

## CONVERGENCE OF NONAUTONOMOUS EVOLUTIONARY ALGORITHM

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**Abstract.** We present a general criterion guaranteeing the stochastic convergence of a wide class of nonautonomous evolutionary algorithms used for finding the global minimum of a continuous function. This paper is an extension of paper [6], where autonomous case was presented. Our main tool here is a cocycle system defined on the space of probabilistic measures and its stability properties.

**1. Introduction.** This paper concerns the problem of numerically finding a point or points at which a given function attains its global minimum (maximum). Let  $f: A \rightarrow \mathbb{R}$  be a function and assume that its minimum value is zero,  $A \subset \mathbb{R}^d$ . Let  $A^* = \{x \in A : f(x) = 0\}$  be the set of all the solutions of the problem. We are interested in the class of stochastic methods that are known as *evolutionary algorithms*. A general form of such an algorithm is as follows

$$x_n = T(n, x_{n-1}, y_n), \quad x_0 \in A, \quad n = 1, 2, 3 \dots$$

Here  $T$  is a given operator,  $\{x_n\}$  is a sequence of approximations of the problem and  $\{y_n\}$  is a random factor,  $n$  represents time. Our aim is to establish a criterion for the stochastic convergence of the sequence  $\{x_n\}$  to the set  $A^*$ . The same problem, when  $T$  does not depend on time  $n$ , was considered in [6] and, generally speaking, a sufficient condition is

$$\int f(T(x, y)) dy < f(x).$$

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In this paper we extend the above results onto the case of the operator  $T$  depending on time by means of some dynamical system, namely

$$x_n = T(\theta^n p, x_{n-1}, y_n), \quad x_0 \in A, \quad p \in P, \quad n = 1, 2, 3, \dots,$$

where  $\theta : P \rightarrow P$  is a map,  $\theta^n$  is its  $n$ -th iteration. If  $P = \{p\}$  is a singleton, we have situation as in [6].

We may, for example, apply our approach to methods that are changed cyclically. In fact, assume there are  $k$  operators  $\{T_1, T_2, \dots, T_k\}$  and put:  $P = \{1, 2, \dots, k\}$ ,  $\theta(p) = p + 1$  for  $p = 1, 2, \dots, k - 1$ ,  $\theta(k) = 1$  and  $T(q, x, y) = T_q(x, y)$  for  $q \in P$ .

As in [6], we express our problem in terms of some system defined on the space of probabilistic measures on  $A$ . This allow us to use some classical results from the theory of dynamical system.

**2. Basic definitions and preliminaries.** Let  $(A, d_A)$  be a compact metric space,  $B = A^l$ , for some fixed  $d, l \in \mathbb{N}$ ,  $f : A \rightarrow \mathbb{R}$  be a continuous function having its global minimum  $\min f$  on  $A$ . Without loss of generality, we may assume that  $\min f = 0$ . Let  $(\Omega, \Sigma, \text{Prob})$  be a probability space and  $(P, \mathbb{N}, \theta)$  a semi-dynamical system on a compact metric space  $(P, d_P)$ . Let  $A^* = \{x \in A : f(x) = 0\}$  be the set of all the solutions of the global minimization problem. We define a *nonautonomous evolutionary algorithm* as an algorithm finding points from  $A^*$ , given by the formula

$$(1) \quad X_n = T(\theta^n p, X_{n-1}, Y_n), \quad n = 1, 2, 3, \dots,$$

Here  $p \in P$  is an initial value of dynamical system  $\theta$ ,  $X_0$  is a fixed random variable with a known distribution on  $A$ ,  $X_0 \sim \lambda$ .  $Y_n$  is a random variable with a known distribution on  $B$ ,  $Y_n \sim \nu$ , for  $n = 1, 2, 3, \dots$ . We assume that  $X_0, Y_1, Y_2, Y_3, \dots$  are independent.  $T : P \times A \times B \rightarrow A$  is an operator identifying the algorithm, that is a measurable function. Thus,  $X_n$  is a random variable with the distribution  $\mu_n$  for  $n = 1, 2, 3, \dots$ . Let  $\mathcal{B}(A), \mathcal{B}(B)$  denote the  $\sigma$ -algebras of Borel subsets of the space  $A$  and  $B$ , respectively. As all the variables  $X_n, n = 1, 2, 3, \dots$  are defined on  $\Omega$ , there is

$$\mu_n(C) = \text{Prob}(X_n \in C) \quad \text{for each } C \in \mathcal{B}(A).$$

