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Abstract	<p>The process of the mathematization of physical situations through differential calculus requires an understanding of the justification for and the meaning of the differential in the context of physics. In this work, four different conceptions about the differential in physics are identified and assessed according to their utility for the mathematization process. We also present an empirical study to probe the conceptions about the differential that are used by students in physics, as well students' perceptions of how they are expected to use differential calculus in physics. The results support the claim that students have a quasi-exclusive conception of the differential as an infinitesimal increment and that they perceive that their teachers only expect them to use differential calculus in an algorithmic way, without a sound understanding of what are they doing and why. These results are related to the lack of attention paid by</p>	

conventional physics teaching to the mathematization process. Finally, some proposals for action are put forward.

Footnote Information



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3 **Obstacles to Mathematization in Physics: The Case** 4 **of the Differential**

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Abstract The process of the mathematization of physical situations through differential
9 calculus requires an understanding of the justification for and the meaning of the differ-
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1 Problem Statement: Why Differentials? Why is it Important to Analyse How Differentials are Taught and Learned in Physics Classes?

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Differential calculus (DC) provides quantitative research methods for studying the process of change and the dependency of one variable on others (Aleksandrov et al. 1956). Its invention in the seventeenth century represented a major boost for many branches of science, and it has been considered as ‘the most powerful theoretical tool ever constructed by men throughout history’ (Rossi 1997, p. 199) and ‘the main quantitative tool for the research of scientific problems for the last three centuries (...) without which physics and modern technology would not exist’ (Kleiner 2001, p. 138). The calculus is one of the great triumphs of modern civilization (Dray and Manogue 2010), it lies at the foundation of our scientific world view and it is important for an understanding of who we are as a society (Bressoud 1992).

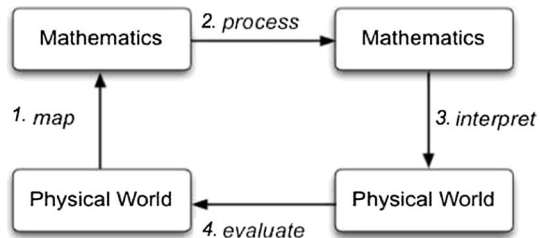
Because of its relevance in the progress of scientific knowledge, DC is first used in Spain in the teaching of maths and physics in the final years of high school (in the technoscientific branch, for 17- to 18-year-old students) and becomes an essential part of university education. Typically, DC in physics is not a straightforward application of the ideas already learned in maths. A reinterpretation of these ideas in the physics context is required (Meredith and Marrongelle 2008; Uhden et al. 2012). Maths used in physics, and pure maths, have distinct objectives because the aim of physics is the description and understanding of a physical system, and not the resolution of equations and the expressing of the most abstract possible relationships (Redish 2006). This latter author proposes a model to describe the bare bones of how we use maths in physics (see Fig. 1).

Once the physical analysis of a real-world problem or situation has been made, step #1 is a mathematization process or mathematical modelling that consists of expressing the ideas of the initial analysis through mathematical concepts and relationships. Uhden et al. (2012) distinguish different levels of mathematization and propose a greater intertwining of physics and mathematics models.

This ‘translation’ from the physical context to the abstract level of the mathematical one requires an understanding of the mathematical framework to be used, that is to say, the mathematical concepts and their relationships (White and Mitchelmore 1996). Usually, conventional teaching at all levels pays no attention to the first step and focuses on step #2: how to operate with the initial mathematical expressions to get a numerical or algebraic result. This imbalance has clear consequences for students: on the one hand, it leads them to turn physics problem-solving into maths problem-solving, and on the other hand, it affects their beliefs on how they are expected to use mathematics in physics (Redish 2006).

When using DC in physics situations, the basic concept that appears in the mathematization process is the one of the differential calculus, referring to independent variables

Fig. 1 A model for the use of maths in science (Redish 2006)





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59 and as part of differential expressions. For instance, $F \cdot dx$ is used to introduce the concept
60 of work; $dq = \rho \cdot dx$ for calculating the charge of a unidimensional charged rod; $dN = -$
61 $\lambda \cdot N \cdot dt$ to study nuclear decay; $dI = R^2 \cdot dM$ to calculate the moment of inertia; and $dT = -$
62 $\alpha \cdot (T - T_a) \cdot dt$ to obtain the functional expression for the cooling of an object over time. If we
63 want students to learn how to mathematize real-world problems when DC is required, a
64 conceptual understanding of this kind of expression and the situations in which these
65 expressions are necessary is essential.

66 But differential is a polysemic concept. Its meaning and role are different in maths and
67 physics—in maths, the differential is concerned with substantiating and formalizing calcu-
68 lus beyond the physical context, and in physics, it is centred on the productive use of a
69 set of concepts and reasoning regardless of rigour (Artigue and Viennot 1987; Dunn and
70 **AQ4** Barbanel 2000). This polysemic character of the differential is also found in education. It is
71 not only its role and meaning that are different in the teaching of maths and physics, but
72 even possible to acknowledge different conceptions within these areas (Alibert et al. 1989;
73 Artigue and Viennot 1987; Dray and Manogue 2010).

74 Our aim in this paper is to identify the conception or conceptions of the differential that
75 are used by students in physics and assess whether these conceptions are the most ap-
76 propriate to allow them to mathematize or, on the contrary, whether they are only useful to
77 allow students to handle in a mathematical way expressions that have already been given
78 mathematically.

79 This search for conceptual transparency must be pursued from the very introduction of
80 DC in physics teaching (as it is, in our experience, in physics courses in upper high school
81 and in the first courses of college physics). Precisely because of their introductory nature,
82 when the physics situations that are being studied are not very complex, reasoning, and a
83 clear justification of what we do when we face real-world physics problems, should be
84 distinctive characteristics of these physics courses. Otherwise, it might happen that the
85 teaching never addresses the necessary requirements for mathematizing physics situations
86 and, in this way, we might encourage mechanical behaviour in students and incomplete or
87 incoherent conceptions about the use of DC in physics, with consequent feelings of in-
88 security when students try to mathematize physics problems.

89 This is the reason why we have studied how students in their final year of high school,
90 after the conventional introduction of differential calculus in physics, conceive and justify
91 differential expressions, what perceptions they have of what they are expected to be able to
92 do with DC in physics and, also, the extent to which these ideas and perceptions remain
93 unchanged among university students.

94 This work is organized in two parts. In the first part, we will summarize some findings
95 **AQ5** of other studies on the usual approach to teaching and learning differential calculus [2].
96 Then, we will introduce [3, 4] some different conceptions of the differential from the point
97 of view of both physics and mathematics (our students are affected by both). And in the
98 last section of the first part, we will assess the possible usefulness and shortcomings of
99 these conceptions [5] for helping in the mathematization process.

100 In the second part, we will describe the experimental design [6], and we will analyse the
101 data [7] on students' ideas about the use of the differential, and the justification for that use,
102 and their perceptions of how they are supposed to use DC in physics. In the last section [8],
103 the main conclusions and implications for teaching that are derived from the results are
104 summarized.



2 Some Prior Work About the Teaching and Students' Use of Differential Calculus

In this section, we will try to give a brief summary of some contributions of the literature on how DC is taught and learned in mathematics and how it is used in physics.

Research in mathematics education has highlighted the poor conceptual understanding that students, and also teachers, have of DC.¹ That poor understanding affects not only the main ideas of calculus, like derivative or integral, but also other related concepts such as variable, function, limit and infinite. Some ideas have been identified as acting as barriers to good learning, such as conceiving the tangent as a straight line that touches the curve at only one point (Ferrini-Mundy and Lauten 1994; Speiser and Walter 1994), considering the surface to be generated by the accretion of lines or the volume by the accretion of surfaces (Schneider 1991; Turégano 1998), rejecting the existence of so-called mental objects as opposed to empirical ones (Schneider 1992; Speiser and Walter 1994), or performing known operations on misunderstood symbols (White and Mitchelmore 1996).

