A QUANTILE-BASED PROBABILISTIC MEAN VALUE THEOREM

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Abstract

For nonnegative random variables with finite means we introduce an analogous of the equilibrium residual-lifetime distribution based on the quantile function. This allows to construct new distributions with support $(0,1)$, and to obtain a new quantile-based version of the probabilistic generalization of Taylor's theorem. Similarly, for pairs of stochastically ordered random variables we come to a new quantile-based form of the probabilistic mean value theorem. The latter involves a distribution that generalizes the Lorenz curve. We investigate the special case of proportional quantile functions and apply the given results to various models based on classes of distributions and measures of risk theory. Motivated by some stochastic comparisons, we also introduce the 'expected reversed proportional shortfall order', and a new characterization of random lifetimes involving the reversed hazard rate function.

Short title: A quantile-based probabilistic mean value theorem.

1 Introduction

The quantile function, being the inverse of the cumulative distribution function of a random variable, is often invoked in applied probability and statistics. In certain cases the approach based on quantile functions is more fruitful than the use of cumulative distribution functions, since quantile functions are less influenced by extreme statistical observations. For instance, quantile functions can be properly employed to formulate properties of entropy function and

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other information measures for nonnegative absolutely continuous random variables (see Sunoj and Sankaran [22] and Sunoj *et al.* [23]). They are also employed in problems that ask for comparisons based on variability stochastic orders such as the dilation order, the dispersive order (see Shaked and Shanthikumar [20]) or the TTT transform order (cf. Kochar *et al.* [13]). In addition, several notions of risk theory and mathematical finance are expressed in terms of quantile functions (see, for instance, Belzunce *et al.* [2] and [3]).

In this paper we use the quantile functions in order to build some stochastic models and obtain various results involving distributions with support $(0,1)$. We are motivated by previous researches in which the equilibrium distribution of nonnegative random variables plays a key role and allows to obtain probabilistic generalizations of Taylor's theorem (see [14] and [16]) and of the mean value theorem (see [6]).

In Section 2 we present some preliminary notions on quantile function and Lorenz curve. Then, in Section 3 we obtain a probabilistic generalization of Taylors theorem based on a suitably defined 'quantile analogue' of the equilibrium distribution, whose density is an extension of the Lorenz curve based on stochastically ordered random variables. Moreover, such distribution is involved in a quantile-based version of the probabilistic mean value theorem provided in Section 4. A special case dealing with proportional quantile functions is also discussed. Finally, various examples of applications are considered in Section 5: the first involves typical classes of distributions (NBU and IFR notions) and conditional value-at-risks; the second involves concepts of risk theory, as the proportional conditional value-at-risk; the third and the fourth applications are founded on distribution functions defined as suitable ratios of quantile functions, and involve the notion of average value-at-risk.

We point out that, aiming to obtain useful stochastic comparisons, in this paper we introduce two new concepts that deserve interest in the field of stochastic orders and characterizations of distributions. In Section 4 we propose the 'expected reversed proportional shortfall order', which is dual to a recently proposed stochastic order. In Section 5.3 we provide a new characterization of random lifetimes, expressed by stating that $x\,\tau(x)$ is decreasing for $x>0$, where $\tau(x)$ is the reversed hazard rate function.

Throughout the paper, $[X\,|\,B]$ denotes a random variable having the same distribution as X conditional on B, the terms decreasing and increasing are used in non-strict sense, and g' denotes the derivative of g.

2 Preliminary notions

Given a random variable X, let us denote its distribution function by $F(x) = P(X \leq x)$, $x \in \mathbb{R}$, and its complementary distribution function by $\overline{F}(x) = 1 - F(x)$, $x \in \mathbb{R}$. The quantile function of X, when existing, is given by

$$Q(u) = \inf\{x \in \mathbb{R} : F(x) \geq u\}, \qquad 0 < u < 1. \tag{1}$$

Moreover, if $Q(u)$ is differentiable, the quantile density function of X is given by

$$q(u) = Q'(u), \qquad 0 < u < 1. \tag{2}$$

Definition 1 *We denote by $\mathcal{D}$ the family of all absolutely continuous random variables with finite mean such that the quantile function (1) satisfies $Q(0) = 0$, and the quantile density function (2) exists.*

A random variable in $\mathcal{D}$ is thus nonnegative and may represent a distribution of interest in actuarial applications or in risk theory, such as an income or a loss. If $X \in \mathcal{D}$, it has finite nonzero mean, and the function

$$L(p) = \frac{1}{E[X]} \int_0^p Q(u)\,\mathrm{d}u, \qquad 0 \leq p \leq 1 \tag{3}$$

denotes the Lorenz curve of X. If the individuals of a given population share a common good such as wealth, which is distributed according to X, then $L(p)$ gives the cumulative share of individuals, from the lowest to the highest, owing the fraction p of the common good. Hence, $L(p)$ is often used in insurance to describe the inequality among the incomes of individuals. See, for instance, Singpurwalla and Gordon [21] and Shaked and Shanthikumar [19] for various applications of the Lorenz curve and its connections with stochastic orders.

It is well known that (3) is the distribution function of an absolutely continuous random variable, say X^L, taking values in $(0, 1)$. In the following proposition we express the mean of an arbitrary function of X^L in terms of the quantile function (1). To this aim we recall that if $g : (0, +\infty) \to \mathbb{R}$ is an integrable function then, for all $0 \leq p_1 < p_2 \leq 1$,

$$\frac{1}{p_2 - p_1} \int_{p_1}^{p_2} g(Q(u))\,\mathrm{d}u = E[g(X) \,|\, Q(p_1) < X \leq Q(p_2)]. \tag{4}$$

Proposition 1 *Let $X \in \mathcal{D}$ and let X^L have distribution function* (3)*. If $h : (0, 1) \to \mathbb{R}$ is such that $h \cdot Q$ is integrable in* $(0, 1)$*, then*

$$E[h(X^L)] = \frac{1}{E[X]} \int_0^1 h(u)\, Q(u)\,\mathrm{d}u = \frac{1}{E[X]}\, E[h(F(X))\, X] \tag{5}$$

or, equivalently,

$$E[h(X^L)] = \frac{1}{E[Q(U)]}\, E[h(U)\, Q(U)],$$

where $U = F(X)$ is uniformly distributed in $(0, 1)$.

Proof. Since X^L has distribution function (3), the proof follows from identity $F[Q(u)] = u$, $0 < u < 1$, and from Eq. (4) for $p_1 = 0$ and $p_2 = 1$. $\square$

As an immediate application of Proposition 1 we have that the moments of X^L, when existing, are given by:

$$E[(X^L)^k] = \frac{1}{E[Q(U)]} E[U^k Q(U)], \qquad k = 1, 2, \dots . \tag{6}$$

Remark 1 Let $F(x) = x^\alpha$, $0 \le x \le 1$, $\alpha > 0$. Then, X is identically distributed to X^L if, and only if, α is equal to the reciprocal of the golden number, i.e. $\alpha = (-1 + \sqrt{5})/2$.

