such that*

$$\gamma_1 \|\psi\|_{H^{s_0}}^2 \leq \mathcal{J}_c(\psi) \leq \gamma_2 \|\psi\|_{H^{s_0}}^2.$$

Proof. From (1.3) we have

$$c^2 - c_2^2 \leq c^2 - c_2^2(1+\xi^2)^{\rho/2} \leq c^2 - l(\xi) \leq c^2 - c_1^2(1+\xi^2)^{\rho/2} \leq c^2.$$

Using this inequality and (1.4) with

$$\mathcal{J}_c(\psi) = \frac{1}{2}\int \left(c^2 - l(\xi)\right) b^{-1}(\xi)|\widehat{\psi}(\xi)|^2 d\xi$$

gives

$$\frac{c^2 - c_2^2}{2c_4^2}\|\psi\|_{H^{s_0}}^2 \leq \mathcal{J}_c(\psi) \leq \frac{c^2}{2c_3^2}\|\psi\|_{H^{s_0}}^2.$$

□

Accordingly we define a new variational problem as

$$\tilde{m}_1(c) = \inf\{\mathcal{J}_c(\psi) : \psi \in H^{s_0}, \quad \mathcal{Q}(\psi) = 1\}. \tag{3.32}$$

The proof of the existence of a minimizer of $\tilde{m}_1(c)$ goes along the same lines as the proof of that of $m_1(c)$ in the previous subsection. The only modification we need is in the decomposition of $\mathcal{J}_c(\psi)$. To this end, we express $\mathcal{J}_c(\psi)$ in the form

$$\mathcal{J}_c(\psi) = \frac{1}{2}\|\tilde{K}_c\psi\|^2 + \frac{1}{2}\gamma_c\|\psi\|^2$$

where $\tilde{K}_c$ is a suitable coercive operator with the symbol $\tilde{k}_c(\xi)$ and γ_c is a positive constant again. This time the symbols satisfy

$$\left(c^2 - l(\xi)\right) b^{-1}(\xi) = \tilde{k}_c^2(\xi) + \gamma_c.$$

By choosing $\gamma_c = (c^2 - c_2^2)/(2c_4^2) > 0$ we get

$$\tilde{k}_c^2(\xi) = \left(c^2 - l(\xi)\right) b^{-1}(\xi) - \frac{c^2 - c_2^2}{2c_4^2}.$$

It is clear that with this setting all the lemmas of the previous subsection will hold yielding the existence of minimizers $\tilde{m}_1(c)$.

Any minimizer ψ of the variational problem (3.32) solves the Euler-Lagrange equation

$$(L - c^2 I)B^{-1}\psi + \theta(p+1)|\psi|^{p-1}\psi = 0,$$

where θ is a Lagrange multiplier. Then a function ϕ_c obtained by a suitable scaling of the minimizer ψ will be a weak solution of (3.31). Applying the regularity argument in the proof of Theorem 3.13 we obtain its analogue:

Theorem 3.15. *Assume that $\rho \le 0$ and that either $\rho \le -2$ and $r \ge 2$ or $\rho > -2$ and $\frac{\rho}{2} + r \ge 1$. Let $c^2 > c_2^2$ and $g(u) = |u|^{p-1}u$. Then the traveling wave solutions of (1.1) exist.*

4. Stability of traveling waves: The case $\rho \ge 0$ and $g(u) = -|u|^{p-1}u$

In this section we will discuss stability of traveling waves under the assumptions of Theorem 3.13. The theorem guarantees that traveling waves exist for $c^2 < c_1^2$. We will first consider orbital stability which roughly speaking, means that a solution starting close to a traveling wave remains close to some possibly other traveling wave with the same velocity. As in [27], we will prove that orbital stability occurs for a velocity c if a suitably defined function d is convex in a neighborhood of c. We then study the function $d(c)$ and show that it is not convex for small c^2, in other words, our method will not predict orbital stability for small c^2. Moreover, we show that the standing waves, $c = 0$, are never orbitally stable. To be precise, we prove that for any standing wave we can find initial data arbitrarily close to it such that the corresponding solution of (1.1) blows up in finite time.

Let G_c denote the set of all traveling wave solutions ϕ_c with a fixed wave velocity c of (1.1). We denote the corresponding set of solutions $\Phi_c = (\phi_c, -c\phi_c)$ of the system (2.3)-(2.4) by

$$\mathcal{G}_c = \{\Phi_c = (\phi_c, -c\phi_c) : \ \phi_c \in G_c\}.$$

By Theorem 2.3, for a solution $U = (u, w)$ of the system (2.3)-(2.4), we have $U(t) \in X = H^{s_0} \times H^{s_0 - \frac{\rho}{2}}$. Hence, we will consider $\mathcal{G}_c$ as a subset of X. Notice that the space X is endowed with the norm $\|U\|_X = \|u\|_{H^{s_0}} + \|w\|_{H^{s_0-\frac{\varrho}{2}}}$. We consider orbital stability in the sense of $X-$ stability defined below.

Definition 4.1. *The set $\mathcal{G}_c$ is said to be X-stable, if for any $\epsilon > 0$ there exists some $\delta > 0$ such that whenever*

$$\inf \{\|U_0 - \Phi_c\|_X : \Phi_c \in \mathcal{G}_c\} < \delta,$$

the solution $U(t)$ of the Cauchy problem (2.3)-(2.5) with $U(0) = (u_0(x), w_0(x))$ exists for all $t > 0$, and satisfies

$$\sup_{t>0} \inf \{\|U(t) - \Phi_c\|_X : \Phi_c \in \mathcal{G}_c\} < \epsilon.$$

We recall that $\phi_c = [2m_1(c)]^{\frac{1}{p-1}} \psi_c$ where ψ_c was the minimizer for $m_1(c)$. Then we get $\mathcal{Q}(\phi_c) = 2\mathcal{I}_c(\phi_c) = 2^{\frac{p+1}{p-1}} [m_1(c)]^{\frac{p+1}{p-1}}$. We begin by establishing the following relationship between the conserved quantities $\mathcal{E}$, $\mathcal{M}$ of Section 2 and the functionals $\mathcal{I}_c$, $\mathcal{Q}$ of Section 3.

