

Robust Solutions to Multi-Objective Linear Programs with Uncertain Data*

M.A. Goberna^{†‡}, V. Jeyakumar[§], G. Li[§] and J. Vicente-Pérez[§]

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Abstract

In this paper we examine multi-objective linear programming problems in the face of data uncertainty both in the objective function and the constraints. First, we derive a formula for radius of robust feasibility guaranteeing constraint feasibility for all possible uncertainties within a specified uncertainty set under affine data parametrization. We then present a complete characterization of robust weakly efficient solutions that are immunized against rank one objective matrix data uncertainty. We also provide classes of commonly used constraint data uncertainty sets under which a robust feasible solution of an uncertain multi-objective linear program can be numerically checked whether or not it is a robust weakly efficient solution.

Keywords. Robust optimization. Multi-objective linear programming. Robust feasibility. Robust weakly efficient solutions.

1 Introduction

Consider the deterministic multi-objective linear programming problem

$$(\bar{P}) \quad V\text{-min} \{ \bar{C}x : \bar{a}_j^\top x \geq \bar{b}_j, j = 1, \dots, p \}$$

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[†]Corresponding author. Tel.: +34 965903533. Fax: +34 965903531.

[‡]Dept. of Statistics and Operations Research, Alicante University, 03071 Alicante, Spain.

[§]Dept. of Applied Mathematics, University of New South Wales, Sydney 2052, Australia.

E-mail addresses: mgoberna@ua.es (M.A. Goberna), v.jeyakumar@unsw.edu.au (V. Jeyakumar), g.li@unsw.edu.au (G. Li), jose.vicente@ua.es (J. Vicente-Pérez).

where V-min stands for *vector minimization*, \overline{C} is a real $m \times n$ matrix called *objective matrix*, $x \in \mathbb{R}^n$ is the decision variable, and $(\overline{a}_j, \overline{b}_j) \in \mathbb{R}^{n+1}$, for $j = 1, \dots, p$, are the constraint input data of the problem. The problem (\overline{P}) has been extensively studied in the literature (see e.g. the overviews [2] and [5]), where perfect information is often assumed (that is, accurate values for the input quantities or parameters), despite the reality that such precise knowledge is rarely available in practice for real-world optimization problems.

The data of real-world optimization problems are often uncertain (that is, they are not known exactly at the time of the decision) due to estimation errors, prediction errors or lack of information. Scalar uncertain optimization problems have been traditionally treated via sensitivity analysis, which estimates the impact of small perturbations of the data in the optimal value, while robust optimization, which provides a deterministic framework for uncertain problems ([1],[6]), has recently emerged as a powerful alternative approach.

Particular types of uncertain multi-objective linear programming problems have been studied, e.g. [12] considers changes in one objective function via sensitivity analysis, [10] and [11] consider changes in the whole objective function $x \mapsto \overline{C}x$, and [8] deals with changes in the constraints, the latter three works using different robustness approaches. The purpose of the present work is to study multi-objective linear programming problems in the face of data uncertainty both in the objective function and constraints from a robustness perspective.

Following the robust optimization framework, the multi-objective problem (\overline{P}) in the face of *data uncertainty* both in the objective matrix and in the data of the constraints can be captured by a parameterized multi-objective linear programming problem of the form

$$(P_w) \quad \text{V-min} \{Cx : a_j^\top x \geq b_j, j = 1, \dots, p\}$$

where the input data, the rows of C and (a_j, b_j) , $j = 1, \dots, p$, are uncertain vectors. The sets \mathcal{U} and \mathcal{V}_j , $j = 1, \dots, p$, are specified uncertainty sets that are bounded, but often infinite sets, $C \in \mathcal{U} \subset \mathbb{R}^{m \times n}$ and $(a_j, b_j) \in \mathcal{V}_j \subset \mathbb{R}^{n+1}$, $j = 1, \dots, p$. So, the uncertain parameter is $w := (C, (a_1, b_1), \dots, (a_p, b_p)) \in \mathcal{W} := \mathcal{U} \times \prod_{j=1}^p \mathcal{V}_j$. By enforcing the constraints for all possible uncertainties within \mathcal{V}_j , $j = 1, \dots, p$, the uncertain problem becomes the following uncertain multi-objective linear semi-infinite programming problem

$$(P_C) \quad \text{V-min} \{Cx : a_j^\top x \geq b_j, \forall (a_j, b_j) \in \mathcal{V}_j, j = 1, \dots, p\}$$

where the data uncertainty occurs only in the objective function and

$$X := \{x \in \mathbb{R}^n : a_j^\top x \geq b_j, \forall (a_j, b_j) \in \mathcal{V}_j, j = 1, \dots, p\},$$

is the robust feasible set of (P_w) .

Following the recent work on robust linear programming (see [1]), some of the key questions of multi-objective linear programming under data uncertainty include:

- I. (*Guaranteeing robust feasibility*) How to guarantee non-emptiness of the robust feasible set X for specified uncertainty sets $\mathcal{V}_j, j = 1, \dots, p$?
- II. (*Defining and identifying robust solutions*) How to define and characterize a robust solution that is immunized against data uncertainty for the uncertain multi-objective problem (P_C) ?
- III. (*Numerical tractability of robust solutions*) For what classes of uncertainty sets robust solutions can be numerically checked?

In this paper, we provide some answers to the above questions for the multi-objective linear programming problem (P_C) in the face of data uncertainty. In particular, we derive a formula for the radius of robust feasibility guaranteeing non-emptiness of the robust feasible set X of (P_C) under affinely parameterized data uncertainty. Then, we establish complete characterizations of robust weakly efficient solutions under rank one objective matrix data uncertainty (the same type of uncertainty considered in [11] for efficient solutions of similar problems with deterministic constraints). We finally provide classes of commonly used uncertainty sets under which robust feasible solutions can be numerically checked whether or not they are robust weakly efficient solutions.

2 Radius of robust feasibility

In this section, we first discuss the feasibility of our uncertain multi-objective model under affine constraint data perturbations. In other words, for any given matrix $\bar{C} \in \mathbb{R}^{m \times n}$, we study the feasibility of the problem

$$(P_\alpha) \quad \begin{aligned} & \text{V-min } \bar{C}x \\ & \text{s.t. } a_j^\top x \geq b_j, \forall (a_j, b_j) \in \mathcal{V}_j^\alpha, j = 1, \dots, p, \end{aligned}$$

for $\alpha \geq 0$, where the uncertain set-valued mapping \mathcal{V}_j^α takes the form

$$\mathcal{V}_j^\alpha := (\bar{a}_j, \bar{b}_j) + \alpha \mathbb{B}_{n+1}, \quad j = 1, \dots, p, \quad (1)$$

with $\{x \in \mathbb{R}^n : \bar{a}_j^\top x \geq \bar{b}_j\} \neq \emptyset$, and \mathbb{B}_{n+1} denotes the closed unit ball for the Euclidean norm $\|\cdot\|$ in \mathbb{R}^{n+1} .

Let $\mathcal{V} := \prod_{j=1}^p \mathcal{V}_j$. The *radius of feasibility* associated with \mathcal{V}_j , $j = 1, \dots, p$, as in (1) is defined to be

$$\rho(\mathcal{V}) := \sup \{ \alpha \in \mathbb{R}_+ : (P_\alpha) \text{ is feasible for } \alpha \}. \quad (2)$$

To establish the formula for the radius of robust feasibility, we first note a known useful characterization of feasibility of an infinite inequality system in terms of the closure of the convex cone generated by its set of coefficient vectors.

Lemma 1 ([9, Theorem 4.4]). *Let T be an arbitrary index set. Then, $\{x \in \mathbb{R}^n : a_t^\top x \geq b_t, t \in T\} \neq \emptyset$ if and only if $(0_n, 1) \notin \text{cl cone}\{(a_t, b_t) : t \in T\}$.*

Using the above Lemma, we first observe that the radius of robust feasibility $\rho(\mathcal{V})$ is a non-negative number since, given $j = 1, \dots, p$, $(0_n, 1) \in (\bar{a}_j, \bar{b}_j) + \alpha \mathbb{B}_{n+1}$ for a positive large enough α , in which case the corresponding problem (P_α) is not feasible.

