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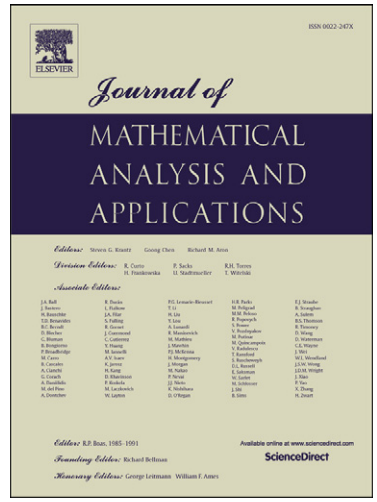

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On geometry of cones and some applications

F. García Castaño[a,*], M. A. Melguizo Padial[a], V. Montesinos[b]

[a] *Departamento de Matemática Aplicada*
Escuela Politécnica Superior
Universidad de Alicante
03080 San Vicente del Raspeig, Alicante, Spain
[b] *Instituto de Matemática Pura y Aplicada*
Universitat Politècnica de València
C/Vera, S/N, 46022 Valencia, Spain.

Abstract

In this work we prove that in any normed space, the origin is a denting point of a pointed cone if and only if it is a point of continuity for the cone and the closure of the cone in the bidual space respect to the weak* topology is pointed. Other related results and consequences are also stated. For example, a criterion to know whether a cone has a bounded base, an unbounded base, or does not have any base; and a result on the existence of super efficient points in weakly compact sets.

Keywords: Denting points, points of continuity, bases for cones, quasi interior points, strictly positive functionals, quasi relative interior
2010 MSC: 46B20, 46B22, 46B40

1. Introduction

The notion of denting point goes back to the early studies of sets with the Radon-Nikodým property in [3]. It has also been applied to renorming theory (e.g. [8] and the references therein) and to optimization ([6]). The notion of point of continuity is a generalization of the former one. It was initially used to provide a geometric proof of the Ryll-Nardzewski fixed point theorem in [16]. Later on, it was used for geometric purposes in [3], and it was applied

*corresponding author
Email addresses: `Fernando.gc@ua.es` (F. García Castaño), `ma.mp@ua.es` (M. A. Melguizo Padial), `vmontesinos@mat.upv.es` (V. Montesinos)

to optimization in [9]. B. L. Lin, P. K. Lin, and S. Troyanski showed in [15] that both notions become equivalent at extreme points of closed, convex, and bounded subsets of Banach spaces (see also [21, Proposition 3.3]).

Regarding cones, X. H. Gong asked in [9, Conclusions] a question which can be restated in the following way: *The property that the origin in a normed space be a point of continuity for a closed and pointed cone, is really weaker than that the origin be a denting point of the cone?* (The original statement was stated in terms of bounded bases instead of denting points). Later on, A. Daniilidis asked negatively such a question (into the frame of Banach spaces) noting the following consequence of the theorem of Lin-Lin-Troyanski, [6, Corollary 2]: *given a closed and pointed cone C in a Banach space X, the origin (0_X for short) is a denting point of C if and only if it is a point of continuity for C.* In addition, the former characterization allowed Daniilidis to prove the equivalence (into the frame of Banach spaces) between two density results of Arrow, Barankin and Blackwell's type, one due to M. Petschke [18, Corollary 4.2] and another due to Gong [9, Theorem 3.2 (a)].

Daniilidis' characterization [6, Corollary 2] is not true for non closed cones, as Example 1.5 in the next subsection shows. Thus, the answer to Gong's question is positive for non closed cones. In this line, C. Kountzakis and I. A. Polyrakis showed the following result, [14, Theorem 4]: *in any normed space X such that the set of quasi-interior positive elements of X^* is non empty, 0_X is a denting point of a pointed cone C if and only if it is a point of continuity for C.* The former characterization provides a partial answer to Gong's question in the context of non closed cones. In addition, it has applications in the theory of Pareto optimization, see [14].

In this work, we continue the research line of [14] and prove the following: *in any normed space X, 0_X is a denting point of a pointed cone C if and only if it is a point of continuity for C and the closure of the cone in the bidual space respect to the weak* topology is pointed.* It corresponds to the equivalence between (i) and (iii) in Theorem 1.1 below. Let us note that we have changed the assumption in Kountzakis and Polyrakis' theorem [14, Theorem 4] which affects to the whole X^*, by another which only affects to the particular cone we are considering. Our characterization also provides a partial answer to Gong's question in the context of non closed cones. On the other hand, if X is reflexive, then the closure of the cone in the bidual space respect to the weak* topology coincides with its closure respect to (X, weak). Moreover, using Mazur's theorem, it is easily seen that the last set is equal to the closure of C in $(X, \|\cdot\|)$. Thus, for reflexive Banach spaces, our characterization is equivalent to Daniilidis' characterization [6, Corollary 2]. Then, in some way, our characterization can be interpreted as a generalization of [6, Corollary 2] for normed spaces. Some consequences and other related results are also stated and proved in this manuscript. They are stated in Subsection 1.2. In the following subsection we have compiled the definitions of most of the notions which appear in the work.

1.1. Notation and main definitions

We will denote by X a normed space, by $\|\cdot\|$ the norm of X, by X^* the dual space of X, by $\|\cdot\|_*$ the norm of X^*, by 0_X the origin of X, and by $\mathbb{R}_+$ the set of non negative real numbers. A non empty convex subset C of X is called a *cone* if $\alpha C \subset C$, $\forall \alpha \in \mathbb{R}_+$. In what follows, $C \subset X$ stands for a cone. C is called *pointed* if $C \cap (-C) = \{0_X\}$. The cone