Lemma 4.2. *Every $\Phi_c \in \mathcal{G}_c$ is a minimizer for $\mathcal{E}(U) + c\mathcal{M}(U)$ with constraint*

$$\mathcal{Q}(u) = 2^{\frac{p+1}{p-1}} [m_1(c)]^{\frac{p+1}{p-1}}.$$

Proof. Combining (2.6)-(2.7) with (3.16)-(3.17) yields

$$\mathcal{E}(U) + c\mathcal{M}(U) = \frac{1}{2} \left\| B^{-1/2} (w + cu) \right\|_{L^2}^2 + \mathcal{I}_c(u) - \frac{1}{p+1} \mathcal{Q}(u).$$

Then

$$\mathcal{E}(U) + c\mathcal{M}(U) \geq \mathcal{I}_c(u) - \frac{1}{p+1} \mathcal{Q}(u) \geq \mathcal{I}_c(\phi_c) - \frac{1}{p+1} \mathcal{Q}(\phi_c) = \mathcal{E}(\Phi_c) + c\mathcal{M}(\Phi_c) \quad (4.1)$$

and the result follows. □

It is worth pointing out that Φ_c is also a minimizer for $\mathcal{E}(U) + c\mathcal{M}(U)$ subject to the constraint

$$\left\| L^{1/2} B^{-1/2} u \right\|_{L^2}^2 - c^2 \left\| B^{-1/2} u \right\|_{L^2}^2 - \|u\|_{L^{p+1}}^{p+1} = 2\mathcal{I}_c(u) - \mathcal{Q}(u) = 0, \qquad u \neq 0 \qquad (4.2)$$

(see [8] for more details).

We now define the function $d(c)$ by

$$d(c)=\inf\left\{\mathcal{E}(U)+c\mathcal{M}(U):U\in X,\quad \mathcal{Q}(u)=2^{\frac{p+1}{p-1}}\left[m_1(c)\right]^{\frac{p+1}{p-1}}\right\}. \tag{4.3}$$

From Lemma 4.2 it follows that

$$d(c)=\mathcal{E}(\Phi_c)+c\mathcal{M}(\Phi_c),$$

or

$$d(c)=\left(\frac{p-1}{p+1}\right)\mathcal{I}_c(\phi_c)=\frac{1}{2}\left(\frac{p-1}{p+1}\right)\mathcal{Q}(\phi_c)=2^{\frac{2}{p-1}}\left(\frac{p-1}{p+1}\right)\left[m_1(c)\right]^{\frac{p+1}{p-1}}. \tag{4.4}$$

Lemma 4.3. *Suppose d is differentiable; then $d'(c)=\mathcal{M}(\Phi_c)$.*

Proof. We have

$$\begin{aligned} d'(c) &= \frac{d}{dc}\int\left[\frac{1}{2}\left(L^{1/2}B^{-1/2}\phi_c\right)^2-\frac{c^2}{2}\left(B^{-1/2}\phi_c\right)^2-\frac{1}{p+1}|\phi_c|^{p+1}\right]dx,\\ &= \int\left[\left(L-c^2I\right)B^{-1}\phi_c-|\phi_c|^{p-1}\phi_c\right]\frac{d\phi_c}{dc}dx-\int c\left(B^{-1/2}\phi_c\right)^2dx. \end{aligned}$$

Since $(L-c^2I)B^{-1}\phi_c-|\phi_c|^{p-1}\phi_c=0$ (see 3.30), we have the desired result;

$$d'(c)=-\int c\left(B^{-1/2}\phi_c\right)^2dx=\mathcal{M}(\Phi_c). \tag{4.5}$$

□

As $\mathcal{M}(\Phi_c)=-c\left\|B^{-1/2}\phi_c\right\|_{L^2}^2$, it follows from (4.5) that, whenever differentiable on some interval not containing the origin, the function $d(c)$ is monotone on the interval. We can state now the main result on orbital stability.

Theorem 4.4. *Let $\rho\geq 0$, $r+\frac{\rho}{2}\geq 1$, $(\rho,r)\neq(0,1)$ and $c^2<c_1^2$. Suppose d is differentiable and strictly convex on some interval J containing c. Then the set $\mathcal{G}_c$ is $X-$stable.*

Proof. Suppose that $\mathcal{G}_c$ is $X-$unstable. Then there are some $\epsilon>0$, initial data $U_n(0)$ and points $t_n>0$ such that

$$\inf_{\Phi_c\in\mathcal{G}_c}\|U_n(0)-\Phi_c\|_X<\frac{1}{n}$$

but

$$\inf_{\Phi_c\in\mathcal{G}_c}\|U_n(t_n)-\Phi_c\|_X\geq\epsilon,$$

where $U_n(t) = (u_n(t), w_n(t))$ is the solution of the Cauchy problem (2.3)-(2.5) with $U_n(0) = (u_n(0), w_n(0))$. By continuity of $U_n(t)$ we can take ϵ sufficiently small and choose t_n such that

$$\inf_{\Phi_c \in \mathcal{G}_c} \|U_n(t_n) - \Phi_c\|_X = \epsilon.$$

In addition to this, we also choose $\Phi_c^n \in \mathcal{G}_c$ such that

$$\lim_{n\to\infty} \|U_n(0) - \Phi_c^n\|_X = 0.$$

Since the invariants $\mathcal{E}$ and $\mathcal{M}$ are continuous on X, we have

