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THE DIMENSION OF THE CATEGORY OF MAXIMAL COHEN-MACAULAY MODULES OVER HYPERSURFACES

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Abstract

The dimension of subcategories of abelian categories was introduced by Dao and Takahashi in 2015 as an analogue of the dimension of triangulated categories. In this paper we investigate the category consisting of maximal Cohen-Macaulay R-modules $\mathrm{CM}(R)$ and we find an upper bound of the dimension of $\mathrm{CM}(R)$ for a given local hypersurface R.

1. Introduction

The dimension of triangulated category was introduced by Rouquier ([13]), which is defined as the number of triangles necessary to build the category from a single object, up to finite direct sum, direct summand and shift. As an analogue of this concept, Dao and Takahashi defined the dimension of subcategories of abelian categories with enough projective objects by comparing short exact sequences to triangles([6]). For a Cohen-Macaulay local ring R, it is natural to ask what is the dimension of the category of maximal Cohen Macaulay R-modules $\mathrm{CM}(R)$, in the sense of Dao and Takahashi. When R is a Gorenstein local ring, it is known that the stable category of Cohen-Macaulay R-modules $\underline{\mathrm{CM}}(R)$ forms a triangulated category, so one can define the dimension of $\underline{\mathrm{CM}}(R)$, in the sense of Rouquier. There exists some results about $\dim \underline{\mathrm{CM}}(R)$. For example, Bergh, Iyengar, Krause and Opperman proved that $\dim \underline{\mathrm{CM}}(R)$ is at least $\operatorname{codim} R-1$ if R is a complete intersection ([2]). Kobayashi gave an example of a complete intersection R of dimension one with $\dim \underline{\mathrm{CM}}(R)=n$ for a given positive integer n([11]). In addition, there are many kinds of upper bounds of $\dim \mathrm{CM}(R)$ under the condition that R has an isolated singularity. For example [7] by Dugas and Leuschke, [3] by Ballard, Favero and Katzarkov.

It is proved that if R is a local hypersurface, then $\dim \mathrm{CM}(R)$ as a subcategory of mod R is equal to $\dim \underline{\mathrm{CM}}(R)$ as a triangulated category([6]). Let R be a complete local hypersurface over an algebraically closed field k of characteristic zero. If R has finite or countable Cohen-Macaulay representation type, then R is isomorphic to $k[\![x_0, x_1, \dots, x_d]\!]/(f)$ and the form of the defining equation f is completely characterized. For the detail, see [4, 10, 14]. In particular, $R=k[\![x, y]\!]/(x^2)$ has countable Cohen-Macaulay representation type and it is proved that the dimension of $\mathrm{CM}(R)$ is equal to 1 by Araya, Iima and Takahashi([1]). On the other hand, it is proved that $R=k[\![x, y]\!]/(x^n)$ with $n \geq 3$ has uncountable maximal Cohen-Macaulay modules by Buchweitz, Greuel and Schreyer ([4]). In addition, Kawasaki, Nakamura and the author proved that the dimension of $\mathrm{CM}(R)$ is also equal to 1 for any $n \geq 3$([9]). In this paper, we consider $\mathrm{CM}(R)$ for a local hypersurface R and describe an upper bound of $\dim \mathrm{CM}(R)$ using the information of $\mathrm{CM}(R/f_i R)$ where f is the defining

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equation of R and $f=f_1 f_2 \cdots f_n$ is a factorization of f. As a simple conclusion of our argument, when R/f_iR has finitely Cohen-Macaulay representation type, it follows that $\dim \mathrm{CM}(R) \leq n \dim R$. We note that our main theorem is available for a local hypersurface which does not have an isolated singularity.

We denote by mod R the category of finitely generated R-modules for a noetherian local ring R. In this paper, we assume that all modules are finitely generated. When R is a Cohen-Macaulay local ring, we denote by $\mathrm{CM}(R)$ (resp. $\underline{\mathrm{CM}}(R)$) the category of maximal Cohen-Macaulay R-modules (resp. the stable category of maximal Cohen-Macaulay R-modules). For $M \in \mathrm{mod}\, R$, $\Omega^i_R M$ stands for the i^{th} syzygy module of M over R. When $i=1$, we simply denote it by $\Omega_R M$.