$$C^* := \{f \in X^* \colon f(c) \geq 0\,, \forall c \in C\},$$

is called the *dual cone* for C, and the set

$$C^\# := \{f \in X^* \colon f(c) > 0\,, \forall c \in C \setminus \{0_X\}\},$$

the *quasi-relative interior* of the dual cone for C or *the set of all strictly positive functionals*. The interior of C^*, $\mathrm{Int}C^*$, is contained in $C^\#$ and $\mathrm{Int}C^* = C^\#$ whenever the first one is non empty. Any $c \in C$ is said to be a *quasi-interior point of* X or *a quasi-interior positive element of* X if $\overline{\cup_{n\in\mathbb{N}}[-nc, nc]} = X$, where $[-nc, nc]$ is the order interval (given by the cone order) $\{x \in X \colon -nc \leq x \leq nc\}$. The set of all quasi-interior points is denoted by $\mathrm{qi}C$. If C has non empty interior, then the concepts of interior point of C and quasi-interior point of X coincide [17]. Besides, $\mathrm{qi}C$ is either empty or dense in C. In general $qiC^* \subset C^\#$, and $qiC^* = C^\#$ in the context of normed lattices [1]. A non empty convex subset B of C is called a *base* for C, if $0 \notin \overline{B}$ and each element $c \in C \setminus \{0\}$ has a unique representation of the form $c = \lambda b$, with $\lambda > 0$ and $b \in B$. A base B is called a *bounded base* if it is a bounded subset of X. It is well known that $B \subset C$ is a base if and only if there is a continuous strictly positive linear functional f of X^* such that $B = f^{-1}(1) \cap C$. Any cone which has a base is necessarily pointed.

If C is now a cone in X^*, the cone

$$C_* := \{x \in X \colon f(x) \geq 0, \, \forall f \in C\},$$

is called *the polar cone* for C.

A set H is called a *half space* of X if it is a weakly open set of the form $H = \{x \in X : f(x) < \lambda\}$ for some $f \in X^* \setminus \{0_{X^*}\}$ and $\lambda \in \mathbb{R}$. We denote H briefly by $\{f < \lambda\}$. Finite intersections of half spaces form a basis for the weak topology. A *slice* of C is a non empty intersection of C with an open half space of X. If $M \subset X$, then $\text{conv}(M)$ stands for the *convex hull* of M, that is, the smallest (in the sense of inclusion) convex subset of X containing M. Similarly, $\overline{\text{conv}}(M)$ stands for the closed convex hull of M. We denote by $B_\varepsilon(x)$ the open ball with centre $x \in X$ and radius $\varepsilon > 0$ and by $\overline{B_\varepsilon(x)}$ the corresponding closed ball. By B_X we denote the open unit ball with centre 0_X, by $\overline{B_X}$ the closed unit ball with centre 0_X, and by S_X the unit sphere. Let A be a convex subset of X, $a \in A$ is said to be a *denting point* of A if $a \notin \overline{\text{conv}}(A \setminus B_\varepsilon(a))\ \forall \varepsilon > 0$. By the Hahn-Banach theorem, a is a denting point of A if and only if for every $\varepsilon > 0$ there exists a slice S of A containing a with diameter less than ε. We denote the diameter of S briefly by diam (S). We call "weak" the weak topology on X and "weak*" the weak star topology on its dual X^*. Let A be a convex subset of X, $a \in A$ is said to be a *point of continuity* for A if the identity map $(A, \text{weak}) \to (A, \|\cdot\|)$ is continuous at a, that is, for every open ball $B_\varepsilon(a)$, there exists a weakly open U such that $a \in U \cap A \subset B_\varepsilon(a) \cap A$. It is clear that every denting point of A is a point of continuity for this set. Let A be a convex subset of $(X^*, \|\cdot\|_*)$, $a \in A$ is said to be a *weak* point of continuity* for A if the identity map $(A, \text{weak}^*) \to (A, \|\cdot\|_*)$ is continuous at a.

Let A be a convex set of X. A point a of a convex set A is called an *extreme point* of A if a does not belong to any non degenerate line segment in A. It is a simple matter to show that a cone C is pointed if and only if 0_X is an extreme point of C. A point $a \in A$ is called a *strongly extreme point* (resp. *weakly strongly extreme point*) of A if given two sequences $(a_n)_n$ and $(\tilde{a}_n)_n$ in A such that $\lim_n(a_n + \tilde{a}_n) = 2a$, then $\lim_n a_n = a$ (resp. weak-$\lim_n a_n = a$). A point $a \in A$ is called a *strongly exposed point* of A if there exists $f \in X^*$ such that $f(a) = \sup_A f$ and $\lim_n a_n = a$ for all sequences $\{a_n\}_n \subset A$ such that $\lim_n f(a_n) = \sup_A f$.

1.2. Main Results

After the first paragraphs of the Introduction the following question raises in a natural way. Does the classic equivalences for 0_X being a denting point of a cone into the frame of Banach spaces and closed cones remain true for normed spaces and cones non necessarily closed? The classic equivalences at which we refer can be found in [6, Theorem 5] and [12, Theorem 2.1]. The answer is positive, and we have stated them in our main result, Theorem 1.1 below. They correspond to Assertions (iv), (v), (vi), (vii) regarding C^*, and (x). In the statement of Theorem 1.1 (and throughout this work) we will denote by Int$A$ the interior respect to the topology of the norm and by $\widetilde{A}$ the closure of A in X^{**} respect to the weak* topology. The last notation is from [21]. The relationship between A and $\widetilde{A}$ is deeply studied there. Let us mention, for example, that $\widetilde{A} \cap X$ is the closure of A by the weak topology on X. Some other properties and results from [21] will be used in our proofs. We refer the reader to the next subsection for the rest of definitions and notation used in this work. Now, our main result.

Theorem 1.1. *Let X be a normed space and $C \subset X$ a pointed cone. The following are equivalent:*

(i) 0_X is a denting point of C.

(ii) 0_X is a point of continuity and a weakly strongly extreme point of C.

(iii) 0_X is a point of continuity for C and $\widetilde{C} \subset X^{**}$ is pointed.

(iv) C has a bounded slice (Property weak (π)).

(v) 0_X is a strongly exposed point for C.

(vi) C has a bounded base.

(vii) Int$C^{\#} \neq \emptyset$ equivalently Int$C^* \neq \emptyset$ (C^* is a solid cone).

(viii) There exists $f \in S_{X^*}$ such that $\inf\limits_{S_X \cap C} f > 0$.