3 An analogous of Taylor's theorem

It is well-known that the equilibrium distribution arises as the limiting distribution of the forward recurrence time in a renewal process. Its role in applied contexts has been largely investigated (see, as example, Gupta [10] and references therein). For instance, we recall that the iterates of equilibrium distributions have been used

- to characterize family of distributions (see Unnikrishnan Nair and Preeth [24]),
- to construct sequences of stochastic orders (see Fagiuoli and Pellerey [8]),
- to determine properties related to the moments of random variables of interest in risk theory (see Lin and Willmot [15]).

Moreover, a probabilistic generalization of Taylor's theorem (studied by Massey and Whitt [16] and Lin [14]) allows to express the expectation of a functional of random variable in terms of suitable expectations involving the iterates of its equilibrium distribution.

We recall that for a random variable $X \in \mathcal{D}$ the density of the equilibrium distribution of X and the density of X^L are given respectively by

$$f_{X_e}(x) = \frac{\overline{F}(x)}{E[X]}, \quad 0 < x < +\infty, \qquad f_{X^L}(u) = \frac{Q(u)}{E[X]}, \quad 0 < u < 1. \tag{7}$$

On the ground of the analogy between such densities, in this section we obtain an analogous of the probabilistic generalization of Taylor's theorem which involves X^L.

Theorem 1 *Let $X \in \mathcal{D}$; if $g : (0,1) \to \mathbb{R}$ is a differentiable function such that $g' \cdot Q$ is integrable on $(0,1)$, then*

$$E[\{g(1) - g(U)\}\, q(U)] = E\left[g'(X^L)\right] E[X], \tag{8}$$

where U is uniformly distributed in $(0,1)$.

Proof. From the mean value theorem, Eq. (2), and condition $Q(0) = 0$ we have

$$\begin{aligned} E[\{g(1) - g(U)\}\, q(U)] &= E\left[q(U) \int_0^1 g'(u)\, \mathbf{1}_{\{U \le u\}}\, \mathrm{d}u\right] \\ &= \int_0^1 g'(u)\, E[q(U)\, \mathbf{1}_{\{U \le u\}}]\, \mathrm{d}u \\ &= \int_0^1 g'(u)\, Q(u)\, \mathrm{d}u. \end{aligned}$$

The proof of (8) then follows from the first equality in (5). □

Example 1 Let X have Lomax distribution, with quantile function $Q(u) = \lambda[(1-u)^{-1/\alpha} - 1]$, $0 < u < 1$, and mean $E[X] = \lambda/(\alpha - 1)$, for $\alpha > 1$ and $\lambda > 0$. Then, under the assumptions of Theorem 1 we have

$$E[\{g(1) - g(U)\}\, (1 - U)^{-1-1/\alpha}] = E\left[g'(X^L)\right]\, \frac{\alpha}{\alpha - 1},$$

where X^L has density $f_{X^L}(u) = (\alpha - 1)[(1 - u)^{-1/\alpha} - 1]$, $0 < u < 1$.

Hereafter we extend the result of Theorem 1 to a more general case, in which the right-hand-side of (8) is expressed in an alternative way. Let $g^{(n)}$ denote the n-th derivative of g, for $n \ge 1$, and let $g^{(0)} = g$.

Theorem 2 *Let $X \in \mathcal{D}$; if $g : (0, 1) \to \mathbb{R}$ is n-times differentiable and $g^{(n)} \cdot Q$ is integrable on $(0, 1)$, for any $n \ge 1$, then*

$$\begin{aligned} E[\{g(1) - g(U)\}\, q(U)] = &\sum_{k=1}^{n-1} \frac{1}{k!} E\left[g^{(k)}(U)(1 - U)^k q(U)\right] \\ &+ \frac{1}{(n-1)!} E\left[g^{(n)}(X^L)(1 - X^L)^{n-1}\right] E[X], \end{aligned} \tag{9}$$

where U is uniformly distributed in $(0, 1)$.

Proof. For $n = 1, 2, \ldots$, consider the function

$$R_n g(u) := g(1) - \sum_{k=0}^{n-1} \frac{g^{(k)}(u)}{k!}\, (1 - u)^k, \qquad 0 < u < 1. \tag{10}$$

It has the following properties, for fixed $u \in (0, 1)$:

$$\begin{aligned} R_1 g(u) &= g(1) - g(u), \\ \frac{\partial}{\partial u} R_n g(u) &= -\frac{g^{(n)}(u)}{(n-1)!}\, (1 - u)^{n-1}, \qquad n \ge 1. \end{aligned} \tag{11}$$

Applying Theorem 1 to the function $g^*(u) := R_n g(u)$ we have

$$E[\{R_n g(U) - R_n g(1)\}\, q(U)] = -E\left[\frac{\partial}{\partial u} R_n g(u)\big|_{u=X^L}\right] E[X].$$

Hence, noting that $R_n g(1) = 0$ and making use of Eq. (11) we obtain

$$E[R_n g(U)\, q(U)] = \frac{1}{(n-1)!}\, E\left[g^{(n)}(X^L)(1-X^L)^{n-1}\right] E[X].$$

Finally, substituting (10) in the left-hand-side and rearranging the terms, Eq. (9) follows. □

We note that Eq. (9) reduces to (8) when $n = 1$.

Corollary 1 *Under the assumptions of Theorem 2, for* $\alpha > 0$ *we have*

$$E[(1-U^\alpha)\, q(U)] = \sum_{k=1}^{n-1} \binom{\alpha}{k} E\left[U^{\alpha-k}(1-U)^k q(U)\right] + n\binom{\alpha}{n} E\left[(X^L)^{\alpha-n}(1-X^L)^{n-1}\right] E[X],$$

where $\binom{\alpha}{k} = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)/k!$ *is the generalized binomial coefficient.*

Proof. The proof follows from Theorem 2 by setting $g(u) = u^\alpha$, $0 < u < 1$. □

For instance, making use of Corollary 1, when $\alpha = 1$ we have

$$E[(1-U)\, q(U)] = E[X] = E[Q(U)],$$

whereas when $\alpha = 2$ the following indentities follow from Eq. (6):

$$E[(1-U^2)\, q(U)] = 2E[X^L]\, E[X] = 2E[U\, Q(U)],$$

$$E[(1-U)^2\, q(U)] = 2E[1-X^L]\, E[X] = 2E[(1-U)\, Q(U)].$$

4 An analogous of the mean value theorem

A suitable transformation investigated in Section 3 of Di Crescenzo [6] allows to construct new probability densities via differences of (complementary) distribution functions of two stochastically ordered random variables. We now aim to construct similarly new densities from differences of quantile functions, thus extending (3) to a more general case. This allows us to obtain a quantile-based version of the probabilistic mean value theorem, given in Theorem 3 below.