Let R be a noetherian local ring. In the rest of this section, we recall the definition of the dimension of subcategories of mod R. To state the definition and some properties, we follow the notation used by Dao and Takahashi in [5] and [6]. Let R be a noetherian ring. For a subcategory $\boldsymbol{X}$ of mod R, we denote by $[\boldsymbol{X}]$ the smallest full subcategory of mod R that contains R and is closed under finite direct sums, direct summands and syzygies, i.e.,

$$[\boldsymbol{X}]=\mathrm{add}\{R, \Omega^i_R X \mid i \geq 0, X \in \boldsymbol{X}\}$$

We note that $[\boldsymbol{X}]$ depends on the choice of the ground ring R. We write $[\boldsymbol{X}]^R$ when we should specify the ground ring R. When $\boldsymbol{X}$ consists of a single object X, we simply denote $[\boldsymbol{X}]$ by $[X]$. For subcategories $\boldsymbol{X}$ and $\boldsymbol{Y}$ of mod R we denote by $\boldsymbol{X} \circ \boldsymbol{Y}$ the full subcategory of mod R consisting of objects M which fit into an exact sequence $0 \to X \to M \to Y \to 0$ in mod R with $X \in$ X and $Y \in \boldsymbol{Y}$. We set $\boldsymbol{X} \bullet \boldsymbol{Y}=[[\boldsymbol{X}] \circ [\boldsymbol{Y}]]$. For a subcategory $\boldsymbol{X}$ of mod R we define inductively the iteration of the operator $\bullet$ as follows:

$$[\boldsymbol{X}]_r=\begin{cases}[\boldsymbol{X}] & (r=1)\\ [\boldsymbol{X}]_{r-1} \bullet [\boldsymbol{X}] & (r \geq 2)\end{cases}$$

When $\boldsymbol{X}$ consists of a single object X, we simply denote it by $[X]_r$. We write $[X]^R_r$ when we should specify the ground ring R. Then we have the following properties. See [5, Proposition 2.2] for the detail.

Proposition 1. 1. *Let R be a noetherian local ring and $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{C}$ subcategories of* mod R.

(1) *For $M \in$* mod R, *M belongs to $\boldsymbol{X} \bullet \boldsymbol{Y}$ if and only if there exist an exact sequence of the form $0 \to A \to M \oplus U \to B \to 0$ with $A \in \boldsymbol{X}$, $B \in \boldsymbol{Y}$ and $U \in$* mod *R.*

(2) $(\boldsymbol{X} \circ \boldsymbol{Y}) \circ \boldsymbol{C}=\boldsymbol{X} \circ (\boldsymbol{Y} \circ \boldsymbol{C})$ *and* $(\boldsymbol{X} \bullet \boldsymbol{Y}) \bullet \boldsymbol{C}=\boldsymbol{X} \bullet (\boldsymbol{Y} \bullet \boldsymbol{C})$

(3) *For integers n, $m>0$, $[\boldsymbol{X}]_n \bullet [\boldsymbol{X}]_m=[\boldsymbol{X}]_{n+m}$.*

Definition 1. 2. Let R be a noetherian ring, and let $\boldsymbol{X}$ be a subcategory of mod R. We define the *dimension* of $\boldsymbol{X}$, denoted by $\dim \boldsymbol{X}$, as the infimum of the integers $n \geq 0$ such that $\boldsymbol{X}=[G]_{n+1}$ for some $G \in \mathrm{mod}\, R$. By definition, $\dim \boldsymbol{X} \in \mathbb{N} \cup \{0, \infty\}$.

