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A few problems connected with invariant measures of Markov maps - verification of some claims and opinions that circulate in the literature

Abstract: It is well known that C^2-transformation φ of the unit interval into itself with a Markov partition (2.1) $\pi = \{ I_k : k \in K \}$ admits φ-invariant density g ($g \geq 0$, $\|g\| = 1$) if: (2.2) $|(\varphi^n)'| \geq C_1 > 1$ for some n (*expanding condition*); (2.3) $|\varphi''(x)/(\varphi'(y))^2| \leq C_2 < \infty$ (*second derivative condition*); and (2.4) $\#\pi < \infty$ or $\varphi(I_k) = [0, 1]$, for each $I_k \in \pi$. If (2.4) is deleted, then the situation dramatically changes. The cause of this fact was elucidated in connection with so-called *Adler's Theorem* ([1] and [2]).

However after that time in the literature occur claims and opinions concerning the existence of invariant densities and their properties for Markov Maps, which satisfy (2.2), (2.3) and do not satisfy (2.4), revealing unacquaintance with that question.

In this note we discuss the problems arising from the mentioned claims and opinions. Some solutions of that problems are given, in a systematic way, on the base of the already published results and by providing appropriate examples.

Keywords: Markov maps; invariant measure; expanding transformation; recurrence; aperiodicity

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1 Introduction

We begin with the celebrated result of [3]. The authors were well aware that their result cannot be extended to expanding transformations with countably many one-to-one pieces in a simple way (see Th. 2, and the comment below on Cond. (17) there). The real task in that period of time was to find reasonable additional conditions which would guarantee the existence of density invariant under the action of expanding map with countably many one-to-one pieces. Several attempts was made to accomplish that task (for more details see e.g. a review article [4], and also [5], or [6], Sect. 6).

One of the mentioned attempts was published in [7], as *Adler's Theorem*. Since no proof was given there, the question arose whether it is true [8]. A solution was published in [1], and [2].

After the above two notes and a few other ones, related with them, were published, some further claims and opinions concerning the existence of invariant densities and their lower and upper bounds for Markov Maps appear in the literature.

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Those claims and opinions reveal that their authors were unacquainted with the essence of the problem. That problem is rather of delicate nature. It involves, among other things, the so-called *measure-theoretic recurrence property*.

In this note we clear up, in a systematic way, the essence of the problems with the aid of examples, comments and some published results.

2 The problems and examples

2.1 Existence of invariant densities

We begin with the following

Definition 2.1. Let $\varphi : I \to I$ be countably piecewise one-to-one and C^2, here $I \subset \mathbb{R}$ denotes an interval. It is called of *Markov type* if there exists a partition (mod 0) $\pi = \{I_k : k \in K\}$ such that: each I_k is an interval, $\varphi_k := \varphi_{|I_k}$ is one-to-one and C^2 from I_k onto $J_k = \varphi(I_k)$ and the following condition holds:

$$\text{for each } j, k \in K, \text{ if } \varphi(I_j) \cap I_k \neq \emptyset, \text{ then } I_k \subseteq \varphi(I_j). \tag{2.1}$$

The following result follows from Ths. 1, and 2 in [3]:

Corollary 2.2. *Assume that φ is of Markov type and satisfies:*

$$\text{for some } n : \quad (\varphi^n)'(x) \geq C_1 > 1 \text{ whenever } (\varphi^n)' \text{ is defined;} \tag{2.2}$$

$$|\varphi''(x)/(\varphi'(y))^2| \leq C_2 < \infty \text{ whenever } \varphi''(x) \text{ and } \varphi'(y) \text{ are defined;} \tag{2.3}$$

$$and \quad \#\pi < \infty \text{ or } \varphi(I_k) = [0, 1], \text{ for each } I_k \in \pi. \tag{2.4}$$

Then:
$$there \ exists \ \varphi - invariant \ density \ g \ (g \geq 0, \ \|g\| = 1). \tag{2.5}$$

Now we delete (2.4) and pose, at the beginning, the following question:

Problem 2.3. *Assume that φ is of Markov type, in the sense of Def.* 2.1, *and satisfies: Conditions* (2.2), *and* (2.3). *Does* (2.5) *holds?*

The following simple example gives negative answer to the question in Problem 2.3:

Example 2.4. Let $I_k = [1 - 2^{-(k-1)}, \ 1 - 2^{-k}]$ $(k = 1, 2, ...)$ and $\varphi : [0, 1] \to [0, 1]$ such that every I_k is mapped linearly on $I_k \cup I_{k+1}$. Then $|\varphi'| \geq 3/2$, φ is of Markov type and fulfils (2.3). Nevertheless (2.5) does not hold because each point moves to the right under the action of φ.