(ix) There exist $f \in S_{X^*}$ and $0 < \delta < 1$ such that

$$\{f \leq \lambda\} \cap C \subset \frac{\lambda}{\delta} B_X, \forall \lambda > 0.$$

(x) There exists $f \in S_{X^*}$ and $0 < \delta < 1$ such that

$$f(c) > \delta \parallel c \parallel, \forall c \in C \setminus \{0\} \quad \textit{(Angle property).}$$

Assertions (ii) and (iii) connect with results of the geometry of the unit ball in a Banach space which can be found in [10]. The condition $\widetilde{C} \subset X^{**}$ *is pointed* in (iii) is the same as the condition 0_X *is a preserved extreme point of* C used in [10]. Example 1.5 shows that the assumption $\widetilde{C}$ *is pointed* cannot be dropped in (iii). In addition, (ii) and (iii) provide (together with (vi)) a new criterion for the existence of a bounded base for a cone. It is worth noting that in the statement of the above-mentioned theorem of Petschke [18, Corollary 4.2] appears the assumption of bounded base. On the other hand, Gong proved [9, Theorem 3.2 (a)] by means of the technique of approximating cones, cones which were constructed from a base in the initial cone of the space. Assertion (vii) is a restatement of the property of solid cone. Assertion (ix) gives a formula to measure the diameter of bounded slices in terms of its "border", i. e., of $\lambda > 0$. Assertion (viii) is a variational restatement of the notion of denting point.

Next, we state a couple of consequences of Theorem 1.1. The first one is a criterion to know whether a cone has a bounded base, an unbounded base, or does not have any base. Assertion (i) is not new, [14, Theorem 10]. However, in this work, we provide an alternative proof.

Corollary 1.2. *Let X be a normed space and C a pointed cone. The following statements hold.*

(i) If C has a bounded base, then every sequence in C which weakly converges to 0_X also converges to 0_X in norm. The reverse is true when $qiC^* \neq \emptyset$.

(ii) Assume that $qiC^* \neq \emptyset$. Then C has a base but does not have bounded base if and only if $C \cap S_X$ has a sequence which weakly converges to 0_X.

(iii) C does not have any base if and only if for every $f \in X^*$ there exists a non constant sequence in $Ker f \cap C$ which converges to 0_X in norm.

The assumption about the existence of a normalized weakly null sequence in a cone, the requirement that each weakly null sequence also converges in norm, and their relationships with the bases of cones are studied for a special class of cones in Banach spaces in [5].

The second consequence of Theorem 1.1 is a result on Pareto efficiency. It is obtained by applying Theorem 1.1 to [2, Proposition 2.4].

Corollary 1.3. *Let X be a normed space and C a pointed cone. If 0_X is a point of continuity for C and $\widetilde{C} \subset X^{**}$ is pointed, then each weakly compact subset of X has super efficient points.*

Let us pay attention to the results of Daniilidis [6, Corollary 2] and of Kountzakis and Polyrakis [14, Theorem 4] again. Both results were proved by different methods. However, the next proposition allows us to obtain each of them as a consequence of Theorem 1.1.

Proposition 1.4. *Let X be a normed space and $C \subset X$ a pointed cone. Assume that at least one of the following two conditions holds:*

(i) X is a Banach space, C is closed, and 0_X is a point of continuity for C;

(ii) $\text{qi}C^* \neq \emptyset$.

Then $\widetilde{C} \subset X^{**}$ is pointed.

The following example shows that we can not replace $\text{qi}C^*$ by $C^\#$ in Statement (ii) in Proposition 1.4 . Moreover, it provides a negative answer to [14, Problem 5] which asked: If the origin is a point of continuity for a cone in a normed space, is $\text{qi}C^* \neq \emptyset$? Let us note that a positive answer to [14, Problem 5] would imply that [6, Corollary 2] is true for cones non necessarily closed. However, this example also shows that [6, Corollary 2] is not true, if we drop the assumption *the cone is closed*.

Example 1.5. Let us define $X := \mathbb{R}^2$ and $C := \mathbb{R} \times (0, +\infty) \cup \{(0, 0)\}$ which is a pointed cone. Then $C^\# \neq \emptyset$, $\widetilde{C}$ is not pointed, 0_X is point of continuity for C, and $\text{qi}C^* = \emptyset$.

The following result goes in the line of some other obtained in [4]. We will see how an strengthening of Assertion (iii) of Theorem 1.1 allows us to obtain slices with weakly compact closure in Banach spaces. The existence of those slices was studied and connected with the notion of locally weakly compact cone in [11, Theorem 2.3].

Theorem 1.6. *Let X be a Banach space and C a pointed and closed cone on X. Then $C^\# \subset X^*$ is a nonempty open subset if and only if C has bounded slices and each of them has, in fact, weakly compact closure.*

To our knowledge, the following still remains as an open problem. Let us note that a positive answer would imply a negative answer to Gong's question.

Problem 1.7. *Let X be a normed space and $C \subset X$ a closed and pointed cone such that 0_X is a point of continuity. Is $\text{qi}C^*$ non empty?*

The paper is organized as follows. Section 2 is rather technical and it is devoted to prove our main results, stated in this subsection. In addition, we state some others which are direct consequence of some of our main results. Namely, we provide two restatements of Corollary 1.2, one into the frame of separable normed lattices (Corollary 2.11), and another one into the frame of a subclass of normed spaces with separable dual (Corollary 2.12). On the other hand, we also state three results regarding the property of reflexivity. The first one (Theorem 2.15) is a characterization for a Banach space to be reflexive. After that, we state Corollary 2.16, which is a restatement in terms of dentability of a known characterization for reflexivity. Finally, another restatement of a known result provides Corollary 2.17, which is a characterization for 0_{X^*} to be a denting point in a dual cone of a reflexive Banach space.

2. Auxiliary results and proofs

The aim of the first part of this section is to prove Theorem 1.1. For this purpose we need to state and prove some preliminary results. Let us begin with a lemma (without proof) on half spaces.

Lemma 2.1. *Let X be a normed space and $H \subset X$ an open half space. Then $0_X \in H$ if and only if there exists $f \in X^* \setminus \{0_{X^*}\}$ and $\lambda > 0$ such that $H = \{f < \lambda\}$.*

Next, a sequence of results regarding suitable bounded subsets of cones.

Lemma 2.2. *Let X be a normed space, $C \subset X$ a cone, and S a bounded slice of C. Then $0_X \in S$ and C is pointed.*

Proof. Let us fix $f \in X^* \setminus \{0_{X^*}\}$ and $\lambda \in \mathbb{R}$ in such a way that the half space $H := \{f < \lambda\}$ provides the slice S, i. e., $S = C \cap H$.