Let  $\mathcal{M}$  be the set of all probabilistic measures on  $\mathcal{B}(A)$ . It is obvious that  $\lambda, \mu_n \in \mathcal{M}$  for  $n = 1, 2, \dots$ . We check the properties of the sequence  $\{X_n\}$  by observing the behavior of the sequence  $\{\mu_n\}$ . Thus, we recall some facts about the topological properties of  $\mathcal{M}$ . It is known (see [7]) that  $\mathcal{M}$  with the Fortet-Mourier metric is a compact metric space and its topology is determined by the weak convergence of the sequence of measures as follows. The sequence  $\mu_n \in \mathcal{M}$  converges to  $\mu_0 \in \mathcal{M}$  if and only if for any continuous (so bounded,

by the compactness of  $A$ ) function  $h: A \rightarrow \mathbb{R}$  :

$$(2) \quad \int_A h(x)\mu_n(dx) \longrightarrow \int_A h(x)\mu_0(dx), \quad \text{as } n \rightarrow \infty.$$

A useful condition for weak convergence (see [2]) is as follows:

$$(3) \quad \mu_n(C) \longrightarrow \mu_0(C), \quad \text{as } n \rightarrow \infty,$$

for every  $C \in \mathcal{B}(A)$  such that  $\mu_0(\partial C) = 0$ . We are interested in the convergence of the sequence  $\{X_n\}$  to the set  $A$  in the stochastic sense, i.e.,

$$(4) \quad \forall \varepsilon > 0 \quad \lim_{n \rightarrow \infty} \text{Prob} \left( d_A(X_n, A) < \varepsilon \right) = 1.$$

In the sequel, we show sufficient conditions for such convergence. Algorithm (1) induces a specific nonautonomous system on the space  $\mathcal{M}$ , called a *cocycle system*. In Section 3, we show that the sequence  $\{\mu_n\}$  is an orbit of this system. In Section 4, we introduce some asymptotic properties of cocycle systems and prove a theorem corresponding to the Lyapunov Theorem for dynamical systems (Theorem 4.2). It gives sufficient conditions for a set  $X^* \subset X$  to be asymptotically stable under a cocycle defined on  $X$ . In Section 5, we apply Theorem 4.2 to our case, by constructing the Lyapunov function for the set  $\mathcal{M}^*$  which denotes the set of all the measures  $\mu \in \mathcal{M}$  that are supported on  $A^*$ . Theorem 5.2 is the main result, and it gives sufficient conditions on  $T$  for the asymptotic stability of  $\mathcal{M}^*$ . Theorem 5.3 is a corollary of Theorem 5.2 and gives sufficient conditions for the stochastic convergence of every  $\{X_n\}$  to the set  $A^*$ .

**3. Cocycle systems.** Now we recall the concept of a cocycle system. It is a triple  $(X, \psi, (P, \mathbb{N}, \theta))$ , where  $X$  is a metric space,  $(P, \mathbb{N}, \theta)$  is a semi-dynamical system, and the cocycle mapping  $\psi: \mathbb{N} \times P \times X \rightarrow X$  satisfies the conditions:

- (C1)  $\psi(0, p, x) = x$  for each  $p \in P, x \in X$ ,
- (C2)  $\psi(n + m, p, x) = \psi(n, \theta^m p, \psi(m, p, x))$  for each  $p \in P, x \in X, n, m \in \mathbb{N}$ ,
- (C3)  $(p, x) \mapsto \psi(n, p, x)$  is a continuous mapping for all  $n \in \mathbb{N}$ .

Let us fix  $q \in P$  for a moment and let  $X_n = T(q, X_{n-1}, Y_n)$ . It has been proved (see [4, 5, 6]) that for every set  $C \in \mathcal{B}(A)$

$$(5) \quad \mu_n(C) = \int_A \left( \int_B I_C(T(q, x, y)) \nu(dy) \right) \mu_{n-1}(dx),$$

and that the above equality defines the *Foias operator*  $S_q: \mathcal{M} \rightarrow \mathcal{M}$  such that  $\mu_n = S_q(\mu_{n-1})$ . Here  $I_C$  is the indicator function of a set  $C$ . Let us define a

new operator  $S : P \times \mathcal{M} \rightarrow \mathcal{M}$  such that  $S(q, \mu) = S_q(\mu)$ . For each fixed  $q$ , it is the Foias operator. By (1) and (5), we get

$$\mu_n = S(\theta^n p, \mu_{n-1}) = S(\theta^n p, S(\theta^{n-1} p, \mu_{n-2})),$$

and by induction,

$$(6) \quad \mu_n = (S(\theta^n p, \cdot) \circ S(\theta^{n-1} p, \cdot) \circ \dots \circ S(\theta p, \cdot))(\lambda).$$