Notice that φ in this example has the following defective property:

$$\bigcup_{j=1}^{\infty} \varphi^j(I_k) = \bigcup_{j=k}^{\infty} I_j \searrow \emptyset \quad \text{as} \quad k \to \infty, \quad \text{for} \quad k \geq 2.$$

The transformation given in [9] is another example which gives negative answer to the question in Problem 2.3.

Let us now eliminate that defect by imposing the so-called indecomposability condition (see [10], Definitions 2.2, 2.3, and Cond. 2.[$M_1 4$]):

$$\bigcup_{j=1}^{\infty} \varphi^j(I_k) = [0, 1], \quad \text{for each} \quad I_k \in \pi. \tag{2.6}$$

Note that indecomposability condition (2.6) is equivalent to the following so-called *transitivity condition*:

$$\text{for every} \quad I_i, I_j \in \pi \quad \text{one has:} \quad I_i \subset \varphi^n(I_j) \quad \text{for some} \quad n \geq 1. \tag{2.7}$$

Definition 2.5. A transformation φ of Markov type which satisfies Cond. (2.6) is called a *Markov Map*.

Now we pose, analogously as before, the following question:

Problem 2.6. *Assume that φ is of Markov type, in the sense of Def. 2.1, and satisfies Conditions:* (2.2), (2.3) *and* (2.6).
Does (2.5) *holds?*

In this case the following example gives negative answer to the question in Problem 2.6 [11]:

Example 2.7. Let $A = \{0\} \cup \{1/k : k = 1, 2, \dots\}$ and $I_k = (1/(k+1), 1/k]$ for $k = 1, 2, \dots$ Then we define φ as follows:

$\varphi_{|I_1}(x) = 2x - 1$;
and for any $k = 2, 3, \dots, \quad \varphi_{|I_k}$ is the increasing linear such that
$\varphi(I_k) = (0, 1/(k-1)]$;
$\varphi(0) = 0$.

Proof. Clearly, φ is of Markov type and satisfies all the assumptions of Problem 2.6. Nevertheless there is no φ-invariant density. This is so because it is ergodic and has σ-finite absolutely invariant measure concentrated on the whole I (there exists piecewise constant, not integrable, and φ-invariant function).   □

Notice that φ in this example fulfils even more restrictive condition than Cond. (2.6). Namely, it fulfils

$$\varphi^k(I_k) = (0, 1] \quad \text{for each} \quad I_k \in \pi. \tag{2.8}$$

Finally, let us consider instead of Cond. (2.6) its more restrictive version; the following condition:

$$\text{there exists} \quad j \geq 2 \quad \text{such that} \quad I_i \subset \bigcup_{r=1}^{j} \varphi^r(I_k) \quad \text{for every} \quad I_i, I_k \in \pi. \tag{2.9}$$

Once more we pose the following question:

Problem 2.8. *Assume that φ is of Markov type in the sense of Def. 2.1, and satisfies Conditions:* (2.2), (2.3) *and* (2.9).
Does (2.5) *holds?*

The so-called *Adler's Theorem* asserts that the answer to the question in Problem 2.8 is positive [7]. But no proof is given there. Further, the comments on that theorem in [12] are restricted to a history of the theorem. However, it was noted that *Adler's Theorem* may not hold in [8].

The counterexamples, published in [1] and [2], disprove *Adler's Theorem*, i.e. they give negative answer to the question in Problem 2.8.