Let us recall some useful definitions of stochastic orders (see, for instance, Shaked and Shanthikumar [20]). To this purpose, we denote by $Q_X(u)$ and $Q_Y(u)$ the quantile functions of two random variables X and Y, defined as in Eq. (1). We say that X is smaller than Y

- in the usual stochastic order (denoted by $X \le_{st} Y$) if $E[\phi(X)] \le E[\phi(Y)]$ for all non-decreasing functions ϕ for which the expectations exist, or equivalently if $\overline{F}_X(t) \le \overline{F}_Y(t)$ for all $t \in \mathbb{R}$, i.e. $Q_X(u) \le Q_Y(u)$ for all $0 < u < 1$;

• in the hazard rate order (denoted by $X \leq_{hr} Y$) if $\overline{F}_X(t)/\overline{F}_Y(t)$ is decreasing in t;

• in the reversed hazard rate order (denoted by $X \leq_{rh} Y$) if $F_X(t)/F_Y(t)$ is decreasing in t;

• in the likelihood ratio order (denoted by $X \leq_{lr} Y$) if $f_X(t)/f_Y(t)$ is decreasing in t;

• in the star order (denoted by $X \leq_{*} Y$) if $Q_X(u)/Q_Y(u)$ is decreasing in $u \in (0,1)$ (see Section 4.B of Shaked and Shanthikumar [20]);

• in the expected proportional shortfall order (denoted by $X \leq_{PS} Y$) if $\int_u^1 Q_X(p)\,\mathrm{d}p / \int_u^1 Q_Y(p)\,\mathrm{d}p$ is decreasing in $u \in (0,1)$ (see Belzunce *et al.* [2] for some equivalent conditions for this order).

The following result is analogous to Proposition 3.1 of Di Crescenzo [6].

Proposition 2 *Let X and Y be random variables taking values in $\mathbb{R}$, and such that $-\infty < E[X] < E[Y] < +\infty$. Then*

$$f_{Z^L}(u) = \frac{Q_Y(u) - Q_X(u)}{E[Y] - E[X]}, \qquad 0 < u < 1 \tag{12}$$

is the probability density function of an absolutely continuous random variable Z^L taking values in $(0,1)$ if, and only if, $X \leq_{st} Y$.

Proof. The proof immediately follows from the definition of the usual stochastic order. $\square$

We remark that, due to (7), the densities f_{X_e} and f_{X^L} are respectively monotonic decreasing and increasing, whereas density (12) is not necessarily monotonic. We also note that f_{Z^L} can be expressed as a linear combination of two densities. Indeed, under the assumptions of Proposition 2, for $0 < u < 1$ we have

$$f_{Z^L}(u) = c\, f_{Y^L}(u) + (1-c)\, f_{X^L}(u), \qquad c = \frac{E[Y]}{E[Y] - E[X]}. \tag{13}$$

Note that c can be negative, in particular $c < 1$ if and only if $E[X] < 0$, and $c > 1$ if and only if $E[X] > 0$.

Remark 2 Let

$$L_{X,Y}(p) = \frac{1}{E[Y] - E[X]} \int_0^p [Q_Y(u) - Q_X(u)]\mathrm{d}u, \qquad 0 \leq p \leq 1$$

be the distribution function corresponding to density (12). This is a suitable extension of the Lorenz curve (3). Indeed, assume that the individuals of a certain population share a common good such as wealth, distributed according to Y. Suppose that the income received by the individuals is subject to losses due to various reasons (e.g. taxes, faults, damages, etc.), distributed according to X. Hence, $Q_Y(u) - Q_X(u)$ is the *net* income received by the poorest fraction u of the population, and thus $L_{X,Y}(p)$ gives the portion of the *net* wealth held by a portion p of the population. Note that $Q_Y(u) - Q_X(u)$ is not necessarily increasing, and thus generally it is not a quantile function.

Let us now introduce the operator Ψ^L on the set of all pairs of random variables X and Y defined as in Proposition 2, such that $\Psi^L(X,Y)$ denotes an absolutely continuous random variable having density (12). Hence, the random variable $X^L = \Psi^L(0,X)$ has distribution function (3), and the distribution of $Y^L = \Psi^L(0,Y)$ is similarly defined.

Example 2 (i) Let X and Y be exponentially distributed with parameters λ_X and λ_Y, respectively, with $\lambda_X > \lambda_Y$. Then, $\Psi^L(X,Y)$ is exponentially distributed with parameter 1.
(ii) Let X and Y be uniformly distributed in $(0,\alpha_X)$ and $(0,\alpha_Y)$, respectively, with $\alpha_X < \alpha_Y$. Then, $\Psi^L(X,Y)$ is uniformly distributed in $(0,1)$.

Aiming to focus on some stochastic comparisons, we now introduce a new stochastic order based on the quantile function, which is dual to the expected proportional shortfall order.

Definition 2 *We say that X is smaller than Y in the* expected reversed proportional shortfall order *(denoted by $X \le_{RPS} Y$) if $\int_0^u Q_X(p)\,\mathrm{d}p / \int_0^u Q_Y(p)\,\mathrm{d}p$ is decreasing in $u \in (0,1)$.*

Results and properties of such an order go beyond the scope of this article, and thus will be the object of future investigation. The proof of the following results follows from the definitions of the involved notions and some straightforward calculations, and thus is omitted.

Proposition 3 *Under the assumptions of Proposition 2, for $X^L = \Psi^L(0,X)$, $Y^L = \Psi^L(0,Y)$ and $Z^L = \Psi^L(X,Y)$ we have:*
(i) If $X \le_* Y$, then $X^L \le_{lr} Z^L$ and $Y^L \le_{lr} Z^L$.
(ii) If $X \le_{PS} Y$, then $X^L \le_{hr} Z^L$ and $Y^L \le_{hr} Z^L$.
(iii) If $X \le_{RPS} Y$, then $X^L \le_{rh} Z^L$ and $Y^L \le_{rh} Z^L$.
(iv) The following conditions are equivalent:
- $X^L \le_{st} Y^L$,
- $X^L \le_{st} Z^L$,
- $Y^L \le_{st} Z^L$.

According to (2), hereafter q_X and q_Y denote respectively the quantile density functions of X and Y. The next result can be viewed as a quantile-based analogue of the probabilistic mean value theorem given in Theorem 4.1 of Di Crescenzo [6].

Theorem 3 *Let $X, Y \in \mathcal{D}$ and such that $X \le_{st} Y$. Moreover, let $g : (0,1) \to \mathbb{R}$ be a differentiable function, and let $g' \cdot Q_X$ and $g' \cdot Q_Y$ be integrable on $(0,1)$. Then, for $Z^L = \Psi^L(X,Y)$ we have that $E\left[g'(Z^L)\right]$ is finite, and*

$$E[\{g(1) - g(U)\}\{q_Y(U) - q_X(U)\}] = E\left[g'(Z^L)\right]\{E[Y] - E[X]\}, \tag{14}$$

where U is uniformly distributed in $[0,1]$.

Proof. From Theorem 1 and Eq. (13) we obtain

$$\begin{aligned} E[\{g(1)-g(U)\}\{q_Y(U)-q_X(U)\}] &= E\left[g'(Y^L)\right]E[Y]-E\left[g'(X^L)\right]E[X] \\ &= \left\{c\,E\left[g'(Y^L)\right]+(1-c)\,E\left[g'(X^L)\right]\right\}\{E[Y]-E[X]\}, \end{aligned}$$

with $c = E[Y]/(E[Y]-E[X])$. This immediately gives Eq. (14). □

As example, under the assumptions of Theorem 3, for $g(u)=u^\alpha$, $0<u<1$, we have:

$$E[(1-U^\alpha)\{q_Y(U)-q_X(U)\}] = \alpha\,E\left[(Z^L)^{\alpha-1}\right]\{E[Y]-E[X]\}, \qquad \alpha>0.$$

In particular, when $\alpha=1$ we get the indentity

$$E[(1-U)\{q_Y(U)-q_X(U)\}] = E[Y]-E[X],$$

which does not depend on Z^L.