Most of the above-mentioned authors point to the merely algorithmic approach used in teaching [that is, the exclusively pragmatic perspective of the mathematics (Uhden et al. 2012)] as the basis of these shortcomings. After all, a large number of maths teachers and of maths teachers' trainers explicitly assume an instrumentalist view of maths (Moreno 2001; Mura 1993, 1995; Pereira de Ataíde and Greca 2013). The research of Nagy et al. (1991) highlighted this algorithmic tendency in calculus teaching: an analysis of sessions focused on studying calculus, including exams, showed a clear predominance of the *techniques* category over other categories relating to the meaning of concepts, when and why they should be used, etc. This operational view is absorbed by the students, who end up believing that doing maths is restricted to performing specific operations with meaningless symbols (Habre and Abboud 2006; Porter and Masingila 2000), and this does not necessarily lead to greater procedural confidence (Engelbrecht et al. 2005).

If we add to this algorithmic approach the usual tendency to give mathematics only a technical role in physics, emphasizing mathematical manipulations, it seems logical that when students use differential calculus in physics, they know how to perform the calculations, but they have difficulty connecting the physical world with mathematics. As several works have shown, although students know how to calculate integrals in a specific physics problem, they have difficulties related to the conceptual understanding. For instance, at all levels when approaching physical situations, students have difficulties in understanding the integral as a limit and interpreting the result as an exact value, in deciding when it is necessary to use the integral concept, in establishing the integration limits and, especially, in writing down the correct differential expression that represents a concrete physical situation or giving meaning to the product $f(x) \cdot dx$ when constructing an integral.²

The above survey highlights the shortcomings in teaching and the difficulties for students in performing the mathematization process. The relevant role of the differential in this process means that it is necessary to address the understanding of its role and meaning

¹ Artigue et al. (1989), Berry and Nyman (2003), Ferrini-Mundy and Gaudard (1992), Ferrini-Mundy and Graham (1991), Labraña (2001), Mahir (2009), Nagle et al. (2013), Orton (1983a, b), Porter and Masingila (2000), Schneider (1991, 1992), Tall (1985, 1992), Thompson (1994), among many others.

² See, for example, Meredith and Marrongelle (2008), Hu and Rebello (2013), Sealey (2014), Von Korff and Rebello (2014), and Wilcox et al. (2013).



146 from the mathematical perspective, as well as to identify and assess the different con-
147 ceptions about the differential as used in physics.

148 3 The Concept of the Differential in Teaching Mathematics

149 In this section, we will briefly characterize some different, and more frequent, roles and
150 meanings that the differential has in teaching maths.

- 151 1. *The differential as a merely formal instrument with no meaning in itself.* DC teaching
152 has remained loyal to nineteenth century mathematics, represented by the work of
153 Cauchy who, by means of an accurate definition of limit, banished from calculus the
154 ambiguity and lack of rigour that can both be attributed to Leibnitz's differential
155 (Martínez Torregrosa et al. 2006). Cauchy defined the differential as an expression
156 involving the derivative: $df = f'(x) \cdot dx$, with an arbitrary (big or small) increment dx of
157 the variable, and it thus became a simple formal instrument, necessary for the
158 abbreviation of certain proofs. The differential was then detached from the ambiguity
159 of the infinitely small quantities, but it was devoid of all physical meaning: it was just
160 the result of multiplying the derivative by the increment of the independent variable.
161 As Freudenthal (1973, p. 550) says: 'Useless differentials can readily be dismissed. If
162 dy and dx occur only in the combination dy/dx , or under the integral sign after the
163 integrand, the question as to what dx and dy mean individually is as meaningful as to
164 ask what the "l", "o", "g" in "log" mean'.
165 2. *The differential as a linear approximation (but never used in practice).* Modern
166 calculus texts, if they introduce the differential and assign it some meaning, usually do
167 this as a linear approximation of the increment: $\Delta y \approx f' dx = dy$. However, after that,
168 in practice, the differential only appears as part of algorithmic developments and plays
169 a similar role to that in conception #1. This conception has been criticized because,
170 although it usefully refers to the idea of linear approximation, the identification with
171 the differential is unnecessary (Dray and Manogue 2010). These authors suspect that
172 this unnecessary identification is done to avoid any risk of identifying differentials
173 with infinitesimals.

174 In recent decades, two conceptions of the differential have been proposed that give it
175 back its central role in the structure of DC: the Fréchet differential and the infinitesimal
176 differential.

- 177 3. *The differential as a tangential linear estimate.* The Fréchet differential, whose
178 original definition in 1911 had its origin in the analysis of functions of infinite
179 variables, is used in some textbooks to introduce the analysis of functions of one
180 variable (Del Castillo 1980; Hallez 1989, p. 67). For this particular case, Fréchet
181 would define differentiability and the differential in this way:

182 A function $f(x)$ admits a differential, in my sense, in point x_0 if there is a
183 homogeneous and linear function of the increment, let it be $A \cdot \Delta x$, that does not
184 differ from the increment of the function Δf , that starts from the value $f(x_0)$, in more
185 than an infinitely small value in relation with Δx . The differential is then, by
186 definition, $df = A \cdot \Delta x$.

187 (...) This definition is expressed by the formula: $\Delta f = df + \varepsilon \cdot \Delta x$, where ε goes to
188 zero when Δx goes to zero. It reminds us of the old definition, as the principal part,



189 and it presents all its advantages, but it overcomes the objections of lack of rigour
 190 that quite correctly had been put forward to it (Artigue 1989, p. 34).

192 This then means that df is an approximation of Δf and that it is linear respect to
 193 Δx . However, it cannot be any linear approximation but it is the one that meets an
 194 additional condition: $(\Delta f - df)$ must be infinitely small in relation to Δx . This does not mean
 195 that Δf or df is infinitely small. The condition imposed on the differential can also be
 196 expressed by saying that $(\Delta f - df)$ goes to zero faster than Δx . This last condition is
 197 equivalent to either of the two following ones:

- 198 • The Riemann sum of the approximations (df) must coincide with the increment:
 199 $\lim_{N \rightarrow \infty} df_i = \lim_{N \rightarrow \infty} \Delta f_i = \Delta f$, that is, $\int df = \Delta f$. This step from the approximation (df) to the
 200 exactness (Δf) is only possible if the approximation fulfils Fréchet's condition. In fact,
 201 for the Riemann sum $\int (df - \Delta f)$ to be zero, the limit of $N \cdot (df - \Delta f)$ when N goes to infinity
 202 must be zero, that is, the limit of $(df - \Delta f) / \Delta x$ must be zero when Δx goes to zero, but this
 203 is a precise condition required by Fréchet for the differential.
- 204 • The differential quotient df/dx equals the derivative f' . We will see that if the
 205 approximation df fulfils Fréchet's condition, then its slope will coincide with the
 206 derivative. In effect, as df is a linear function of the increment, its slope $df/\Delta x$ has a
 207 constant value and therefore the Fréchet condition can be expressed as:
 208 $\lim_{\Delta x \rightarrow 0} \frac{(df - \Delta f)}{\Delta x} = 0 \rightarrow \frac{df}{dx} - \lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x} = 0 \rightarrow \frac{df}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x}$. The second member of this last
 209 equality is the derivative of the function, if it exists, and since for a single independent
 210 variable $\Delta x = dx$, then: $f' = df/dx$.