$$\begin{aligned}\lim_{n\to\infty} \mathcal{E}(U_n(t_n)) &= \lim_{n\to\infty} \mathcal{E}(U_n(0)) = \mathcal{E}(\Phi_c^n),\\ \lim_{n\to\infty} \mathcal{M}(U_n(t_n)) &= \lim_{n\to\infty} \mathcal{M}(U_n(0)) = \mathcal{M}(\Phi_c^n),\end{aligned}$$

noting that the terms on the right-hand side are independent of n. By taking ϵ to be sufficiently small, we can make the values of $u_n(t_n)$ arbitrarily close to ϕ_c^n and consequently the values of $\mathcal{Q}(u_n(t_n))$ arbitrarily close to $\mathcal{Q}(\phi_c^n) = 2\left(\frac{p+1}{p-1}\right) d(c)$. Since $d(c)$ is monotone on J, for each n, there is a unique c_n satisfying

$$\mathcal{Q}(u_n(t_n)) = \mathcal{Q}(\phi_{c_n}) = 2\left(\frac{p+1}{p-1}\right) d(c_n),$$

for the traveling wave solution ϕ_{c_n}. This means $\mathcal{Q}(u_n(t_n)) = \mathcal{Q}(\phi_{c_n}) = 2^{\frac{p+1}{p-1}} [m_1(c_n)]^{\frac{p+1}{p-1}}$. By Lemma 4.2 we have

$$\mathcal{E}(U_n(t_n)) + c_n \mathcal{M}(U_n(t_n)) \geq d(c_n). \tag{4.6}$$

On the other hand, we can write

$$d(c_n) = d(c) + d'(c)(c_n - c) + \int_c^{c_n} [d'(s) - d'(c)]\, ds. \tag{4.7}$$

By assumption, d is strictly convex and consequently d' is strictly increasing. From this, it follows that the integral on the right-hand side is positive for $c \neq c_n$. Using Lemma 4.3, we have

$$\begin{aligned} d(c) + d'(c)(c_n - c) &= \mathcal{E}\left(\Phi_c^n\right) + c\mathcal{M}\left(\Phi_c^n\right) + \mathcal{M}\left(\Phi_c^n\right)(c_n - c)\\ &= \mathcal{E}\left(\Phi_c^n\right) + c_n \mathcal{M}\left(\Phi_c^n\right). \end{aligned}$$

Combining this with (4.6) and (4.7) yields

$$\mathcal{E}(U_n(t_n)) + c_n\mathcal{M}(U_n(t_n)) \geq \mathcal{E}\left(\Phi_c^n\right) + c_n\mathcal{M}\left(\Phi_c^n\right) + \int_c^{c_n} \left[d'(s) - d'(c)\right] ds,$$

or

$$\mathcal{E}(U_n(t_n)) - \mathcal{E}\left(\Phi_c^n\right) + c_n\left(\mathcal{M}(U_n(t_n)) - \mathcal{M}(\Phi_c^n)\right) \geq \int_c^{c_n} \left[d'(s) - d'(c)\right] ds.$$

But as $n \to \infty$, the left-hand side of the inequality converges to zero. As $d'(s)$ is strictly increasing this is possible only when $\lim_{n\to\infty} c_n = c$. Continuity of d implies that

$$\lim_{n\to\infty} \mathcal{Q}(u_n(t_n)) = \lim_{n\to\infty} 2\left(\frac{p+1}{p-1}\right) d(c_n) = 2\left(\frac{p+1}{p-1}\right) d(c) = \mathcal{Q}(\phi_c^n).$$

Taking the limit of both sides of the following inequality as $n \to \infty$

$$\mathcal{I}_c(u_n(t_n)) - \frac{1}{p+1}\mathcal{Q}(u_n(t_n)) \leq \mathcal{E}(U_n(t_n)) + c\mathcal{M}(U_n(t_n)),$$

and using (4.4) we get

$$\lim_{n\to\infty} \mathcal{I}_c(u_n(t_n)) \leq \lim_{n\to\infty} \frac{2}{p-1} d(c_n) + d(c) = \frac{p+1}{p-1} d(c)$$

or

$$\lim_{n\to\infty} \mathcal{I}_c(u_n(t_n)) \leq \mathcal{I}_c(\phi_c).$$

This result implies that $\{u_n(t_n)\}$ is a minimizing sequence. By the existence theorem of traveling waves solutions, Theorem 3.13, there is a shifted subsequence that converges in H^{s_0} to some $\phi_c^0 \in G_c$. We further note that

$$\frac{1}{2}\left\|B^{-1/2}\left(w_n(t_n) + cu_n(t_n)\right)\right\|_{L^2}^2 = \mathcal{E}(U_n(t_n)) + c\mathcal{M}(U_n(t_n)) + \frac{1}{p+1}\mathcal{Q}(u_n(t_n)) - \mathcal{I}_c(u_n(t_n))$$

converges to zero as $n \to \infty$. This gives $\lim_{n\to\infty}\left(w_n(t_n) + cu_n(t_n)\right) = 0$ in $H^{s_0 - \frac{\rho}{2}}$. Therefore, a shifted subsequence of $U_n(t_n)$ converges in X to $\Phi_c^0 = (\phi_c^0, -c\phi_c^0)$. In conclusion, we have

$$\inf_{\phi \in G_c} \|U_n(t_n) - \Phi_c\|_X = 0,$$

which contradicts our assumption. Note that $s_0 = \frac{r}{2} + \frac{\rho}{2} > \frac{1}{2}$ when $(\rho, r) \neq (0, 1)$. Hence Theorem 2.3 guarantees local well-posedness in $H^{s_0} \times H^{s_0 - \frac{\rho}{2}}$. The above argument, at first attempt, can only hold locally, i.e. for $0 \leq t < T$. On the other hand, the same argument shows that $U(t)$ stays bounded in $H^{s_0} \times H^{s_0 - \frac{\rho}{2}}$; hence can be continued beyond T. This in fact shows that $U(t)$ is indeed global and stays close to the orbit for all times. □