4.1 Proportional quantile functions

Let $X,Y\in\mathcal{D}$ have proportional quantile functions. For instance (see Escobar and Meeker [7]) such assumption leads to a scale-accelerated failure-time model. Let

$$Q_X(u)=\alpha_X\,\varphi(u), \qquad Q_Y(u)=\alpha_Y\,\varphi(u), \qquad \forall u\in(0,1), \tag{15}$$

with $\alpha_X>0$, $\alpha_Y>0$, where $\varphi(u)$ is a suitable increasing and differentiable function such that $\eta := \int_0^1 \varphi(u)\,\mathrm{d}u$ is finite and $\varphi(0)=0$. In other terms, X and Y belong to the same scale family of distributions, with

$$F_X(x)=F_Y\left(\frac{\alpha_Y}{\alpha_X}\,x\right), \qquad \forall x\in\mathbb{R}.$$

Proposition 4 *The random variables $X^L=\Psi^L(0,X)$ and $Y^L=\Psi^L(0,Y)$ are identically distributed if, and only if, X and Y have proportional quantile functions as specified in (15).*

Proof. Since the distribution function of X^L is given by (3), the proof thus follows. □

Proposition 5 *If the quantile functions of X and Y are proportional as expressed in Eq. (15), with $\alpha_Y>\alpha_X>0$, then $X\le_{st}Y$. Moreover, $Z^L=\Psi^L(X,Y)$ is identically distributed to X^L and Y^L, with density $f_{Z^L}(u)=\varphi(u)/\eta$, $0<u<1$, and the following equality holds:*

$$E[\{g(1)-g(U)\}\,\varphi'(U)] = \eta\,E\left[g'(Z^L)\right], \tag{16}$$

where U is uniformly distributed in $(0,1)$, and $g:(0,1)\to\mathbb{R}$ is a differentiable function.

Proof. The proof follows from Propositions 2 and 4, and from Theorem 3. □

We remark that the variables considered in Example 2 satisfy the assumptions of Proposition 5. Other cases are shown hereafter.

Example 3 (i) Let X and Y have the following distribution functions, with $\beta > 0$:

$$F_X(x) = \left(\frac{x}{\alpha_X}\right)^\beta, \qquad 0 \le x \le \alpha_X, \qquad F_Y(x) = \left(\frac{x}{\alpha_Y}\right)^\beta, \qquad 0 \le x \le \alpha_Y.$$

If $\alpha_Y > \alpha_X > 0$, then the assumptions of Proposition 5 are satisfied. Hence, Eq. (16) holds, with $\varphi(u) = u^{1/\beta}$, $0 < u < 1$, and $\eta = \frac{\beta}{1+\beta}$, so that $Z^L = \Psi^L(X, Y)$ has density $f_{Z^L}(u) = \frac{\beta+1}{\beta} u^{1/\beta}$, $0 < u < 1$.
(ii) Let X and Y have Pareto (Type I) distribution, with

$$F_X(x) = 1 - \left(\frac{\alpha_X}{x}\right)^\beta, \qquad x \ge \alpha_X > 0, \qquad F_Y(x) = 1 - \left(\frac{\alpha_Y}{x}\right)^\beta, \qquad x \ge \alpha_Y > 0,$$

for $\beta > 0$. If $\alpha_Y > \alpha_X > 0$ and if $\beta > 1$, then the hypotheses of Proposition 5 hold. Relation (16) is thus fulfilled, with $\varphi(u) = (1-u)^{-1/\beta}$, $0 < u < 1$, and $\eta = \frac{\beta}{\beta-1}$, by which $Z^L = \Psi^L(X, Y)$ has density $f_{Z^L}(u) = \frac{\beta-1}{\beta(1-u)^{1/\beta}}$, $0 < u < 1$.

5 Applications

Let us now analyse various applications of the results given in the previous section.

5.1 Classes of distributions

Among the classes of probability distributions, wide attention is given to the following notions. Let X be a nonnegative random variable; then

(i) X is NBU (new better that used) $\Leftrightarrow X_t \le_{st} X$ for all $t \ge 0$, i.e. $\overline{F}(s)\overline{F}(t) \ge \overline{F}(s+t)$ for all $s \ge 0$ and $t \ge 0$,

(ii) X is IFR (increasing failure rate) $\Leftrightarrow X_t \le_{st} X_s$ for all $t \ge s \ge 0$, i.e. $\overline{F}$ is logconcave,

where $X_t := [X - t \,|\, X > t]$, for $t \ge 0$. The above notions can be expressed also in terms of the quantiles. Consider the residual of X evaluated at $Q(p)$, i.e.

$$X_{Q(p)} = [X - Q(p) \,|\, X > Q(p)] \qquad \text{for } 0 < p < 1. \tag{17}$$

In risk theory $X_{Q(p)}$ describes the losses exceeding $Q(p)$. Indeed, in a population of losses distributed as X, then $X_{Q(p)}$ denotes the residual of a loss whose level is equal to the pth quantile, for $0 < p < 1$. If X has a strictly increasing quantile function $Q(p)$, then

(i) X is NBU $\Leftrightarrow \overline{F}(Q(p)+Q(r)) \le (1-p)(1-r)$ for all $p, r \in (0,1)$,

(ii) X is IFR $\Leftrightarrow \frac{\overline{F}(Q(s)+Q(p))}{\overline{F}(Q(s)+Q(r))} \le \frac{1-p}{1-r}$ for all $0 < p < r < 1$ and $0 < s < 1$.

It is worth noting that comparisons of variables defined as in (17) allow to define the lr-order of the dispersion type (see Belzunce *et al.* [1]). We recall that, if $X \in \mathcal{D}$, the *conditional value-at-risk* of X is given by (see Belzunce *et al.* [2], or Denuit *et al.* [5]):

$$CVaR[X;p] := E[X_{Q(p)}] = E[X - Q(p) \,|\, X > Q(p)] = \frac{1}{1-p}\int_{Q(p)}^{+\infty} \overline{F}(y)\,\mathrm{d}y, \qquad 0 < p < 1, \tag{18}$$

with $CVaR[X;0] = E[X]$. (Note that in the literature some authors give different definitions for the conditional value-at-risk.) In the context of reliability theory the function given in (18) is also named 'mean residual quantile function', since

$$CVaR[X;p] = \mathrm{mrl}(Q(p)) = \frac{1}{1-p}\int_p^1 [Q(t) - Q(p)]\,\mathrm{d}t, \qquad 0 < p < 1,$$

where $\mathrm{mrl}(t) = E[X_t] = E[X - t \,|\, X > t]$ is the *mean residual life* of a lifetime X evaluated at age $t \ge 0$. Furthermore, we point out that the conditional value-at-risk is also related to the *right spread function* of X through the following identity: $CVaR[X;p] = S_X^+(p)/(1-p)$ (see, for instance, Fernandez-Ponce *et al.* [9] for several results on $S_X^+(p)$). Finally, the integral in the right-hand-side of (18) is known as the 'excess wealth transform', and plays an essential role in the excess wealth order (cf. Section 3.C of [20]).