211 The combination of these two forms of expressing the Fréchet condition leads directly
 212 to the fundamental theorem: $\int \square \cdot dx$ equals ΔF if and only if: $\square \cdot dx = dF$, that is, $\square = dF /$
 213 $dx = F'$. We have addressed these ideas in more detail in other works (Martínez Torre-
 214 grosa et al. 2002, 2006).

215 4. *The differential as an infinitesimal.* This last conception brings back the original idea
 216 of Leibniz: the differential as an infinitesimal, but this time in a rigorous way based on
 217 the non-standard analysis introduced by Robinson in the 1960s. Robinson built up a
 218 large set of numbers that includes real numbers, infinitesimals and infinite numbers.
 219 The infinitesimals are nonzero numbers that are lower than any real number. There are
 220 a few introductory calculus texts based on Robinson's ideas; among these, the text of
 221 Keisler (2000) is worth quoting. Keisler introduces the main DC ideas (derivative,
 222 differential, integral and the fundamental theorem) by means of this new set of
 223 numbers and without the concept of limit. Next, we briefly present the conception of
 224 the differential as an infinitesimal, but we recommend the study of Keisler's text.

225 If Δx is an infinitesimal, then Δf is also an infinitesimal and the derivative f' is defined, if
 226 it exists, as the nearest real number to the quotient $\Delta f / \Delta x$ (which is called the standard part
 227 of this quotient) (Keisler 2000, pp. 55–57). The differential is defined as the product of a
 228 real number times an infinitesimal: $df = f' \cdot \Delta x$, which is also an infinitesimal. Here again,
 229 for the independent variable, $\Delta x = dx$, and thus, we can write: $df = f' \cdot dx$. The so-called
 230 theorem of the increment demonstrates that if Δx is an infinitesimal, then there exists
 231 another infinitesimal ε that satisfies: $\Delta f = df + \varepsilon \cdot \Delta x$ (Keisler 2000, pp. 55–57).

232 We highlight some characteristics of this conception to avoid any possible inadequate
 233 interpretations. The differential is an infinitesimal, but not an infinitesimal increment. df is
 234 an approximation of the increment Δf ; however, it is not just any approximation but it is the



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235 one that meets an additional condition related to the quotient $(\Delta f - df)/\Delta x$. For the in-
236 finitesimal differential, the condition is expressed by saying that this quotient must be an
237 infinitesimal, while in the case of the Fréchet differential, it is said that the limit of this
238 quotient goes to zero when Δx goes to zero.

239 The fundamental difference between the two is that for the infinitesimal differential, the
240 emphasis is on its value (which is lower than that of any real number), while the Fréchet
241 differential puts the emphasis on the linear nature of this estimation of Δf , highlighting
242 what the French mathematician Dieudonné (1960, p. 145) considered to be *the funda-*
243 *mental idea of calculus*: the approximation of any function by linear functions (Artigue and
244 Viennot 1987; Martínez Torregrosa et al. 2006).

245 Sofronas et al. (2011) have shown that the conventional teaching of calculus does not
246 incorporate the idea of approximation. Similarly, the analysis of French calculus texts by
247 Alibert et al. (1989, p. 10) concludes that, although the idea of approximation associated with
248 the differential in mathematics was introduced in the 1960s, there were no changes in
249 practice, and it continued to be considered as a merely formal instrument, useful for devel-
250 oping operations.

251 4 The Differential Concept in Physics Teaching

252 In this section, we will briefly describe some different conceptions that are used or could be
253 used in physics teaching. This description is based on the results of an analysis of high
254 school physics texts (in Spain) and university physics texts for the first courses of science
255 and engineering degrees (López-Gay 2002), on our experience as physics teachers in high
256 schools and for the first courses of engineering degrees, and on the study presented above
257 from the mathematical perspective. From now on, we will continue to use the mathematical
258 notation (f, x, \dots) when referring to general physics equations and magnitudes, and we will
259 only change this notation for specific physics situations.

- 260 1. *The meaningless differential.* A few (fortunately!) physics textbooks use differentials
261 without assigning them any meaning, whether explicit or implicit. It seems that they
262 consider differentials to be part of an intermediate routine that leads to derivatives and
263 integrals, in a similar way to conceptions #1 and #2 of the previous section. The
264 following fragment comes from an upper high school standard physics textbook:
265 ‘According to the equation $F = q \cdot v \cdot B \cdot \sin \alpha$, the force exerted by the magnetic field on
266 the charge dq is: $dF = dq \cdot v \cdot B \cdot \sin \alpha$ (...) that is the force that an *element* of electric
267 current will suffer inside a magnetic field’.
- 268 2. *The differential as an infinitesimal increment.* According to this conception, df is equal to
269 an infinitesimal Δf , produced by an also infinitesimal Δx . The terms infinitesimal, very
270 small, tiny or extremely small are used to express the same idea. Indeed, explicit
271 definitions of this conception are sometimes expressed: ‘ df is the limit of Δf when Δx goes
272 to zero’. But if this is the case, if $f(x)$ is continuous, then df will always be zero.

273 This intuitive conception is similar to the original idea of Leibniz, an ambiguous idea
274 removed from calculus in the nineteenth century. As we have already seen, infinitesimals
275 were reinserted into mathematics, free of any suspicion, in the 1960s, although their use in
276 DC teaching continues to be marginal. Moreover, from the mathematical perspective, the
277 infinitesimal differential is not equal to Δf . Therefore, modern calculus legitimates the use
278 of infinitesimals but not the interpretation of df as an infinitesimal Δf .



279 3. *The differential as an infinitesimal approximation.* This conception considers that df is an
280 infinitesimal that is very close to the value of Δf when Δx is infinitesimal. Here again, the
281 terms very small or extremely small are used to express the same idea. In this conception,
282 in the expression $df = \square \cdot dx$, dx is an infinitesimal Δx , so small that \square practically remains
283 unchanged, so that df is practically identical to the extremely small Δf .

284 This conception acknowledges the idea of approximation as well as the linear depen-
285 dence of df on Δx (although in other words: ' \square practically remains unchanged'), but, at the
286 same time, it uses infinitesimals or very small quantities in order to assert that $(df - \Delta f)$ is
287 negligible in practice; that is, df can be replaced by Δf without any error. This is a very
288 similar conception to the mathematical one of the differential as an infinitesimal (Sect. 3,
289 #4), but it fails to state the additional condition that must be fulfilled by the linear
290 approximation.

291 4. *The differential as linear estimate.* In this conception, df is an approximation of Δf , an
292 approximation that consists in supposing that Δf changes uniformly with Δx , without
293 any reference to whether Δx , Δf , df or $(\Delta f - df)$ have big or small values. According to
294 this idea, in the expression $df = \square \cdot dx$, dx (independent variable) is a Δx and df is an
295 approximation of the corresponding Δf that is calculated by assuming that \square is kept
296 constant along that Δx .

297 This conception highlights the idea of linear approximation by clearly stating the dif-
298 ference between df and Δf , and it does not emphasize the idea that they can be inter-
299 changed. It is similar to the Fréchet conception (Sect. 3, #3) but, in this case, there is no
300 explicit expression of the additional condition that must be fulfilled by this linear estimate.

301 The last three conceptions have in common that they consider df as a change, as Δf or as
302 an approximation of Δf . Some studies indicate that in certain physical situations, the
303 interpretation is different: df can be considered as an amount and not as a change (Von
304 Korff and Rebello 2014). Therefore, when they write $d\phi = B \cdot dS$, they interpret $d\phi$ as the
305 amount of flux passing through an amount of surface. However, this distinction is not
306 necessary because $d\phi$ can be considered as a change in the amount of flux passing through
307 a surface, when this surface changes by an amount dS . A different issue is that many
308 physical examples are developed in terms of accumulation of quantities rather than ac-
309 cumulation of changes. We do not argue in this work about whether it is adequate to
310 proceed in this way, but, if this were the case, the change from the interpretation as change
311 to that as amount can be done without difficulties. Therefore, from now on, we will
312 interpret the differential as a change.