For any measurable function  $h : A \rightarrow \mathbb{R}$ , we define the function  $Uh : P \times A \rightarrow \mathbb{R}$  as:

$$Uh(q, x) = \int_B h(T(q, x, y)) \nu(dy).$$

It is known (see [4, 5, 6]) that if  $q \in P$  is fixed, then for every measure  $\mu \in \mathcal{M}$  and measurable function  $h : A \rightarrow \mathbb{R}$  there holds

$$(7) \quad \int_A h(x) S(q, \mu)(dx) = \int_A Uh(q, x) \mu(dx) \quad \text{for each } q \in P,$$

and hence

$$(8) \quad \mu_n(C) = \int_A UI_C(q, x) \mu_{n-1}(dx).$$

We say that an operator  $T$  is  $\nu$ -almost everywhere continuous ( $\nu$ -a.e. continuous) when the following two conditions hold:

- 1) for each  $q \in P, x_0 \in A, x_k \rightarrow x_0 : T(q, x_k, y) \rightarrow T(q, x_0, y)$  a. e.  $\nu$ ,
- 2) for each  $x \in A, q_0 \in P, q_k \rightarrow q_0 : T(q_k, x, y) \rightarrow T(q_0, x, y)$  a. e.  $\nu$ .

We now prove the following

LEMMA 3.1. *Let  $T$  be  $\nu$ -a.e. continuous. Then  $S$  is continuous.*

PROOF. As  $P \times \mathcal{M}$  is compact, we can prove the continuity of  $S$  with respect to each of the variables separately. First, let us fix  $\mu \in \mathcal{M}$ . Let  $h : A \rightarrow \mathbb{R}$  be a continuous function (thus measurable),  $q_n \rightarrow q_0$ . We prove that  $S(q_n, \mu) \rightarrow S(q_0, \mu)$  in the sense of (2). By the continuity of  $h$  and  $T$ , for each  $x \in A$ , there is

$$h(T(q_n, x, y)) \longrightarrow h(T(q_0, x, y)) \quad \text{a. e. } \nu.$$

By the Lebesgue Dominated Convergence Theorem ( $X, P$  - compact),

$$\int_B h(T(q_n, x, y))(dy) \longrightarrow \int_B h(T(q_0, x, y))(dy).$$

This means that  $Uh(q_n, \cdot) \rightarrow Uh(q_0, \cdot)$ . Again by the Lebesgue Dominated Convergence Theorem and by (7), for each continuous function  $h$ , there holds

$$\int_A h(x) dS(q_n, \mu) = \int_A Uh(q_n, x) d\mu \longrightarrow \int_A Uh(q_0, x) d\mu = \int_A h(x) dS(q_0, \mu),$$

which proves the continuity of  $S$  with respect to the first variable.

Now fix  $q \in P$ . Let  $\mu_n \rightarrow \mu_0$ . We prove that  $S(q, \mu_n) \rightarrow S(q, \mu_0)$  in the sense of (2). Let  $h: A \rightarrow \mathbb{R}$  be a continuous function. From the continuity of  $T$  we get

$$Uh(q, x_n) = \int_B h(T(q, x_n, y))(dy) \longrightarrow \int_B h(T(q, x_0, y))(dy) = Uh(q, x_0),$$

for each sequence  $x_n \rightarrow x_0$ . It means that the function  $Uh(q, \cdot): A \rightarrow \mathbb{R}$  is continuous. So from (7), there follows

$$\int_A h(x)dS(q, \mu_n) = \int_A Uh(q, x)d\mu_n \longrightarrow \int_A Uh(q, x)d\mu_0 = \int_A h(x)dS(q, \mu_0),$$

which proves that  $S(q, \mu_n) \rightarrow S(q, \mu_0)$ .  $\square$

We now prove the main result of this section.