Remark 3 Given two nonnegative random variables X and Y, one has $\frac{1}{\alpha_X}CVaR[X;p] = \frac{1}{\alpha_Y}CVaR[Y;p]$ for all $p \in (0,1)$ if and only if the proportional quantile functions model holds as specified in (15).

Let us now provide a result involving NBU random variables.

Proposition 6 *Let $X \in \mathcal{D}$ be NBU and such that $CVaR[X;p] < E[X]$ for all $p \in (0,1)$. If $g : (0,1) \to \mathbb{R}$ is a differentiable function, and if U is uniformly distributed in $(0,1)$, then for all $p \in (0,1)$ we have*

$$E[\{g(1) - g(U)\}\{q(U) - q(1-(1-p)(1-U))(1-p)\}] = E[g'(Z^L)]\{E[X] - CVaR[X;p]\}, \tag{19}$$

where $Z^L := \Psi^L(X_{Q(p)}, X)$ has density

$$f_{Z^L}(u) = \frac{Q(u) + Q(p) - Q(1-(1-p)(1-u))}{E[X] - CVaR[X;p]}, \qquad 0 \le u < 1.$$

Proof. The mean of $X_{Q(p)}$ is expressed in (18), whereas due to (17) its quantile function and quantile density for $p \in (0,1)$ and $u \in (0,1)$ are given respectively by

$$Q_{X_{Q(p)}}(u) = Q(1-(1-u)(1-p)) - Q(p), \tag{20}$$

and

$$q_{X_{Q(p)}}(u) = q(1-(1-u)(1-p))(1-p).$$

The assertion then follows from Proposition 2 and Theorem 3. □

We remark that the quantile function given in (20) is often used to model reliability data, since it represents the uth percentile residual life expressed in terms of quantile, as shown in Eq. (2.7) of Unnikrishnan Nair and Sankaran [25].

In the line of Proposition 6 we now provide a similar result for IFR random variables.

Proposition 7 *Let $X \in \mathcal{D}$ be IFR and such that $CVaR[X;p]$ is strictly decreasing for $0 < p < 1$. Then, for all $0 < r < p < 1$ we have*

$$E[\{g(1)-g(U)\}\{q(1-(1-r)(1-U))(1-r) - q(1-(1-p)(1-U))(1-p)\}] \\ = E[g'(Z^L)]\{CVaR[X;r] - CVaR[X;p]\},$$

where U is uniformly distributed in $(0,1)$, and $Z^L := \Psi^L(X_{Q(p)}, X_{Q(r)})$ has density

$$f_{Z^L}(u) = \frac{Q(1-(1-r)(1-u)) - Q(r) - Q(1-(1-p)(1-u)) + Q(p)}{CVaR[X;r] - CVaR[X;p]}, \qquad 0 < u < 1. \quad (21)$$

Proof. Since X is IFR, we have $X_{Q(p)} \le_{st} X_{Q(r)}$ for all $0 < r < p < 1$. The proof thus proceeds similarly as Proposition 6. □

Example 4 Let $X = \max\{T_1, \ldots, T_N\}$, where T_i, $i \ge 1$, are independent, identically Exp(λ)-distributed random variables, and where N is a geometric random variable independent of X_i, $i \ge 1$, and with parameter $\delta \in (0,1)$. Then, X is IFR (see Example 7.2 of Ross *et al.* [18]). Its quantile function and quantile density are respectively given by

$$Q(u) = \frac{1}{\lambda} \ln\left(\frac{\delta + (1-\delta)u}{\delta(1-u)}\right), \qquad q(u) = [\lambda(1-u)(\delta + (1-\delta)u)]^{-1}, \qquad 0 < u < 1.$$

From (18) we see that the conditional value-at-risk of X is:

$$CVaR[X;p] = -\frac{\ln(p + (1-p)\delta)}{\lambda(1-p)(1-\delta)}, \qquad 0 < p < 1.$$

We note that $CVaR[X;p]$ is strictly decreasing in $p \in (0,1)$. Hence, from Proposition 7 we have, for $0 < r < p < 1$,

$$E\left[\frac{g(1)-g(U)}{1-U}\left\{\frac{1}{U(1-r)(1-\delta) + r(1-\delta) + \delta} - \frac{1}{U(1-p)(1-\delta) + p(1-\delta) + \delta}\right\}\right] \\ = E[g'(Z^L)]\,\frac{1}{1-\delta}\left\{\frac{\ln(p + \delta(1-p))}{1-p} - \frac{\ln(r + \delta(1-r))}{1-r}\right\},$$

where U is uniformly distributed in $(0,1)$. The density of $Z^L = \Psi^L(X_{Q(p)}, X_{Q(r)})$ can be obtained from (21) and the above given expressions.

We remark that Propositions 6 and 7 provide identities holding for specific ranges of the involved parameters. However, a 'local version' of such results can be easily stated under mild assumptions. For instance Eq. (19) holds for a fixed $p \in (0,1)$, provided that $X_{Q(p)} \leq_{st} X$ and $CVaR[X;p] < E[X]$ for such fixed p. An example in which these conditions hold for some $p \in (0,1)$ is provided when X is the maximum of two independent exponential distributions with unequal parameters, whose distribution is IFRA (increasing failure rate in average) and thus NBU, but not IFR (see, for instance, Klefsjö [12]).

5.2 Risks

When comparing risks, the quantile function $Q(p)$ of X plays a very important role. In fact, in this context it is known as *value-at-risk* and is denoted by $VaR[X;p] \equiv Q(p)$, $0 < p < 1$. However, to avoid discrepancies we adopt the notation $Q(p)$. Given a nonnegative random variable X with finite mean, we define

$$\widetilde{X}_p := \left\{ \frac{X - Q(p)}{Q(p)} \middle| X > Q(p) \right\} \tag{22}$$

for all $p \in S_X := \{u \in (0,1) : Q(u) > 0\}$. The random variable $\widetilde{X}_p$ is useful to compare risks of different nature, and can be viewed as *proportional conditional value-at-risk* because it measures the conditional upper tail from $Q(p)$ on, but proportional to $Q(p)$. Moreover, from (17) and (22) we have $\widetilde{X}_p = X_{Q(p)}/Q(p)$ for all $p \in S_X$. Hence, Eqs. (18) and (22) yield

$$E[\widetilde{X}_p] = \frac{CVaR[X;p]}{Q(p)}, \qquad p \in S_X. \tag{23}$$

In this case, we can consider conditions similar to NBU and IFR properties which are defined in terms of (22):

(i) $\widetilde{X}_p \leq_{st} X$ for all $0 < p < 1 \Leftrightarrow \overline{F}((1+x)Q(p)) \leq \overline{F}(x)(1-p)$ for all $0 < p < 1$ and $x \geq 0$.

(ii) $\widetilde{X}_p \leq_{st} \widetilde{X}_r$ for all $0 < r < p < 1 \Leftrightarrow \frac{\overline{F}((1+x)Q(p))}{\overline{F}((1+x)Q(r))} \leq \frac{1-p}{1-r}$ for all $0 < r < p < 1$ and $x \geq 0$.