313 5 Critical Assessment of the Different Conceptions of the Differential 314 in Physics Teaching

315 Our concern for the differential is caused by its important role in the mathematization
316 process of physics problems and situations requiring the use of DC for their solution, that
317 is, in the first step of Redish's schema about the use of mathematics in physics. Such a
318 process, from our standpoint, requires an understanding of the role and meaning of the
319 differential in the physical context. This means that students should be able to answer the
320 following questions in *concrete* physics situations: Why is it necessary to use differentials?
321 What is the meaning of the differential in physics? Why do we write down exactly that



322 differential expression and not another one? Next, we will appraise the different concep-
323 tions of the differential according to the answers to those questions.

324 5.1 Why is it Necessary to Use Differentials?

- 325 • From conception 4#1, the meaningless differential, there is no answer to this question;
326 in fact, 'differential' is used when terms like 'elemental' or 'element' appear as cues in
327 the text to introduce the differential.
- 328 • From conception 4#2, the differential as an infinitesimal increment, we need
329 differentials because the magnitudes are very small. Differentials are in fact written
330 just to indicate that we are in the realm of the *very small*; hence, we write the
331 expression $dv = a \cdot dt$ because we are referring to a very tiny Δt .
- 332 • From conception 4#4, the differential as a linear estimate of the increment, the reason
333 we must use the differential is that the change in a magnitude is non-uniform with
334 changes in the other variable, which prevents us from calculating the Δf produced by a
335 Δx . Differentials are, in fact, written to indicate a non-real change, the change that
336 would occur if the change were uniform along the Δx . Thus, the expression $dv = a \cdot dt$
337 is written because the speed does not change uniformly with time, that is, because the
338 acceleration is not constant during that time interval.
- 339 • From conception 4#3, the differential as an infinitesimal approximation, differentials
340 are needed for a combination of the reasons given for conceptions 4#2 and 4#4.

341 5.2 What is the Meaning of the Differential in Physics?

- 342 • For conception 4#1, this is a nonsensical question since the differential is considered to
343 be an instrument without any meaning. When certain cues like *elemental displacement*
344 appear, differentials must be used.
- 345 • For conception 4#2, if $dv = a \cdot dt$, dv is the very small change in the velocity produced
346 in the very small time interval dt . This conception assumes that is impossible to assign
347 numerical values to dv and dt because they can always be even smaller; thus, it is
348 difficult to interpret the numerical value of the acceleration at any *instant*, for instance,
349 $a(t = 7) = 2$ (SI units).
- 350 • For conception 4#3, if $dv = a \cdot dt$, dv is an approximation of the change in the velocity
351 in a very small time interval dt , so small that the acceleration can be considered
352 constant in that interval; hence, dv is practically equal to the small Δv in that small
353 Δt . The same difficulties as in 4#2 appear when assigning numerical values to dv and
354 dt , or interpreting the numerical value of the acceleration at an instant.
- 355 • For conception 4#4, if $dv = a \cdot dt$, dv is an estimate of the velocity change that would
356 occur in any time interval dt ; this estimate is made by assuming that the acceleration is
357 constant throughout that time interval. The numerical value of the acceleration in an
358 instant, for instance, $a(t = 7) = 2$ (SI units) means that, from $t = 7$ s, if the
359 acceleration is kept constant, the speed would change by 2 m/s every second. Similarly,
360 if $dt = 5$ s, then $dv = 10$ m/s, and this would be the change in speed from $t = 7$ to
361 $t = 12$ if the acceleration was kept constant.



362 5.3 Why Exactly that Differential Expression and not Another One?

- 363 • For conception 4#1, there is no answer: any expression could be written, if it is called
364 elemental and we use the differential symbol.
- 365 • For conception 4#2, which identifies the differential as an infinitesimal increment,
366 when we are addressing a physical situation in which we start from an already defined
367 expression for finite increments ($\Delta x = v \cdot \Delta t$, $\Delta m = \rho \cdot dV \dots$), the differential expression
368 will be exactly the same because it only indicates that those same changes are now very
369 small ($dx = v \cdot dt$, $dm = \rho \cdot dV \dots$). The trouble with this conception arises in physical
370 situations in which an expression for finite increments has not been previously defined.
371 Let us, for instance, look at the disintegration process of a radioactive sample. In this
372 case, $dN = -\lambda \cdot N \cdot dt$ is written with confidence, as if it were possible to know what
373 happens in very small time intervals but not in 'normal' time intervals. In fact, starting
374 from that expression, via integration, we arrive at: $\Delta N = N \cdot (1 - e^{-\lambda \cdot \Delta t})$, and,
375 therefore, in order to match this conception, the differential expression should be
376 written as $dN = N \cdot (1 - e^{-\lambda \cdot dt})$. Starting from a particular premise, a conclusion that
377 contradicts that premise is obtained.
- 378 • For conception 4#4, in physical situations of the first kind (we have already defined
379 expressions for the uniform cases: $\Delta x = v \cdot \Delta t$, $\Delta m = \rho \cdot dV \dots$), the corresponding
380 differential expression is similar, because the differential approximation consists
381 precisely in assuming that there is linear behaviour even though we know that actually
382 there is not; hence, $dx = v \cdot dt$, $dm = \rho \cdot dV \dots$. In relation to the second kind of physical
383 situation mentioned above, we must search for an expression that represents uniform
384 behaviour although we know that actually the behaviour is not uniform. From this
385 perspective, the following differential expressions could represent the corresponding
386 linear estimates for the radioactive disintegration process: $dN = -\lambda \cdot N \cdot dt$, $dN = -$
387 $\lambda \cdot N^2 \cdot dt$, $dN = -\lambda \cdot dt / N \dots$. Any linear function of dt could be a good mathematical
388 candidate. However, we know that only one of them complies with the Fréchet
389 condition, $dN/dt = N'$, but as we do not know $N(t)$, any of them may remain valid. As
390 in many other physical situations, we must conceive each of the linear estimates as a
391 plausible hypothesis, work mathematically on it ('via integration') and find a functional
392 expression for ΔN or of $N(t)$, the validity of which must be empirically probed in the
393 physical situation at hand or by its coherence with other findings in the same field. In
394 such cases, we must say that differential expressions (initially) have a hypothetical
395 nature.
- 396 • For conception 4#3, the answer to this question is similar to that for conception 4#2 or
397 for conception 4#4, depending on whether the emphasis is on the infinitesimal value or
398 on the character of the differential as a linear approximation. Our experience is that the
399 infinitesimal value is emphasized. This conception is often based on the intuition,
400 which is apparently correct, that the sum of many thousands of very tiny
401 approximations ends up giving an accurate result because the error in each
402 approximation is practically zero. In fact, when university physics students and high
403 school physics teachers are asked to analyse a mathematical development that begins
404 with a 'reasonable' differential expression but produces a physically or geometrically
405 erroneous result, almost none of them regards the differential expression (called by us
406 the 'differential hypothesis') as doubtful. They usually check the mathematical
407 operations again and again from beginning to the end, without finding the 'mistake'.



408 Some contradictory situations like this, in the context of the calculus of surfaces and the
409 volumes of regular geometric objects, have been addressed by certain authors (Artigue
410 et al. 1989, pp. 31–38; Schneider 1991). When the infinitesimal character is emphasized
411 instead of the linear estimation one, it is difficult to justify why some differential ex-
412 pressions lead to a correct result and others do not.

413 5.4 Which is the more Suitable Concept of the Differential for Teaching 414 Introductory Physics Courses?