**THEOREM 3.2.** *Let  $T$  be  $\nu$ -a.e. continuous. Then triple  $(\mathcal{M}, \psi, (P, \mathbb{N}, \theta))$ , where  $\psi: \mathbb{N} \times P \times \mathcal{M} \rightarrow \mathcal{M}$  is given by the formula  $\psi(n, p, \lambda) = \mu_n$ , is a cocycle system.*

**PROOF.** We prove conditions **(C1)**–**(C3)** from the definition of a cocycle system. Condition **(C1)** is obvious. We prove condition **(C2)**. From (6), for all  $n, m \in \mathbb{N}, p \in P, \mu \in \mathcal{M}$

$$\psi(n + m, p, \lambda) = (S(\theta^{n+m}p, \cdot) \circ \dots \circ S(\theta^{m+1}p, \cdot) \circ S(\theta^m p, \cdot) \circ \dots \circ S(\theta p, \cdot))(\lambda).$$

Then, by properties of the dynamical system  $\theta$ ,

$$\psi(n + m, p, \lambda) = S(\theta^n \theta^m p, \cdot) \circ S(\theta^{n-1} \theta^m p, \cdot) \circ \dots \circ S(\theta \theta^m p, \mu_m),$$

and again by (6), we get

$$\psi(n + m, p, \lambda) = \psi(n, \theta^m p, \mu_m) = \psi(n, \theta^m p, \psi(m, p, \lambda)).$$

The continuity (condition **(C3)**) of the cocycle  $\psi$  follows from Lemma 3.1, (8) and (6), as  $\psi$  is a composition of continuous mappings.  $\square$

**4. Stability in cocycle systems.** Let  $(X, \psi, (P, \mathbb{N}, \theta))$  be a nonautonomous dynamical system (NDS) and let  $d_H$  denote the Hausdorff distance (semi-metric) on the space  $2^X$ , i.e.,

$$d_H(A, B) = \sup_{a \in A} \inf_{b \in B} d_X(a, b).$$

The following notions are taken from [3]. A function  $\hat{A}: P \ni p \mapsto A(p)$  taking values in the set of nonempty (compact) subsets of  $X$  is called a *nonautonomous (compact) set*. A nonautonomous set  $\hat{A}$  is called *forward invariant* under NDS  $\psi$ , if for each  $p \in P, n \in \mathbb{N} : \psi(n, p, A(p)) \subset A(\theta^n p)$ . A nonautonomous compact set  $\hat{C}$  is called a *neighborhood* of a set  $\hat{A}$  if for each  $p \in P : A(p) \subset \text{int } C(p)$ .

A nonautonomous set  $\widehat{A}$ , compact and forward invariant under  $\psi$  is called:

- (i) *stable* if for every  $\varepsilon > 0$  there exists a nonautonomous compact, forward invariant set  $\widehat{C}$  which is a neighborhood of  $\widehat{A}$  and such that

$$d_H(C(p), A(p)) \leq \varepsilon \quad \text{for each } p \in P;$$

- (ii) *attractor* of  $\psi$  if for every  $p \in P, x \in X$

$$(9) \quad \lim_{n \rightarrow \infty} d_X(\psi(n, p, x), A(\theta^n p)) = 0;$$

- (iii) *asymptotically stable* if it is an attractor and is stable.

Let  $\widehat{A}$  be a nonautonomous compact set, forward invariant under  $\psi$ .

A function  $V: P \times X \mapsto \mathbb{R}$  is called a *Lyapunov function* for  $\widehat{A}$  if

(L1)  $V$  is continuous,

(L2)  $V(p, x) = 0$  for  $x \in A(p)$ ,  $V(p, x) > 0$  for  $x \notin A(p)$ ,

(L3)  $V(\theta^n p, \psi(n, p, x)) < V(p, x)$  for each  $p \in P, n \in \mathbb{N}, x \notin A(p)$ .

The following lemma and its proof are taken from [1].

LEMMA 4.1. *Let  $X$  and  $P$  be compact metric spaces,  $V$  a Lyapunov function for a nonautonomous compact set  $\widehat{A}$ , forward invariant under  $\psi$ . Then, for each  $\delta > 0$ , the set  $\widehat{C}_\delta$  such that*

$$C_\delta(p) = \overline{V^{-1}(p, [0, \delta))} = \overline{\{x \in X : V(p, x) < \delta\}},$$

*is a compact nonautonomous set, forward invariant under  $\psi$ .*

PROOF. Let us first note that for each  $p \in P, \delta > 0$ , the set  $C_\delta(p)$  is compact as a closed subset of a compact set. It remains to show that

$$(10) \quad \psi(n, p, C_\delta(p)) \subset C_\delta(\theta^n p) \quad \text{for each } \delta > 0, p \in P, n \in \mathbb{N}.$$