We first prove that $0_X \in H$. We assume that $\lambda \leq 0$, and pick $x_0 \in S$ such that $f(x_0) < \lambda$. Hence $f(nx_0) = nf(x_0) < n\lambda \leq \lambda$, $\forall n \geq 1$, which contradicts the fact that S is bounded. Therefore $\lambda > 0$ and $0 \in H$.

Next, we check that C is a pointed cone. We assume now that there exists $x_0 \in C \cap (-C)$, $x_0 \neq 0_X$. There is no loss of generality in assuming $0 \leq f(x_0) < \lambda$. This is because if $f(x_0) < 0$, we can consider $-x_0$ instead and, if $f(x_0) > \lambda$, we can choose some $\delta \in (0, 1)$ such that $y_0 := \delta x_0 \in C \cap (-C)$ and $0 < f(y_0) < \lambda$. From $x_0 \in -C$ it follows that $-x_0 \in C$ and $-nx_0 \in C$,

$\forall n \geq 1$. Moreover, $f(-nx_0) = -nf(x_0) \leq f(x_0) < \lambda$ which contradicts the boundedness assumption on S again. For this reason $x_0 = 0_X$ and we conclude that C is pointed. □

Proposition 2.3. *Let X be a normed space and C a cone on X. Assume that there exists $n_0 \geq 1$, $\{f_i\}_{i=1}^{n_0} \subset X^*$, and $\lambda > 0$ such that $\cap_{i=1}^{n_0}\{f_i < \lambda\} \cap C$ is bounded. Then 0_X is a point of continuity for C.*

Proof. It is sufficient to show that given $\varepsilon > 0$, there exists a weakly open W which contains to 0_X such that diam $(W \cap C) < \varepsilon$. Let us denote $L := \cap_{i=1}^{n_0}\{f_i < \lambda\}$ and fix $M > 0$ such that diam $(L \cap C) \leq M$. We choose $m_0 \geq 1$ such that $M/m_0 < \varepsilon$. Then, diam $(\cap_{i=1}^{n_0}\{f_i < \lambda/m_0\} \cap C) \leq \varepsilon$ because if we fix x and y in $\cap_{i=1}^{n_0}\{f_i < \lambda/m_0\} \cap C$, both m_0x and m_0y belong to $\cap_{i=1}^{n_0}\{f_i < \lambda\} \cap C$. Thus $\| m_0x - m_0y \| \leq M$, which yields $\| x - y \| \leq M/m_0 \leq \varepsilon$. □

Proposition 2.4. *Let X be a normed space, C a pointed cone on X, $f \in X^*$, and $\lambda > 0$. If $\{f \leq \lambda\} \cap C$ is bounded, then 0_X is a denting point of C and $f \in IntC^{\#}$.*

Proof. The Hahn-Banach theorem together with the proof of Proposition 2.3 for the case $n_0 = 1$ show that 0_X is a denting point of C. On the other hand, if we assume that there exists $c \in C \setminus \{0_X\}$ such that $f(c) \leq 0$, then the unbounded sequence $\{nc\}_{n\geq 1} \subset S := \{f \leq \lambda\} \cap C$ contradicts the boundedness hypothesis on S. Then $f \in C^{\#}$. Moreover, we will check that $f \in IntC^{\#}$. We can certainly assume that $\lambda = 1$ and $f \in B_{X^*}$, since otherwise we can replace f by $\frac{f}{\lambda\|f\|_*}$. Let us denote by M the diameter of the set $\{f = 1\} \cap C$. We will check that the ball (in X^*), $B_{\frac{1}{2M}}(f)$, is contained in $C^{\#}$. Fix any $g \in B_{\frac{1}{2M}}(f)$. It is sufficient to show that $g(c) > 0$, for every c in the base $\{f = 1\} \cap C$. If c_0 belongs to the former set, then $g(c_0) = f(c_0) + (g(c_0) - f(c_0)) \geq 1 - \frac{1}{2M} > 0$. □

The following result is a reformulation of [10, Proposition 2.2] for a cone (instead of the unit ball) in a normed space X. The proof there also works in this case. However, for the convenience of the reader, we will show those implications used in the proof of our main result, thus making our exposition self contained.

Proposition 2.5. *Let X be a normed space and $C \subset X$ a pointed cone. Consider the following properties.*

(i) 0_X *is a weakly strongly extreme point of* C*;*

(ii) $\widetilde{C} \subset X^{**}$ *is pointed (i.e.,* 0_X *is a preserved extreme point of* C*);*

(iii) For any $R > 0$ *and* $C_R := C \cap \overline{B_R(0_X)}$*, the family of open slices containing* 0_X *forms a neighbourhood base for* 0_X *relative to* $(C_R, weak)$*.*

Then we have (i)⇒(ii)⇒(iii). If C *is also closed, then the three properties above are equivalent.*

Proof.

(i)⇒(ii). Since 0_X is a weakly strongly extreme point of C, the following property holds. Given any $f \in X^*$, for every $\varepsilon > 0$ there exists $\delta(\varepsilon, f) > 0$ such that if $\{c, c^*\} \subset C$ and $\| c + c^* \| < \delta(\varepsilon, f)$, then $\max\{|f(c)|, |f(\tilde{c})|\} < \varepsilon$. Assume that $\widetilde{C}$ is not pointed. Then there exists $x^{**} \in \widetilde{C} \cap (-\widetilde{C})$, $x^{**} \neq 0_X$. We choose $f_0 \in X^*$ and $\varepsilon_0 > 0$ such that $x^{**}(f_0) > \varepsilon_0$. Fix a net $(c_i)_{i\in I} \subset C$ such that weak*-$\lim_{i\in I} c_i = x^{**}$ and $f(c_i) > \varepsilon_0$, $\forall i \in I$. We claim that there exists $\tilde{c} \in \operatorname{conv}\{c_i : i \in I\}$ such that $f(\tilde{c}) < \varepsilon_0$, which contradicts the choice of the net $(c_i)_{i\in I}$. Let us show the claim and the contradiction. Fix another net $(c_i^*)_{i\in I} \subset C$ weak* converging to $-x^{**}$. Obviously weak-$\lim_i (c_i + c_i^*) = 0_X$. Now, by Mazur's theorem, $0_X \in \overline{\operatorname{conv}}\{c_i + c_i^* : i \in I\}$. Thus, fixed $\delta(\varepsilon_0, f_0)$ (from the property of the beginning of the proof), there exists $\sum_{j=1}^n \lambda_j (c_{i_j} + c_{i_j}^*) \in C$, for some $n \geq 1$ and $(\lambda_j)_{j=1}^n \subset [0, 1]$ such that $\sum_{j=1}^n \lambda_j = 1$ and $\| \sum_{j=1}^n \lambda_j (c_{i_j} + c_{i_j}^*) \| < \delta(\varepsilon_0, f_0)$. Then $\tilde{c} := \sum_{j=1}^n \lambda_j c_{i_j}$ verifies $\varepsilon_0 > f(\tilde{c}) = \sum_{j=1}^n \lambda_j f(c_{i_j}) > \sum_{j=1}^n \lambda_j \varepsilon_0 = \varepsilon_0$, a contradiction.