The next result provides a formula for the radius of feasibility which involves the so-called *hypographical set* ([3]) of the system $\{\bar{a}_j^\top x \geq \bar{b}_j, j = 1, \dots, p\}$, defined as

$$H(\bar{a}, \bar{b}) := \text{conv} \{(\bar{a}_j, \bar{b}_j), j = 1, \dots, p\} + \mathbb{R}_+ \{(0_n, -1)\}, \quad (3)$$

where $\bar{a} := (\bar{a}_1, \dots, \bar{a}_p) \in (\mathbb{R}^n)^p$ and $\bar{b} := (\bar{b}_1, \dots, \bar{b}_p) \in \mathbb{R}^p$. We observe that $H(\bar{a}, \bar{b})$ is the sum of the polytope $\text{conv} \{(\bar{a}_j, \bar{b}_j), j = 1, \dots, p\}$ with the closed half-line $\mathbb{R}_+ \{(0_n, -1)\}$, so that it is a polyhedral convex set.

Lemma 2. *Let $(\bar{a}_j, \bar{b}_j) \in \mathbb{R}^n \times \mathbb{R}$, $j = 1, \dots, p$, and $\alpha \geq 0$. Suppose that*

$$(0_n, 1) \in \text{cl cone} \left(\{(\bar{a}_j, \bar{b}_j), j = 1, \dots, p\} + \alpha \mathbb{B}_{n+1} \right).$$

Then, for all $\delta > 0$, we have

$$(0_n, 1) \in \text{cone} \left(\{(\bar{a}_j, \bar{b}_j), j = 1, \dots, p\} + (\alpha + \delta) \mathbb{B}_{n+1} \right).$$

Proof. Let $\delta > 0$. To see the conclusion, we assume by contradiction that

$$(0_n, 1) \notin \text{cone} \left(\{(\bar{a}_j, \bar{b}_j), j = 1, \dots, p\} + (\alpha + \delta) \mathbb{B}_{n+1} \right).$$

Then, the separation theorem implies that there exists $(\xi, r) \in \mathbb{R}^{n+1} \setminus \{0_{n+1}\}$ such that for all $(y, s) \in \text{cone} \left(\{(\bar{a}_j, \bar{b}_j), j = 1, \dots, p\} + (\alpha + \delta) \mathbb{B}_{n+1} \right)$ one has

$$r = \langle (\xi, r), (0_n, 1) \rangle \leq 0 \leq \langle (\xi, r), (y, s) \rangle, \quad (4)$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product, i.e. $\langle (\xi, r), (y, s) \rangle = \xi^\top y + rs$. Recall that $(0_n, 1) \in \text{clcone}(\{(\bar{a}_j, \bar{b}_j), j = 1, \dots, p\} + \alpha \mathbb{B}_{n+1})$. So, there exist sequences $\{(y_k, s_k)\}_{k \in \mathbb{N}} \subset \mathbb{R}^n \times \mathbb{R}$, $\{\mu_k^j\}_{k \in \mathbb{N}} \subset \mathbb{R}_+$, and $\{(z_k^j, t_k^j)\}_{k \in \mathbb{N}} \subset \mathbb{B}_{n+1}$, $j = 1, \dots, p$, such that $(y_k, s_k) \rightarrow (0_n, 1)$ and

$$(y_k, s_k) = \sum_{j=1}^p \mu_k^j ((\bar{a}_j, \bar{b}_j) + \alpha(z_k^j, t_k^j)).$$

If $\{\sum_{j=1}^p \mu_k^j\}_{k \in \mathbb{N}}$ is a bounded sequence, by passing to subsequence if necessary, we have

$$(0_n, 1) \in \text{cone}(\{(\bar{a}_j, \bar{b}_j), j = 1, \dots, p\} + \alpha \mathbb{B}_{n+1}).$$

Thus, the claim is true whenever $\{\sum_{j=1}^p \mu_k^j\}_{k \in \mathbb{N}}$ is a bounded sequence. So, we may assume that $\sum_{j=1}^p \mu_k^j \rightarrow +\infty$ as $k \rightarrow \infty$. Let $(y, s) \in \mathbb{B}_{n+1}$ be such that $\langle (y, s), (\xi, r) \rangle = \|(\xi, r)\|$. Note that

$$\sum_{j=1}^p \mu_k^j ((\bar{a}_j, \bar{b}_j) + \alpha(z_k^j, t_k^j) - \delta(y, s)) \in \text{cone}(\{(\bar{a}_j, \bar{b}_j), j = 1, \dots, p\} + (\alpha + \delta) \mathbb{B}_{n+1}).$$

Then, (4) implies that

$$\begin{aligned} r \leq 0 &\leq \langle (\xi, r), \sum_{j=1}^p \mu_k^j ((\bar{a}_j, \bar{b}_j) + \alpha z_k^j) \rangle - \left(\sum_{j=1}^p \mu_k^j \right) \delta \|(\xi, r)\| \\ &= \langle (\xi, r), (y_k, s_k) \rangle - \left(\sum_{j=1}^p \mu_k^j \right) \delta \|(\xi, r)\|. \end{aligned}$$

Passing to the limit, we arrive to a contradiction as $(\xi, r) \neq 0_{n+1}$, $\delta > 0$, $\sum_{j=1}^p \mu_k^j \rightarrow +\infty$ and $(y_k, s_k) \rightarrow (0_n, 1)$. \square

We now provide our promised formula for the radius of robust feasibility. Observe that, since $0_{n+1} \notin H(\bar{a}, \bar{b})$ by Lemma 1, $d(0_{n+1}, H(\bar{a}, \bar{b}))$ can be computed minimizing $\|\cdot\|^2$ on $H(\bar{a}, \bar{b})$ (i.e. by solving a convex quadratic program).

Theorem 3 (Radius of robust feasibility). *For (P_α) , let $(\bar{a}_j, \bar{b}_j) \in \mathbb{R}^n \times \mathbb{R}$, $j = 1, \dots, p$, with $\{x \in \mathbb{R}^n : \bar{a}_j^\top x \geq \bar{b}_j, j = 1, \dots, p\} \neq \emptyset$. Let $\mathcal{V}_j := (\bar{a}_j, \bar{b}_j) + \alpha \mathbb{B}_{n+1}$, $j = 1, \dots, p$, and $\mathcal{V} := \prod_{j=1}^p \mathcal{V}_j$. Let $\rho(\mathcal{V})$ be the radius of robust feasibility as given in (2) and let $H(\bar{a}, \bar{b})$ be the hypographical set as given in (3). Then, $\rho(\mathcal{V}) = d(0_{n+1}, H(\bar{a}, \bar{b}))$.*

Proof. If a given $(v, w) \in (\mathbb{R}^n)^p \times \mathbb{R}^p$ is interpreted as a perturbation of $(\bar{v}, \bar{w}) \in (\mathbb{R}^n)^p \times \mathbb{R}^p$, we can measure the size of this perturbation as the supremum of the distances between the vectors of coefficients corresponding to the same index. This can be done by endowing the parameter space $(\mathbb{R}^n)^p \times \mathbb{R}^p$ with the metric \tilde{d} defined by

$$\tilde{d}((v, w), (p, q)) := \sup_{j=1, \dots, p} \|(v_j, w_j) - (p_j, q_j)\|, \text{ for } (v, w), (p, q) \in (\mathbb{R}^n)^p \times \mathbb{R}^p.$$

Let $\bar{a} \in (\mathbb{R}^n)^p$ and $\bar{b} \in \mathbb{R}^p$ be as in (3). Denote the set consisting of all inconsistent parameters by Θ_i , that is,

$$\Theta_i = \{(v, w) \in (\mathbb{R}^n)^p \times \mathbb{R}^p : \{x \in \mathbb{R}^n : v_j^\top x \geq w_j, j = 1, \dots, p\} = \emptyset\}.$$

We now show that

$$\tilde{d}((\bar{a}, \bar{b}), \Theta_i) = d(0_{n+1}, H(\bar{a}, \bar{b})). \quad (5)$$

By Lemma 1, $d(0_{n+1}, H(\bar{a}, \bar{b})) > 0$. Let $(a, b) \in H(\bar{a}, \bar{b})$ be such that $\|(a, b)\| = d(0_{n+1}, H(\bar{a}, \bar{b}))$. Then, $0_{n+1} \in H_1$ where

$$H_1 := H(\bar{a}, \bar{b}) - (a, b) = \text{conv} \{(\bar{a}_j - a, \bar{b}_j - b), j = 1, \dots, p\} + \mathbb{R}_+ \{(0_n, -1)\}.$$

So, there exist $\lambda_j \geq 0$ with $\sum_{j=1}^p \lambda_j = 1$ and $\mu \geq 0$ such that

$$0_{n+1} = \sum_{j=1}^p \lambda_j (\bar{a}_j - a, \bar{b}_j - b) + \mu (0_n, -1).$$

