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WELL-POSED PROBLEMS FOR THE FRACTIONAL LAPLACE EQUATION WITH INTEGRAL BOUNDARY CONDITIONS

NIYAZ TOKMAGAMBETOV, BERIKBOL T. TOREBEK

Abstract. In this remark we study the boundary-value problems for a fractional analogue of the Laplace equation with integral boundary conditions in rectangular and half-strip domains. We prove the existence and uniqueness of solutions by using the spectral decomposition method.

1. Introduction

In [10], a fractional analogue of the classical Sturm-Liouville problem was found. Moreover, it stands for a symmetric fractional differential operator of order 2α, $(1/2 < \alpha < 1)$. Using the extension theory, we described a class of self-adjoint boundary-value problems associated with the fractional Sturm-Liouville equation.

Here, we aim at studying fractional operators in two dimensional cases, that is, a fractional Laplace equation. The main difference of the fractional Laplace equation, that we are going to introduce, from an operator made of the Laplacian by taking it in a fractional power is that the last one is a pseudo–differential operator with the symbol $(\xi_1^2 + \xi_2^2)^\beta$ for some $\beta \in \mathbb{R}$ nevertheless the first one is not.

The purpose of this paper is to study two boundary value problems for the fractional Laplace equation. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, -\infty < a < y < b < \infty\}$ and $\Omega_\infty = \{(x, y) \in \mathbb{R}^2 : 0 < x < +\infty, -\infty < a < y < b < \infty\}$. Now, we consider the equation

$$\mathcal{D}^\alpha_{x,0+}\mathcal{D}^\alpha_{x,0+}u(x, y) - \mathcal{D}^\beta_{y,a+}D^\beta_{y,b-}u(x, y) = 0, \tag{1.1}$$

in Ω, or in Ω_∞, where $0 < \alpha < 1$, $1/2 < \beta < 1$,

$$\mathcal{D}^\delta_{t,p+}u(t, z) = \frac{1}{\Gamma(1-\delta)}\int_p^t (t-s)^{-\delta}\frac{\partial u}{\partial s}(s, z)ds, \quad -\infty \le p < t < q \le \infty$$

is the left Caputo derivative of order $\delta \in (0, 1]$ of u with respect to t, and

$$D^\omega_{z,d-}u(t, z) = -\frac{1}{\Gamma(1-\omega)}\frac{\partial}{\partial z}\int_z^d (\xi - z)^{-\omega}u(r, \xi)d\xi, \quad -\infty \le c < z < d \le \infty$$

is the right Riemann-Liouville derivative of order $\omega \in (0, 1]$ of u with respect to z, [4].

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We say that the function $u \in C(\bar{\Omega})$ is a regular solution of (1.1) if u satisfies (1.1) and

$$\mathcal{D}^{\alpha}_{x,0+}u \in C(\Omega), \quad \mathcal{D}^{\alpha}_{x,0+}\mathcal{D}^{\alpha}_{x,0+}u \in C(\Omega), \quad \mathcal{D}^{\beta}_{y,a+}D^{\beta}_{y,b-}u \in C(\Omega).$$

Since for $\alpha = 1$, $\beta = 1$ one has

$$\mathcal{D}^{1}_{x,0+}\mathcal{D}^{1}_{x,0+} - \mathcal{D}^{1}_{y,a+}D^{1}_{y,b-} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \Delta,$$

Equation (1.1) is a fractional generalization of the Laplace equation.

Problem 1.1. Find in the domain Ω a regular solution of Equation (1.1), satisfying the following boundary value conditions:

$$u(0,y) = \varphi(y), u(1,y) = \psi(y), a \le y \le b, \tag{1.2}$$

$$I^{1-\beta}_{b-,y}u(x,a) = 0, \quad I^{1-\beta}_{b-,y}u(x,b) = 0, \quad 0 \le x \le 1. \tag{1.3}$$

Here $\varphi(y)$ and $\psi(y)$ are given sufficiently smooth functions.