In the former paper was also proposed a correction of *Adler's Theorem*. Namely, in the case of bounded interval I the following additional condition was proposed (see Cond. (1.H3) there):

$$\inf\{|\varphi(I_k)| : k \in K\} > 0; \quad \textit{where} \quad |\cdot| \quad \textit{denotes Lebesgue measure}. \tag{2.10}$$

While in the case of unbounded interval I, it was proposed the following (see Cond. (1.H4) there):

$$\lim_{n\to\infty} R(n) = 0 \quad where \quad R(n) = \sup_{k\in K} \int\limits_{I\setminus V_n} \tilde{\sigma}_k(x)\, dx, \quad \tilde{\sigma}_k(x) = \frac{\sigma_k(x)}{|I_k|}, \tag{2.11}$$

$$\sigma_k(x) = |(\varphi_k^{-1})'(x)|\mathbf{1}_{\varphi(I_k)}(x), \quad \{V_n\}_{n=1}^{\infty} \ is\ a\ sequence\ of\ subsets\ of\ I,$$

and each V_n is the union of a finite number of I_k's such that $V_n \subset V_{n+1}$,
$\bigcup\limits_{n=1}^{\infty} V_n = I \,(\mathrm{mod}\,|\cdot|)$.

Note that under the assumptions of Problem 2.8 the two Conditions (2.10) and (2.11) are equivalent.

A more efficient than the last two above conditions is the following one:

$$\int h_1\, dx > 0 \quad where \quad h_1 = \inf_s \sum_{k\in K} \tilde{\sigma}_k \int\limits_{I_k} \tilde{\sigma}_s(x)\, dx, \tag{2.12}$$

and $\tilde{\sigma}_k$ is defined in (2.11).

Note that Condition (2.12) is an analogue of the widely known condition from the theory of Markov Chain, the analogue is explained in [13]. Its efficiency is illustrated by examples in ([13], Ex. 2.1) and in ([6], Exs. 4.3.1, and 4.3.2). One has also to underline, that

Remark 2.9. Condition (2.12) additionally assures aperiodicity but Condition (2.10) does not (see Example 2.13, below).

The role which each of the last three conditions plays in the problem of the existence of invariant density consists in guaranteeing that the needed *measure-theoretic recurrence property* holds.

Since the transformations given in Example 2.4 and in [9] have global attractors (single point and the Cantor set, respectively), they are without that property. Note also that they do not satisfy the simple Condition (2.10).

On the other hand, it is not easy to decide, without Condition (2.12), whether or not the above mentioned transformations of Examples 4.3.1, or 4.3.2 in [6] have the needed *measure-theoretic recurrence property*.

Theorems stated in [14] as Theorem 1.2 and, in more abstract setting, as Theorem 1.3 contain the theorem questioned in [8].

There is also given a proof of Th. 1.3 which is incorrect (the thesis of the Lemma 1.5 does not hold, in general). That fact is not noted in [15].

Theorem 2.2 in [16] is a version of Th. 1.3 from [14]. It is stated under the indispensable Condition (2.10). This condition is incorporated, as Condition c), in the definition of Markov Map (Definition p. 353).

However, in connection with the Assertion c) of that theorem and the opinion on transitivity Assumption contained in Remark 4c) p. 354, here Condition (2.7), one has to raise two questions. The first question reads:

Problem 2.10. *Assume that φ is of Markov type in the sense of Def.* 2.1. *What is the essential role played by Condition* (2.7) *in the theory of Markov Maps?*

We begin with example of a Markov type map without property (2.7) (see also [10], Example 2.1, *W-transformation*):

Example 2.11. Let $0 < a < 1$, and then let $\psi : I = [0, 1] \to I$ be defined by

$$\psi(x) = \begin{cases} x/a & if \quad 0 \le x < a, \\ -(1+a)x + (a^2 + a + 1) & if \quad a \le x < 1. \end{cases}$$

Proof. Clearly, ψ is a transformation of *Markov-type* with respect to the following intervals: $I_1 = [0, a^2)$, $I_2 = [a^2, a)$, $I_3 = [a, 1)$.