This shows that

$$(0_n, 1) = \sum_{j=1}^p \frac{\lambda_j}{\mu + \frac{1}{k}} (\bar{a}_j - a, \bar{b}_j - b + \frac{1}{k}), \quad k \in \mathbb{N}.$$

So, $\{x : (\bar{a}_j - a)^\top x \geq \bar{b}_j - b + \frac{1}{k}, j = 1, \dots, p\} = \emptyset$. Thus, $(\bar{a} - a, \bar{b} - b + \frac{1}{k}) \in \Theta_i$, and so, $(\bar{a} - a, \bar{b} - b) \in \text{cl } \Theta_i$. It follows that

$$\tilde{d}((\bar{a}, \bar{b}), \Theta_i) = \tilde{d}((\bar{a}, \bar{b}), \text{cl } \Theta_i) \leq \|(a, b)\| = d(0_{n+1}, H(\bar{a}, \bar{b})).$$

To see (5), we suppose on the contrary that $d((\bar{a}, \bar{b}), \Theta_i) < d(0_{n+1}, H(\bar{a}, \bar{b}))$. Then, there exist $\varepsilon_0 > 0$, with $\varepsilon_0 < \|(a, b)\|$, and $(\hat{a}, \hat{b}) \in \text{bd } \Theta_i$ such that $\tilde{d}((\bar{a}, \bar{b}), (\hat{a}, \hat{b})) = \tilde{d}((\bar{a}, \bar{b}), \Theta_i) < \|(a, b)\| - \varepsilon_0$. Then, one can find $\{(\hat{a}^k, \hat{b}^k)\}_{k \in \mathbb{N}} \subset \Theta_i$ such that $(\hat{a}^k, \hat{b}^k) \rightarrow (\hat{a}, \hat{b})$. So, Lemma 1 gives us that

$$(0_n, 1) \in \text{cl cone}\{(\hat{a}_j^k, \hat{b}_j^k) : j = 1, \dots, p\} = \text{cone}\{(\hat{a}_j^k, \hat{b}_j^k) : j = 1, \dots, p\}.$$

Thus, there exist $\lambda_j^k \geq 0$ such that $(0_n, 1) = \sum_{j=1}^p \lambda_j^k (\hat{a}_j^k, \hat{b}_j^k)$. Note that $\sum_{j=1}^p \lambda_j^k > 0$, and so,

$$0_{n+1} = \sum_{j=1}^p \frac{\lambda_j^k}{\sum_{j=1}^p \lambda_j^k} (\hat{a}_j^k, \hat{b}_j^k) + \frac{1}{\sum_{j=1}^p \lambda_j^k} (0_n, -1).$$

Then as $k \rightarrow \infty$,

$$\left\| \sum_{j=1}^p \frac{\lambda_j^k}{\sum_{j=1}^p \lambda_j^k} (\hat{a}_j, \hat{b}_j) + \frac{1}{\sum_{j=1}^p \lambda_j^k} (0_n, -1) \right\| = \left\| \sum_{j=1}^p \frac{\lambda_j^k}{\sum_{j=1}^p \lambda_j^k} (\hat{a}_j - \hat{a}_j^k, \hat{b}_j - \hat{b}_j^k) \right\| \rightarrow 0.$$

So, $0_{n+1} \in \text{cl } H(\hat{a}, \hat{b}) = H(\hat{a}, \hat{b})$. It then follows that there exist $\lambda_j \geq 0$ with $\sum_{j=1}^p \lambda_j = 1$ and $\mu \geq 0$ such that

$$0_{n+1} = \sum_{j=1}^p \lambda_j (\hat{a}_j, \hat{b}_j) + \mu (0_n, -1).$$

Thus, we have

$$\begin{aligned} & \left\| \sum_{j=1}^p \lambda_j (\bar{a}_j, \bar{b}_j) + \mu (0, -1) \right\| \\ &= \left\| \left(\sum_{j=1}^p \lambda_j (\bar{a}_j, \bar{b}_j) + \mu (0, -1) \right) - \left(\sum_{j=1}^p \lambda_j (\hat{a}_j, \hat{b}_j) + \mu (0_n, -1) \right) \right\| \\ &= \left\| \sum_{j=1}^p \lambda_j ((\bar{a}_j, \bar{b}_j) - (\hat{a}_j, \hat{b}_j)) \right\| \\ &\leq \tilde{d}((\bar{a}, \bar{b}), (\hat{a}, \hat{b})) < \|(a, b)\| - \varepsilon_0, \end{aligned}$$

where the first inequality follows from the definition of \tilde{d} and $\lambda_j \geq 0$ with $\sum_{j=1}^p \lambda_j = 1$. Note that $\sum_{j=1}^p \lambda_j (\bar{a}_j, \bar{b}_j) + \mu (0_n, -1) \in H(\bar{a}, \bar{b})$. We see that $H(\bar{a}, \bar{b}) \cap (\|(a, b)\| - \varepsilon_0) \mathbb{B}_{n+1} \neq \emptyset$. This shows that $d(0_{n+1}, H(\bar{a}, \bar{b})) \leq \|(a, b)\| - \varepsilon_0$ which contradicts the fact that $d(0_{n+1}, H(\bar{a}, \bar{b})) = \|(a, b)\|$. Therefore, (5) holds.

Let $\alpha \in \mathbb{R}_+$ so that (P_C) is feasible for α . Then, $(a, b) \in \Theta_i$ implies that $\tilde{d}((\bar{a}, \bar{b}), (a, b)) > \alpha$. Therefore, (5) gives us that $d(0_{n+1}, H(\bar{a}, \bar{b})) = \tilde{d}((\bar{a}, \bar{b}), \Theta_i) \geq \alpha$. Thus, $\rho(\mathcal{V}) \leq d(0_{n+1}, H(\bar{a}, \bar{b}))$.

We now show that $\rho(\mathcal{V}) = d(0_{n+1}, H(\bar{a}, \bar{b}))$. To see this, we proceed by the method of contradiction and suppose that $\rho(\mathcal{V}) < d(0_{n+1}, H(\bar{a}, \bar{b}))$. The, there exists $\delta > 0$ such that $\rho(\mathcal{V}) + 2\delta < d(0_{n+1}, H(\bar{a}, \bar{b}))$. Let $\alpha_0 := \rho(\mathcal{V}) + \delta$. Then, by the definition of $\rho(\mathcal{V})$, (P_{α_0}) is not feasible, that is,

$$\{x \in \mathbb{R}^n : a^\top x \geq b, (a, b) \in \bigcup_{j=1}^p \{(\bar{a}_j, \bar{b}_j) + \alpha \mathbb{B}_{n+1}\}\} = \emptyset.$$

Hence, it follows from Lemma 1 that

$$(0_n, 1) \in \text{clcone} \left\{ \bigcup_{j=1}^p \{(\bar{a}_j, \bar{b}_j) + \alpha \mathbb{B}_{n+1}\} \right\}.$$

By applying Lemma 2, we can find $\mu_j \geq 0$ and $(z_j, t_j) \in \mathbb{B}_{n+1}$, $j = 1, \dots, p$, such that

$$(0_n, 1) = \sum_{j=1}^p \mu_j \left((\bar{a}_j, \bar{b}_j) + (\alpha_0 + \delta) (z_j, t_j) \right).$$

Let $(a_j, b_j) = (\bar{a}_j, \bar{b}_j) + (\alpha_0 + \delta) (z_j, t_j)$, $j = 1, \dots, p$, $a := (a_1, \dots, a_p) \in (\mathbb{R}^n)^p$ and $b := (b_1, \dots, b_p) \in \mathbb{R}^p$. Then, $\tilde{d}((\bar{a}, \bar{b}), (a, b)) \leq \alpha_0 + \delta$ and

$$(0_n, 1) = \sum_{j=1}^p \mu_j (a_j, b_j) \in \text{cone} \{(a_j, b_j), j = 1, \dots, p\}.$$

So, Lemma 1 implies that $\{x \in \mathbb{R}^n : (a_j, b_j), j = 1, \dots, p\} = \emptyset$ and hence $(a, b) \in \Theta_i$. Thus,

$$\tilde{d}((\bar{a}, \bar{b}), \Theta_i) \leq \tilde{d}((\bar{a}, \bar{b}), (a, b)) \leq \alpha_0 + \delta = \rho(\mathcal{V}) + 2\delta.$$

Thus, from (5), we see that $d(0_{n+1}, H(\bar{a}, \bar{b})) \leq \tilde{d}((\bar{a}, \bar{b}), \Theta_i) \leq \rho(\mathcal{V}) + 2\delta$. This contradicts the fact that $\rho(\mathcal{V}) + 2\delta < d(0_{n+1}, H(\bar{a}, \bar{b}))$. So, the conclusion follows. \square

Remark 4. *We would like to note that we have given a self-contained and simple proof for Theorem 3 by exploiting the finiteness of the linear inequality system. A semi-infinite version of Theorem 3 under a regularity condition was presented in [8, Theorem 3.3], where the proof relies on several results in [3] and [4].*

In the following example we show how the radius of robust feasibility of (P_α) can be calculated using Theorem 3.