Remark 4.5. *In the case $(\rho, r) = (0,1)$, namely, $s_0 = \frac{1}{2}$, the above proof shows that we have a weaker version of orbital stability in the following sense: If the initial data $U(0) \in H^s \times H^s$ (for some $s > \frac{1}{2}$) is close to the orbit in the weaker $H^{\frac{1}{2}} \times H^{\frac{1}{2}}$ norm, then the solution, as long as as it exists, remains close to the orbit in the same norm.*

We now discuss convexity of $d(c)$. To this end we investigate more closely the properties of $m_1(c)$. Let M_c denote the set of minimizers for $m_1(c)$:

$$M_c = \{\psi \in H^{s_0} : \ \mathcal{Q}(\psi) = 1, \quad \mathcal{I}_c(\psi) = m_1(c)\}\,.$$

As $m_1(c)$ is an even function, it suffices to consider the interval $[0, c_1)$.

Lemma 4.6. *On the interval $[0, c_1)$ where c_1 is the coercivity constant of L, the following statements hold.*

(i) The map $m_1(c)$ is strictly decreasing.

(ii) The maps

$$\alpha^-(c) = \inf\left\{\left\|B^{-1/2}\psi_c\right\|_{L^2}^2 : \psi_c \in M_c\right\}, \quad \alpha^+(c) = \sup\left\{\left\|B^{-1/2}\psi_c\right\|_{L^2}^2 : \psi_c \in M_c\right\}$$

are strictly increasing.

(iii) Except for countably many points, $\alpha^-(c) = \alpha^+(c)$ hence $\left\|B^{-1/2}\psi_c\right\|_{L^2}^2$ is constant on M_c.

(iv) The map $m_1(c)$ is continuous on $[0, c_1)$, is differentiable and $m_1'(c) = -c\left\|B^{-1/2}\psi_c\right\|_{L^2}^2$ at all points where $\alpha^-(c) = \alpha^+(c)$.

(v) The map $m_1(c)$ is concave.

Proof. Let $\tilde{c} \in [0, c_1)$ such that $c \neq \tilde{c}$. Suppose that ψ_c and $\psi_{\tilde{c}}$ are two minimizers corresponding to c and $\tilde{c}$, respectively. Then we have

$$\begin{aligned} m_1(c) &= \mathcal{I}_c(\psi_c) = \frac{1}{2}\left\|L^{1/2}B^{-1/2}\psi_c\right\|_{L^2}^2 - \frac{c^2}{2}\left\|B^{-1/2}\psi_c\right\|_{L^2}^2 \\ &= \mathcal{I}_{\tilde{c}}(\psi_c) + \frac{\tilde{c}^2 - c^2}{2}\left\|B^{-1/2}\psi_c\right\|_{L^2}^2 \\ &> m_1(\tilde{c}) + \frac{\tilde{c}^2 - c^2}{2}\left\|B^{-1/2}\psi_c\right\|_{L^2}^2. \end{aligned}$$

By symmetry we get

$$\frac{\tilde{c}^2 - c^2}{2}\left\|B^{-1/2}\psi_c\right\|_{L^2}^2 < m_1(c) - m_1(\tilde{c}) < \frac{\tilde{c}^2 - c^2}{2}\left\|B^{-1/2}\psi_{\tilde{c}}\right\|_{L^2}^2.$$

This proves assertions (i) and (ii) of the lemma. It also implies that $m_1(c)$ is continuous. From (ii) we conclude that $\alpha^+(c)$ and $\alpha^-(c)$ are continuous except for countably many points in $[0, c_1)$. For (iii) notice that the intervals $[\alpha^-(c), \alpha^+(c)]$ have disjoint interior; this is possible only if $\alpha^-(c) = \alpha^+(c)$ except for countably many c, implying (iii). Take some c where α^- is continuous and $\alpha^-(c) = \alpha^+(c)$. For $c > \tilde{c}$,

$$-\frac{\tilde{c}+c}{2}\left\|B^{-1/2}\psi_c\right\|_{L^2}^2 < \frac{m_1(c)-m_1(\tilde{c})}{c-\tilde{c}} < -\frac{\tilde{c}+c}{2}\left\|B^{-1/2}\psi_{\tilde{c}}\right\|_{L^2}^2,$$

with the reverse inequality holding for $c < \tilde{c}$. Then

$$m_1'(c) = \lim_{\tilde{c}\to c}\frac{m_1(c)-m_1(\tilde{c})}{c-\tilde{c}} = -c\left\|B^{-1/2}\psi_c\right\|_{L^2}^2$$

as was predicted in Lemma 4.3. Then, by assertion (ii), $m_1'(c)$, whenever it exists, is strictly decreasing for $c > 0$. At the points where $m_1'(c)$ does not exist we have corners with the slopes decreasing as we pass through the corners. Thus $m_1(c)$ is strictly concave. We also note that $m_1'(0) = 0$. □

We obtain from (4.4) that $d'(c) = 2^{\frac{2}{p-1}}[m_1(c)]^{\frac{2}{p-1}}m_1'(c)$. Both $m_1(c)$ and $m_1'(c)$ are decreasing for $c > 0$. Since $m_1(c) > 0$, $m_1'(0) = 0$ and $m_1'(c) < 0$ we observe that d' decreases when c is near zero. This means that $d(c)$ will not be convex for small c. Therefore, the stability result of Theorem 4.4 will not apply to traveling waves with small velocity. In fact, following the approach in [28], we now show that there is instability by blow up in the case $c = 0$. To that end we state Theorem 3.5 of [8] in the following form:

Theorem 4.7. *Let $U_0 = (u_0, w_0)$ with $u_0 = (v_0)_x$ for some $v_0 \in L^2$. Suppose $\mathcal{E}(U_0) < d(0)$ and $2\mathcal{I}_0(u_0) - \mathcal{Q}(u_0) < 0$. Then the solution $U(t)$ of the Cauchy problem (2.3)-(2.5) with initial data U_0 blows up in finite time.*