(ii)⇒(iii). Let us fix W a weak-neighbourhood of 0_X in X. It is not a restriction to assume that there exist $n_0 \geq 1$, $\{f_i\}_{i=1}^{n_0} \subset X^*$, and $\alpha > 0$ such that $W = \cap_{i=1}^{n_0} \{f_i \leq \alpha\}$. Consider $R > 0$ and the respective set C_R. Then $\widetilde{C_R} \subset X^{**}$ is a weakly* compact set that contains 0_X as an extremal point. On the other hand, $\widetilde{W \cap C_R}$ is a weak*-neighbourhood of 0_X in $\widetilde{C_R}$. Now, by Choquet's Lemma, there exist $f \in X^*$ and $\lambda > 0$ such that the set $V := \{y \in \widetilde{C_R} : f(y) \leq \lambda\}$ is contained in $\widetilde{W \cap C_R}$. Thus, $\{f \leq \lambda\} \cap C_R$ is contained in $\widetilde{W \cap C_R} \cap X = \overline{W \cap C_R} \subset W \cap \overline{C}_R$ which leads to $\{f \leq \lambda\} \cap C_R \subset W \cap C_R$. □

To our knowledge, the following question remains open.

Problem 2.6. *Is the implication (iii)⇒(i) in the former proposition true for cones not necessarily closed?*

Now, we have all the tools we need in order to prove our main result.

Proof of Theorem 1.1.

(i)⇒(ii) It is evident that 0_X is a point of continuity for C. Let us show that 0_X is a weakly strongly extreme point of C. Fix $(c_n)_n$ and $(c_n^*)_n$ two sequences in C such that $\lim_n \frac{c_n+c_n^*}{2} = 0$. We will show that $\lim_n c_n = 0$. For every $\varepsilon > 0$, there exists $n_\varepsilon \geq 1$ and a slice $S_\varepsilon \subset C$ (containing 0_X) with diameter not bigger than ε such that $\frac{c_n+c_n^*}{2} \in S_\varepsilon$, $\forall n \geq n_\varepsilon$. By convexity, either $c_n \in S_\varepsilon$ or $c_n^* \in S_\varepsilon$, $\forall n \geq n_\varepsilon$. Fix an arbitrary $n_0 \geq n_\varepsilon$ and assume that $c_{n_0}^* \in S_\varepsilon$. Then $\frac{\|c_{n_0}\|}{2} \leq \| \frac{c_{n_0}+c_{n_0}^*}{2} \| + \frac{\|c_{n_0}^*\|}{2} \leq 2\varepsilon$. Then $\| c_{n_0} \| \leq 4\varepsilon$. As n_0 was arbitrary we have that $\lim_n c_n = 0_X$.

(ii)⇒(iii) This is Proposition 2.5.

(iii)⇒(iv) Since 0_X is a point of continuity for C, there exist $0 < r < R$ and a weak neighbourhood W of zero such that $W \cap C \subset C_r \subset C_R$. Applying Proposition 2.5 to W and $R > 0$, one can assert that there exist $f \in X^*$ and $\lambda > 0$ such that $\{f < \lambda\} \cap C_R \subset W \cap C_R \subset C_r$. We claim that $\{f < \lambda\} \cap C \subset C_R$. Otherwise, we could choose $c \in C$ such that $\| c \| > R$ and $f(c) < \lambda$. Then we pick

$$c_0 := \frac{r+R}{2 \| c \|} c \in \{f < \lambda\} \cap C_R \setminus C_r,$$

a contradiction.

(iv)⇒(v) Let $S = \{f \leq \lambda\} \cap C$ be the bounded slice. By Proposition 2.4, it follows that $f \in \text{Int}C^{\#}$, and so $0 = \sup_C(-f)$. Fix a sequence $\{c_n\}_n \subset C$ such that $\lim_n(-f(c_n)) = 0$. Hence $\lim_n f(c_n) = 0$. We will show that $\lim_n c_n = 0_X$. Fix $\delta > 0$ such that $f + \delta \overline{B_{X^*}} \subset C^{\#}$, a sequence of integers $(k_n)_n$ diverging to $+\infty$ such that $f(c_n) = \frac{1}{k_n}$, and a sequence $(g_n)_n \subset S_{X^*}$ such that $g(c_n) = \| c_n \|$, $\forall n \geq 1$. Then, for every $n \geq 1$, $f - \delta g_n \in C^{\#}$. Thus $f(c_n) - \delta g(c_n) > 0$, which implies $\frac{1}{k_n} - \delta \| c_n \| > 0$. Therefore $\| c_n \| < \frac{1}{\delta k_n}$, $\forall n \geq 1$ and the proof is over.

(v)⇒(i) Let $g \in X^*$ be the functional which strongly exposes 0_X, consider $f := -g$, and an arbitrary $\lambda > 0$. Assume that $S := \{f \leq \lambda\} \cap C$ is not bounded. It is not a restriction to assume that there exists a sequence $(c_n)_n \subset S$ such that $\| c_n \| = n$ for every $n \geq 1$. Then $\lim_n f(\frac{c_n}{\sqrt{n}}) = 0$, which implies that $\lim_n \frac{c_n}{\sqrt{n}} = 0_X$. However $\| \frac{c_n}{\sqrt{n}} \| = \sqrt{n}$, $\forall n \geq 1$, a contradiction.

Therefore, S must be bounded. Now Proposition 2.4 applies to give that 0_X is a denting point.