Example 5. (Calculating radius of robust feasibility) *Consider (P_α) with $n = 3$, $p = 5$ and \mathcal{V}_j^α as in (1), with*

$$\{(\bar{a}_j, \bar{b}_j), j = 1, \dots, 5\} = \left\{ \begin{pmatrix} -2 \\ -1 \\ -2 \\ -6 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ -2 \\ -6 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ -3 \end{pmatrix} \right\}. \quad (6)$$

The minimum of $\|\cdot\|^2$ on $H(\bar{a}, \bar{b})$, whose linear representation

$$\left\{ \begin{array}{l} x_1 + x_2 - x_3 \geq -1 \\ 3x_1 + 3x_2 + 3x_3 - 4x_4 \geq 9 \\ -x_1 - x_2 - x_3 \geq 1 \\ -3x_1 + x_2 + x_3 \geq -1 \\ x_1 - 3x_2 + x_3 \geq -1 \\ -x_1 - x_2 + 3x_3 \geq -3 \end{array} \right\}$$

is obtained from (3) and (6) by Fourier-Motzkin elimination, is attained at $(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -3)$. So,

$$\rho(\mathcal{V}) = \left\| \left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -3 \right) \right\| = \sqrt{\frac{28}{3}}.$$

3 Rank-1 Objective Matrix Uncertainty

In this section we assume that the matrix C in the objective function is uncertain and it belongs to the one-dimensional compact convex uncertainty set in $\mathbb{R}^{m \times n}$ given by

$$\mathcal{U} = \{ \bar{C} + \rho uv^\top : \rho \in [0, 1] \},$$

where \bar{C} is a given $m \times n$ matrix while $u \in \mathbb{R}_+^m$ and $v \in \mathbb{R}^n$ are given vectors. This data uncertainty set was introduced and examined in [11, Section 3].

Recall that the normal cone of a closed convex set X at $\bar{x} \in X$ is

$$N(X, \bar{x}) := \{ u \in \mathbb{R}^n : u^\top(x - \bar{x}) \leq 0, \forall x \in X \}.$$

Moreover, the simplex Δ_m is defined as $\Delta_m := \{ \lambda \in \mathbb{R}_+^m : \sum_{i=1}^m \lambda_i = 1 \}$. Recall that given $x, y \in \mathbb{R}^m$, we write $x \leq y$ ($x < y$) when $x_i \leq y_i$ ($x_i < y_i$, respectively) for all $i \in I := \{1, \dots, m\}$. Moreover, we write $x \leq y$ when $x \leq y$ and $x \neq y$.

Robust efficiency means in [12] and [11, Section 4], where the constraints are deterministic, the preservation of the corresponding property for all $C \in \mathcal{U}$. So, this concept is very restrictive unless the uncertainty set \mathcal{U} is small in some sense (e.g. segments emanating from \bar{C}). In our general framework of uncertain objectives and constraints the following definition of robust weak efficiency is referred to the set X of robust feasible solutions.

Definition 6 (Robust weakly efficient solution). *We say that $\bar{x} \in \mathbb{R}^n$ is a robust weakly efficient solution of (P_C) if there is no $x \in X$ such that $Cx < C\bar{x}$, for all $C \in \mathcal{U}$.*

The next characterization of the robust weakly efficient solutions in terms of multipliers involves the so-called *characteristic cone* ([9, p. 81]) of the constraint system of (P_C) , defined as

$$C(\mathcal{V}) := \text{cone} \left(\bigcup_{j=1}^p \mathcal{V}_j \right) + \mathbb{R}_+ \{(0_n, -1)\}.$$

If \mathcal{V}_j is a polytope for all $j = 1, \dots, p$, then $C(\mathcal{V})$ is generated by the extreme points of the sets \mathcal{V}_j , $j = 1, \dots, p$, together with the vector $(0_n, -1)$. So, $C(\mathcal{V})$ is a polyhedral convex cone.

If \mathcal{V}_j is a compact convex set for all $j = 1, \dots, p$ and the strict robust feasibility condition

$$\{x \in \mathbb{R}^n : a_j^\top x > b_j, \forall (a_j, b_j) \in \mathcal{V}_j, j = 1, \dots, p\} \neq \emptyset \quad (7)$$

holds, then, according to [9, Theorem 5.3 (ii)], $\text{cone} \left(\bigcup_{j=1}^p \mathcal{V}_j \right)$ is closed, and this in turn implies that $C(\mathcal{V})$ is closed too.

Theorem 7 (Robust weakly efficient solutions). *The point $\bar{x} \in X$ is a robust weakly solution of (P_C) if and only if there exist $\lambda, \tilde{\lambda} \in \Delta_m$ such that*

$$\bar{C}^\top \lambda \in -N(X, \bar{x}) \text{ and } (\bar{C} + uv^\top)^\top \tilde{\lambda} \in -N(X, \bar{x}).$$

Moreover, if \mathcal{V}_j is convex, $j = 1, \dots, p$, and $C(\mathcal{V})$ is closed, then the robust weak efficiency of $\bar{x} \in X$ is further equivalent to the condition that there exist $\lambda, \tilde{\lambda} \in \Delta_m$ and $(a_j, b_j), (\tilde{a}_j, \tilde{b}_j) \in \mathcal{V}_j$, $\mu_j, \tilde{\mu}_j \geq 0$, $j = 1, \dots, p$, such that

$$\bar{C}^\top \lambda = \sum_{j=1}^p \mu_j a_j \quad \text{and} \quad \mu_j (a_j^\top \bar{x} - b_j) = 0, j = 1, \dots, p,$$

and

$$(\bar{C} + uv^\top)^\top \tilde{\lambda} = \sum_{j=1}^p \tilde{\mu}_j \tilde{a}_j \quad \text{and} \quad \tilde{\mu}_j (\tilde{a}_j^\top \bar{x} - \tilde{b}_j) = 0, j = 1, \dots, p.$$

Proof. Let $\bar{x} \in X$ be a robust weakly efficient solution. Then, we have for each $C \in \mathcal{U}$, there exist no $x \in X$ such that $Cx < C\bar{x}$. By [7, Prop. 18 (iii)], this is equivalent to the fact that

$$(\forall C \in \mathcal{U}), (\exists \lambda \in \mathbb{R}_+^m \setminus \{0_m\})(C^\top \lambda \in -N(X, \bar{x})).$$

As $N(X, \bar{x})$ is a cone, by normalization, we may assume that $\lambda \in \Delta_m$, and so, \bar{x} is a robust weakly efficient solution if and only if

$$(\forall C \in \mathcal{U}), (\exists \lambda \in \Delta_m)(C^\top \lambda \in -N(X, \bar{x})). \quad (8)$$

To see the first assertion, it suffices to show that (8) is further equivalent to

$$(\exists \lambda, \tilde{\lambda} \in \Delta_m)(\overline{C}^\top \lambda \in -N(X, \bar{x}) \text{ and } (\overline{C} + uv^\top)^\top \tilde{\lambda} \in -N(X, \bar{x})). \quad (9)$$

To see the equivalence, we only need to show that (9) implies (8) when $u \neq 0_m$ (otherwise \mathcal{U} is a singleton set). To achieve this, suppose that (9) holds and fix an arbitrary $C \in \mathcal{U}$. Then there exists $\alpha \in [0, 1]$ such that $C = \overline{C} + \alpha uv^\top$.