Using Theorem 4.7, we now prove that the set of standing waves, $\mathcal{G}_0$, is unstable by blow-up, namely:

Theorem 4.8. *Let $\epsilon > 0$ and $\Phi_0 \in \mathcal{G}_0$. There exists initial data $U_0 \in X$ with $\|U_0 - \Phi_0\|_X < \epsilon$ for which the solution $U(t)$ of the Cauchy problem (2.3)-(2.5) with initial data U_0 blows up in finite time.*

Proof. First, for $\lambda > 1$, consider $\lambda\Phi_0 = (\lambda\phi_0, 0)$. Then

$$\begin{aligned}\mathcal{E}(\lambda\Phi_0) &= \lambda^2\mathcal{I}_0(\phi_0) - \frac{\lambda^{p+1}}{p+1}\mathcal{Q}(\phi_0) \\ &= \left(\frac{\lambda^2}{2} - \frac{\lambda^{p+1}}{p+1}\right)\mathcal{Q}(\phi_0) \\ &< \left(\frac{1}{2} - \frac{1}{p+1}\right)\mathcal{Q}(\phi_0) = d(0).\end{aligned}$$

Also

$$2\mathcal{I}_0(\lambda\phi_0) - \mathcal{Q}(\lambda\phi_0) = 2\lambda^2\mathcal{I}_0(\phi_0) - \lambda^{p+1}\mathcal{Q}(\phi_0) = (\lambda^2 - \lambda^{p+1})\mathcal{Q}(\phi_0) < 0.$$

Next, as in [28], we define v_0 via Fourier transform:

$$\widehat{v_0}(\xi) = \frac{1}{i\xi}\widehat{\phi_0}(\xi) \ \text{ for } \ |\xi| \geq h, \qquad \text{and} \qquad \widehat{v_0}(\xi) = 0 \ \text{ for } \ |\xi| < h.$$

Then $v_0 \in L^2$. In fact, since $\phi_0 \in H^{s_0}$, we have $v_0 \in H^{s_0+1}$ and thus $(v_0)_x \in H^{s_0}$. For any $\epsilon > 0$ we can choose h sufficiently small such that $\|(v_0)_x - \phi_0\|_{H^{s_0}} < \epsilon$. For $\lambda > 1$ we let $U_0 = (\lambda(v_0)_x, 0)$. Since $\mathcal{E}$, $\mathcal{I}_0$, and $\mathcal{Q}$ are continuous on H^{s_0} for λ sufficiently close to 1, we get $\|U_0 - \Phi_0\|_X < \epsilon$, $\mathcal{E}(U_0) < d(0)$ and $2\mathcal{I}_0(u_0) - \mathcal{Q}(u_0) < 0$. But then U_0 satisfies the conditions of Theorem 4.7, and hence $U(t)$ will blow up in finite time. □

The next example illustrates the application of the above procedure to the Boussinesq equation.

Example 1. (The Boussinesq Equation) If we set $L = I - \partial_x^2$ and $B = I$, we end up with (3.1) and consequently with (3.2) for which the solitary waves exist for $c^2 < 1$. Combining these with (4.4), after a straightforward calculation, we obtain the corresponding function $d(c)$ in the form

$$d(c) = d(0)(1 - c^2)^{\frac{p+3}{2(p-1)}}$$

where $d(0) = \frac{1}{2}\left(\frac{p-1}{p+1}\right)\left(\|\psi\|_{L^2}^2 + \|\psi'\|_{L^2}^2\right)$. Here the function ψ satisfies $\psi'' - \psi + |\psi|^{p-1}\psi = 0$. Then we have

$$d''(c) = 4d(0)\frac{p+3}{(p-1)^2}(1 - c^2)^{\frac{7-3p}{2(p-1)}}\left(c^2 - \frac{p-1}{4}\right).$$

So, when

$$\frac{p-1}{4} < c^2 < 1 \quad \text{and} \quad 1 < p < 5,$$

$d(c)$ is convex and by Theorem 4.4 the solitary wave solutions of (3.1) are orbitally stable. This is exactly the same result which was obtained by Bona and Sachs [12] for the stability of solitary wave solutions of (3.1). On the other hand, Theorem 4.4 is not applicable for small values of c since the convexity assumption is not valid. But Theorem 4.8 tells us that, for suitable initial data close to the standing wave, solutions of (3.1) blow up in finite time. For a more general case, Liu [13] proved that the solitary waves of (3.1) are orbitally unstable in suitable function spaces if either

$$c^2 \leq \frac{p-1}{4} \quad \text{and} \quad 1 < p < 5,$$

or

$$c^2 < 1 \quad \text{and} \quad p \geq 5.$$

As we have already mentioned, Liu [28] showed that for $c = 0$, the solitary waves are strongly unstable by blow-up, that is, certain solutions with initial data sufficiently close to ϕ_0 blow up in finite time. This result was extended to the case of a small nonzero wave velocity in [29] and to the case of

$$0 < c^2 < \frac{p-1}{2(p+1)}$$

in [30]. For a recent discussion of these issues in the case of non-power nonlinearities, we refer the reader to [32].

We now consider the double dispersion equation as a special case.

Example 2. (The Double Dispersion Equation) When $L = (I - a_1\partial_x^2)^{-1}(I - a_2\partial_x^2)$ and $B = (I - a_1\partial_x^2)^{-1}$ for two positive constants a_1 and a_2, (1.1) reduces to (3.7). Since $\rho = 0$, both regimes defined by (3.13) and (3.14) occur for the double dispersion equation. That is, solitary waves exist either for $c^2 < 1$ and $g(u) = -|u|^{p-1}u$ (i.e., the case $\rho \geq 0$ in Subsection 3.1) or for $c^2 > 1$ and $g(u) = |u|^{p-1}u$ (i.e., the case $\rho \leq 0$ in Subsection 3.2). Regarding the stability properties of solitary waves, the comments made for the first regime are also valid for the double dispersion equation. We refer the reader to [33] for a strong instability result obtained in the first regime for that equation.