2. Main theorem

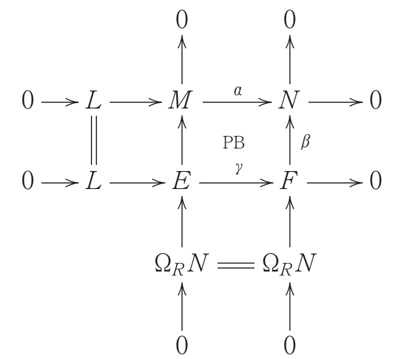
We begin with the following lemma. It might be a known result, but we give the proof for the sake of completeness.

Lemma 2. 1. *Let R be a noetherian local ring and $0 \to L \to M \xrightarrow{\alpha} N \to 0$ an exact sequence in* mod R. *Then there exist a finitely generated free R-module F and the following sequence*

$$0 \to \Omega_R N \to L \oplus F \to M \to 0$$

is exact.

Proof. Since $\Omega_R N$ is the first syzygy module of N, we can take a free R-module F and an exact sequence: $0 \to \Omega_R N \to F \xrightarrow{\beta} N \to 0$. Taking the pull back of α and β, we obtain the following diagram:



Since F is a free R-module, γ is a split epimorphism. It means that E is isomorphic to $L \oplus F$ and the assertion follows. □

The next proposition is a key of our argument.

Proposition 2. 2. *Let R be a noetherian local ring of dimension d and I an ideal of R. We assume R/I is a Cohen-Macaulay local ring of dimension d. Let $\boldsymbol{X}$ be a subcategory of* mod R *with $R/I \in \boldsymbol{X}$ such that* $\mathrm{CM}(R/I) \subseteq [\boldsymbol{X}]^R_r$ *for some integer r. Let M be an R/I-module and set $t=d-$* depth M. *Then $\Omega^t_R M \in [\boldsymbol{X}]^R_{r+t}$.*

Proof. We prove the following claim by induction on i.

The conclusion when $i=t$ is the required result.

Claim. For all $0\leq i\leq t$, it follows that $\Omega_R^i(\Omega_{R/I}^{t-i}M)\in[\boldsymbol{X}]_{r+t}^R$.

Let $i=0$. It is clear that $\Omega_{R/I}^t M\in \mathrm{CM}(R/I)\subseteq[\boldsymbol{X}]_r^R$.

Let $i>0$ and we assume the claim is true for $i-1$. We take the following exact sequence:

$$0\to\Omega_{R/I}^{t-i+1}M\to G\to\Omega_{R/I}^{t-i}M\to 0$$

of R/I-modules, where G is a free R/I-module. Applying the horseshoe lemma to the above exact sequence, we obtain the following one:

$$0\to\Omega_R^{i-1}(\Omega_{R/I}^{t-i+1}M)\to(\Omega_R^{i-1}G)\oplus F_1\to\Omega_R^{i-1}(\Omega_{R/I}^{t-i}M)\to 0,$$

where F_1 is a free R-module. We apply Lemma 2.1 to this exact sequence, then we get

$$0\to\Omega_R^i(\Omega_{R/I}^{t-i}M))\to(\Omega_R^{i-1}(\Omega_{R/I}^{t-i+1}M))\oplus F_2\to\Omega_R^{i-1}G\oplus F_1\to 0,$$

where F_2 is a free R-module. Applying Lemma 2.1 again, we get

$$0\to\Omega_R^iG\to(\Omega_R^i(\Omega_{R/I}^{t-i}M))\oplus F_3\to\Omega_R^{i-1}(\Omega_{R/I}^{t-i+1}M))\oplus F_2\to 0,$$

where F_3 is a free R-module. The left side object belongs to $[R/I]^R$, which is contained in $[\boldsymbol{X}]^R$ since $R/I\in\boldsymbol{X}$. The right side object belongs to $[\boldsymbol{X}]_{r+i-1}^R$ by the hypothesis of induction. By Proposition 2.1, it follows that the middle object belongs to $[\boldsymbol{X}]_{r+i}^R$. Hence, its direct summand $\Omega_R^i(\Omega_{R/I}^{t-i}M)$ belongs to $[\boldsymbol{X}]_{r+i}^R$. □

Remark 2.3. Under the situation of Proposition 2.2, if $s\geq t=d-\operatorname{depth} M$, then we have $\Omega_R^sM\in[\boldsymbol{X}]_{r+t}^R$ because $[\boldsymbol{X}]_{r+t}^R$ is a category being closed under taking syzygies.