Define $\tau := \frac{(1-\alpha)\tilde{\lambda}^\top u}{(1-\alpha)\tilde{\lambda}^\top u + \alpha\lambda^\top u}$ and $\gamma := \tau\lambda + (1-\tau)\tilde{\lambda} \geq 0_m$. As $\lambda, \tilde{\lambda} \in \Delta_m$ and $u \in \mathbb{R}_+^m$, we see that $\tau \in [0, 1]$ and $\gamma \in \Delta_m$. Moreover, we have

$$\begin{aligned} & \tau\alpha(uv^\top)^\top \lambda - (1-\alpha)(1-\tau)(uv^\top)^\top \tilde{\lambda} \\ = & \frac{(1-\alpha)\tilde{\lambda}^\top u}{(1-\alpha)\tilde{\lambda}^\top u + \alpha\lambda^\top u} \alpha(uv^\top)^\top \lambda - \frac{\alpha\lambda^\top u}{(1-\alpha)\tilde{\lambda}^\top u + \alpha\lambda^\top u} (1-\alpha)(uv^\top)^\top \tilde{\lambda} \\ = & \frac{(1-\alpha)\tilde{\lambda}^\top u}{(1-\alpha)\tilde{\lambda}^\top u + \alpha\lambda^\top u} \alpha(u^\top \lambda)v - \frac{\alpha\lambda^\top u}{(1-\alpha)\tilde{\lambda}^\top u + \alpha\lambda^\top u} (1-\alpha)(u^\top \tilde{\lambda})v = 0_m. \end{aligned} \quad (10)$$

Now,

$$\begin{aligned} C^\top \gamma &= (\overline{C} + \alpha uv^\top)^\top (\tau\lambda + (1-\tau)\tilde{\lambda}) \\ &= \tau\overline{C}^\top \lambda + \tau\alpha(uv^\top)^\top \lambda + (1-\tau)(\overline{C} + \alpha uv^\top)^\top \tilde{\lambda} \\ &= \tau\overline{C}^\top \lambda + \tau\alpha(uv^\top)^\top \lambda + (1-\tau)(\overline{C} + uv^\top)^\top \tilde{\lambda} - (1-\alpha)(1-\tau)(uv^\top)^\top \tilde{\lambda} \\ &= \tau\overline{C}^\top \lambda + (1-\tau)(\overline{C} + uv^\top)^\top \tilde{\lambda} \in N(X, \bar{x}). \end{aligned}$$

where the fourth equality follows from (10) and the last relation follows from (9) and the convexity of $N(X, \bar{x})$.

To see the second assertion, we assume that \mathcal{V}_j is convex, $j = 1, \dots, p$, and $C(\mathcal{V})$ is closed. We only need to show

$$N(X, \bar{x}) = \left\{ -\sum_{j=1}^p \mu_j a_j : (a_j, b_j) \in \mathcal{V}_j, \mu_j \geq 0 \text{ and } \mu_j(a_j^\top \bar{x} - b_j) = 0, j = 1, \dots, p \right\}.$$

The system $\{a^\top x \geq b, (a, b) \in T\}$, with $T = \left(\bigcup_{j=1}^p \mathcal{V}_j\right)$, is a linear representation of X . Thus, $u \in N(X, \bar{x})$ if and only if the inequality $-u^\top x \geq$

$-u^\top \bar{x}$ is consequence of $\{a^\top x \geq b, (a, b) \in T\}$ if and only if (by the Farkas Lemma, [9, Corollary 3.1.2])

$$-(u, u^\top \bar{x}) \in \text{cone}T + \mathbb{R}_+ \{(0_n, -1)\}.$$

This is equivalent to assert the existence of a finite subset S of T , corresponding non-negative scalars λ_s , $s \in S$, and $\mu \geq 0$, such that

$$-(u, u^\top \bar{x}) = \sum_{(a,b) \in S} \lambda_{(a,b)} (a, b) + \mu (0_n, -1). \quad (11)$$

Multiplying by $(\bar{x}, -1)$ both members of (11) we get $\mu = 0$, so that (11) is equivalent to

$$-u = \sum_{(a,b) \in S} \lambda_{(a,b)} a \text{ and } \lambda_{(a,b)} (a^\top \bar{x} - b) = 0, (a, b) \in S. \quad (12)$$

Finally, since $S \subset \bigcup_{j=1}^p \mathcal{V}_j$, we can write $S = \bigcup_{j=1}^p S_j$, with $S_j \subset \mathcal{V}_j$, $j = 1, \dots, p$, and $S_i \cap S_j = \emptyset$ when $i \neq j$. Let $\mu_j := \sum_{(a,b) \in S_j} \lambda_{(a,b)}$, $j = 1, \dots, p$. If $\mu_j \neq 0$ one has, by convexity of \mathcal{V}_j ,

$$(a_j, b_j) := \frac{\sum_{(a,b) \in S_j} \lambda_{(a,b)} (a,b)}{\mu_j} \in \mathcal{V}_j.$$

Take $(a_j, b_j) \in \mathcal{V}_j$ arbitrarily when $\mu_j = 0$. Then we get from (12) that

$$-u = \sum_{j=1}^p \mu_j a_j \text{ and } \mu_j (a_j^\top \bar{x} - b_j) = 0, j = 1, \dots, p.$$

Thus, the conclusion follows. \square

In the definition of the rank-1 objective data uncertainty set, $\mathcal{U} = \{\bar{C} + \rho uv^\top : \rho \in [0, 1]\}$, we require that $u \in \mathbb{R}_+^m$. The following example (inspired in [11, Example 3.3]) illustrates that if this non-negativity requirement is dropped, then the above solution characterization in Theorem 7 may fail.

Example 8 (Non-negativity requirement for rank-1 objective data uncertainty). *Let*

$$\bar{C} = \begin{pmatrix} -3 & -1 & -2 \\ 0 & -1 & -2 \end{pmatrix}, \quad u = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \notin \mathbb{R}_+^2 \quad \text{and} \quad v = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix}.$$

Consider the uncertain multiobjective optimization problem

$$V\text{-min} \{Cx : a_j^\top x \geq b_j, \forall (a_j, b_j) \in \mathcal{V}_j, j = 1, \dots, 4\}, \quad (13)$$

where the objective data matrix C is an element of

$$\{\bar{C} + \rho uv^\top : \rho \in [0, 1]\} = \left\{ \begin{pmatrix} -3 & -1 & -2 \\ 0 & -1 & -2 \end{pmatrix} + \rho \begin{pmatrix} 0 & 3 & 0 \\ 0 & -3 & 0 \end{pmatrix} : \rho \in [0, 1] \right\}$$

and the uncertainty sets for the constraints are the convex polytopes

$$\mathcal{V}_1 = \text{conv} \left\{ \begin{pmatrix} -2 \\ -1 \\ -2 \\ -6 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ -2 \\ -6 \end{pmatrix} \right\} \text{ and } \mathcal{V}_2 = \text{conv} \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ -3 \end{pmatrix} \right\}.$$

Note that the robust feasible set is

$$X = \{x \in \mathbb{R}^n : a_j^\top x \geq b_j, \forall (a_j, b_j) \in \mathcal{V}_j, j = 1, 2\} = \{\bar{a}_j^\top x \geq \bar{b}_j, j = 1, \dots, 5\},$$

where $\{\bar{a}_j^\top x \geq \bar{b}_j, j = 1, \dots, 5\}$ is the set in (6). It can be checked that $\bar{x} = (1, 1, 3/2) \in X$ and so,

$$N(X, \bar{x}) = \left\{ \mu_1 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \mu_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} : \mu_1 \geq 0, \mu_2 \geq 0 \right\}.$$

Let $\lambda = (2/3, 1/3)^\top$ and $\tilde{\lambda} = (1/3, 2/3)^\top$. Then, we have

$$\bar{C}^\top \lambda \in -N(X, \bar{x}) \text{ and } (\bar{C} + uv^\top)^\top \tilde{\lambda} \in -N(X, \bar{x}).$$

On the other hand, for

$$C = \begin{pmatrix} -3 & -1 & -2 \\ 0 & -1 & -2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 3 & 0 \\ 0 & -3 & 0 \end{pmatrix} = \begin{pmatrix} -3 & \frac{1}{2} & -2 \\ 0 & -\frac{5}{2} & -2 \end{pmatrix} \in \mathcal{U},$$

and $x = (0, 0, 3)^\top \in X$, we see that

$$Cx = \begin{pmatrix} -6 \\ -6 \end{pmatrix} < \begin{pmatrix} -\frac{11}{2} \\ -\frac{11}{2} \end{pmatrix} = C\bar{x}.$$

So, \bar{x} is not a weakly efficient solution of (13). Thus, the above solution characterization fails.

In the case where the constraints are uncertainty free, i.e. the sets \mathcal{V}_j are all singletons, we obtain the following solution characterization for robust multiobjective optimization problem with rank-one objective uncertainty.