We conclude this section with the following remark regarding the case $\rho \leq 0$.

Remark 4.9. *When $\rho \leq 0$, although ϕ_c is a minimizer for $\mathcal{J}_c$ (or a maximizer for $\mathcal{I}_c$) under a certain constraint, a variant of Lemma 4.2 does not hold. In fact, at ϕ_c we have* *a saddle point of* $\mathcal{E}(U) + c\mathcal{M}(U)$. *This can be observed easily from* $\mathcal{E}(U) + c\mathcal{M}(U) = \frac{1}{2}\left\|B^{-1/2}(w+cu)\right\|_{L^2}^2 - \mathcal{J}_c(u) - \frac{1}{p+1}\mathcal{Q}(u)$. *This is the main reason that the method used above for the case* $\rho \geq 0$ *will not work for the present case. In fact the case* $\rho \leq 0$ *corresponds to the "bad case" in [24]. We now briefly indicate the results currently available in the literature for the the improved Boussinesq equation which provides a prototype equation for the case* $\rho \leq 0$. *Pego and Weinstein [14] proved that solitary waves of (3.4) are linearly unstable in* $H^1 \times H^2$ *if*

$$1 < c^2 < \frac{3(p-1)}{2(p+1)} \quad \textit{and} \quad p > 5.$$

When $p = 2$, *the linear instability of periodic traveling waves has recently been shown in [34].*

In the next section we study stability properties of the traveling waves for the case $L = I$.

5. An example: A regularized Klein-Gordon-type equation

The previous section shows that orbital stability depends on the convexity of $d(c)$. In particular cases, for instance, in the case of the Boussinesq-type equations considered in the previous section, $d(c)$ can be computed explicitly using either the explicit form of the traveling wave solution ϕ_c or a Pohozaev-type identity, but both of these approaches will not work for the general case we deal with. In other words, we cannot get $d(c)$ explicitly unless we make further assumptions on L and/or B. In this section we consider the particular case $L = I$ for which $\rho = 0$ and $c_1 = c_2 = 1$. Note that $s_0 = s_0 - \frac{\rho}{2} \equiv \frac{r}{2}$. We will restrict our attention to the regime $c^2 < 1$ and $g(u) = -|u|^{p-1}u$. Taking $L = I$ allows us to compute $d(c)$ explicitly and hence determine the stability interval. Moreover, we are able to improve the instability result given in Theorem 4.8 to get an almost complete characterization for stability of solitary waves in the first regime. When $L = I$, (1.1) reduces to

$$u_{tt} - u_{xx} = B(-|u|^{p-1}u)_{xx}, \tag{5.1}$$

with the general pseudo-differential operator B of order $-r$. Due to the smoothing effect of B, (5.1) can be considered as a regularized Klein-Gordon-type equation. Note that due to Theorem 4.4 we need to take $r > 1$. We now give a full characterization of the orbital stability/instability of traveling waves for (5.1) below.

Theorem 5.1. *Let $L = I$, $r > 1$, $c^2 < 1$ and $g(u) = -|u|^{p-1}u$. Then*

(i) For $c^2 > \frac{p-1}{p+3}$, the traveling wave solutions of (2.3)-(2.5) with velocity c are orbitally stable.

(ii) For $c^2 < \frac{p-1}{p+3}$, the traveling wave solutions of (2.3)-(2.5) with velocity c are unstable by blow up; namely, for any $\epsilon > 0$ and $\Phi_c \in \mathcal{G}_c$ there exists initial data $U_0 \in X$ with $\|U_0 - \Phi_c\|_X < \epsilon$ for which the solution $U(t)$ of the Cauchy problem (2.3)-(2.5) with initial data U_0, blows up in finite time.

We first note from (3.16) that, for $L = I$

$$\mathcal{I}_c(u) = \frac{1}{2}(1-c^2)\left\|B^{-1/2}u\right\|_{L^2} = (1-c^2)\mathcal{I}_0(u).$$

So all the minimizers and hence ϕ_c traveling wave solutions are certain multiples of ϕ_0, namely $\phi_c = (1-c^2)^{\frac{1}{p-1}}\phi_0$. From (4.4) we have $d(c) = d(0)(1-c^2)^{\frac{p+1}{p-1}}$. Having disposed of this preliminary step, we can now easily prove the first assertion of Theorem 5.1. A straightforward computation gives

$$d''(c) = d(0)\frac{2(p+1)}{(p-1)^2}(1-c^2)^{\frac{3-p}{p-1}}\left((p+3)c^2 - p + 1\right).$$

Since $d(c)$ is strictly convex for $c^2 > \frac{p-1}{p+3}$, it follows from Theorem 4.4 that traveling waves are orbitally stable for $c^2 > \frac{p-1}{p+3}$. This completes the proof of assertion (i) of Theorem 5.1.

The rest of this section will be devoted to the proof of assertion (ii) of Theorem 5.1. That is, we will prove that, when $c^2 < \frac{p-1}{p+3}$, we can find initial data arbitrarily close to traveling wave solutions such that the solution of the corresponding Cauchy problem blows up in finite time. Before proving the assertion, we need some preliminary definitions and results. Let us first define a set $\Sigma_-(c)$ as follows.

$$\Sigma_-(c) = \{(u,w) \in H^{s_0} \times H^{s_0-\frac{\varrho}{2}} : \quad \mathcal{E}(u,w) + c\mathcal{M}(u,w) < d(c), \quad 2\mathcal{I}_c(u) - \mathcal{Q}(u) < 0\}.$$

The following lemma from [8] shows that, for $L = I$ and $c^2 < 1$, the set $\Sigma_-(c)$ is invariant under the flow generated by (2.3)-(2.5).