The interval I_1 is the so-called inessential interval; the remainder two intervals I_2, I_3 are essential [10]. Transformation ψ restricted to the last two intervals satisfies already condition (2.7). Consequently, the invariant density is supported by $I_2 \cup I_3$. ☐

In general, any transformations of M-type can be decomposed into transformations with property (2.7) and an inessential part (see for details [10]).

The second question:

Problem 2.12. *Assume that φ is of Markov type in the sense of Def.* 2.1 *which satisfies:* (2.5) *and the so-called transitivity condition* (2.7).
Does the invariant density is necessarily exact in the sense of Rochlin ([17])?

As it shows the transformation of the below Example 2.13, *transitivity condition* (2.7) does not assure, in general, exactness. It has to be complemented to *exclude periodicity of Markov Map*. However, as it is noted in Remark 2.9, Condition (2.12) involves already aperiodicity.

The role of such complementary condition plays Condition (1.H5) in ([18], Remark (1.1)) (see also Cond. (3.$H_1$4) in [13]), or Cond. (4.1.$H_1$3) in [6]). That condition reads:

$$There\ exist\ an\ integer\ \tilde{n} \geq 1\ and\ I_{\tilde{k}}\ such\ that\ \varphi^{\tilde{n}}(I_{\tilde{k}}) = [0, 1]. \tag{2.13}$$

Note that it is a weak version of Cond. (2.16) below.

Example 2.13. (of a Markov map with properties (2.7) and not exact)
Put $I_k = [k, k+1)$ for $k = 0, 1, 2, 3$. Let $\chi_k : I_k \to I_2 \cup I_3$ for $k = 0, 1$ be linear, increasing, and onto. Analogously, let $\chi_k : I_k \to I_0 \cup I_1$ for $k = 2, 3$ be linear, increasing, and onto.

Finally, define $\chi : I = [0, 4) \to I$ by $\chi(x) = \chi_k(x)$ iff $x \in I_k$.

Then χ trivially fulfils the Conds. (2.2), and (2.3) of Corollary 2.2 and is a Markov Map which satisfies Condition (2.7) (actually Condition (2.9), for $j = 3$) and Condition (2.10), but it is not exact. Therefore it is a counterexample to the Assertions c) and d) of Theorem 2.2 in [16].

The proof of the last fact is based on the following criterion of exactness [17]:

Let $(I, \mathfrak{F}, \varphi : I \to I; d\mu)$ *where I is a space, $\mathfrak{F}$ is a σ-algebra of its subsets, φ transformation with μ invariant measure. Then*

$$\varphi\ is\ exact\ \Leftrightarrow\ for\ every\ A \in \mathfrak{F},\ \lim_{n\to\infty} \mu(\varphi^n(A)) = 1. \tag{2.14}$$

Proof. Now we are going to show that χ is not exact. Note first that $d\mu = 1/4\, dx$ is the unique invariant density.

Further $\chi(I_k) = I_2 \cup I_3$ for $k = 0, 1$ and $\chi(I_k) = I_0 \cup I_1$ for $k = 2, 3$. From these relations it follows that

$$\mu(\varphi^n(I_k)) = 1/2\quad for\ every\ I_k,\ k = 0, 1, 2, 3,\ and\ n = 1, 2, 3, \ldots, \tag{2.15}$$

and therefore the criterion (2.14) is not fulfilled. ☐

Finally, one needs to complete the opinion on Cond. (2.10) expressed in the Remark 4c), p. 354 of the cited book [16]. The authors claim that it can be somewhat weakened but it is certainly not possible to dispense with it altogether if we want to have that Markov Maps necessarily have invariant density.

The essential role played by Cond. (2.10) for the existence of invariant density has been already explained above.

As for the weakening of that condition, it is in general less efficient than Condition (2.12) (see: Convergence Theorem; Coroll. 1.1 in [18] – 1-dimensional case; or 3.1. Theorem; 3.1. Coroll. in [13] – multi-dimensional case). The efficiency of Condition (2.12) is also shown below by Examples 2.19 and 2.20.

In the introduction of [19] is noted that in Chapter 7, Section 4 of the book [20], in English, the proof appears to have an error.