415 The answer depends on the aim of the teaching. If we wanted to focus only on the
416 mathematical operations, starting from expressions that are already known, then concep-
417 tion 4#1 could be sufficient. We are committed to helping students to learn the modelling
418 process, so we discard that conception.

419 Conception 4#2 is intuitive and, in our experience, is the most frequently used one in
420 physics teaching. However, we have identified some shortcomings in this conception:

- 421 • It does not allow one to identify the situations in which it is necessary to use the
422 differential, since it does not justify the need take infinitesimals when approaching a
423 concrete problem.
- 424 • It does not allow one to assign numerical values to explain the meaning of the
425 differential, a didactic exercise that, in general, promotes understanding; neither does it
426 provide an easy explanation of the numerical values of derivatives as quotients since,
427 again, it is necessary to refer to the quotient of two quantities that do not admit
428 numerical values.
- 429 • It makes it impossible to acknowledge the hypothetical nature of the differential
430 expression in some physical situations, because there is no criterion for deciding
431 between one expression and another.
- 432 • It fails in its internal logic: starting with an expression for an infinitesimal Δf , which is
433 considered to be correct, it permits one, via integration, to obtain another different
434 expression that is also valid for an infinitesimal Δf .

435 For these reasons (besides its incorrectness from a mathematical viewpoint, including
436 the viewpoint of the non-standard calculus), conception 4#2 is not the most suitable one for
437 the mathematization of physical situations.

438 Conception 4#3 preserves the intuitive character and the reference to infinitesimals that
439 is so frequent in physics teaching and, besides, avoids some of the shortcomings mentioned
440 before:

- 441 • It clearly identifies the situations that require that a differential expression be written:
442 these are when there is non-uniform or nonlinear behaviour.
- 443 • It highlights the character of the differential as an approximation and imposes a
444 condition that must be fulfilled to justify the transition from approximation to
445 exactitude via *integral* that the slope of the approximation (df/dx) should coincide with
446 the derivative (f'), that is to say, with the quotient of two infinitesimals.
- 447 • It permits one to acknowledge the hypothetical nature of the differential expressions in
448 some physical situations that have already been described.
- 449 • Contradictions disappear.

450 However, some drawbacks that come from the identification of differentials with in-
451 finitesimals remain. One of them is that this conception does not permit numerical values
452 to be assigned and, in this way, an explanation of the meaning of differential and the



453 interpretation of concrete values of the derivative to be facilitated. The other shortcoming
454 is that it is extremely difficult to understand the meaning in physics of any differential
455 expression in the realm of the infinitely small, and this becomes a major drawback in
456 selecting a hypothetical differential expression when required. Moreover, this drawback
457 sometimes goes unnoticed as a result of the intuitive, but erroneous, idea that any ap-
458 proximation will end up giving the exact result if sufficiently small Δx are taken.

459 And lastly, conception 4#4 maintains the advantages of conception 4#3 while avoiding
460 the objections noted above. In particular, the differential expression makes sense for any
461 Δx whether large or small, which allows the physical meaning of the approximation to be
462 interpreted more clearly and hypothetical differential expressions to be written through
463 physical analysis, without taking extremely small values for which it seems that, in the end,
464 'anything goes'.

465 In accordance with these assessments, conceptions of the differential that highlight the
466 idea of approximation, whether as an infinitesimal approximation or as a linear estimate,
467 are the most suitable for facilitating the mathematization of physical situations through
468 DC. Between the two, we find some important additional advantages in the conception of
469 the differential as a linear estimate of the increment. However, we are aware that these
470 appraisals depend on our aim and, therefore, that the assessment and selection can be
471 different from another perspective and for other purposes (Ostebee and Zorn 1997).

472 6 Objectives and Experimental Design

473 Our experimental study seeks to gather data on how final-year high school students justify
474 the use of the differential in physics, assign meaning to differential expressions and per-
475 ceive how they are supposed to use DC in physics, and on the extent to which these aspects
476 evolve in university students. In this way, we expect to identify students' conceptions
477 and the persistence of these conceptions, for a better understanding of the difficulties students
478 can have in mathematizing physical situations requiring DC.

479 We have designed a set of instruments that includes both qualitative and quantitative
480 tools:

- 481 • *Four written questions (3 closed and 1 open).* These are designed for getting data on
482 students' conceptions of the justification and meaning of differential expressions and on
483 their perceptions of how they are expected to use DC in physics. They are measured
484 using a Likert-type scale. When the results are presented, the content of the questions
485 will be explained.
- 486 • *Three problems to be solved by students,* on physics topics with which the students
487 were familiar. In two of these problems, paragraphs were included to remind students
488 of the necessary conceptual grounds in physics for solving the problems. The wording
489 of each problem explicitly requested students to write explanatory comments,
490 especially each time they used differential calculus in solving the problem.
- 491 • *An individual semi-structured interview on one written solved problem* from which the
492 explanations had been removed. We divided the solution with horizontal lines into a
493 total of seven sections that were gradually revealed to the student during the interview.
494 For each section, the student was asked the corresponding question that appears in the
495 Appendix. The data obtained from questions 4–6 are not used in this work. The
496 objective of this audio-recorded interview was to obtain complementary qualitative



497 information to illustrate and support the interpretation of the quantitative results
498 obtained from the students' written responses to the questions and problems.

499 The considerable time period for our work has provided us with large sample sizes. In
500 total, the sample was made up of 488 students belonging to six high schools and four
501 Spanish universities and fulfilled a diversity of criteria with regard to both the origin of the
502 students and that of the teachers of the physics or maths courses. The respondents were
503 divided into three subgroups: 190 final-year physics high school students (in the techno-
504 scientific branch, aged 17–18 years), 153 university freshmen and 145 second- to fourth-
505 year university students. Over half the university students included in the sample were
506 physics undergraduates and the rest were studying for other scientific or engineering
507 degrees. All the students in the sample were taking a physics subject and, for the youngest,
508 a maths subject that included differential calculus topics.

509 Because our interest was in the teaching of physics in high school (when DC is in-
510 troduced for the first time), seven bright students in the final year of high school were
511 selected by their own teachers to be interviewed. We also interviewed four recent gra-
512 duates in physics or technical studies who are training to become high school teachers. We
513 chose these four recent graduates, who are now our students in a Master's programme on
514 Teaching Physics and Chemistry at secondary level, because we believe them to be rep-
515 resentative of university students.

516 In order to decide whether there were significant differences between the results ob-
517 tained by the different sample subgroups, the Student's *t* test was used with a significance
518 level lower than 5 %.

519 7 Results and Discussion

520 The results, instruments and interpretations will be grouped around the three objectives
521 mentioned above. The interpretations will be accompanied by verbatim fragments from the
522 interviews. A more detailed and thorough analysis can be found in López-Gay (2002).

523 7.1 Results on When and Why it is Necessary to Use Differentials

524 These results come from data obtained from two closed questions and from the analysis of
525 the problems solved by the students (each of the three problems was different, and adapted,
526 for each subgroup).

527 The first question aims to discover what students consider to be the best reason to justify
528 the change from increments to differentials. Each of the response items is related to one of
529 the conceptions of the differential as used in physics [4]. We used item (d) to distinguish
530 between students who justify the use of the differential by the existence of a dependency
531 relationship (Meredith and Marrongelle 2008) and those who refer to the non-uniform
532 behaviour during Δt (see Table 1).