(iv)⇒(vi) Let us fix a bounded slice $S \subset C$. By Lemma 2.2, $0_X \in S$. By Lemma 2.1, there exists $f \in X^*$ and $\lambda > 0$ such that $S = \{f < \lambda\} \cap C$. Now, by Proposition 2.4, $f \in C^\#$. It is clear that $B := \{f = \lambda\} \cap C$ is a bounded base for C.

(vi)⇒(vii) Let $B \subset C$ be a bounded base for C. By the characterization given after the definition of bounded base in Subsection 1.1, there exists $f \in C^\#$ and $\delta > 0$ such that $B = \{f = \delta\} \cap C$. It is clear that $S = \{f \leq \delta\} \cap C$ is a bounded slice. The fact that $\text{Int}C^\# \neq \emptyset$ follows from Proposition 2.4. Let us check that $\text{Int}C^\# \neq \emptyset \Leftrightarrow \text{Int}C^* \neq \emptyset$. Clearly $C^\# \subset C^*$, it suffices to show that $\text{Int}C^* \subset C^\#$. Fix $f \in \text{Int}C^*$ and $\delta > 0$ such that $B_\delta(f) \subset C^* \subset X^*$. It suffices to prove that $f(c) \neq 0$ for every $c \in C \setminus \{0_X\}$. For that purpose we fix an arbitrary $c_0 \in C \setminus \{0_X\}$. There exist $g \in X^*$ and $\alpha > 0$ such that $g(c_0) > 0$ and $\| \alpha g \|_* < \delta$. Set $h := f - \alpha g \in X^*$. Then $h \in B_\delta(f)$. Thus $h(c_0) = f(c_0) - \alpha g(c_0) \geq 0$ which leads to $f(c_0) \geq \alpha g(c_0) > 0$.

(vi)⇒(iv) is obvious.

(vii)⇒(viii) Let $f \in \text{Int}C^\#$. Assume $\inf_{S_X \cap C} f = 0$. Fix $\varepsilon > 0$ such that $\overline{B_{2\varepsilon}(f)} \subset C^\#$. Then we can find $x \in S_X \cap C$ such that $f(x) < \varepsilon$. Find $g \in S_{X^*}$ such that $g(x) = 1$. Then $(f - 2\varepsilon g)(x) < \varepsilon - 2\varepsilon = -\varepsilon$, and so $f - 2\varepsilon g \notin C^\#$. Since $f - 2\varepsilon g \in \overline{B_{2\varepsilon}(f)}$, we reach a contradiction.

(viii)⇒(ix)⇒(x)⇒(viii) is obvious.

(viii)⇒(vi) Define $\delta := \inf\limits_{S_X \cap C} f > 0$. In particular $f \in C^\#$. Then the set $B := \{c \in C : f(c) = \frac{\delta}{2}\}$ is a base for C. Moreover, if $b \in B$, then $\delta \leq f\left(\frac{b}{\|b\|}\right) = \frac{\delta}{2\|b\|}$. Hence $\| b \| \leq \frac{1}{2}$ and so B is bounded. □

Corollary 2.7. *Let X be a normed space and C a pointed cone. The following properties hold for any $f \in \text{Int}C^\# \cap S_{X^*}$.*

a) $\{f \leq \lambda\} \cap C$ *is bounded,* $\forall \lambda > 0$.

b) $\exists \delta \in (0,1)$ *such that* $f(c) > \delta \| c \|$, $\forall c \in C \setminus \{0\}$.

c) $\inf\limits_{S_X \cap C} f > 0$.

Proof. Fix $\delta > 0$ such that $f + \delta\overline{B_{X^*}} \subset C^{\#}$. We will show the inclusion $\{f = \lambda\} \cap C \subset \frac{\lambda}{\delta} B_X$, $\forall \lambda > 0$ (statement (ix) in Theorem 1.1). In fact, we will show that $\| c \| < \lambda/\delta$, $\forall \lambda > 0$ and $c \in \{f = \lambda\} \cap C$. For that purpose, we fix $\lambda > 0$ and an arbitrary $c \in \{f = \lambda\} \cap C$. Now, let $g \in S_{X^*}$ be such that $g(c) = \| c \|$. Then $f - \delta g \in C^{\#}$, therefore $f(c) - \delta g(c) > 0$ which implies $\lambda - \delta \| c \| > 0$. □

The following objective is to prove Corollary 1.2. In this, the set qiC^* plays a crucial role. This makes necessary to show several preliminary results on it.

Lemma 2.8. *Let X be a normed space and $C \subset X$ a cone . Then qi$C^* \subset C^{\#}$.*

Proof. Fix $f \in$ qiC^* and $c \in C \setminus \{0_X\}$. Assume that $f(c) \leq 0$, which yields $g(c) \leq 0$ for every $g \in \cup_n[-nf, nf]$. Next, we choose $h \in X^*$ such that $h(c) = \lambda > 0$ and $\varepsilon \in (0, \lambda \| c \|)$. Then, there exists $g \in \cup_n[-nf, nf]$ such that $\| h - g \| < \varepsilon / \| c \|$. Hence, $h(c) \leq h(c) - g(c) = |h(c) - g(c)| < \lambda$, which is a contradiction. □

Lemma 2.9. *Let X be a normed space and C a pointed cone. Assume that there exists $f \in$ qiC^* and $\lambda \in \mathbb{R}$ such that $\{f = \lambda\} \cap C$ is unbounded. Then $C \cap S_X$ has a sequence which weakly converges to 0_X.*

Proof. For every $n \geq 1$ we consider $\bar{c}_n \in nS_X \cap C$ such that $f(\bar{c}_n) = \lambda$. Next, for every n, we define $c_n := \bar{c}_n/n \in S_X$. We claim that the sequence $\{c_n\}_n$ weakly converges to 0_X. From the definition of each c_n and Lemma 2.8 we obtain that $\lim_n f(c_n) = 0$. Fix $g \in X^*$ and $\varepsilon > 0$. $f \in$ qiC^*, and so we can choose $n_0 \geq 1$ and $h \in [-n_0 f, n_0 f]$ such that $\| h - g \|_* < \varepsilon$. Thus $-n_0 f(c_n) \leq h(c_n) \leq n_0 f(c_n)$ for all n, which means $\lim_n h(c_n) = 0$. Then $|g(c_n)| \leq \varepsilon + \frac{1}{n}$, for n big enough. As $\varepsilon > 0$ was taken arbitrary we have $\lim_n g(c_n) = 0$. □

If 0_X is a denting point of C, then every sequence in C which weakly converges to 0_X also converges to 0_X in norm (Theorem 1.1, (v)). The reverse does not hold true in general. However, making use of the set qiC^*, we have the following result.