Proposition 8 *Let $X \in \mathcal{D}$ be such that $\widetilde{X}_p \leq_{st} X$ and $CVaR[X;p] < E[X]\,Q(p)$ for all $p \in S_X$. If $g : (0,1) \to \mathbb{R}$ is a differentiable function and if U is uniformly distributed in $(0,1)$, then for all $p \in S_X$ we have*

$$E\left[\{g(1) - g(U)\}\left\{q(U) - \frac{1}{Q(p)}\,q(1-(1-p)(1-U))(1-p)\right\}\right] = E[g'(Z^L)]\left\{E[X] - \frac{1}{Q(p)}\,CVaR[X;p]\right\},$$

where $Z^L = \Psi^L(\widetilde{X}_p, X)$ is a random variable with density function

$$f_{Z^L}(u) = \frac{(1+Q(u))Q(p) - Q(1-(1-p)(1-u))}{E[X]Q(p) - CVaR[X;p]}, \qquad 0 < u < 1.$$

Proof. Since the mean of $\widetilde{X}_p$ is (23), and the quantile function and quantile density are given respectively by

$$Q_{\widetilde{X}_p}(u) = \frac{1}{Q(p)}\,Q(1-(1-p)(1-u)) - 1, \qquad q_{\widetilde{X}_p}(u) = \frac{1}{Q(p)}\,q(1-(1-p)(1-u))(1-p)$$

for each $p \in S_X$ and $0 < u < 1$, the proof follows from Proposition 2 and Theorem 3. □

An extension of Proposition 8 to a more general case is given hereafter. The proof is analogous, and then is omitted.

Proposition 9 *Let $X \in \mathcal{D}$ be such that $\widetilde{X}_p \le_{st} \widetilde{X}_r$ and $CVaR[X;p]/Q(p)$ is strictly decreasing for all $p \in S_X$. If $g : (0,1) \to \mathbb{R}$ is a differentiable function and if U is uniformly distributed in $(0,1)$, then for all $r, p \in S_X$ such that $0 < r < p < 1$ we have*

$$E\left[\{g(1)-g(U)\}\left\{\frac{q(1-(1-r)(1-U))(1-r)}{Q(r)} - \frac{q(1-(1-p)(1-U))(1-p)}{Q(p)}\right\}\right] = E[g'(Z^L)]\left\{\frac{CVaR[X;r]}{Q(r)} - \frac{CVaR[X;p]}{Q(p)}\right\},$$

where $Z^L = \Psi^L(\widetilde{X}_p, \widetilde{X}_r)$ is a random variable having density function

$$f_{Z^L}(u) = \frac{Q(p)\,Q(1-(1-r)(1-u)) - Q(r)\,Q(1-(1-p)(1-u))}{Q(p)\,CVaR[X;r] - Q(r)\,CVaR[X;p]}, \qquad 0 < u < 1. \tag{24}$$

Example 5 Let X have Rayleigh distribution, $F(x) = 1 - e^{-\alpha x^2}$, $x \ge 0$, with $\alpha > 0$. Hence, the quantile function and the quantile density of X are:

$$Q(u) = \sqrt{|\ln(1-u)|/\alpha}, \qquad q(u) = \sqrt{\alpha}\left[2(1-u)\sqrt{|\ln(1-u)|}\right]^{-1}, \qquad 0 < u < 1.$$

Due to (18) the conditional value-at-risk of X is given by

$$CVaR[X;p] = \frac{\sqrt{\pi}\,\mathrm{erfc}\left[\sqrt{|\ln(1-p)|}\right]}{2\sqrt{\alpha}\,(1-p)}, \qquad 0 < p < 1,$$

where erfc[·] denotes the complementary error function. It is not hard to verify that $CVaR[X;p]$ is strictly decreasing in $p \in (0,1)$. Moreover, since X is IFR, from Proposition 7 we have, for all $0 < r < p < 1$,

$$E\left[\frac{g(1)-g(U)}{1-U}\left\{\frac{1}{\sqrt{|\ln[(1-r)(1-U)]|}} - \frac{1}{\sqrt{|\ln[(1-p)(1-U)]|}}\right\}\right] = E[g'(Z^L)]\frac{\sqrt{\pi}}{\alpha}\left\{\frac{\mathrm{erfc}\left[\sqrt{|\ln(1-r)|}\right]}{1-r} - \frac{\mathrm{erfc}\left[\sqrt{|\ln(1-p)|}\right]}{1-p}\right\},$$

where $g : (0,1) \to \mathbb{R}$ is a differentiable function, U is uniformly distributed in $(0,1)$, and the density of $Z^L = \Psi^L(X_{Q(p)}, X_{Q(r)})$ can be obtained from (21). Furthermore, $CVaR[X;p]/Q(p)$

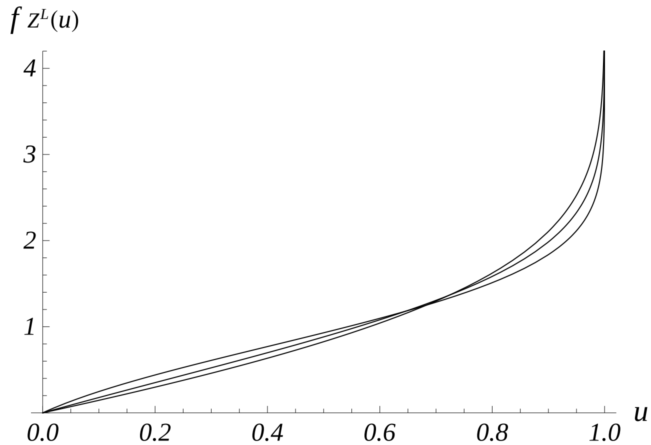

Figure 1: Density (25), for $(r,p) = (0.1, 0.3), (0.4, 0.6), (0.7, 0.9)$ (from top to bottom near the origin).

is strictly decreasing in $p \in (0,1)$. Hence, Proposition 9 yields the following indentity, for $0 < r < p < 1$,

$$E\left[\frac{g(1)-g(U)}{1-U}\left\{\frac{1}{\sqrt{\ln(1-r)\ln[(1-r)(1-U)]}} - \frac{1}{\sqrt{\ln(1-p)\ln[(1-p)(1-U)]}}\right\}\right]$$
$$= E[g'(Z^L)]\frac{\sqrt{\pi}}{\alpha}\left\{\frac{\mathrm{erfc}\left[\sqrt{|\ln(1-r)|}\right]}{(1-r)\sqrt{|\ln(1-r)|}} - \frac{\mathrm{erfc}\left[\sqrt{|\ln(1-p)|}\right]}{(1-p)\sqrt{|\ln(1-p)|}}\right\}.$$

In this case, owing to (24), for $0 < u < 1$ the density of $Z^L = \Psi^L(\widetilde{X}_p, \widetilde{X}_r)$ is

$$f_{Z^L}(u) = \frac{2(1-p)(1-r)\left\{\sqrt{\ln(1-r)\ln[(1-p)(1-u)]} - \sqrt{\ln(1-p)\ln[(1-r)(1-u)]}\right\}}{\sqrt{\pi}\left\{(1-r)\,\mathrm{erfc}\left[\sqrt{|\ln(1-p)|}\right]\sqrt{|\ln(1-r)|} - (1-p)\,\mathrm{erfc}\left[\sqrt{|\ln(1-r)|}\right]\sqrt{|\ln(1-p)|}\right\}}. \tag{25}$$

We remark that such density does not depend on α. Some plots of (25) are given in Figure 1.