Problem 1.2. Find in the domain Ω_∞ a regular solution of (1.1), satisfying the following boundary value conditions:

$$u(0,y) = \phi(y), \lim_{x\to+\infty} |u(x,y)| \to 0, \quad a \le y \le b, \tag{1.4}$$

$$I^{1-\beta}_{b-,y}u(x,a) = 0, \quad I^{1-\beta}_{b-,y}u(x,b) = 0, \quad 0 \le x \le +\infty. \tag{1.5}$$

where $\phi(y)$ is a sufficiently smooth function.

Note that Problems 1.1 and 1.2 for (1.1) when $\beta = 1$ were studied in [11, 5]. Some questions of solvability of boundary value problems with fractional analogues of the Laplace operator were studied in [6, 2].

The meed to study boundary-value problems for (1.1) is determined by using the fractal Laplace equations to describe the production processes in mathematical modeling of socio-economic systems [8]. We also note that in [8] an attention was drawn to the fact that the problem of finding a generalized two-factor Cobb-Douglas function is reduced to the classical boundary value problems for a generalized Laplace equation of a fractional order.

2. Auxiliary statements

In this section we start by recalling the definitions that we need later.

Definition 2.1. The left and right Riemann-Liouville fractional integrals I^{α}_{a+} and I^{α}_{b-} of order $\alpha \in \mathbb{R}$ ($\alpha > 0$) are defined as

$$I^{\alpha}_{a+}[f](t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad t \in (a,b],$$

$$I^{\alpha}_{b-}[f](t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s)ds, \quad t \in [a,b),$$

respectively. Here Γ stands for the Euler gamma function.

Definition 2.2. The left Riemann-Liouville fractional derivative D^{α}_{a+} of order $\alpha \in \mathbb{R}$ ($0 < \alpha < 1$) is given by

$$D^{\alpha}_{a+}[f](t) = \frac{d}{dt} I^{1-\alpha}_{a+}[f](t), \quad \forall t \in (a,b].$$

Analogously, the right Riemann-Liouville fractional derivative D^α_{b-} of order $\alpha \in \mathbb{R}$ $(0 < \alpha < 1)$ is defined as

$$D^\alpha_{b-}[f](t) = -\frac{d}{dt} I^{1-\alpha}_{b-}[f](t), \quad \forall t \in [a, b).$$

Definition 2.3. The left and right Caputo fractional derivatives of order $\alpha \in \mathbb{R}$ $(0 < \alpha < 1)$ are given by

$$\mathcal{D}^\alpha_{a+}[f](t) = D^\alpha_{a+}[f(t) - f(a)], \quad t \in (a, b],$$
$$\mathcal{D}^\alpha_{b-}[f](t) = D^\alpha_{b-}[f(t) - f(b)], \quad t \in [a, b),$$

respectively.

Let λ be a positive real number, $I = (0, 1)$, $\bar{I} = [0, 1]$. Consider the problem

$$\mathcal{D}^\alpha_{0+}\mathcal{D}^\alpha_{0+}\nu(x) - \lambda\nu(x) = 0, \quad t \in I, \tag{2.1}$$
$$\nu(0) = a_0,\ \nu(1) = a_1, \tag{2.2}$$

where a_0 and a_1 are real numbers.

We recall that the solution of problem (2.1)-(2.2) is a function $\nu \in C(\bar{I})$, such that $\mathcal{D}^\alpha_{0+}\nu \in C(\bar{I})$, $\mathcal{D}^\alpha_{0+}\mathcal{D}^\alpha_{0+}\nu \in C(I)$.