Corollary 9. Let $\mathcal{V}_j = \{(\bar{a}_j, \bar{b}_j)\}$, $j = 1, \dots, p$, and $\bar{x} \in X$. Then, the following statements are equivalent:

- (i) \bar{x} is a robust weakly efficient solution;
- (ii) there exist $\lambda, \tilde{\lambda} \in \Delta_m$ such that

$$\bar{C}^\top \lambda \in -N(X, \bar{x}) \text{ and } (\bar{C} + uv^\top)^\top \tilde{\lambda} \in -N(X, \bar{x});$$

- (iii) there exist $\lambda, \tilde{\lambda} \in \Delta_m$ and $\mu_j, \tilde{\mu}_j \geq 0$, $j = 1, \dots, p$, such that

$$\bar{C}^\top \lambda = \sum_{j=1}^p \mu_j \bar{a}_j \text{ and } \mu_j (\bar{a}_j^\top \bar{x} - \bar{b}_j) = 0, j = 1, \dots, p,$$

and

$$(\bar{C} + uv^\top)^\top \tilde{\lambda} = \sum_{j=1}^p \tilde{\mu}_j \bar{a}_j \text{ and } \tilde{\mu}_j (\bar{a}_j^\top \bar{x} - \bar{b}_j) = 0, j = 1, \dots, p;$$

- (iv) \bar{x} is a weakly efficient solution for the problems

$$(P_0) \quad V\text{-min} \quad \bar{C}x \\ \text{s.t.} \quad \bar{a}_j^\top x \geq \bar{b}_j, j = 1, \dots, p,$$

and

$$(P_1) \quad V\text{-min} \quad (\bar{C} + uv^\top)x \\ \text{s.t.} \quad \bar{a}_j^\top x \geq \bar{b}_j, j = 1, \dots, p.$$

Proof. Let $\mathcal{V}_j = \{(\bar{a}_j, \bar{b}_j)\}$, $j = 1, \dots, p$. The equivalences (i) \Leftrightarrow (ii) \Leftrightarrow (iii) come from Theorem 7, taking into account that all the uncertainty sets \mathcal{V}_j are polytopes. Note that (i) \Rightarrow (iv) always holds. Finally, the implication (iv) \Rightarrow (ii) is immediate by the usual characterization for weakly efficient solutions (e.g. see [7, Prop. 18(iii)]). Thus, the conclusion follows. \square

Remark 10. The equivalence (i) \Leftrightarrow (iii) in Corollary 9, on robust weakly efficient solutions of uncertain vector linear programming problems, can be seen as a counterpart of [11, Theorem 3.1], on robust efficient solutions of the same type of problems.

4 Tractable Classes of Robust Multi-Objective LPs

In this Section, we provide various classes of commonly used uncertainty sets determining the robust feasible set

$$X = \{x \in \mathbb{R}^n : a_j^\top x \geq b_j, \forall (a_j, b_j) \in \mathcal{V}_j, j = 1, \dots, p\},$$

under which one can numerically check whether a robust feasible point is a robust weakly efficient solution or not. Throughout this Section we assume that the objective function of (P_C) satisfies the rank-1 matrix data uncertainty, as defined in Section 3. We begin with the simple box constraint data uncertainty.

4.1 Box constraint data Uncertainty

Consider

$$\mathcal{V}_j = [\underline{a}_j, \bar{a}_j] \times [\underline{b}_j, \bar{b}_j], \quad (14)$$

where $\underline{a}_j, \bar{a}_j \in \mathbb{R}^n$ and $\underline{b}_j, \bar{b}_j \in \mathbb{R}$, $j = 1, \dots, p$. Denote the extreme points of $[\underline{a}_j, \bar{a}_j]$ by $\{\hat{a}_j^{(1)}, \dots, \hat{a}_j^{(2^n)}\}$.

Theorem 11. *Let \mathcal{V}_j be as in (14), $j = 1, \dots, p$. The point $\bar{x} \in X$ is a robust weakly efficient solution of (P_C) if and only if there exist $\lambda, \tilde{\lambda} \in \Delta_m$ and $\mu_j^{(l)}, \tilde{\mu}_j^{(l)} \geq 0$ such that*

$$\bar{C}^\top \lambda = \sum_{j=1}^p \sum_{l=1}^{2^n} \mu_j^{(l)} \hat{a}_j^{(l)} \quad \text{and} \quad \mu_j^{(l)} \left((\hat{a}_j^{(l)})^\top \bar{x} - \bar{b}_j \right) = 0, \quad j = 1, \dots, p, l = 1, \dots, 2^n,$$

and

$$(\bar{C} + uv^\top)^\top \tilde{\lambda} = \sum_{j=1}^p \sum_{l=1}^{2^n} \tilde{\mu}_j^{(l)} \hat{a}_j^{(l)} \quad \text{and} \quad \tilde{\mu}_j^{(l)} \left((\hat{a}_j^{(l)})^\top \bar{x} - \bar{b}_j \right) = 0, \quad j = 1, \dots, p, l = 1, \dots, 2^n.$$

Proof. Let \bar{x} be a robust weakly efficient solution of (P_C) . Note that X can be rewritten as

$$\begin{aligned} X &= \{x \in \mathbb{R}^n : a_j^\top x - b_j \geq 0 \text{ for all } (a_j, b_j) \in [\underline{a}_j, \bar{a}_j] \times [\underline{b}_j, \bar{b}_j]\} \\ &= \left\{ x \in \mathbb{R}^n : (a_j^{(l)})^\top x - \bar{b}_j \geq 0, l = 1, \dots, 2^n, j = 1, \dots, p \right\}. \end{aligned}$$

Then, we have

$$N(X, \bar{x}) = \left\{ - \sum_{j=1}^p \sum_{l=1}^{2^n} \mu_j^{(l)} \hat{a}_j^{(l)} : \mu_j^{(l)} \left((\hat{a}_j^{(l)})^\top \bar{x} - \bar{b}_j \right) = 0, \mu_j^{(l)} \geq 0, \forall l, \forall j \right\}.$$

Since \mathcal{V}_j is a convex polytope for $j = 1, \dots, p$, the conclusion follows from Theorem 7. \square

It is worth noting, from Theorem 11, that one can determine whether or not a given robust feasible point \bar{x} of (P_C) under the box constraint data uncertainty is a robust weakly efficient solution by solving finitely many linear equalities.

4.2 Norm constraint data uncertainty

Consider the constraint data uncertainty set

$$\mathcal{V}_j = \{\bar{a}_j + \delta_j \bar{v}_j : \bar{v}_j \in \mathbb{R}^n, \|Z_j \bar{v}_j\|_s \leq 1\} \times [\underline{b}_j, \bar{b}_j], \quad (15)$$

where $\bar{a}_j \in \mathbb{R}^n$, $\bar{b}_j \in \mathbb{R}$, Z_j is an invertible symmetric $(n \times n)$ matrix, $j = 1, \dots, p$, and let $\|\cdot\|_s$ denote the s -norm, $s \in [1, +\infty]$, defined by

$$\|x\|_s = \begin{cases} \sqrt[s]{\sum_{i=1}^n |x_i|^s} & \text{if } s \in [1, +\infty), \\ \max\{|x_i| : 1 \leq i \leq n\} & \text{if } s = +\infty. \end{cases}$$

Moreover, we define $s^* \in [1, +\infty]$ to be the number so that $\frac{1}{s} + \frac{1}{s^*} = 1$. The following simple facts about s -norms will be used later on. First, the dual norm of the s -norm is the s^* -norm, that is,

$$\sup_{\|x\|_s \leq 1} u^\top x = \|u\|_{s^*} \text{ for all } u \in \mathbb{R}^n.$$

Second, $\partial(\|\cdot\|_{s^*})(u) = \{v : \|v\|_s \leq 1, v^\top u = \|u\|_{s^*}\}$ where $\partial f(x)$ denotes the usual convex subdifferential of a convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at $x \in \mathbb{R}^n$, i.e.

$$\partial f(x) = \{z \in \mathbb{R}^n : z^\top (y - x) \leq f(y) - f(x) \forall y \in \mathbb{R}^n\}.$$

In this case, we have the following characterization of robust weakly efficient solutions.