5.3 A model involving the average value-at-risk

Let $X \in \mathcal{D}$ have quantile function $Q(p)$. One can introduce a new family of random variables X_v^*, $0 < v < 1$, having distribution function

$$F_v^*(x) = \frac{Q(vx)}{Q(v)}, \qquad 0 < x < 1.$$

If X represents a risk, the *average value-at-risk* of X is defined as

$$AVaR[X;v] = E[X \mid X \le Q(v)] = \frac{1}{v}\int_0^v Q(u)\,\mathrm{d}u, \qquad 0 < v < 1. \tag{26}$$

Note that the risk measure given in (26) represents the conditional expected loss given that the loss X is less than its value-at-risk. See Chapter 6 of Rachev *et al.* [17] for results, properties and applications in mathematical finance of $AVaR[X;v]$. The average value-at-risk plays a significant role also in stochastic orders of interest in risk theory (see Jewitt [11]). In addition, $AVaR[X;v]$ can be viewed as the L-moment of order 1 of the reversed quantile function (see Unnikrishnan Nair and Vineshkumar [26]).

We can easily show that the mean of X_v^* can be expressed in terms of $AVaR[X;v]$ as

$$E[X_v^*] = 1 - \frac{1}{Q(v)}\, AVaR[X;v], \qquad 0 < v < 1. \tag{27}$$

Then, we have the following result, where f denotes the density of X.

Proposition 10 *Let $X \in \mathcal{D}$ be such that $X_v^* \le_{st} X_w^*$ for $0 < v < w < 1$, and $\frac{1}{Q(v)}\, AVaR[X;v]$ is strictly decreasing in $v \in (0,1)$. If $g : (0,1) \to \mathbb{R}$ is a differentiable function and if U is uniformly distributed in $(0,1)$, then for all $0 < v < w < 1$ we have*

$$E\left[\{g(1) - g(U)\}\left\{\frac{1}{w} f(Q(w)\,U)\,Q(w) - \frac{1}{v} f(Q(v)\,U)\,Q(v)\right\}\right] = E[g'(Z^L)]\left\{\frac{AVaR[X;v]}{Q(v)} - \frac{AVaR[X;w]}{Q(w)}\right\},$$

where $Z^L = \Psi^L(X_v^*, X_w^*)$ is a random variable with density function

$$f_{Z^L}(u) = \frac{\frac{1}{w} F(Q(w)\,u) - \frac{1}{v} F(Q(v)\,u)}{\frac{1}{Q(v)}\, AVaR[X;v] - \frac{1}{Q(w)}\, AVaR[X;w]}, \qquad 0 < u < 1.$$

Proof. We recall that the mean of X_v^* is (27). Moreover, its quantile function and quantile density for $v \in (0,1)$ are respectively given by

$$Q_v^*(u) = \frac{1}{v} F(Q(v)\,u), \qquad q_v^*(u) = \frac{1}{v} f(Q(v)\,u)\,Q(v), \qquad 0 < u < 1. \tag{28}$$

The proof thus follows applying Proposition 2 and Theorem 3. □

Let us now provide an equivalent condition for $X_v^* \le_{st} X_w^*$, $0 < v < w < 1$, which was considered in Proposition 10.

Proposition 11 *Let $X \in \mathcal{D}$. Then, $X_v^* \le_{st} X_w^*$ for $0 < v < w < 1$ if, and only if, $x\,\tau(x)$ is decreasing for $x > 0$, where $\tau(x) = \frac{\mathrm{d}}{\mathrm{d}x} \log F(x)$ is the reversed hazard rate function of X.*

Proof. Given $0 < v < w < 1$, we have $X_v^* \le_{st} X_w^*$ if, and only if, $Q_v^*(u) \le Q_w^*(u)$ for all $u \in (0,1)$. Hence, due to the first of (28), this property is equivalent to the following condition:

$$\frac{F(x)}{F(\frac{x}{u})} \text{ is increasing in } x > 0 \text{ for all } u \in (0,1). \tag{29}$$

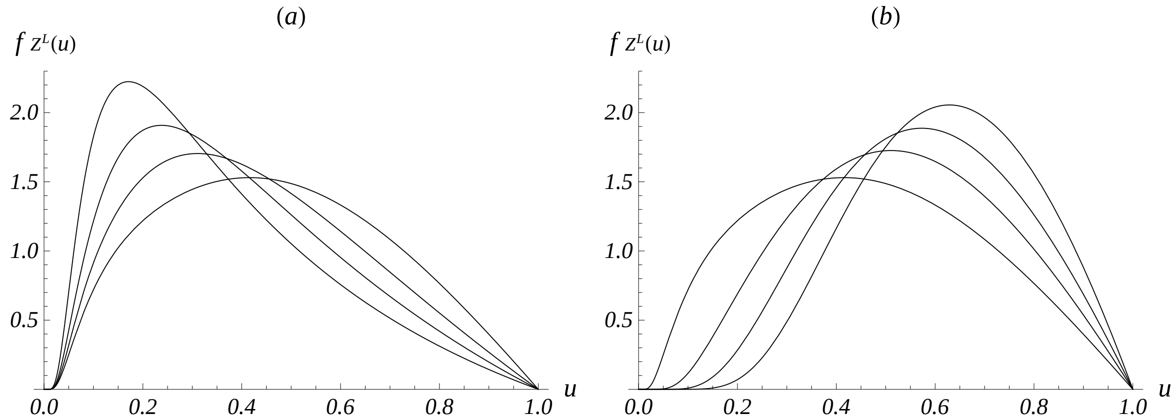

Figure 2: Density (30), for (a) $w = 0.9$, $v = 0.1, 0.3, 0.5, 0.7$ (from bottom to top near the origin), and (b) $v = 0.1$, $w = 0.3, 0.5, 0.7, 0.9$ (from top to bottom near $u = 1$).

Since $F(x) = \exp\{-\int_x^{+\infty} \tau(z)\,\mathrm{d}z\}$, $x > 0$, condition (29) holds if, and only if,

$$\int_x^{x/u} \tau(z)\,\mathrm{d}z \quad \text{is decreasing in } x > 0 \text{ for all } u \in (0,1).$$

By differentiation we see that this condition is satisfied if, and only if, $u\,\tau(x) \geq \tau(\frac{x}{u})$ for all $x > 0$ and $u \in (0,1)$. Finally, by setting $u = x/y$, with $x < y$, we obtain $x\,\tau(x) \geq y\,\tau(y)$, this giving the proof. □

Example 6 Let X have distribution function $F(x) = \exp\{-cx^{-\gamma}\}$, $x > 0$, with $c, \gamma > 0$. In this case, $\tau(x) = c\gamma x^{-(\gamma+1)}$ and thus $x\,\tau(x)$ is decreasing. Moreover, from (26) we have $AVaR[X; v] = c^{1/\gamma}\Gamma(1 - 1/\gamma, -\ln v)$, where $\Gamma(\cdot,\cdot)$ is the incomplete gamma function. Hence, recalling Proposition 11, the assumptions of Proposition 10 are satisfied. For instance, by setting for simplicity $\gamma = 1$, for all $0 < v < w < 1$ we have

$$E\left[\frac{g(1) - g(U)}{U^2}\left\{w^{1/U-1}|\ln w| - v^{1/U-1}|\ln v|\right\}\right] = E[g'(Z^L)]\left\{\frac{1}{v}\,\mathrm{li}(v)\,\ln v - \frac{1}{w}\,\mathrm{li}(w)\,\ln w\right\},$$

where $\mathrm{li}(x) = \int_0^x (\ln t)^{-1}\,\mathrm{d}t$ is the logarithmic integral function. Moreover $Z^L = \Psi^L(X_v^*, X_w^*)$ has density function

$$f_{Z^L}(u) = \frac{w^{1/u-1} - v^{1/u-1}}{\frac{1}{v}\,\mathrm{li}(v)\,\ln v - \frac{1}{w}\,\mathrm{li}(w)\,\ln w}, \qquad 0 < u < 1. \tag{30}$$

Some plots of $f_{Z^L}(u)$ are given in Figure 2.