Lemma 2.4 ([5]). *The solution of problem* (2.1)-(2.2) *exists, and is unique. Moreover, it can be written in the form*

$$\nu(x) = a_0 C(\lambda x) + a_1 S(\lambda x), \tag{2.3}$$

where

$$C(\lambda x) = \frac{E_{\alpha,1}(\sqrt{\lambda})E_{\alpha,1}(-\sqrt{\lambda}x^\alpha) - E_{\alpha,1}(-\sqrt{\lambda})E_{\alpha,1}(\sqrt{\lambda}x^\alpha)}{2\sqrt{\lambda}E_{2\alpha,\alpha+1}(\lambda)}, \tag{2.4}$$

$$S(\lambda x) = \frac{x^\alpha E_{2\alpha,\alpha+1}(\lambda x^{2\alpha})}{E_{2\alpha,\alpha+1}(\lambda)}. \tag{2.5}$$

Here

$$E_{\alpha,\mu}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \mu)}$$

is the Mittag - Leffler type function [4]*.*

It is easy to see that the function $E_{\alpha,1}(\pm\sqrt{\lambda}x^\alpha)$ for $0 < \alpha < 1$ satisfies the equation

$$\nu''(x) \mp \lambda D^{2-\alpha}_{0+}\nu(x) = 0, x \in I. \tag{2.6}$$

Lemma 2.5 ([9]). *If the function $\nu \in C(\bar{I}) \cap C^2(I)$, $\nu(x) \neq Const$ is a solution of Equation* (2.6)*, then it can not attain its positive maximum (negative minimum) within the segment $\bar{I}$.*

Lemma 2.6 ([4]). *For $E_{\alpha,\beta}(z)$ as $|z| \to \infty$ the following asymptotic estimation holds*

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{\frac{(1-\beta)}{\alpha}} e^{z^{\frac{1}{\alpha}}} - \sum_{k=1}^p \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O(\frac{1}{|z|^{p+1}}), \tag{2.7}$$

where $|\arg z| \le \rho_1\pi$, $\rho_1 \in (\frac{\alpha}{2}, \min\{1, \alpha\}), \alpha \in (0, 2)$, and for $\arg z = \pi$

$$E_{\alpha,\beta}(z) = \frac{1}{1 + |z|}, |z| \to \infty. \tag{2.8}$$

It is easy to show that functions C_k and S_k are solutions of (2.6) and

$$\begin{aligned} C_k(0) = 1, \quad C_k(1) = 0, \\ S_k(0) = 0, \quad S_k(1) = 1. \end{aligned} \tag{2.9}$$

Lemma 2.7. *For any $x \in [0,1]$ the following inequalities hold:*

$$0 \leq S(\lambda x), \quad C(\lambda x) \leq 1.$$

An application of the Fourier method to Problem 1.1 leads to the eigenvalue problem

$$\mathcal{L} := \mathcal{D}^{\beta}_{y,a+} D^{\beta}_{y,b-} \tau(y) = \lambda \tau(y), \quad a < y < b, \tag{2.10}$$

with the conditions

$$I^{1-\beta}_{y,b-} \tau(a) = 0, \quad I^{1-\beta}_{y,b-} \tau(b) = 0. \tag{2.11}$$

For the fractional Sturm-Liouville problem (2.10)-(2.11) the following assertions are true [10].

Lemma 2.8. *The fractional Sturm-Liouville problem* (2.10)-(2.11) *is self-adjoint and positive in $L^2(a,b)$.*

Lemma 2.9. *The spectrum of the fractional Sturm-Liouville problem* (2.10)-(2.11) *is discrete and positive, and the system of eigenfunctions is a complete orthogonal basis in $L^2(a,b)$.*

It is not difficult to show that the eigenvalue problem (2.10)-(2.11) is equivalent to the integral equation

$$\mathcal{L}^{-1} \tau(y) := \int_a^b \mathcal{K}(y,\xi) \tau(\xi) d\xi = \lambda^{-1} \tau(y), \tag{2.12}$$

where $\mathcal{K}(y,\xi) = \int_{\max\{y,\xi\}}^b \frac{(\zeta-y)^{\beta-1}(\zeta-\xi)^{\beta-1}}{\Gamma^2(\beta)} d\zeta$.