Lemma 5.2. *(Lemma 3.2 of [8]) Suppose* $(u_0, w_0) \in \Sigma_-(c)$*, and let* $(u(t), w(t))$ *be the solution of the Cauchy problem (2.3)-(2.5) with initial data* (u_0, w_0)*. Then* $(u(t), w(t)) \in \Sigma_-(c)$ *for* $0 < t < T_{\max}$*.*

We also need the following lemma:

Lemma 5.3. *Suppose* $2\mathcal{I}_c(u) - \mathcal{Q}(u) < 0$*. Then* $\frac{p+1}{p-1} d(c) < \mathcal{I}_c(u)$*.*

Proof. Recall from (3.20) that $m_1(c) = \inf\{\mathcal{I}_c(u) : \mathcal{Q}(u) = 1\}$. By homogeneity one gets

$$[m_1(c)]^{\frac{p+1}{2}} \leq \frac{[\mathcal{I}_c(u)]^{\frac{p+1}{2}}}{\mathcal{Q}(u)}$$

whenever $u \neq 0$. If $2\mathcal{I}_c(u) - \mathcal{Q}(u) < 0$ then

$$2[m_1(c)]^{\frac{p+1}{2}} \mathcal{I}_c(u) < [m_1(c)]^{\frac{p+1}{2}} \mathcal{Q}(u) \leq [\mathcal{I}_c(u)]^{\frac{p+1}{2}}.$$

Combining this with (4.4) yields

$$\frac{p+1}{p-1} d(c) = 2^{\frac{2}{p-1}} [m_1(c)]^{\frac{p+1}{p-1}} < \mathcal{I}_c(u).$$

□

We are now ready to prove the second assertion of Theorem 5.1:

Proof. Let $c^2 < \frac{p-1}{p+3}$ and $\Phi_c = (\phi_c, -c\phi_c) \in \mathcal{G}_c$. We will follow the approach in Theorem 4.8 to construct initial data arbitrarily close to Φ_c such that the solution of the corresponding Cauchy problem blows up in finite time. For $\lambda > 1$ consider $\lambda\Phi_c = (\lambda\phi_c, -c\lambda\phi_c)$. Then, just as in the proof of Theorem 4.8, we obtain

$$\begin{aligned} \mathcal{E}(\lambda\Phi_c) + c\mathcal{M}(\lambda\Phi_c) &= \lambda^2 \mathcal{I}_c(\phi_c) - \frac{\lambda^{p+1}}{p+1} \mathcal{Q}(\phi_c) \\ &= \left(\frac{\lambda^2}{2} - \frac{\lambda^{p+1}}{p+1}\right) \mathcal{Q}(\phi_c) \\ &< \left(\frac{1}{2} - \frac{1}{p+1}\right) \mathcal{Q}(\phi_c) = d(c), \end{aligned}$$

and

$$2\mathcal{I}_c(\lambda\phi_c) - \mathcal{Q}(\lambda\phi_c) = 2\lambda^2 \mathcal{I}_c(\phi_c) - \lambda^{p+1} \mathcal{Q}(\phi_c) = (\lambda^2 - \lambda^{p+1}) \mathcal{Q}(\phi_c) < 0.$$

These two results show that $\lambda\Phi_c = (\lambda\phi_c, -c\lambda\phi_c) \in \Sigma_-(c)$. Moreover,

$$\begin{aligned} -c\mathcal{M}(\lambda\Phi_c) &= -c\lambda^2\mathcal{M}(\Phi_c) = c^2\lambda^2 \left\| B^{-1/2}\phi_c \right\|_{L^2}^2 = \frac{2c^2\lambda^2}{1-c^2}\mathcal{I}_c(\phi_c) \\ &> \frac{2c^2}{1-c^2}\left(\frac{p+1}{p-1}\right) d(c) \end{aligned}$$

where we have used (4.4). Next, as in the proof of Theorem 4.8, we choose some $v_0 \in H^{s_0+1}$ such that $\|(v_0)_x - \phi_c\|_{H^{s_0}} < \epsilon$. For $\lambda > 1$ we let $U_0 = (u_0, w_0) = (\lambda(v_0)_x, -c\lambda(v_0)_x)$. Since $\mathcal{E}$, $\mathcal{I}_c$, and $\mathcal{Q}$ are continuous on H^{s_0} for λ sufficiently close to 1, one gets: $\|U_0 - \Phi_c\|_X < \epsilon$, $U_0 \in \Sigma_-(c)$ and

$$-c\mathcal{M}(U_0) > \frac{2c^2}{1-c^2}\left(\frac{p+1}{p-1}\right) d(c). \tag{5.2}$$

Let $U(t) = (u(t), w(t))$ be the solution of the Cauchy problem (2.3)-(2.5) with $L = I$. The rest of the proof is quite similar to the one of Theorem 3.5 of [8]. We then have $u = v_x$ with

$$v(.,t) = \lambda v_0 + \int_0^t w(.,\tau)d\tau.$$

With an easy computation this yields

$$\left\| B^{-1/2}v(t) \right\|_{L^2} \le \lambda \left\| B^{-1/2}v_0 \right\|_{L^2} + \int_0^t \left\| B^{-1/2}w(\tau) \right\|_{L^2} d\tau.$$