Proposition 2.10. *Let X be a normed space and C a pointed cone on X such that qi$C^* \neq \emptyset$. If every sequence in C which weakly converges to 0_X also converges to 0_X in norm, then 0_X is a denting point of C.*

Proof. By Theorem 1.1, we only need to show that C has a bounded slice. Fix $f \in \text{qi}C^*$ and $\alpha > 0$. We claim that the slice $\{f < \alpha\} \cap C$ is bounded. If not, then $\{f = \alpha\} \cap C$ is unbounded. By Lemma 2.9, we can pick a sequence in $C \cap S_X$ weakly converging to 0_X, which contradicts the hypothesis of the statement. □

Proof of Corollary 1.2.

(i) The first part is a consequence of Theorem 1.1, assertions (vi) and (v). The reverse is nothing but Proposition 2.10.

(ii) Assume C has a base but does not have a bounded base. Fix $f \in \text{qi}C^*$ and $\lambda > 0$. By Theorem 1.1 the slice $\{f < \lambda\} \cap C$ is unbounded. Thus the set $\{f = \lambda\} \cap C$ is also unbounded. Now Lemma 2.9 applies and the proof is over. For the reverse implication we fix $f \in \text{qi}C^*$ and $\lambda > 0$. We consider the nonempty slice $\{f < \lambda\} \cap C$. Suppose the slice is bounded. Applying Theorem 1.1 (v) we get that every sequence in C that weakly converges to 0_X converges to 0_X in norm. But this is a contradiction with the hypothesis. The proof finishes by noting that the unbounded set $\{f = 1\} \cap C$ is an unbounded base for C.

(iii) If C does not have any base, then $C^\# = \emptyset$. Now, fixed any $f \in X^*$, there exists $c \in C \setminus \{0_X\}$ such that $f(c) = 0$. Then $\{c/n\}_n$ is the desired sequence. For the reverse implication, note that from the assumption, it is clear that $C^\# = \emptyset$. Then C does not have any base. □

Next, a version of Corollary 1.2 for separable normed lattices. It is a consequence of [13, Theorem 3.38] and of [1, Theorem 3.9].

Corollary 2.11. *Let X be a separable normed lattice and C a closed and pointed cone. The following statements hold.*

(i) C has a bounded base if and only if every sequence in C which weakly converges to 0_X also converges to 0_X in norm.

(ii) C has a base but not bounded base if and only if $C \cap S_X$ has a sequence which weakly converges to 0_X.

Out of the frame of normed lattices we can state the following result, which is a consequence of [17, Proposition 4.6].

Corollary 2.12. *Let X be a normed and C a pointed cone. Assume that the dual space X^* is separable, the dual cone C^* is complete, and $C^* - C^*$ is dense in X^*. The following statements hold.*

(i) C has a bounded base if and only if every sequence in C which weakly converges to 0_X also converges to 0_X in norm.

(ii) C has a base but not a bounded base if and only if $C \cap S_X$ has a sequence which weakly converges to 0_X.

Now is time for the proof of Proposition 1.4, which we will provide after a couple of propositions on weak* points of continuity. The first one can be obtained by an easy adaptation of the proof of Proposition 2.3.

Proposition 2.13. *Let X be a normed space and C a cone on X. If 0_X has a bounded weak*-neighbourhood in $\widetilde{C} \subset X^{**}$, then 0_X is a weak* point of continuity for $\widetilde{C}$.*

Proposition 2.14. *Let X be a normed space and $C \subset X$ a cone. If 0_X is a point of continuity for C, then it is also a weak* point of continuity for $\widetilde{C} \subset X^{**}$.*

Proof. If 0_X is a point of continuity for C, then there exist $n_0 \geq 1$, $\{f_i\}_{i=1}^{n_0} \subset X^*$, and $\lambda > 0$ such that $W := \cap_{i=1}^{n_0} \{f_i < \lambda\}$ verifies $W_C := W \cap C$ is bounded. Moreover, $\widetilde{W_C}$ is weak*-bounded. Hence, by [7, Theorem 3.15], $\widetilde{W_C}$ is bounded, and in this way, a bounded weak*-neighbourhood of 0_X in $\widetilde{C} \subset X^{**}$. Thus, by Proposition 2.13, 0_X is also a weak* point of continuity for $\widetilde{C}$. □

Proof of Proposition 1.4.

Case (i) By Proposition 2.14, 0_X is a weak* point of continuity for $\widetilde{C}$. Thus 0_X is a weak* point of continuity for every $\widetilde{C_R}$, where $C_R = C \cap B_R(0_X)$, $\forall R > 0$. It suffices to prove that 0_X is an extreme point of each set $\widetilde{C_R}$, $\forall R > 0$. Fix $R > 0$. Assume that there exists distinct $y, z \in \widetilde{C_R}$ such that $0 = \alpha z + (1-\alpha) y$ with $\alpha \in (0, 1)$. By [21, Lemma 3.2], z and y are weak* points of continuity for $\widetilde{C_R}$. As C is closed and X is a Banach space we have that z and y are also points of continuity for C_R, [21, Fact on page 30]. However, this contradicts our assumption that C is pointed.