533 Fifty-eight percentage of the secondary school students justify the step from increments
534 to differentials because these are infinitely small values (conceptions 4#2 and 4#3), and this
535 percentage increases significantly for the university degree students, with 82 % of students
536 on these higher courses choosing this option. Overall, 69 % of students chose this option.
537 By contrast, the option that justifies the step from increments to differentials because of the
538 non-uniform behaviour (conceptions 4#3 and 4#4) was chosen by only 11 % of the high
539 school students and by only 3 % of the university students. Finally, the statement related to



Table 1 Wording and results obtained from the first closed question about *justification**

In a text on kinematics, one reaches the following expression: $\Delta v = a \cdot \Delta t$, which is then written as follows: $dv = a \cdot dt$ Tick (✓) for which of the following reasons you think justifies more accurately the need for taking this step	Last year high school (N = 149) % (SD)	1st U. (N = 92) % (SD)	≥2nd U. (N = 90) % (SD)
a. Because we want to finish with a derivative or an integral	20.1 (3.3)	15.2 (3.8)	15.2 (3.5)
b. Because infinitely small times are being considered	57.7 (4.1)	73.9 (4.6)	82.2 (4.1)
c. Because acceleration depends on time	10.7 (2.5)	2.2 (1.5)	3.3 (1.9)
d. Because speed depends on time	17.4 (3.1)	10.9 (3.3)	6.7 (2.6)
e. I don't know	4.7 (1.7)	4.3 (2.1)	2.2 (1.6)

* Although we asked students to choose only one statement, some of them chose more than one. This is the reason for the sum of percentages of each column being higher than 100 %

540 conception 4#1 (“Because we want to finish with a derivative or an integral”) was chosen
 541 by 20 % of the high school students and a slightly lower percentage of university students.

542 The second closed question was used to obtain information on the students not through
 543 what they say but through what they do: we wanted to know their criteria for using DC in a
 544 physics situation. This would give us information on the implicit justification for using
 545 calculus. The wording of the question and the results obtained are presented in Table 2.

546 Only 15 % of the high school students correctly identify that it is necessary to use DC
 547 when nonlinear relationships appear (conceptions 4#3 and 4#4); this percentage sig-
 548 nificantly decreases in university students, falling to 5 % of the students in higher years of
 549 university courses. Overall, 89 % of the students do not know the characteristics of a
 550 situation in which DC is needed (as would be expected if they were thinking with con-
 551 ceptions 4#1 and 4#2), and this percentage is higher as the education level increases.

552 At high school, 5 % of students consider it necessary to use DC in all situations, and this
 553 percentage increases significantly during university education, reaching 30 % of students
 554 in the higher years of university courses. On the other hand, 59 % of high school students
 555 and a higher percentage of university students consider that it is necessary to use DC in all
 556 cases in which position depends on time.

557 The analysis of the problems solved by students provides information on what students
 558 say and do when they use differentials. The results related to justification and meaning are
 559 presented in Table 3.

Table 2 Wording and results obtained from the second closed question about *justification*

We know the position equation of four different moving objects. We want to calculate the instantaneous speed of each object. Tick (✓) for the cases in which the use of differential calculus (derivatives, differentials, integrals ...) is necessary	Last year high school (N = 117) % (SD)	1st U. (N = 89) % (SD)	≥2nd U. (N = 43) % (SD)
a. $x = 12$	5.1 (2.0)	11.2 (3.4)	30.2 (7.1)
b. $x = 8 + 3t^2$	87.2 (3.1)	91.0 (3.0)	97.7 (2.3)
c. $x = 6t - 2$	72.6 (4.1)	80.9 (4.2)	93.0 (3.9)
d. $x = 5 \cos 3t$	87.2 (3.1)	91.0 (3.0)	100 (-)
Tick all the options (a, b, c, d)	3.4 (1.7)	10.1 (3.2)	30.2 (7.1)
Tick cases where $x = f(t)$ (b, c, d)	59.0 (4.6)	68.5 (5.0)	62.8 (7.5)
Tick only nonlinear cases (b, d)	14.5 (3.3)	9.0 (3.1)	4.7 (3.3)



Table 3 Results on the justification and explicit meaning of the differential when solving physics problems

When they solve problems in which they are specifically asked to include explanatory comments each time, they use differential calculus	Last year high school ($N = 57$) % (SD)	1st U. ($N = 95$) % (SD)	≥ 2 nd U. ($N = 105$) % (SD)
They use differentials <i>Of them</i>	17.5 (5.1)	50.5 (5.2)	44.8 (4.9)
Try to justify why they use differentials	10.0 (10.0)	4.2 (2.9)	10.7 (4.6)
Write any meaning of the differential	0 (–)	6.4 (3.6)	21.2 (6.0)

560 The low percentage of students who use DC to solve the problems is striking. For high
 561 school students, this could be due to two main difficulties: mathematizing and using DC.
 562 However, it is even harder to explain why nearly half of the university students of all
 563 courses do not use DC when it is necessary to solve the problem, despite knowing the
 564 physics of the problem and being given a reminder of the main physics ideas. Moreover,
 565 only one out of ten students who use DC to solve the problem tries to justify its use, despite
 566 being clearly asked for an explanation for the use of DC. Given the low number of students
 567 who try to justify the use of DC, we do not consider it useful to distinguish between the
 568 different kinds of justifications.

569 The results of Tables 1, 2, and 3 show that, when using the differential or DC to solve
 570 a physics situation, most of the students cannot identify which characteristics of the
 571 situation at hand lead them to use it. Moreover, when they are given certain statements,
 572 their justifications refer to infinitesimal values and not to the idea of linear ap-
 573 proximation. We interpret these results as showing the predominance—more marked in
 574 higher education levels—of the conception of the differential as an infinitesimal in-
 575 crement, as seen in both their explicit statements and the absence of criteria for when to
 576 use the differential.

577 The following extracts from interviews illustrate the difficulties that students have in
 578 justifying the step from increments to differentials:

582 Extract 1. Pedro, a bright upper high school pupil

583 P: *The top one $[\Delta m/\Delta V]$ is the definition of density and the bottom one $[dm/dV]$ is the same but for very,
 very small pieces... It would be the same but maybe here mass could be written depending on the
 volume...*

584 In: And couldn't you do the same thing using the top expression?

585 P: *Yes, I suppose, but I don't really know...*

588 Extract 2. Maria, teacher in training

589 M: *He has turned the increments into differentials, and I don't know why he does that. What I don't
 understand is why he doesn't directly replace an increment. He has gone from increments to
 differential...*

590 In: Why? Couldn't he have done the same with increments?

591 M: *I don't know why he goes to differentials*

592



593 **7.2 Results on the Meaning of Differential Expressions**

594 The open question is aimed at obtaining information on the meaning that students give to
 595 the differential in a context of a familiar physics situation that is briefly described. The
 596 complete wording of the question is shown in Table 6.

597 We analysed the responses to this question in order to see which conception of the
 598 differential appears from the responses of each student. In the light of the responses given,
 599 we have added a new unexpected category: the differential is an increment, without any
 600 reference to its value (Table 4).

601 We did not consider category *b* as a new conception, but as an incomplete view of what
 602 category *c* represents. Taking into account this criterion, 61 % of the students (58 % of
 603 high school and university students in higher years and 70 % of freshmen) expressed their
 604 conception of the differential as a small or very small infinitesimal increment. Only *one* out
 605 of 122 students expressed the conception of the differential as an infinitesimal ap-
 606 proximation (4#3).

607 Category *a* includes those students who do not know any meaning of 'differential', as
 608 well as those who think that differential has no meaning. Hence, we cannot distinguish how
 609 many students have the conception of the differential as a formal instrument.

610 When they are directly asked, between 58 and 72 % of students are able to express a
 611 conception of the differential (Table 4). In contrast, the results obtained when analysing
 612 the problems solved by students (Table 3) show how little sense students make when
 613 explaining the meaning of differentials when they use them, despite being asked for
 614 explanations. This could be interpreted as an effect of the algorithmic approach of calculus.