This inequality tells us that $\left\|B^{-1/2}w(t)\right\|_{L^2}$, equivalently $\|w(t)\|_{H^{r/2}}$, and thus $U(t)$ blows up in finite time whenever the functional $H(t) = \frac{1}{2}\left\|B^{-1/2}v(t)\right\|_{L^2}^2$ does so. Therefore the proof is completed by showing that $H(t)$ blows up in finite time. Thanks to Levine's Lemma [35]. It says that if $H'(t_0) > 0$ for some $t_0 > 0$, and $HH'' - (1+\nu)(H')^2 \ge 0$

for some $\nu > 0$ then $H(t)$ will blow up in finite time. We proceed to show that

$$
\begin{aligned}
H'(t) &= \left\langle B^{-1/2}v, B^{-1/2}v_t \right\rangle, \\
H''(t) &= \left\| B^{-1/2}v_t \right\|_{L^2}^2 + \left\langle B^{-1/2}v, B^{-1/2}v_{tt} \right\rangle \\
&= \left\| B^{-1/2}v_t \right\|_{L^2}^2 + \int vB^{-1}v_{tt}dx \\
&= \left\| B^{-1/2}v_t \right\|_{L^2}^2 + \int v\left(B^{-1}v_{xx} - \left(|v_x|^{p-1}v_x\right)_x\right)dx \\
&= \left\| B^{-1/2}v_t \right\|_{L^2}^2 - \int v_x\left(B^{-1}v_x - \left(|v_x|^{p-1}v_x\right)\right)dx \\
&= \left\| B^{-1/2}w \right\|_{L^2}^2 - \left\| B^{-1/2}u \right\|_{L^2}^2 + \mathcal{Q}(u) \\
&= \|B^{-1/2}(w+cu)\|_{L^2}^2 - \left(1+c^2\right)\left\| B^{-1/2}u \right\|_{L^2}^2 - 2c\mathcal{M}(u,w) + \mathcal{Q}(u) \\
&= \|B^{-1/2}(w+cu)\|_{L^2}^2 - \frac{2\left(1+c^2\right)}{1-c^2}\mathcal{I}_c(u) - 2c\mathcal{M}(u,w) + \mathcal{Q}(u).
\end{aligned}
$$

By (2.6), (2.7), (3.16) and (3.17) we have

$$
Q(u) = \frac{p+1}{2}\left\| B^{-1/2}(w+cu) \right\|_{L^2}^2 + (p+1)\mathcal{I}_c(u) - (p+1)[\mathcal{E}(u,w) + c\mathcal{M}(u,w)].
$$

Substituting this result into the above equation we get

$$
\begin{aligned}
H''(t) &= \frac{p+3}{2}\left\| B^{-1/2}(w+cu) \right\|_{L^2}^2 + \left(p+1-\frac{2(1+c^2)}{1-c^2}\right)\mathcal{I}_c(u) \\
&\quad - 2c\mathcal{M}(u,w) - (p+1)[\mathcal{E}(u,w) + c\mathcal{M}(u,w)]. \qquad (5.3)
\end{aligned}
$$

Note that the coefficient of $\mathcal{I}_c(u)$ is positive since $c^2 < \frac{p-1}{p+3}$. So the estimate of Lemma 5.3, i.e. $\frac{p+1}{p-1}d(c) < \mathcal{I}_c(u)$, can be employed above. Furthermore, using the conservation laws we get

$$
\mathcal{E}(U) + c\mathcal{M}(U) = \mathcal{E}(U_0) + c\mathcal{M}(U_0) = d(c) - \delta < d(c)
$$

for some $\delta > 0$, and by (5.2)

$$
-c\mathcal{M}(U) = -c\mathcal{M}(U_0) > \frac{2c^2}{1-c^2}\left(\frac{p+1}{p-1}\right)d(c).
$$

Combining these with (5.3) we obtain

$$
\begin{aligned}
H''(t) \quad > \quad & \frac{p+3}{2}\left\|B^{-1/2}(w+cu)\right\|_{L^2}^2+\left(p+1-\frac{2\left(1+c^2\right)}{1-c^2}\right)\left(\frac{p+1}{p-1}\right)d(c)\\
& +\frac{4c^2}{1-c^2}\left(\frac{p+1}{p-1}\right)d(c)-(p+1)d(c)+(p+1)\,\delta\\
= \quad & \frac{p+3}{2}\left\|B^{-1/2}(w+cu)\right\|_{L^2}^2+\left(p+1-\frac{2\left(1+c^2\right)}{1-c^2}\right.\\
& \left.+\frac{4c^2}{1-c^2}\right)\left(\frac{p+1}{p-1}\right)d(c)-(p+1)d(c)+(p+1)\delta\\
= \quad & \frac{p+3}{2}\left\|B^{-1/2}(w+cu)\right\|_{L^2}^2+(p+1)\,\delta.
\end{aligned}
$$

So, $H''(t)>(p+1)\,\delta$ which in turn implies that $H'(t_0)>0$ for some $t_0>0$. Thus, one of the two conditions of Levine's Lemma holds. What is left is to show that the second condition is also satisfied. Note that as D_x commutes with $B^{-1/2}$ we have

$$
\langle B^{-1/2}v, B^{-1/2}u\rangle=\int\left(B^{-1/2}v\right)\left(B^{-1/2}v\right)_x dx=\frac{1}{2}\int\frac{\partial}{\partial x}\left(B^{-1/2}v\right)^2dx=0.
$$

Since

$$
\langle B^{-1/2}v, B^{-1/2}w\rangle=\langle B^{-1/2}v, B^{-1/2}\,(w+cu)\rangle
$$

we have

$$
\left(H'(t)\right)^2=\left(\langle B^{-1/2}v, B^{-1/2}(w+cu)\rangle\right)^2\leq\|B^{-1/2}v\|_{L^2}^2\|B^{-1/2}(w+cu)\|_{L^2}^2.
$$

Finally, with $1+\nu=\frac{p+3}{4}$ we have

$$
H(t)\,H''(t)-\frac{p+3}{4}\left(H'(t)\right)^2\geq\frac{p+1}{2}\|B^{-1/2}v\|_{L^2}^2\delta=(p+1)H(t)\delta\geq 0
$$

This completes the proof of assertion (ii) of Theorem 5.1 □

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