Let  $x \in \psi(n, p, C_\delta(p))$ . This means that there exists a  $y \in C_\delta(p)$  such that  $x = \psi(n, p, y)$  and  $V(p, y) \leq \delta$ . From the properties of a Lyapunov function it follows that  $V(\theta^n p, \psi(n, p, y)) \leq V(p, y)$ . Therefore,

$$V(\theta^n p, \psi(n, p, y)) = V(\theta^n p, x) \leq \delta,$$

and hence  $x \in C_\delta(\theta^n p)$ . The proof is complete.  $\square$

Now we prove the main result of this section; the result gives sufficient conditions for the asymptotic stability of nonautonomous sets of the form  $A(p) = A^*$  for some compact subset  $A^*$  of the set  $X$  and for each  $p \in P$ .

THEOREM 4.2. *Let  $(X, \psi, (P, \mathbb{N}, \theta))$  be an NDS and let  $X$  and  $P$  be compact. If there exists a Lyapunov function  $V$  for a nonautonomous compact set  $\widehat{A}$ , forward invariant under  $\psi$ , of the form  $A(p) = A^*$  for each  $p \in P$ , then the set  $\widehat{A}$  is asymptotically stable under  $\psi$ .*

PROOF. We begin with showing the stability of  $\widehat{A}$ . From condition **(L2)** we conclude that the nonautonomous set  $\widehat{C}_\delta$  given by Lemma 4.1 is a neighborhood of  $\widehat{A}$ . By the forward invariance of  $\widehat{C}_\delta$  it remains to show that for each  $\varepsilon > 0$ , we find  $\delta > 0$  such that  $d_H(C_\delta(p), A(p)) < \varepsilon$  for each  $p \in P$ . Let us suppose for the contrary that:

$$\exists \varepsilon_0 \forall n \in \mathbb{N} \forall p_n \in P \exists x_n \in X : x_n \in C_{\frac{1}{n}}(p_n), d_X(x_n, A(p_n)) \geq \varepsilon_0.$$

From the definition of  $\widehat{C}_\delta$ , there follows  $V(p_n, x_n) < \frac{1}{n}$ . By the compactness of  $X$  and  $P$ , without loss of generality, we may assume that  $x_n \rightarrow x_0, p_n \rightarrow p_0$  for some  $x_0 \in X, p_0 \in P$ . Therefore, by continuity of  $V$ , we get  $V(p_0, x_0) = 0$ .

On the other hand, by  $A(p) = A^*$ , we get  $d_X(x_0, A(p_0)) \geq \varepsilon_0$ , hence  $x_0 \notin A(p_0)$ . Again by **(L2)**, we get  $V(p_0, x_0) > 0$ . This contradicts the above condition:  $V(p_0, x_0) = 0$ . Thus we have proved the stability of  $\widehat{A}$ .

Now we are going to show (9). Define the  $\omega$ -limit set

$$\omega(p, x) = \{(q, y) \in P \times X : \exists n_k \rightarrow \infty, \theta^{n_k} p \rightarrow q, \psi(n_k, p, x) \rightarrow y\}.$$

By the compactness of  $P$  and  $X$ , the  $\omega$ -limit set is nonempty for each  $(p, x)$ . We show that  $V$  is constant on  $\omega(p, x)$ . Indeed, let  $(q, y), (r, z) \in \omega(p, x)$ . This means that there exist sequences  $\{n_k\}, \{m_k\}$  divergent to infinity such that

$$\theta^{n_k} p \rightarrow q, \psi(n_k, p, x) \rightarrow y, \theta^{m_k} p \rightarrow r, \psi(m_k, p, x) \rightarrow z.$$

Without loss of generality we may assume that  $n_k < m_k < n_{k+1} < m_{k+1}$  for each  $k \in \mathbb{N}$ . Then from property **(L3)** we get

$$\begin{aligned} V(\theta^{n_k} p, \psi(n_k, p, x)) &\leq V(\theta^{m_k} p, \psi(m_k, p, x)) \\ &\leq V(\theta^{n_{k+1}} p, \psi(n_{k+1}, p, x)) \leq V(\theta^{m_{k+1}} p, \psi(m_{k+1}, p, x)). \end{aligned}$$

By the continuity of  $V$  (property **(L1)**):

$$V(q, y) \leq V(r, z) \leq V(q, y) \leq V(r, z),$$

and hence  $V(q, y) = V(r, z)$ .