Now we state the following theorem proved by Delgado and Ruzhansky [3]

Theorem 2.10. *Let M be a closed manifold of dimension n. Let K belongs to the Sobolev space $H^{\mu}(M \times M)$ for some index $\mu > 0$. Then the integral operator T on $L^2(M)$, defined by*

$$(Tf) = \int_M K(x,s) f(s) ds,$$

is in the Schatten classes $S_p(L^2(M))$ for $p > \frac{2n}{n+2\mu}$.

Corollary 2.11. *The operator $\mathcal{L}^{-1}$, defined on $L^2(a,b)$ by* (2.12) *is in the Schatten classes $S_p(L^2(a,b))$ for $p > \frac{2}{1+4\beta}$.*

The above corollary provides a useful spectral property; that is,

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k^p} < \infty \tag{2.13}$$

for any $p > \frac{2}{1+4\beta}$.

3. Well-posedness of Problem 1.1

Theorem 3.1. *Let* $0 < \delta < 1$, $\mathcal{D}^{\beta}_{y,a+}D^{\beta}_{y,b-}\varphi(y) \in C^{1+\delta}[a,b]$, $\mathcal{D}^{\beta}_{y,a+}D^{\beta}_{y,b-}\psi(y) \in C^{\delta}[a,b]$ *and*

$$I^{1-\beta}_{y,b-}\varphi(a) = I^{1-\beta}_{y,b-}\varphi(b) = 0,$$
$$I^{1-\beta}_{y,b-}\psi(a) = I^{1-\beta}_{y,b-}\psi(b) = 0.$$

Then the solution of Problem 1.1 exists and is unique. Moreover, it can be written in the form

$$u(x,y) = \sum_{k=1}^{\infty}[\varphi_k C(\lambda_k x) + \psi_k S(\lambda_k x)]\tau_k(y), \tag{3.1}$$

where $\varphi_k = (\varphi(y), \tau_k(y))$, $\psi_k = (\psi(y), \tau_k(y))$ *and* $\tau_k(y)$ *are eigenfunctions of the problem* (2.10)-(2.11) *form an orthonormal basis in* $L^2(a,b)$.

Proof. **Existence of the solution.** Since the system of eigenfunctions $\{\tau_k(y)\}_{k\in\mathbb{N}}$ of the fractional Sturm-Liouville problem (2.10)-(2.11) forms an orthonormal basis in $L^2(a,b)$, the function u can be represented as follows

$$u(x,y) = \sum_{k=1}^{\infty}\nu_k(x)\tau_k(y), \quad \text{in} \quad \Omega, \tag{3.2}$$

where $\nu_k(x)$ are unknown functions. It is well known that if $\varphi(y)$ and $\psi(y)$ satisfy the conditions of Theorem 3.1, then they can be uniquely represented in uniformly and absolutely convergent Fourier series by $\{\tau_k(y)\}$:

$$\varphi(y) = \sum_{k=1}^{\infty}\varphi_k\tau_k(y),$$
$$\psi(y) = \sum_{k=1}^{\infty}\psi_k\tau_k(y),$$

where $\varphi_k = (\varphi, \tau_k)$, $\psi_k = (\psi, \tau_k)$.

Putting (3.2) into (1.1) and boundary conditions (1.2), for unknown functions $\nu_k(x)$, we obtain the problem

$$\mathcal{D}^{\alpha}_{0+}\mathcal{D}^{\alpha}_{0+}\nu_k(x) - \lambda_k\nu_k(x) = 0, \quad 0 < x < 1, \tag{3.3}$$
$$\nu_k(0) = \varphi_k, \quad \nu_k(1) = \psi_k. \tag{3.4}$$

By Lemma 2.4 the solution of (3.3)-(3.4) exists, is unique and it can be written in the form

$$\nu_k(x) = \varphi_k C(\lambda_k x) + \psi_k S(\lambda_k x),$$

where $C(\lambda_k x)$ and $S(\lambda_k x)$ are defined by (2.4) and (2.5), respectively. Furthermore, according to Lemma 2.7 inequalities

$$0 \le S(\lambda_k x),\, C(\lambda_k x) \le 1, x \in [0,1]$$

are true.