Actually the theorem contained in Chapter 7, Section 4 of that book does not hold. This is so because the theorem in question is stated under somewhat less restrictive conditions than that of the so-called *Adler's Theorem*. Therefore the above mentioned counterexamples in [1], and [2] disprove that theorem as well.

In a review [21] of the book [22], in Polish, the reviewer claims that the proof of the Theorem 1, § 4, Section 7 on p. 164 is not correct.

This problem is clear up in [23]. It turns out that this is the very same problem as that raised in the introduction of [19].

One has to return to the already mentioned above note [19]. At the end of that note is questioned the double inequality of Remark 1 in [24]. That remark states:

If countably piecewise C^1-Markov Map satisfies conditions:
Cond. (2.2, *for n = 1*), *Cond.* (2.3), *and*

$$\text{there exists one integer } \tilde{n} \text{ such that } \varphi^{\tilde{n}}(I_k) = [0, 1] \text{ for each } I_k \in \pi, \tag{2.16}$$

then the invariant measure is unique and its density is bounded away from 0.

The author also claims in ([24], p. 38) that the density g of the unique invariant measure satisfies the following double inequality:

$$M_1 \le g(x) \le M_2 \quad \text{for some constants} \quad M_1, M_2 > 0. \tag{2.17}$$

On the other hand, the authors in ([19], p. 868) question the above double inequality (2.17).

Remark 2.14. Additionally the authors suggest that the fault is connected with the use of the idea of regularity functional. This is not the case. The regularity functional has been used to get bounds (see e.g. [6], [13], or [18]).

However, the bounds of the double inequality are, in general, not constants as in (2.17), but functions (see: the above cited papers or Coroll. 2.21, below).

Next if $\tilde{n} = 1$ in (2.16), then the remark in question is obviously correct.

Finally, in connection with the discussed Remark 1 in ([24], pp. 37-38), one has to raise two further questions. The first question is still connected with the problem of the existence of invariant densities for Markov Maps:

Problem 2.15. *Assume that φ is of Markov type in the sense of Def.* 2.1 *and satisfies: Cond.* (2.2) *for n = 1, Cond.* (2.3) *and Cond.* (2.16) *for $\tilde{n} \ge 2$.*
Does (2.5) *holds?*

The second question is connected with the problem of the existence of the lower and upper bounds of invariant density. It is delayed until the second subsection.

As for the question in Problem 2.15, first note that Cond. (2.16) is essentially more restrictive than Cond. (2.9) thus it is a restrictive version of *Adler's Theorem*. Nevertheless, the answer to that question is negative too.

It follows from the repeatedly cited counterexamples published in [1] and [2]. More exactly, the defined in [2] Markov Map $\tilde{\tau} : I = [0, 1] \to I$ satisfies

$$\tilde{\tau}^2(\tilde{I}_k) = [0, 1] \quad \text{for each } \tilde{I}_k \text{ (see there Final remark (b))}.$$

2.2 Bounds of invariant densities

We begin this subsection with the question announced at the end of the previous one. It can be formulated as follows:

Problem 2.16. *Assume that φ is of Markov type in the sense of Def. 2.1 which satisfies Cond. (2.5). Does the invariant density satisfies the double inequality (2.17)?*

Regarding the Problem 2.16. As was above noted, the authors in ([19], p. 868) question the above double inequality (2.17) in [24] and suggest that the fault is connected with the use of the idea of regularity functional (see Remark 2.14).

On the other hand, in ([25], Ex. 4) is given an example of Markov Map S in the sense of Definition 2.5 and it is shown that the double inequality (2.17) does not hold.

Remark 2.17.

(a) More specifically, the Markov Map S satisfies Conditions (2.2), (2.3), and (2.10), above and therefore, as it is proved in ([25], Proposition 2), it belongs to a class considered in [26].

Then the author shows that the invariant density h of the Markov Map S satisfies:

$$\lim_{x \to 1} h(x) = 0. \tag{2.18}$$

Therefore it does not satisfy Cond. (2.17) (it is not bounded away from 0).

(b) Note that the relation (2.18) is an immediate consequence of the double inequality of the below Coroll. 2.21.