Now let  $(q, y) \in \omega(p, x), \theta^{n_k} p \rightarrow q, \psi(n_k, p, x) \rightarrow y$ . For some fixed  $n$ , let  $m_k = n_k + n$ . Then from the properties of DS and NDS, we get  $\theta^{m_k} p = \theta^n \theta^{n_k} p \rightarrow \theta^n q$ , and  $\psi(m_k, p, x) = \psi(n_k + n, p, x) = \psi(n, \theta^{n_k} p, \psi(n_k, p, x)) \rightarrow \psi(n, q, y)$ . By the definition of an  $\omega$ -limit set, it means that  $(\theta^n q, \psi(n, q, y)) \in \omega(p, x)$ .

Now from the above we get  $V(\theta^n q, \psi(n, q, y)) = V(q, y)$ . Hence, by property **(L3)**,  $y \in A(q) = A^*$ . As  $X$  and  $P$  are compact, for every sequence  $\{x_k\}$  in  $X$  there exists a convergent subsequence  $\{x_{k_i}\}$  and, by the above,  $x_{k_i} = \psi(n_{k_i}, p, x) \rightarrow A^*$ . Therefore,

$$d_X(\psi(n_{k_i}, p, x), A(\theta^n p)) = d_x(\psi(n_{k_i}, p, x), A^*) \rightarrow 0,$$

for each  $p, x$ . The proof is complete.  $\square$

**5. Main result.** Assume that  $\psi$  is the cocycle defined by Theorem 3.2. Let  $\mathcal{M}^*$  denote the set of all the measures  $\mu \in \mathcal{M}$  supported on  $A^*$ . Let  $\widehat{\mathcal{M}}$  denote the nonautonomous set of the form  $\mathcal{M}(p) = \mathcal{M}^*$  for each  $p \in P$ .

LEMMA 5.1. *Let  $T$  be  $\nu$ -a.e. continuous and assume that:*

$$(11) \quad T(q, x, y) \in A^* \quad \text{for all } x \in A^*, q \in P, y \in Y.$$

*Then  $\widehat{\mathcal{M}}$  is a compact nonautonomous set, forward invariant under  $\psi$ .*

PROOF. In Section 2, we noted that  $\mathcal{M}$  is compact. We prove that  $\mathcal{M}^* \subset \mathcal{M}$  is closed. Indeed, let  $\mu_n \in \mathcal{M}^*$  and  $\mu_n \rightarrow \mu_0$ . Then from the continuity of  $f$  there follows

$$0 = \int_A f(x) \mu_n(dx) \longrightarrow \int_A f(x) \mu_0(dx).$$

Therefore,  $\int_A f(x) \mu_0(dx) = 0$  and  $\mu_0 \in \mathcal{M}^*$ .

As  $\mathcal{M}(p) = \mathcal{M}^*$  for each  $p \in P$ , it remains to show that  $\psi(n, p, \mathcal{M}^*) \subset \mathcal{M}^*$ , for each  $n \in \mathbb{N}, p \in P$ . By (6), it remains to show that  $S(q, \mathcal{M}^*) \subset \mathcal{M}^*$  for each  $q \in P$ .

Let  $q \in P$  and  $\mu \in \mathcal{M}^*$ . We want to show that  $S(q, \mu) \in \mathcal{M}^*$ .

Let us first note that from (11) there follows

$$I_{A^*}(T(q, x, y)) \geq I_{A^*}(x) \quad \text{for each } x \in A, q \in P, y \in Y.$$

By (5) and the above, we get

$$\begin{aligned} S(q, \mu)(A^*) &= \int_A \left( \int_B I_{A^*}(T(q, x, y)) \nu(dy) \right) \mu(dx) \\ &\geq \int_A \left( \int_B I_{A^*}(x) \nu(dy) \right) \mu(dx). \end{aligned}$$

By Fubini's Theorem ( $\nu$  and  $\mu$  are probabilistic measures), and by the assumption  $\mu \in \mathcal{M}^*$ ,

$$S(q, \mu)(A^*) \geq \int_B \left( \int_A I_{A^*}(x) \mu(dx) \right) \nu(dy) = \int_B 1 \nu(dy) = 1.$$

Therefore,  $S(q, \mu)(A^*) = 1$ , which means that  $\text{supp } S(q, \mu) \subset A^*$ , and the assertion follows.  $\square$

Now we prove the main result of this paper.