Hereafter we show an example of distribution function that does not satisfy the conditions of Proposition 11.

Counterexample 1 Let X a random variable with distribution function and reversed hazard rate given respectively by (cf. Section 2 of Block *et al.* [4])

$$F(x) = \begin{cases} \exp\{-1-\frac{1}{x}\}, & 0 < x < 1 \\ \exp\{\frac{x^2-5}{2}\}, & 1 \le x < 2 \\ \exp\{-\frac{1}{x}\}, & 2 \le x < \infty, \end{cases} \qquad \tau(x) = \begin{cases} \frac{1}{x^2}, & 0 < x < 1 \\ x, & 1 \le x < 2 \\ \frac{1}{x^2}, & 2 \le x < \infty. \end{cases}$$

It is not hard to see that $x\,\tau(x)$ is not decreasing, and thus Proposition 10 cannot be applied.

5.4 A model based on increasing variables

Let $X \in \mathcal{D}$ have density $f(x)$ and quantile function $Q(p)$. We now consider a random variable $\widehat{X}_v$, $v \in (0,1)$, with support $(0,v)$ and distribution function

$$\widehat{F}_v(x) := \frac{Q(x)}{Q(v)}, \qquad 0 \le x \le v. \tag{31}$$

From (31) it is not hard to see that

$$E[\widehat{X}_v] = v\left[1 - \frac{1}{Q(v)}\, AVaR[X;v]\right], \qquad 0 < v < 1, \tag{32}$$

where $AVaR[x;v]$ is the average value-at-risk defined in (26).

Proposition 12 *Let $X \in \mathcal{D}$. If $g : (0,1) \to \mathbb{R}$ is a differentiable function and if U is uniformly distributed in $(0,1)$, then for all $0 < v < w < 1$ we have*

$$E[\{g(1)-g(U)\}\{f[Q(w)\,U]\,Q(w) - f[Q(v)\,U]\,Q(v)\}] \\ = E[g'(Z^L)]\left\{w\left(1 - \frac{AVaR[X;w]}{Q(w)}\right) - v\left(1 - \frac{AVaR[X;v]}{Q(v)}\right)\right\},$$

where $Z^L = \Psi^L(\widehat{X}_v, \widehat{X}_w)$ is a random variable with density function

$$f_{Z^L}(u) = \frac{F[Q(w)\,u] - F[Q(v)\,u]}{w\left(1 - \frac{1}{Q(w)}\,AVaR[X;w]\right) - v\left(1 - \frac{1}{Q(v)}\,AVaR[X;v]\right)}, \qquad 0 < u < 1.$$

Proof. From (31) it is easy to see that the quantile function and the quantile density of $\widehat{X}_v$, $0 < v < 1$, are respectively given by

$$\widehat{Q}_v(u) = F[Q(v)\,u], \qquad \widehat{q}_v(u) = f[Q(v)\,u]\,Q(v), \qquad 0 < u < 1.$$

Note that $\widehat{X}_v$ is stochastically increasing in $v \in (0,1)$, since $\widehat{X}_v \le_{st} \widehat{X}_w$ for all $0 < v < w < 1$. Moreover, the given assumptions ensure that the mean (32) is strictly increasing in $v \in (0,1)$. The proof thus follows from Proposition 2 and Theorem 3. $\square$

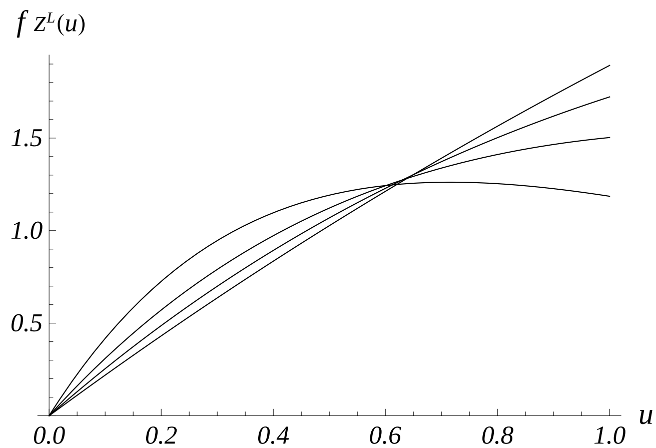

Figure 3: Density (34), for $(v, w) = (0.1, 0.2), (0.3, 0.4), (0.5, 0.6), (0.7, 0.8)$ (from bottom to top near the origin).

Example 7 Let X be an exponentially distributed random variable with parameter $\lambda > 0$. Then, for all $0 < v < 1$ and $0 < u < 1$ we have

$$E[\widehat{X}_v] = \frac{v}{\ln(1-v)} + 1, \qquad \widehat{Q}_v(u) = 1 - (1-v)^u, \qquad \widehat{q}_v(u) = -(1-v)^u \ln(1-v).$$

Note that the above functions do not depend on λ. Therefore, from Proposition 12 it follows

$$E\left[\{g(1) - g(U)\}\left\{(1-v)^U \ln(1-v) - (1-w)^U \ln(1-w)\right\}\right] = E[g'(Z^L)]\left\{\frac{w}{\ln(1-w)} - \frac{v}{\ln(1-v)}\right\}, \quad (33)$$

for all $0 < v < w < 1$. Here $Z^L = \Psi^L(\widehat{X}_v, \widehat{X}_w)$ is a random variable with density function

$$f_{Z^L}(u) = \frac{(1-v)^u - (1-w)^u}{\frac{w}{\ln(1-w)} - \frac{v}{\ln(1-v)}}, \qquad 0 < u < 1. \quad (34)$$

See Figure 3 for some plots of $f_{Z^L}(u)$.

5.5 Concluding remarks

In our view, the main issues of this paper are given in Proposition 2 and Theorem 3. The first result allows us to construct new probability densities with support $(0, 1)$ starting from suitable pairs of stochastically ordered random variables. The second result is useful to obtain equalities involving uniform-$(0, 1)$ distributions and quantile functions. The cases treated in this section give only a partial view of the potentiality of Theorem 3. Indeed, we considered some special cases in which the random variables X and Y involved in Theorem 3 belong to the same family of distributions. Other useful applications are likely to be developed under various choices of such variables, and specific selection of function g. This will be the object of future research.

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