When R is a local hypersurface, we can state the next lemma. See [12, Lemma 2.6] for a proof.

Lemma 2.4. *Let S be a regular local ring and $x, y\in S$ non-zero elements. Let $M\in\mathrm{CM}(S/(x))$. Then for each integer $n>0$, one has an inclusion* $[M]_n^{S/(x)}\subseteq[M\oplus S/(x)\oplus\Omega_{S/(x)}M]_n^{S/(xy)}$. *In particular, if* $\dim\mathrm{CM}(S/(x))\leq r$, *then there exists $N\in\mathrm{CM}(S/(xy))$ such that* $\mathrm{CM}(S/(x))\subseteq[N]_{r+1}^{S/(xy)}$.

The following theorem is the main result of this paper.

Theorem 2.5. *Let S be a regular local ring of dimension $d+1$ with $d>0$ and $x_1,\ldots,x_n\in S$ non-zero non-unit elements. Then*

$$\dim\mathrm{CM}(S/(x_1\cdots x_n))\leq\begin{cases}\sum_{i=1}^n\dim\mathrm{CM}(S/(x_i))+n-1 & (d=1),\\ \sum_{i=1}^n\dim\mathrm{CM}(S/(x_i))+nd-2 & (d>1).\end{cases}$$

Proof. We may assume that $\dim\mathrm{CM}(S/(x_i))<\infty$ for all $i=1,\ldots,n$ and set $r_i=\dim\mathrm{CM}(S/(x_i))$. Then there exists $G_i\in\mathrm{CM}(S/(x_1\cdots x_n))$ such that $\mathrm{CM}(S/(x_i))\subseteq[G_i]_{r_i+1}^{S/(x_1\cdots x_n)}$ by Lemma 2.4. Note that $[\oplus_{i=1}^n G_i]_r^{S/(x_1\cdots x_n)}\subseteq\mathrm{CM}(S/(x_1\cdots x_n))$ for any positive integer r since $G_i\in\mathrm{CM}(S/(x_1\cdots x_n))$ for each $1\leq i\leq n$.

Let $M\in\mathrm{CM}(S/(x_1\cdots x_n))$. Setting $M_i:=(0:_M x_1\cdots x_i)$, we have a filtration $0=M_0\subseteq M_1\subseteq\cdots\subseteq M_n=M$ of S-submodules of M such that M_i/M_{i-1} is an $S/(x_i)$-module. Note that there is an isomorphism $M_i/M_{i-1}\to(0:_{x_1\cdots x_{i-1}M}x_i)$ given by $\bar{z}\mapsto x_1\cdots x_{i-1}z$ for $z\in M_i$. The target is a submodule of M, and hence it has positive depth.

We have the following exact sequences

$$0\to M_1\to M_2\to M_2/M_1\to 0,$$
$$0\to M_2\to M_3\to M_3/M_2\to 0,$$
$$\vdots$$
$$0\to M_{n-1}\to M\to M/M_{n-1}\to 0$$

of $S/(x_1\cdots x_n)$-modules. When $d>1$, $\operatorname{depth} M_i\geq 2$ since $\operatorname{depth} M_i/M_{i-1}>0$. Proposition 2.2 and Lemma 2.4 imply that

$$(*)\begin{cases}\Omega_{S/(x_1\cdots x_n)}^{d-2}M_1\in[G_1]_{r_1+d-1}^{S/(x_1\cdots x_n)}\text{ and}\\ \Omega_{S/(x_1\cdots x_n)}^{d-1}M_i/M_{i-1}\in[G_i]_{r_1+d}^{S/(x_1\cdots x_n)}\text{ for all }2\leq i\leq n.\end{cases}$$