The author claims that the property (2.18) of S is associated with Cond. (2.10). There is however no argumentation given that this is the case.

Actually, that property of invariant density is neither caused by Cond. (2.10) nor by any other condition that assures existence of invariant density.

Example 4 in [25] illustrates in the reality quite another fact. Namely, it shows that assumption (2.22) in Coroll. 2.22 cannot be omitted.

Indeed, it follows from the following

Proposition 2.18.

(a) *There exist two Markov Maps ψ and $\tilde{\psi}$ which do not belong to the class considered in [26]. Precisely, they do not satisfy Cond. ($\star$) of Proposition 2 in ([25], p. 1274) and therefore, a fortiori, they do not satisfy Cond. (2.10).*

(b) *There is ψ-invariant density g_ψ such that:*

$$\liminf_{x \to 1} g_\psi(x) = 0, \tag{2.19}$$

and therefore g_ψ does not satisfy Cond. (2.17) (it is not bounded away from 0).

(c) *There is $\tilde{\psi}$-invariant density $g_{\tilde{\psi}}$ which satisfies Cond. (2.17).*

Proof. The two transformations are given in the following two examples:

Example 2.19. Let $p_k = 1 - 2^{-k}$, and put $I_k = [p_k, p_{k+1})$, for $k = 0, 1, 2, \ldots$
Then define linear mappings $\psi_{2k} : I_{2k} \to [0, p_{2k+3})$ for $k = 0, 1, 2, \ldots$;
$\psi_1 : I_1 \to [0, p_4)$; and $\psi_{2k+1} : I_{2k+1} \to I_{2k}$ for $k = 1, 2, 3, \ldots$
Finally, define $\psi : [0, 1) \to [0, 1)$ by $\psi(x) = \psi_k(x)$ iff $x \in I_k$.

Example 2.20. Markov Map $\tilde{\psi}$ is a result of simple modification of ψ in such a way that its first linear mappings ψ_0 is replaced with the linear mapping $\tilde{\psi}_0$ from I_0 onto the whole $[0, 1]$.

Now we show that the two Markov Maps ψ and $\tilde{\psi}$ have the properties listed in the proposition. Part (a) of Prop. 2.18 follows directly from definitions of the two Markov Maps.

The proof of the remainder two Parts (b) and (c) is based on the following two corollaries:

Corollary 2.21. *Let a Markov map $\varphi : [0, 1] \to [0, 1]$ satisfy: Conditions (2.2) and (2.3) of Corollary 2.2 and additionally Cond. (2.10) or, the more efficient, Cond. (2.12). Then:*

there is a unique φ–invariant density g_0 such that

$$C_1^{-1}\tilde{g} \leq g_0 \leq C_1\tilde{g}, \tag{2.20}$$

where $C_1 > 0$ is a constant and $\tilde{g} > 0$ on $\{g_0 > 0\}$ is given by

$$\tilde{g} = \sum_{k \in K} \tilde{\sigma}_k \int_{I_k} g_0 \, dx, \tag{2.21}$$

and $\tilde{\sigma}_k$ is defined in (2.11).

Proof. (see: Convergence Theorem; Coroll. 1.1 in [18], 1-dimensional case, or 3.1. Theorem, 3.1. Coroll. in [13], multi-dimensional case). □

The second corollary reads:

Corollary 2.22. *If, in particulary, φ satisfies:*

$$\text{there is } \pi_1 \subset \pi, \ \#\pi_1 < \infty \ \text{ such that:}$$

$$\inf_{x \in \varphi(I_k)} \tilde{\sigma}_k(x) > 0 \ \text{ for } I_k \in \pi_1, \ \text{ and } \bigcup_{I_k \in \pi_1} \varphi(I_k) = \{g_0 > 0\} \tag{2.22}$$

then there is a constant $C_0 > 0$ such that $g_0 \geq C_0$.

Proof. This fact is a simple consequence of the assumptions of the previous Corollary 2.21 and the double inequality (2.20) together with (2.21). □

The proof of (2.19) consists of two parts. In the first part it is proved that there exists a unique ψ-invariant density g_ψ; in the second part it is proved that it satisfies (2.19).