If for φ and ψ the conditions of Theorem 3.1 hold then

$$|\varphi_k| \le \frac{C}{\lambda_k^{2+\delta}},\, |\psi_k| \le \frac{C}{\lambda_k^{1+\delta}}, \quad C = \text{const}.$$

For such functions, we obtain

$$|\nu_k(x)| \le C\Big(\frac{1}{\lambda_k^{2+\delta}} + \frac{1}{\lambda_k^{1+\delta}}\Big). \tag{3.5}$$

Then taking into account the property (2.13) the convergence of the series (3.2) is obvious in $u(x,y) \in C(\bar{\Omega})$. Further, using estimates (2.7) and (2.8), we get

$$\begin{gathered} S_k(\lambda_k x) = O(\mathrm{e}^{\lambda_k^{1/\alpha}(x-1)}), \\ C(\lambda_k x) = O(\frac{1}{\sqrt{\lambda_k}}). \end{gathered} \tag{3.6}$$

Applying $\mathcal{D}^{\beta}_{y,a+}D^{\beta}_{y,b-}$ term by term of the series (3.2), one obtains

$$\mathcal{D}^{\beta}_{y,a+}D^{\beta}_{y,b-}u(x,y) = \sum_{k=1}^{\infty} \lambda_k \nu_k(x)\tau_k(y).$$

Then for all $x \ge x_0 > 0$, $a \le y \le b$, by taking into account inequalities (3.5), we have

$$\begin{aligned} |\mathcal{D}^{\beta}_{y,a+}D^{\beta}_{y,b-}u(x,y)| &\le C\sum_{k=1}^{\infty} |\lambda_k||\nu_k(x)| \\ &\le C\sum_{k=1}^{\infty} \lambda^{-1-\delta} + \lambda^{-\delta}e^{-\lambda_k(1-x)}. \end{aligned}$$

Similarly, we can estimate the series

$$\mathcal{D}^{\alpha}_{x,0+}\mathcal{D}^{\alpha}_{x,0+}u(x,y) = \sum_{k=1}^{\infty} \lambda_k \nu_k(x)\tau_k(y).$$

Then $\mathcal{D}^{\beta}_{y,a+}D^{\beta}_{y,b-}u(x,y)$, $\mathcal{D}^{\alpha}_{x,0+}\mathcal{D}^{\alpha}_{x,0+}u(x,y) \in C(\Omega)$.

Uniqueness of the solution. Suppose that there are two solutions $u_1(x,y)$ and $u_2(x,y)$ of Problem 1.1. Denote

$$u(x,y) = u_1(x,y) - u_2(x,y).$$

Then the function $u(x,y)$ satisfies (1.1) and homogeneous conditions (1.2) and (1.3).

Let

$$u_k(x) = \langle u(x,y), \tau_k(y)\rangle, k \in \mathbb{N}. \tag{3.7}$$

Applying the operator $\mathcal{D}^{\alpha}_{0+}\mathcal{D}^{\alpha}_{0+}$ to Equation (3.3), we have

$$\mathcal{D}^{\alpha}_{0+}\mathcal{D}^{\alpha}_{0+}u_k(x) = \langle \mathcal{D}^{\alpha}_{x,0+}\mathcal{D}^{\alpha}_{x,0+}u(x,y), \tau_k(y)\rangle = \langle \mathcal{D}^{\beta}_{a+,y}D^{\beta}_{b-,y}u(x,y), \tau_k(y)\rangle.$$

Integrating by parts and taking into account the homogeneous condition (1.2), we obtain

$$\mathcal{D}^{\alpha}_{0+}\mathcal{D}^{\alpha}_{0+}u_k(x) - \lambda_k u_k(x) = 0, \quad u_k(0) = 0, \quad u_k(1) = 0.$$

Consequently from Lemma 2.4 we get $u_k(x) \equiv 0$.