Applying horseshoe lemma to the above sequences, we get short exact sequences

$$0\to\Omega_{S/(x_1\cdots x_n)}^{d-1}M_{i-1}\to\Omega_{S/(x_1\cdots x_n)}^{d-1}M_i\oplus(S/(x_1\cdots x_n))^{\oplus}\to\Omega_{S/(x_1\cdots x_n)}^{d-1}M_i/M_{i-1}\to 0$$

for all $2\leq i\leq$ n. When $i=2$, it follows that

$$\Omega_{S/(x_1\cdots x_n)}^{d-1}M_2\in[G_1\oplus G_2]_{r_1+r_2+2d-1}^{S/(x_1\cdots x_n)}$$

from $(*)$. Iterating this procedure, we have that

$$\Omega_{S/(x_1\cdots x_n)}^{d-1}M\in[\bigoplus_{i=1}^n G_i]_{\Sigma_{i=1}^n r_i+nd-1}^{S/(x_1\cdots x_n)}.$$

Since $S/(x_1\cdots x_n)$ is a local hypersurface, M is isomorphic to the $(d-1)^{\text{th}}$ or d^{th} syzygy of itself. See [14, Chapter 7] for the detail. Therefore $\mathrm{CM}(S/(x_1\cdots x_n))\subseteq[\oplus_{i=1}^n G_i]_{\Sigma_{i=1}^n r_i+nd-1}^{S/(x_1\cdots x_n)}$.

When $d=1$, $M_i\in\mathrm{CM}(S/(x_i))\subseteq[G_i]_{r_i+1}^{S/(x_1\cdots x_n)}$ for all $1\leq i\leq n$ since $\operatorname{depth} M_i>0$. Therefore M belongs to $[\oplus_{i=1}^n G_i]_{\Sigma_{i=1}^n r_i+n}^{S/(x_1\cdots x_n)}$. □

We will give some examples of Theorem 2.5.

Example 2.6. Under the same notations as in Theorem 2.5, we assume that $S/(x_i)$ has finite Cohen-Macaulay representation type for all $1\leq i\leq n$, i.e., there exist only finitely many isomorphism classes of indecomposable maximal Cohen-Macaulay $S/(x_i)$-modules. Then it is clear that $\dim\mathrm{CM}(S/(x_i))=0$. Therefore, Theorem 2.5 implies that

$$\dim \mathrm{CM}(S/(x_1\cdots x_n))\leq\begin{cases} n-1 (d=1),\\ nd-2 (d>1).\end{cases}$$

For example, when $S=k[\![X_0,\ldots,X_d]\!]$ with $d>1$, where k is a field and $x_1=\cdots=x_n=X_0$, Theorem 2.5 implies that

$$\dim \mathrm{CM}(S/(X_0^n))\leq\begin{cases} n-1 (d=1),\\ nd-2 (d>1).\end{cases}$$

for each positive integer $n>0$.

However, there exists the better upper bound if $\dim S/(x_1,\ldots x_n)=1$ under the same notation as in Theorem 2.5, one has the following theorem.

Theorem 2.7 ([12]). *Let S be a regular local ring of dimension two and $x_1,\ldots x_n\in S$. Then one has*

$$\dim \mathrm{CM}(S/(x_1,\ldots x_n))\leq \sup\{\dim \mathrm{CM}(S/(x_i))\}_{i=1,\ldots n}+1.$$

Example 2.8. Let $S=k[\![X,Y,Z]\!]$, where k is an algebraically closed field with characteristic zero and let $0<n\leq m$ positive integers. We consider $\dim \mathrm{CM}(S/(X^mY^m))$.