To prove the existence of g_ψ we show that ψ satisfies Cond. (2.12). To this end observe that from the inequalities

$$\tilde{\sigma}_{2k}(x) \geq \mathbf{1}_{[0, p_5)}(x), \ \text{ for } i = 0, 1, 2, \ldots, \ \text{ and } \ \tilde{\sigma}_1(x) \geq \mathbf{1}_{[0,p_4)}(x) \tag{2.23}$$

where $\tilde{\sigma}_k$ is defined in (2.11), it follows

$$\sum_{k=0}^{\infty} \tilde{\sigma}_k \mathbf{1}_{\psi(I_k)} \int_{I_k} \tilde{\sigma}_i \mathbf{1}_{\psi(I_i)} \, dx \geq \tilde{\sigma}_1 \mathbf{1}_{\psi_1(I_1)} \int_{I_1} \tilde{\sigma}_i \mathbf{1}_{\psi(I_i)} \, dx + \sum_{k=0}^{\infty} \tilde{\sigma}_{2k} \mathbf{1}_{\psi(I_{2k})} \int_{I_{2k}} \tilde{\sigma}_i \mathbf{1}_{\psi(I_i)} \, dx \tag{2.24}$$

$$\geq \mathbf{1}_{[0, p_4)} \int_{I_1 \cup (\bigcup_{k=0}^{\infty} I_{2k})} \tilde{\sigma}_i \mathbf{1}_{\psi(I_i)} \, dx > 0,$$

for any $\tilde{\sigma}_i$. It implies Cond. (2.12).

As for (2.19), note first that the density $\tilde{g}$ given by (2.21) of Coroll. 2.21 and associated with g_ψ can be written as a sum

$$\tilde{g} = g_1 + g_2, \tag{2.25}$$

where

$$g_1 = \tilde{\sigma}_1 \mathbf{1}_{\psi(I_1)} \int_{I_1} g_\psi \, dx + \sum_{i=1}^{\infty} \tilde{\sigma}_{2i+1} \mathbf{1}_{\psi(I_{2i+1})} \int_{I_{2i+1}} g_\psi \, dx,$$

and

$$g_2 = \sum_{i=0}^{\infty} \tilde{\sigma}_{2i} \mathbf{1}_{\psi(I_{2i})} \int_{I_{2i}} g_\psi \, dx.$$

Further for g_1 one has

$$g_1 = \frac{1}{p_4}\mathbf{1}_{I_0 \cup I_1 \cup I_2 \cup I_3}\int_{I_1} g_\psi\,dx + \frac{1}{|I_2|}\mathbf{1}_{I_2}\int_{I_3} g_\psi\,dx + \sum_{i=2}^{\infty}\frac{1}{|I_{2i}|}\mathbf{1}_{I_{2i}}\int_{I_{2i+1}} g_\psi\,dx, \qquad (2.26)$$

and for g_2 one has

$$
\begin{aligned}
g_2 &= \sum_{i=0}^{\infty}\frac{1}{|\psi(I_{2i})|}\mathbf{1}_{\psi(I_{2i})}\int_{I_{2i}} g_\psi\,dx && (2.27)\\
&\leq \frac{1}{p_3}\left(\mathbf{1}_{I_0 \cup I_1 \cup I_2}\int_{A_0} g_\psi\,dx + \sum_{i=1}^{\infty}\mathbf{1}_{I_{2i+1}\cup I_{2i+2}}\int_{A_{2i}} g_\psi\,dx\right),
\end{aligned}
$$

where

$$A_{2i} = \bigcup_{k=0}^{\infty} I_{2(i+k)}, \quad \text{for } i = 0, 1, 2, \ldots$$

Thus the relation (2.19) follows from (2.20) and (2.21) of Coroll. 2.21, and the relations (2.25), (2.26), and (2.27).

As for the Part (c), note that the simple modification of ψ described in Example 2.20 leads to a Markov Map $\tilde{\psi}$ which satisfies Assumption (2.22) of Coroll. 2.22. Therefore its $\tilde{\psi}$-invariant density $g_{\tilde{\psi}}$ satisfies Cond. (2.17).   □

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