Case (ii) Fix $\tilde{c} \in \widetilde{C} \cap (-\widetilde{C})$, we will check that $\tilde{c} = 0_X$. Let $(c_\alpha)_\alpha \subset C$ be a net weakly* convergent to $\tilde{c} \in X^{**}$. Suppose, contrary to our claim, that $\tilde{c} \neq 0_X$. Then, there exists $g \in X^*$ such that $\lim_\alpha g(c_\alpha) = g(\tilde{c}) \neq 0$. Choose $f \in \mathrm{qi}C^*$. If $f(\tilde{c}) < 0$, then $f(c_{\alpha_0}) < 0$ for some α_0, which contradicts $f \in C^\#$ (Lemma 2.8). If $f(\tilde{c}) > 0$, then $f(-\tilde{c}) < 0$ leading us to the same contradiction. Therefore $0 = f(\tilde{c}) = \lim_\alpha f(c_\alpha)$. In the rest of this proof we will use the same argument as in the final part of Lemma 2.9. We define M to be $\sup_\alpha \parallel c_\alpha \parallel < \infty$. Fix $\varepsilon > 0$. Next, we choose $n_0 \geq 1$ such that $h \in [-n_0 f, n_0 f]$ and $\parallel h - g \parallel_* < \varepsilon$. Then $\lim_\alpha h(c_\alpha) = 0$. Therefore, $|g(c_\alpha)| \leq (M+1)\varepsilon$ for α big enough. But $\lim_\alpha g(c_\alpha) = 0$ since $\varepsilon > 0$ was taken arbitrary, which is impossible. □

In the following proof, we will use the fact that $\overline{\{f < \lambda\} \cap C}$ coincides with $\{f \leq \lambda\} \cap C$ for every $f \in X^*$, $\lambda > 0$, and any closed cone $C \subset X$. Moreover, if $\{f < \lambda\} \cap C$ is bounded, then so is $\{f \leq \lambda\} \cap C$.

Proof of Theorem 1.6. Assume that $C^\# \subset X^*$ is a non empty open subset. Let us fix $f \in \mathrm{Int}C^\#$ and consider the bounded slice $S := \{f \leq 1\} \cap C$, Corollary 2.7 (a). We only need to show that S is weakly sequentially compact, [7, Theorem 4.51]. For that purpose, we consider a sequence $(s_n)_n \subset S$. Then, there exists $r \in [0,1]$ and a subsequence $(s_{n_k})_k$ such that $\lim_k f(s_{n_k}) = r$. We will check that if $r = 0$, then $\lim_k s_{n_k} = 0$ in the weak topology. For that purpose, we fix an arbitrary $g \in X^*$ and we will prove that $\lim_k g(s_{n_k}) = 0$. As $f \in \mathrm{Int}C^\#$, there exists $\delta > 0$ such that $f + \delta B_{X^*} \subset C^\#$. Then, on the one hand, $f(s_{n_k}) + \delta g(s_{n_k}) / \parallel g \parallel > 0$, $\forall k$. On the other hand, $f(s_{n_k}) - \delta g(s_{n_k}) / \parallel g \parallel > 0$, $\forall k$. The two assertions together lead us to the inequality

$$-\frac{\parallel g \parallel}{\delta} f(s_{n_k}) < g(s_{n_k}) < \frac{\parallel g \parallel}{\delta} f(s_{n_k}),\ \forall k,$$

which assures that $\lim_k g(s_{n_k}) = 0$. If $r > 0$ the former argument does not work. In that case, we can certainly assume that none of the $f(s_n)$ is zero. We define a new sequence $(\tilde{s}_n)_n$ by $\tilde{s}_n := [r/f(s_n)]s_n$. Then $(\tilde{s}_n)_n \subset \{f = r\} \cap C$, and the last set is weakly compact, [4, Lemma 3.4]. In order to provide a self-contained argument, we will give an sketch of the proof of the last claim. The idea is to show that every $h \in X^*$ attains its supremum over $T := \{f = r\} \cap C$. Set $M := \sup_T h$. We shall prove the case $M > 0$ (the other are similar). It is not restrictive to assume that $r = 1 = M$. Define $d := f - h \in C^*$. By Corollary 2.7 (b), $d \notin \mathrm{Int}C^\#$ because the definition of supremum provides a sequence $(t_n)_n \subset T$ such that $\| t_n \| > 1/ \| f \|$ and $\lim_n d(t_n) = 0$. Then, $d \notin C^{\#}$, which yields that there exists $c \in C \setminus \{0\}$ such that $d(c) = 0$. Therefore $h(c/f(c)) = \sup_T h$ and $\{f = r\} \cap C$ is weakly compact. From the above it follows that there exists a subsequence $(\tilde{s}_{n_k})_k \subset \{f = r\} \cap C \subset S$ which weakly converges to some $s \in \{f = r\} \cap C \subset S$. But then, clearly, $(s_{n_k})_k \subset S$ also weakly converges to $s \in S$.

Let us prove now the converse implication. Let us fix $f \in X^*$ such that $S := \{f \leq 1\} \cap C$ is weakly compact. Then $f \in \text{Int}C^{\#}$ by Proposition 2.4. In addition, there exists $\delta > 0$ such that $\overline{B_X} \cap C \subset \delta S$. Then $\overline{B_X} \cap C$ is weakly compact and then, by [4, Theorem 3.3], $C^{\#}$ is a non empty open subset of X^*. □

As the next result shows, the existence of weakly compact slices of cones in a Banach space is related to the property of reflexivity. It can be proved by an easy adaptation of the proofs of [4, Theorems 3.5 and 3.6].

Theorem 2.15. *The following are equivalent for a Banach space X.*

(i) X is reflexive.

(ii) There exists a closed and pointed cone C with non empty interior which has a slice with weakly compact closure.

(iii) If C is any closed and pointed cone in X. Then either every slice in C is unbounded or every slice in C has a weakly compact closure.

Hence, the non reflexivity of a Banach space X can be characterized by the existence of a pointed and closed cone $C \subset X$ such that $C^{\#} \subset X^*$ is not open although $\text{Int}C^{\#} \neq \emptyset$. In this situation, C has bounded and unbounded slices. This fact provides embeddings of l^1 and c_0 in X, see [4, Theorem 4.8]

Next, we will restate some results regarding reflexivity of Banach spaces in terms of dentability of cones. Let us note that the notion of base introduced in [19] and [11] is different to that given here. However, it can be easily checked that a closed cone has a bounded base if and only if it has a bounded base in the sense of [19] (or [11]). Therefore, applying Theorem 1.1 to [19, Theorem 1] and to [11, Theorem 2.4], we obtain respectively our last results.

Corollary 2.16. *A Banach space X is reflexive if and only if every closed cone $C \subset X^*$ for which 0_{X^*} is a denting point verifies $IntC_* \neq \emptyset$.*

Corollary 2.17. *Let X be a reflexive Banach space X and $C \subset X$ a closed cone. Then $IntC \neq \emptyset$ if and only if 0_{X^*} is a denting point of C^*.*

In the former result, the implication $\Rightarrow$ is always true, see [12]. However, the reverse is not true without the assumption of reflexivity, as Qiu showed in [20].

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