Further, by the completeness of the system $\{\tau_k(x)\}_{\mathbb{N}}$ in $L^2(a,b)$ we conclude that

$$u(x,t) \equiv 0, \quad 0 \le x \le 1, \quad a \le y \le b.$$

Hence, the uniqueness of the solution of Problem 1.1 is proved. □

4. Well-posedness of Problem 1.2

Theorem 4.1. *Let* $0<\delta<1$, $\mathcal{D}^{\beta}_{y,a+}D^{\beta}_{y,b-}\phi(y)\in C^{1+\delta}[a,b]$ *and*

$$I^{1-\beta}_{y,b-}\phi(a)=I^{1-\beta}_{y,b-}\phi(b)=0.$$

Then the solution of Problem 1.2 exists, is unique and can be represented as

$$u(x,y)=\sum_{k=1}^{\infty}\phi_k E_{\alpha,1}(-\sqrt{\lambda_k}x^{\alpha})\tau_k(y), \tag{4.1}$$

where $\phi_k=(\phi,\tau_k)$, *and* $\{\tau_k(y)\}_{k\in\mathbb{N}}$ *is the system of eigenfunctions of the problem* (2.10)-(2.11) *forms an orthonormal basis in* $L^2(a,b)$.

Proof. By applying the Fourier method to solve Problem 1.2, we lead it to the spectral problem (2.10)–(2.11). The system $\{\tau_k(y)\}_{k\in\mathbb{N}}$ is an orthonormal basis in the space $L^2(a,b)$. Thus, a regular solution of Problem 1.2 for all $x>0$ can be represented as the series

$$u(x,y)=\sum_{k=1}^{\infty}u_k(x)\tau_k(y), \tag{4.2}$$

where $u_k(x)$ is an unknown function. We expand the function $\phi(y)$ into the Fourier series by the system $\{\tau_k(y)\}_{k\in\mathbb{N}}$, that is,

$$\phi(y)=\sum_{k=1}^{\infty}\phi_k\tau_k(y), \tag{4.3}$$

where $\phi_k=(\phi,\tau_k)$.

Let us consider functions

$$u_k(x)=\int_a^b u(x,y)\tau_k(y)dy, \quad k\in\mathbb{N}. \tag{4.4}$$

Applying the operator $\mathcal{D}^{\alpha}_{0+}\mathcal{D}^{\alpha}_{0+}$ to the functions (4.4) and by taking into account Equation (1.1), we have

$$\mathcal{D}^{\alpha}_{0+}\mathcal{D}^{\alpha}_{0+}u_k(x)=\int_a^b \mathcal{D}^{\alpha}_{x,0+}\mathcal{D}^{\alpha}_{x,0+}u(x,y)\tau_k(y)dy=\int_a^b \mathcal{D}^{\beta}_{a+,y}D^{\beta}_{b-,y}u(x,y)\tau_k(y)dy.$$

Twice integrating by parts the last integral and by using the conditions (1.4) and (1.5), we obtain

$$\mathcal{D}^{\alpha}_{x,0+}\mathcal{D}^{\alpha}_{x,0+}u_k(x)-\lambda_k u_k(x)=0, \quad 0<x<+\infty, \tag{4.5}$$

$$u_k(0)=\phi_k, \quad \lim_{x\to+\infty}|u_k(x)|\to 0. \tag{4.6}$$

The general solution of Equation (4.5) has the form

$$u_k(x)=C_1E_{\alpha,1}(\sqrt{\lambda_k}x^{\alpha})+C_2E_{\alpha,1}(-\sqrt{\lambda_k}x^{\alpha}),$$

where C_1 and C_2 are unknown constants. Since $E_{\alpha,1}(\sqrt{\lambda_k}x^{\alpha})$ is completely monotonic [7], that is,

$$E_{\alpha,1}(\sqrt{\lambda_k}x^{\alpha})\to\infty, \quad x\to+\infty,$$

we need to choose $C_1=0$ to have the second condition in (4.6). Then

$$u_k(x)=C_2E_{\alpha,1}(-\sqrt{\lambda_k}x^{\alpha})$$

and by the first condition in (4.6) we have

$$u_k(x) = \phi_k E_{\alpha,1}(-\sqrt{\lambda_k}x^\alpha).$$

Furthermore, the identity (4.4) directly implies the uniqueness of the solution of Problem 1.2: if $\phi(y) = 0$ on $[a, b]$ then $u_k(x) = 0$ on $[0, +\infty)$. Consequently, due to the completeness of the system $\{\tau_k(y)\}_{k\in\mathbb{N}}$ we obtain $u(x, y) = 0$ for all $(x, y) \in \Omega_\infty$.