615 Our experience as teachers leads us to assume that the concept of the differential that
 616 prevails in physics teaching is that of a small or very small infinitesimal increment. Why
 617 then do 'only' 61 % of the students express this openly? We believe that it is likely that the
 618 algorithmic approach of calculus induces inconsistency and a lack of confidence among
 619 students, which prevent them from clearly expressing any assertion about the only, but
 620 nebulous, conception they have.

621 Over the course of the interviews, a lack of meaning appeared in certain cases (extract
 622 3) or an identification of the differential and an increment, *with no additional conditions*
 623 (extract 4). However, the most frequent answer was that it was a matter of *very small*
 624 increments, although the arguments converged on operational ideas (extract 5).

Table 4 Results of the analysis of responses to the open question about the meaning of dN

	Last year high school ($N = 52$) % (SD)	1st U ($N = 37$) % (SD)	≥ 2 nd U. ($N = 33$) % (SD)
a. They do not write any meaning	42.3 (6.9)	27.1 (7.4)	42.4 (8.7)
b. It is ΔN (without more accuracy)	34.6 (6.7)	29.7 (7.6)	36.4 (8.5)
c. It is a very small ΔN (without more accuracy)	23.1 (5.9)	40.5 (8.2)	21.2 (7.2)
d. It is a small ΔN that, so small that any other magnitude is thought to be constant	0 (-)	2.7 (2.7)	0 (-)
e. It is a linear estimation of ΔN	0 (-)	0 (-)	0 (-)

Author Proof



626

627 Extract 3. Isa, a bright high school pupil

628 I: *dm is as if the mass is changed with regard to... I'm getting mixed up!*

629 In: Do you know any meaning of that expression?

630 I: *I know that by doing the integral you remove the "d", but I can't think of one at the moment*

633 Extract 4. Julia, teacher in training

634 J: *I have always taken an increment to mean the difference between the final and the initial mass, and a differential when you do not place any limits between what is varying the mass*

635 In: But, does it vary?

636 J: *Of course, so it has to have limits*

637 In: What difference is there then between increments and differentials?

638 J: *There is no difference*

641 Extract 5. Juan, a bright high school pupil

642 J: *I understand differential to mean when you want to study the parts as "tiny" as you want*

643 In: So what is dm ?

644 J: *Well "tiny" little pieces of the... (unable to finish sentence)*

645 In: Of what?

646 J: *Of the mass...*

647 J: *(...) Every time we use differentials, my teacher says: "in order to study this curve we are going to take the straight lines as small as we please..."*

648 J: *(...) dV , dh are volume increments, height increments... he seems to take it like that*

649 In: And is that what you think?

650 J: *I don't understand it... In truth, I know how to calculate integrals, but I haven't actually understood the differentials that occur, I see them in writing but I don't know what they are... and, why am I going to bother to ask, since they are going to tell me: "these are the little pieces..."*

651

652 7.3 Results on Students' Perceptions of How they are Supposed to Use 653 Differential Calculus in Physics

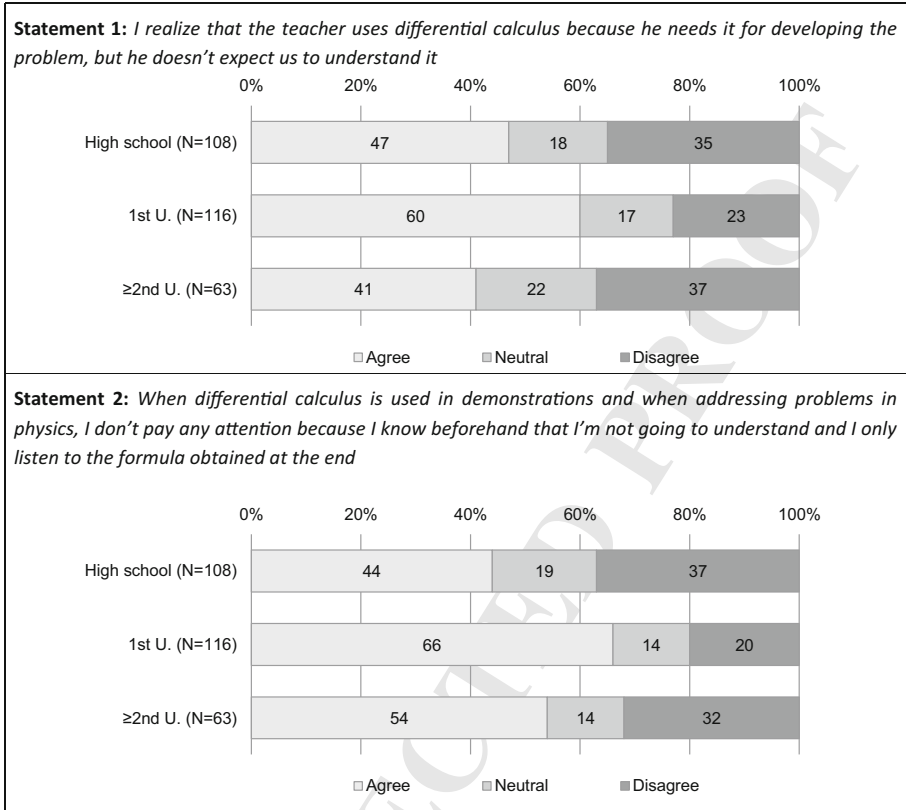
654 Different works on teaching and using differential calculus [2] have revealed the pre-
 655 dominance of the algorithmic approach, so the regular use of calculus in physics does not
 656 take into account the mathematization process. It focuses only on the manipulation of
 657 mathematical expressions. This situation is clearly reflected in the results obtained from the
 658 analysis of the problems solved by students (Table 3): most of the students are unable to
 659 mathematize a familiar physics situation using DC and, if they are, they cannot express
 660 either the reasons that have led them to use DC or the meaning of the differential ex-
 661 pression they write.

662 We believe that this situation will affect students' perception of how they are supposed
 663 to use DC in physics. To obtain information on this perception, we wrote two comple-
 664 mentary statements: one refers to what students perceive from their teachers and the other
 665 refers to their own actions when using calculus. We asked students to express their degree
 666 of agreement with each of these two statements according to a Likert scale from 1 to 5 (1
 667 Totally agree—2 Agree—3 Neutral—4 Disagree—5 Totally disagree).

Author Proof



Table 5 Wording and results for the statements related to students' perception of how they are supposed to use differential calculus in physics



668 The wording of the two statements and the results obtained by grouping the responses
 669 given by students into three categories (1–2 Agree, 3 Neutral, and 4–5 Disagree) are shown
 670 in Table 5.

671 According to Table 5, 47 % of secondary students perceive that their teacher does not
 672 expect them to understand the use of DC and, indeed, 44 % of them refuse to understand it.
 673 This perception is even more widespread among freshmen: 60 % of them perceive that
 674 their teacher does not expect them to understand the use of DC, and 66 % of them refuse to
 675 understand it. Although this growth does not reach as far as university students in higher
 676 years, it is a significant fact that 41 % of them still perceive that the teacher does not expect
 677 ^{AQ6} them to understand the use of DC, and 54 % refuse to understand it (Table 6).

678 We can conclude that the percentage of students who perceive that what is expected of
 679 them is to use DC without understanding it and only to use it mechanically is always
 680 greater than that of students who do not share that perception, independently of the
 681 statement and the subgroup of students.