Theorem 12. *Let \mathcal{V}_j be as in (15), $j = 1, \dots, p$, and suppose that there exists $x_0 \in \mathbb{R}^n$ such that*

$$\bar{a}_j^\top x_0 - \bar{b}_j - \delta \|Z_j^{-1} x_0\|_{s^*} > 0, j = 1, \dots, p. \quad (16)$$

Then, a point $\bar{x} \in X$ is a robust weakly efficient solution of (P_C) if and only if there exist $\lambda, \tilde{\lambda} \in \Delta_m$, $\mu, \tilde{\mu} \in \mathbb{R}_+^p$ and $w_j, \tilde{w}_j \in \mathbb{R}^n$, with $\|w_j\|_s \leq 1$ and $\|\tilde{w}_j\|_s \leq 1$, such that

$$-\lambda^\top \bar{C} \bar{x} = \sum_{j=1}^p \mu_j \bar{b}_j \quad \text{and} \quad \bar{C}^\top \lambda + \sum_{j=1}^p \mu_j (\bar{a}_j - \delta Z_j^{-1} w_j) = 0_n.$$

and

$$-\tilde{\lambda}^\top (\bar{C} + uv^\top) \bar{x} = \sum_{j=1}^p \tilde{\mu}_j \bar{b}_j \quad \text{and} \quad (\bar{C} + uv^\top)^\top \tilde{\lambda} + \sum_{j=1}^p \tilde{\mu}_j (\bar{a}_j - \delta Z_j^{-1} \tilde{w}_j) = 0_n.$$

Proof. Note that X can be rewritten as

$$\begin{aligned} X &= \{x \in \mathbb{R}^n : \bar{a}_j^\top x - b_j + \delta(\bar{v}_j)^\top x \geq 0 \text{ for all } \|Z_j \bar{v}_j\|_s \leq 1, b_j \in [\underline{b}_j, \bar{b}_j], j = 1, \dots, p\} \\ &= \{x \in \mathbb{R}^n : \bar{a}_j^\top x - b_j + \delta(Z_j^{-1} \bar{u}_j)^\top x \geq 0 \text{ for all } \|\bar{u}_j\|_s \leq 1, b_j \in [\underline{b}_j, \bar{b}_j], j = 1, \dots, p\} \\ &= \{x \in \mathbb{R}^n : \bar{a}_j^\top x - \bar{b}_j - \delta\|Z_j^{-1}x\|_{s^*} \geq 0, j = 1, \dots, p\}. \end{aligned}$$

Since \mathcal{V}_j is a compact convex set for $j = 1, \dots, p$ and the strict robust feasibility condition (7) holds as a consequence of (16), the conclusion will follow from Theorem 7 if we show that

$$N(X, \bar{x}) = \left\{ u : \exists \mu_j \geq 0, \|w_j\|_{s^*} \leq 1 \text{ s.t. } -u^\top \bar{x} = \sum_{j=1}^p \mu_j \bar{b}_j \text{ and } u + \sum_{j=1}^p \mu_j (\bar{a}_j - \delta Z_j^{-1} w_j) = 0_n \right\}.$$

To see this, let $u \in N(X, \bar{x})$. Then, \bar{x} is a solution of the following convex optimization problem:

$$\min \{-u^\top x : \bar{a}_j^\top x - \bar{b}_j - \delta\|Z_j^{-1}x\|_{s^*} \geq 0, j = 1, \dots, p\}$$

As the strict feasibility condition (16) holds, by the Lagrangian duality, there exist $\mu_j \geq 0, j = 1, \dots, p$, such that

$$-u^\top \bar{x} = \min_{x \in \mathbb{R}^n} \left\{ (-u^\top x) + \sum_{j=1}^p \mu_j (-\bar{a}_j^\top x + \bar{b}_j + \delta\|Z_j^{-1}x\|_{s^*}) \right\}.$$

As $\bar{x} \in X$, this implies that $\mu_j (-\bar{a}_j^\top \bar{x} + \bar{b}_j + \delta\|Z_j^{-1}\bar{x}\|_{s^*}) = 0, j = 1, \dots, p$, and so, the function $h(x) := (-u^\top x) + \sum_{j=1}^p \mu_j (-\bar{a}_j^\top x + \bar{b}_j + \delta\|Z_j^{-1}x\|_{s^*})$ attains its minimum on X at \bar{x} and $\min_{x \in \mathbb{R}^n} h(x) = -u^\top \bar{x}$. This implies that $0_n \in \partial h(\bar{x})$, and so, there exist $w_j \in \mathbb{R}^n$ with $\|w_j\|_s \leq 1$ such that $w_j^\top (Z_j^{-1} \bar{x}) = \|Z_j^{-1} \bar{x}\|_{s^*}$ and

$$u + \sum_{j=1}^p \mu_j (\bar{a}_j - \delta Z_j^{-1} w_j) = 0_n.$$

This together with $h(\bar{x}) = -u^\top \bar{x}$ gives us that $-u^\top \bar{x} = \sum_{j=1}^p \mu_j \bar{b}_j$. Then, we have

$$N(X, \bar{x}) \subset \left\{ u : \exists \mu_j \geq 0, \|w_j\|_{s^*} \leq 1 \text{ s.t. } -u^\top \bar{x} = \sum_{j=1}^p \mu_j \bar{b}_j \text{ and } u + \sum_{j=1}^p \mu_j (\bar{a}_j - \delta Z_j^{-1} w_j) = 0_n \right\}$$

To see the reverse inclusion, let $u \in \mathbb{R}^n$ with $-u^\top \bar{x} = \sum_{j=1}^p \mu_j \bar{b}_j$ and $u + \sum_{j=1}^p \mu_j (\bar{a}_j - \delta Z_j^{-1} w_j) = 0_n$ for some $\mu_j \geq 0, \|w_j\|_{s^*} \leq 1$. Then, for all

$x \in X$,

$$\begin{aligned}
u^\top(x - \bar{x}) &= u^\top x + \sum_{j=1}^p \mu_j \bar{b}_j = - \left(\sum_{j=1}^p \mu_j (\bar{a}_j - \delta Z_j^{-1} w_j) \right)^\top x + \sum_{j=1}^p \mu_j \bar{b}_j \\
&= \sum_{j=1}^p \mu_j (-\bar{a}_j^\top x + \bar{b}_j + \delta (Z_j^{-1} w_j)^\top x) \\
&\leq \sum_{j=1}^p \mu_j (-\bar{a}_j^\top x + \bar{b}_j + \delta \|Z_j^{-1} x\|_{s^*}) \leq 0,
\end{aligned}$$

where the inequality follows from $\|w_j\|_{s^*} = \max_{\|u\|_s \leq 1} w_j^\top u$ (and hence, $w_j^\top v \leq \|w_j\|_{s^*} \|v\|_s \leq \|v\|_s$ for all $v \in \mathbb{R}^n$). So, $u \in N(X, \bar{x})$ and hence, the conclusion follows. \square

Theorem 12 shows that one can determine whether a robust feasible point \bar{x} under norm data uncertainty is a robust weakly efficient solution or not by solving finitely many sth-order cone systems (that is, linear equations where the variable lies in the ball determined by the $\|\cdot\|_s$ -norm) as long as the strict feasibility condition (16) is satisfied.

4.3 Ellipsoidal constraint data uncertainty

In this subsection we consider the case where the constraint data are uncertain and belong to the ellipsoidal constraint data uncertainty sets

$$\mathcal{V}_j = \left\{ \bar{a}_j^0 + \sum_{l=1}^{q_j} v_j^l \bar{a}_j^l : \|(v_j^1, \dots, v_j^{q_j})\| \leq 1 \right\} \times [\underline{b}_j, \bar{b}_j], \quad (17)$$

where $\bar{a}_j^l \in \mathbb{R}^n$, $l = 0, 1, \dots, q_j$, $q_j \in \mathbb{N}$ and $\underline{b}_j, \bar{b}_j \in \mathbb{R}$, $j = 1, \dots, p$.

Theorem 13. *Let \mathcal{V}_j , $j = 1, \dots, p$, be as in (17) and suppose that there exists $x_0 \in \mathbb{R}^n$ such that*

$$(\bar{a}_j^0)^\top x_0 - \bar{b}_j - \|((\bar{a}_j^1)^\top x_0, \dots, (\bar{a}_j^{q_j})^\top x_0)\| > 0, \quad j = 1, \dots, p. \quad (18)$$

Then, a point $\bar{x} \in X$ is a robust weakly efficient solution of (P_C) if and only if there exist $\lambda, \tilde{\lambda} \in \Delta_m$, $\mu, \tilde{\mu} \in \mathbb{R}_+^p$ and $w, \tilde{w} \in \mathbb{R}^n$ with $\|w\| \leq 1$ and $\|\tilde{w}\| \leq 1$ such that

$$-\lambda^\top \bar{C} \bar{x} = \sum_{j=1}^p \mu_j \bar{b}_j \quad \text{and} \quad -\bar{C}^\top \lambda - \sum_{j=1}^p \mu_j (\bar{a}_j^0 - y_j) = 0_m$$

and

$$-\lambda^\top (\bar{C} + uv^\top) \bar{x} = \sum_{j=1}^p \mu_j \bar{b}_j \quad \text{and} \quad -(\bar{C} + uv^\top)^\top \lambda - \sum_{j=1}^p \mu_j (\bar{a}_j^0 - y_j) = 0_m,$$

where $y_j = ((\bar{a}_j^1)^\top w, \dots, (\bar{a}_j^{q_j})^\top w)^\top$.