When $n=m=1$, it is known that $\dim \mathrm{CM}(S/(XY))=1$ by [1, Proposition 2.2 (1)]. Using this fact and the result of Example 2.6, we have

$$\begin{aligned}\dim \mathrm{CM}(S/(X^nY^m)) &\leq n\cdot\dim \mathrm{CM}(S/(XY))+\dim \mathrm{CM}(S/(Y^{m-n}))+(n+1)\cdot 2-2\\ &\leq n\cdot 1+\{(m-n)\cdot 2-2\}+(n+1)\cdot 2-2\\ &=2m+n-2.\end{aligned}$$

The theory of the matrix factorization enables Example 2.6 and 2.8 to extend the higher dimensional cases. Here we recall the basic facts about matrix factorizations. For detail, we refer to [8, Chapter 6], [10], [14, Chapter 7].

Let $(S,\mathfrak{n})$ be a regular local ring and $f\in\mathfrak{n}$ a non-zero element. Let $R=S/(f)$ and $M\in\mathrm{CM}(R)$. Then we say that (φ,ψ) is a matrix factorization of f corresponding to M if the following conditions hold.

(1) Both φ and ψ are n-th square matrices with entries in S, where n is the number of a minimal generating system of M.

(2) There exists an exact sequence $0\to S^n\xrightarrow{\varphi}S^n\to M\to 0$.

(3) $\varphi\psi=fE_n$, where E_n is the identity matrix of rank n.

We denote by $\mathrm{MF}_S(f)$ the category of the matrix factorizations of f. If $(\varphi,\psi)\in\mathrm{MF}_S(f)$, we obtain an object of $\mathrm{CM}(R)$ as the cokernel of the map $S^n\xrightarrow{\varphi}S^n$. This correspondence gives a functor from $\mathrm{MF}_S(f)$ to $\mathrm{CM}(R)$. Eisenbud proved that this functor induces the equivalence between the quotient categories $\mathrm{MF}_S(f)/\{(1,f),(f,1)\}$ and $\underline{\mathrm{CM}}(R)$ [8, Chapter 6], where $\underline{\mathrm{CM}}(R)$ is the stable category of $\mathrm{CM}(R)$. The following theorem is called Knörrer's periodicity.

Theorem 2.9. ([10, Theorem 3.1]) *Let $S=k[\![x_0,x_1,\ldots,x_d]\!]$ and $T=k[\![x_0,x_1,\ldots,x_n,y,z]\!]$ be formal power series rings over an algebraically closed field k of characteristic zero*
and $f\in(x_0,\ldots,x_d)S$ a non-zero element. The functor $\mathrm{MF}_S(f)\to\mathrm{MF}_T(f+yz)$ given by $(\varphi,\psi)\mapsto\left(\begin{pmatrix}\varphi & y\\ z & -\psi\end{pmatrix},\begin{pmatrix}\psi & y\\ z & -\varphi\end{pmatrix}\right)$ induces an equivalence between the stable categories $\underline{\mathrm{CM}}(S/fS)$ and $\underline{\mathrm{CM}}(T/(f+yz)T)$.

When R is a Gorenstein local ring, $\underline{\mathrm{CM}}(R)$ forms a triangulated category. Therefore we can define $\dim\underline{\mathrm{CM}}(R)$ in the sense of Rouquier [13]. However, if R is a local hypersurface, there exists the equality $\dim\underline{\mathrm{CM}}(R)=\dim\underline{\mathrm{CM}}(R)$ [6, Proposition 3.5 (3)]. Therefore, we have the following inequalities as a corollary of Example 2.6 and 2.8.

Corollary 2.10. *Let n, m and d be integers such that $0<n\leq m$ and $d\geq 0$. Let k be an algebraically closed field of characteristic zero. Then the following inequalities hold.*

(1) $\dim \mathrm{CM}\left(\dfrac{k[\![X_0,X_1,X_2,Y_1,\ldots,Y_{2d}]\!]}{(X_0^n+Y_1^2+Y_2^2+\cdots+Y_{2d}^2)}\right)\leq 2n-2.$

(2) $\dim \mathrm{CM}\left(\dfrac{k[\![X_0,X_1,X_2,Y_1,\ldots,Y_{2d}]\!]}{(X_0^nX_1^m+Y_1^2+Y_2^2+\cdots+Y_{2d}^2)}\right)\leq 2m+n-2.$

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