682 This situation, particularly at university level, would be unsustainable if it were not for
 683 an unspoken agreement between teachers and students: although differential calculus is



Table 6 Open question on the meaning of the differential

A radioactive substance is one whose nuclei are being transformed into other nuclei or particles, that is, they are disintegrating. If we call N the number of nuclei of a radioactive substance at an instant t , this number will decrease in an interval of time due to the number of disintegrations that have taken place

The law of radioactive disintegration refers to the number of disintegrated nuclei in a certain radioactive substance over an interval of time, and its initial mathematical expression is the following: $dN = -\lambda \cdot N \cdot dt$, with λ being a characteristic constant of each radioactive element

In your own words, and as clearly as possible, explain the physical meaning of dN that is deduced from the above mathematical expression

684 used, the teacher does not expect his or her students to understand it, and nor do the
 685 students aspire to understand it; what matters is that the students know how to apply the
 686 rules. The following extracts from interviews illustrate this lack of understanding among
 687 students and a generalized mechanical attitude.
 688

691 Extract 6. Julia, teacher in training

692 In: Did you learn when your physics teachers or textbooks used differential calculus?

693 J: *I learnt how to perform calculus, but not what it actually was, I've never learnt that*

696 Extract 7. Sergio, a bright upper high school pupil

697 S: *Seeing all that on the board is a shock, at first sight it's horrendous; of course it puts you off.*

698 In: Why do you think that is?

699 S: *We are all for being practical, and seeing so many operations well, it frightens you a little, but it
 701 doesn't really because finally what's important is the result, from a practical point of view. The same
 702 thing happens to me, but then I get home and I manage to do it*

702 Extract 8. Juan, a bright upper high school pupil

703 J: *The truth is, when there are some integrals—for example, some that have got a “little zero” in the
 middle that I don't know what they're about—and I can see them, but I don't study them because I
 can waste too much time trying to understand them*

704 In: And the others?

705 J: *If I “get” them quickly, yes*

706 In: But didn't you say you don't know their meaning?

707 J: *Yes, but I know how to do them*

708

709 8 Summary and Implications

710 In this study, we have shown four conceptions of the differential as used in physics: as a
 711 merely formal instrument, as an infinitesimal increment, as an infinitesimal approximation
 712 and as a linear estimate of the increment. We have assessed each of these according to how
 713 useful they are in helping in the mathematization process of physics situations using
 714 differential calculus, a process that generally leads to the use of differentials. The con-
 715 clusion, according to the explicit assessment criteria that we have established [5], is that
 716 the last two conceptions, especially that of a linear estimate of the increment, are the more
 717 suitable. However, global analysis of different results obtained using different instruments
 718 supports the claim that the main conception of students in physics contexts, especially
 719 university students, is the one that identifies the differential with an infinitesimal incre-
 720 ment; this constitutes an obstacle to students' ability to mathematize. We have come to this
 721 conclusion from the students' direct answers when they are asked about the meaning of the



Author Proof

722 differential in a physics context (61 %, see Table 4), from the reasons they give to justify
723 the use of differentials (69 %, see Table 1), as well as from their difficulties in identifying
724 those cases in which it is necessary to use differential calculus, difficulties that are asso-
725 ciated with this conception of the differential (89 %, see Table 2).

726 Furthermore, the data seem to indicate that students perceive that all that is expected of
727 them is a mechanical use of differential calculus in physics. The percentage of students
728 who hold this perception is always greater than that of students who do not hold it, no
729 matter what their academic level or the instrument used to obtain data.

730 Although we have not proved that there is a cause and effect relationship between the
731 prevailing conception of the differential in students and their perception of what is
732 expected of them, we think that the two results are linked. In effect, as we have ex-
733 plained, the idea of the differential as an infinitesimal increment is inappropriate for the
734 mathematization process of physical situations by calculus. The fact that this is the
735 prevailing conception in students can only be explained by concluding that the use of
736 calculus is focused on an operational process (isolate a variable, replace it, solve
737 derivatives and integrals ...) on a mathematical expression that has already been written,
738 in order to get a result, neglecting the process that begins with a physical analysis of the
739 situation at hand and leads to that starting expression. This claim is consistent with the
740 results of different studies about the algorithmic approach of the teaching and use of
741 calculus. In this context, it seems reasonable for students to perceive that they may only
742 use calculus in a mechanical way.

743 In our opinion, if the aim is to teach students the mathematization process when they use
744 calculus in physics, and to change their perceptions, the conception of the differential that
745 is usually used should be changed. To students who are familiar with the idea of the
746 infinitesimal increment, it may be easier to promote this change to the conception of the
747 differential as an infinitesimal approximation. In the case of students who not only are
748 going to start learning about differential calculus but will also be using it to do physics, we
749 think that, even though the idea of an infinitesimal approximation could be valid, the idea
750 of a linear estimation of the increment is better because it allows one to see a clearer
751 relationship between physics analysis and the written differential expressions.

752 Anyway, it is necessary to avoid approaching physics problems and theoretical devel-
753 opments as if the initial steps of mathematization were self-evident or a mechanical re-
754 sponse to cues like 'infinitesimal' or 'elemental'. For our part, we are working on the
755 design and implementation of physics teaching sequences for upper high schools that
756 systematically incorporate the conception of the differential as a linear estimation, to help
757 students to use differential calculus with understanding and good sense.

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760

761 Appendix: Document and Guidelines for the Semi-Structured Interview

762 **PROBLEM STATEMENT:** We know that the density of air (ρ) decreases with height
763 (h) according to the following equation: $\rho = 1.29 \cdot (1 - 0.000125 \cdot h)$. That equation is written
764 for the International System, that is, if h is written in metres, density is obtained in kg/m^3 .
765 The value $h = 0$ represents sea level. *What would be the mass of a cylindrical column of*
766 *air measuring 1 m^2 at the base that rises 2000 m above sea level?*



Author Proof

$\rho = 1.29(1.000125h) \text{ (Kg/m}^3\text{)}$

What's the mass of this air column?

M

$\rho = \frac{\Delta m}{\Delta V} \Rightarrow \Delta m = \rho \cdot \Delta V \xrightarrow{(1)} dm = \rho \cdot dV \quad (2) (3)$

Or: $\rho = \frac{dm}{dV} \xrightarrow{(4)} dm = \rho \cdot dV$

Since: $dV = A \cdot dh \quad (5)$ then: $dm = \rho \cdot A \cdot dh$

By integrating: $\int_0^M dm = \int_0^{2000} \rho \cdot A \cdot dh \quad (6)$

$M = \int_0^{2000} 1.29(1.000125h) \cdot 1 \cdot dh = 1.29 \int_0^{2000} (1.000125h) dh$

$M = 1.29 \left[h - 0.000125 \frac{h^2}{2} \right]_0^{2000} = 1.29 \left[2000 - 0.000125 \frac{(2000)^2}{2} \right] = \underline{\underline{2.2575 \text{ kg}}}$

- 768 1. Why do you take that step? (Be it the step from increment to differential or from
- 769 incremental quotient to differential quotient)
- 770 2. What is the meaning of that expression? (We insist on searching for an explanation
- 771 that goes beyond the use of key words or literal reading)
- 772 3. What could be the value of dm , dV ...? (If they answer with a numerical value, we
- 773 check on its meaning and functional nature)
- 774 4. How must that expression be read? Is it correct to isolate the differential? (We are
- 775 referring to the expression of the derivative, and want to know if they consider it as a
- 776 true differentials quotient)
- 777 5. What is the meaning of those integrals? (They may adhere to the idea of the anti-
- 778 derivative, or go further and identify Riemann sums)
- 779 6. Why is the result of that integral precisely that? (We will inquire to see if they are
- 780 capable of justifying why the integral of a differential is a macroscopic increment, or
- 781 why infinite sums are necessarily calculated using anti-derivatives)
- 782 7. Do you understand properly when your teacher or the textbook use differential
- 783 calculus in physics lessons?
- 784 8. Where have you best learnt the use and meaning of differential calculus, in physics or
- 785 maths lessons?
- 786 9. In general, do you think that the use of differential calculus makes students like physics
- 787 more or less? Why do you think so? And do you think that is the case for you too?

788
789



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