THEOREM 5.2. *Let  $T$  be  $\nu$ -a.e. continuous, satisfy condition (11) and let*

$$(12) \quad \int_B f(T(q, x, y)) \nu(dy) < f(x).$$

*Then  $\widehat{\mathcal{M}}$  is asymptotically stable under  $\psi$ .*



PROOF. By Lemma 5.1, the set  $\widehat{\mathcal{M}}$  is compact and forward invariant. Define a function  $V : P \times \mathcal{M} \rightarrow \mathbb{R}$

$$V(p, \mu) = \int_A f(x) \mu(dx).$$

We show that  $V$  satisfies conditions **(L1)**–**(L3)** from the definition of a Lyapunov function in Section 4.

Condition **(L1)** is obvious as  $f$  is continuous and  $V$  is constant with respect to the variable  $p$ . Let us note that  $V(p, \mu) \geq 0$  for each  $p, \mu$ . If  $\mu \in \mathcal{M}(p) = \mathcal{M}^*$ , then obviously  $V(p, \mu) = 0$ . Let now  $V(p, \mu) = 0$  for some measure  $\mu \in \mathcal{M}$ . Then, by the definition of  $A^*$

$$0 = V(p, \mu) = \int_A f(x) d\mu = \int_{A^*} f(x) d\mu + \int_{A \setminus A^*} f(x) d\mu = \int_{A \setminus A^*} f(x) d\mu.$$

As  $f$  is positive on  $A \setminus A^*$ ,  $\mu(A \setminus A^*) = 0$ , and therefore  $\mu \in \mathcal{M}^*$ . Condition **(L2)** is proved.

It remains to prove **(L3)**. We first prove that

$$(13) \quad \forall \mu \notin \mathcal{M}^*, \forall q \in P \quad V(q, S(q, \mu)) < V(q, \mu).$$

From (12), for each  $x \in A \setminus A^*$ ,

$$Uf(q, x) = \int_B f(T(q, x, y)) \nu(dy) < f(x).$$

The above equality, (7) and the definition of  $A^*$  give

$$\begin{aligned} V(q, S(q, \mu)) &= \int_A f(x) S(q, \mu)(dx) = \int_A Uf(q, x) \mu(dx) \\ &= \int_{A \setminus A^*} Uf(q, x) \mu(dx) < \int_A f(x) \mu(dx) = V(q, \mu), \end{aligned}$$

which proves (13). To show **(L3)** we use (6), the equality  $\mu_k = S(\theta^k p, \mu_{k-1})$ , for  $k = 1, 2, \dots, n$ , and (13) ( $n$  times):

$$V(\theta^n p, \psi(n, p, \mu)) = V(\theta^n p, \mu_n) < V(\theta^n p, \mu_{n-1}) < \dots < V(\theta^n p, \mu).$$

To end the proof, we use the fact that  $V$  is constant with respect to the first variable and Theorem 4.2.  $\square$

The last result is a corollary from the above theorem. It concerns describes the convergence of algorithm (1).

**THEOREM 5.3.** *Under the conditions of Theorem 5.2:*

$$\lim_{n \rightarrow \infty} \text{Prob}(d_A(X_n, A^*) < \varepsilon) = 1 \quad \text{for all } \varepsilon > 0.$$

PROOF. Fix  $\varepsilon > 0$ . Let  $B_\varepsilon(A^*) = \{x \in A : d_A(x, A^*) < \varepsilon\}$  and let  $\mu_n$  be the measure defined in Section 2, i.e.,  $\mu_n \sim X_n$ , for  $n = 1, 2, 3, \dots$ , where  $X_n$  is a random variable generated by algorithm (1). By Theorem 5.2,  $\mu_n \rightarrow \mu_0$ , for some measure  $\mu_0 \in \mathcal{M}^*$ . By (3), it means that  $\mu_n(B_\varepsilon(A^*)) \rightarrow \mu_0(B_\varepsilon(A^*)) = 1$ . Finally, we get

$$\mu_n(B_\varepsilon(A^*)) = \text{Prob}(X_n \in B_\varepsilon(A^*)) = \text{Prob}(d_A(X_n, A^*) < \varepsilon) \longrightarrow 1,$$

which was to be shown.  $\square$

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