Proof. Note that X can be rewritten as

$$\begin{aligned} X &= \{x \in \mathbb{R}^n : (\bar{a}_j^0)^\top x - b_j + \sum_{l=1}^{q_j} v_j^l (\bar{a}_j^l)^\top x \geq 0 \text{ for all} \\ &\quad \|(v_j^1, \dots, v_j^{q_j})\| \leq 1, b_j \in [\underline{b}_j, \bar{b}_j], j = 1, \dots, p\} \\ &= \{x \in \mathbb{R}^n : (\bar{a}_j^0)^\top x - \bar{b}_j - \|((\bar{a}_j^1)^\top x, \dots, (\bar{a}_j^{q_j})^\top x)\| \geq 0, j = 1, \dots, p\}. \end{aligned}$$

The conclusion will follow from Theorem 7 if we show that

$$N(X, \bar{x}) = \left\{ u \in \mathbb{R}^n : \exists \mu_j \geq 0, \|w\| \leq 1 \text{ s.t. } -u^\top \bar{x} = \sum_{j=1}^p \mu_j \bar{b}_j \text{ and } -u - \sum_{j=1}^p \mu_j (\bar{a}_j^0 - y_j) = 0_m \right\}$$

To see this, let $u \in N(X, \bar{x})$. Then, \bar{x} is a solution of the following convex optimization problem:

$$\min \{-u^\top x : (\bar{a}_j^0)^\top x - \bar{b}_j - \|((\bar{a}_j^1)^\top x, \dots, (\bar{a}_j^{q_j})^\top x)\| \geq 0, j = 1, \dots, p\}.$$

As the strict feasibility condition (18) holds, by the Lagrangian duality, there exist $\mu_j \geq 0$, $j = 1, \dots, p$, such that

$$-u^\top \bar{x} = \min_{x \in \mathbb{R}^n} \left\{ (-u^\top x) + \sum_{j=1}^p \mu_j (-(\bar{a}_j^0)^\top x + \bar{b}_j + \|((\bar{a}_j^1)^\top x, \dots, (\bar{a}_j^{q_j})^\top x)\|) \right\}.$$

As $\bar{x} \in X$, this implies that $\mu_j ((\bar{a}_j^0)^\top \bar{x} - \bar{b}_j - \|((\bar{a}_j^1)^\top \bar{x}, \dots, (\bar{a}_j^{q_j})^\top \bar{x})\|) = 0$, $j = 1, \dots, p$, and so, the function $h(x) := (-u^\top x) + \sum_{j=1}^p \mu_j (-(\bar{a}_j^0)^\top x + \bar{b}_j + \|((\bar{a}_j^1)^\top x, \dots, (\bar{a}_j^{q_j})^\top x)\|)$ attains its minimum at \bar{x} and $\min_{x \in \mathbb{R}^n} h(x) = -u^\top \bar{x}$. This implies that $0_n \in \partial h(\bar{x})$, and so, there exists $w \in \mathbb{R}^n$ with $\|w\| \leq 1$ such that

$$-u^\top \bar{x} = \sum_{j=1}^p \mu_j \bar{b}_j \quad \text{and} \quad -u - \sum_{j=1}^p \mu_j \bar{a}_j^0 + \sum_{j=1}^p \mu_j y_j = 0_n,$$

where $y_j = ((\bar{a}_j^1)^\top w, \dots, (\bar{a}_j^{q_j})^\top w)^\top$. Then, we have

$$N(X, \bar{x}) \subset \left\{ u \in \mathbb{R}^n : \exists \mu_j \geq 0, \|w\| \leq 1 \text{ s.t. } -u^\top \bar{x} = \sum_{j=1}^p \mu_j \bar{b}_j \text{ and } -u - \sum_{j=1}^p \mu_j (\bar{a}_j^0 - y_j) = 0_n \right\}$$

To see the reverse inclusion, let $u \in \mathbb{R}^n$ be such that $-u^\top \bar{x} = \sum_{j=1}^p \mu_j \bar{b}_j$ and $-u - \sum_{j=1}^p \mu_j (\bar{a}_j^0 - y_j) = 0_n$ for some $\mu_j \geq 0$, $j = 1, \dots, p$, and $\|u\| \leq 1$. Then, for all $x \in X$,

$$u^\top (x - \bar{x}) = \sum_{j=1}^p \mu_j \bar{b}_j - \sum_{j=1}^p \mu_j (\bar{a}_j^0 - y_j)^\top x \leq \sum_{j=1}^p \mu_j (-(\bar{a}_j^0)^\top x + \bar{b}_j + \|((\bar{a}_j^1)^\top x, \dots, (\bar{a}_j^{q_j})^\top x)\|) \leq 0.$$

Thus, $u \in N(X, \bar{x})$ and so, the conclusion follows. \square

The above robust solution characterization under the constraint ellipsoidal data uncertainty shows that one can determine whether a robust feasible point is a robust weakly efficient solution point or not by solving finitely many second order cone systems as long as the strict robust feasibility condition (18) is satisfied.

Finally, it should be noted that there are other approaches in defining robust solutions for uncertain multiobjective optimization when the data uncertainty $\mathcal{U} \subset \mathbb{R}^{m \times n}$ in the objective matrix is a columnwise objective data uncertainty, that is, $\mathcal{U} = \prod_{i=1}^m \mathcal{U}_i$ where $\mathcal{U}_i \subset \mathbb{R}^n$. In this case, one can define a robust solution of the uncertain multi-objective optimization problem as the solution of the following deterministic multiobjective optimization problem

$$V\text{-min} \left\{ \left(\max_{c_1 \in \mathcal{U}_1} c_1^\top x, \dots, \max_{c_m \in \mathcal{U}_m} c_m^\top x \right) : a_j^\top x \geq b_j, \forall (a_j, b_j) \in \mathcal{V}_j, j = 1, \dots, p \right\}.$$

This approach has been recently examined in the paper [8] for uncertain multiobjective optimization with semi-infinite constraints under columnwise objective data uncertainty.

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References

- [1] A. Ben-Tal, L. El Ghaoui, A. Nemirovski, Robust optimization, Princeton U.P., Princeton, 2009.
- [2] J. Branke, K. Deb, K. Miettinen, R. Slowinski (eds.), Multiobjective optimization: interactive and evolutionary approaches, Lect. Notes in Computer Sci. 5252, Springer, 2008.

- [3] M.J. Cánovas, M.A. López, J. Parra, F.J. Toledo, Distance to ill-posedness and the consistency value of linear semi-infinite inequality systems, *Math Programming* 103A (2005) 95-126.
- [4] M.J. Cánovas, M.A. López, J. Parra, F.J. Toledo, Distance to ill-posedness for linear inequality systems under block perturbations: convex and infinite-dimensional cases, *Optimization* 60 (2011) 925-946.
- [5] M. Ehrgott, *Multicriteria optimization*, 2nd Ed., Springer, Berlin, 2005.
- [6] L. El Ghaoui, H. Lebret, Robust solutions to least-squares problems with uncertain data, *SIAM J. Matrix Anal. Appl.* 18 (1997) 1035-1064.
- [7] M.A. Goberna, F. Guerra-Vázquez, M.I. Todorov, Constraint qualifications in linear vector semi-infinite optimization, *European J. Oper. Res.* 227 (2013) 12-21.
- [8] M.A. Goberna, V. Jeyakumar, G. Li, J. Vicente-Pérez, Robust solutions of uncertain multi-objective linear semi-infinite programming, Preprint, Department of Applied Mathematics, University of New South Wales, Sydney.
- [9] M.A. Goberna, M.A. López, *Linear semi-infinite optimization*, Wiley, Chichester, 1998.
- [10] B.L. Gorissen, D. den Hertog, Approximating the Pareto set of multiobjective linear programs via robust optimization, *Oper. Res. Letters* 40 (2012) 319-324.
- [11] Gr.G. Pando, D.T. Luc, P. Pardalos, Robust aspects of solutions in deterministic multiple objective linear programming, *European J. Oper. Res.* 229 (2013) 29-36.
- [12] S. Sitarz, Postoptimal analysis in multicriteria linear programming, *European J. Oper. Res.* 191 (2008) 7-18.