Therefore, the formal solution of Problem 1.2 can be represented as in (3.1). If the function $\phi(y)$ satisfies conditions of Theorem 4.1, then for the Fourier coefficients we get inequality:

$$|\phi_k| \le \frac{C}{\lambda_k^{1+\delta}}.$$

Then for all $y \in [a, b]$, for each $x \in [0, +\infty)$ we conclude

$$|u(x, y)| \le \sum_{k=1}^{\infty} \frac{C}{\lambda_k^{1+\delta}} < \infty,$$

i.e., the series (3.1) converges uniformly in the domain $[a, b] \cap [0, \infty)$. Therefore, $u \in C(\bar{\Omega}_\infty)$. Similarly, we show that $\mathcal{D}^\alpha_{x,0+}\mathcal{D}^\alpha_{x,0+}u \in C(\Omega_\infty)$, $\mathcal{D}^\beta_{y,a+}D^\beta_{y,b-}u \in C(\Omega_\infty)$. The proof is complete. □

5. Non-Homogeneous case

In this section we study a non-homogeneous fractional Laplace equation

$$\mathcal{D}^\alpha_{x,0+}\mathcal{D}^\alpha_{x,0+}u(x, y) - \mathcal{D}^\beta_{y,a+}D^\beta_{y,b-}u(x, y) = f(x, y), \quad (x, y) \in \Omega, \tag{5.1}$$

with the boundary conditions

$$u(0, y) = 0, \quad u(1, y) = 0, \quad a \le y \le b, \tag{5.2}$$

$$I^{1-\beta}_{b-,y}u(x, a) = 0, \quad I^{1-\beta}_{b-,y}u(x, b) = 0, \quad 0 \le x \le 1, \tag{5.3}$$

for some sufficiently smooth function f.

Theorem 5.1. *Let $0 < \delta < 1$. Assume that $f \in C(\bar{\Omega})$. Then there is a unique solution $u \in C(\bar{\Omega})$ of the problem (5.1)-(5.3) such that*

$$\mathcal{D}^\alpha_{x,0+}u \in C(\Omega), \quad \mathcal{D}^\alpha_{x,0+}\mathcal{D}^\alpha_{x,0+}u \in C(\Omega), \quad \mathcal{D}^\beta_{y,a+}D^\beta_{y,b-}u \in C(\Omega).$$

Moreover, we have the expansion

$$\begin{aligned} u(x, y) &= \sum_{k=1}^{\infty} \tau_k(y) \int_0^x (x - s)^{2\alpha-1} C_k(\lambda_k(x - s)) f_k(s) ds \\ &\quad - \sum_{k=1}^{\infty} \tau_k(y) S(\lambda_k x) \int_0^1 (1 - s)^{2\alpha-1} C_k(\lambda_k(1 - s)) f_k(s) ds. \end{aligned} \tag{5.4}$$

Here, $f_k(x)$ is from

$$f(x, y) = \sum_{k=1}^{\infty} f_k(x)\tau_k(y),$$

where $\{\tau_k\}_{k=1}^\infty$ is an orthonormal basis in $L^2(a, b)$ and a system of eigenfunctions generated by the spectral problem (2.10)–(2.11); that is,

$$\mathcal{D}^\beta_{y,a+}D^\beta_{y,b-}\tau(y) = \lambda\tau(y), \quad a < y < b,$$

with the conditions

$$I^{1-\beta}_{y,b-}\tau(a) = 0, \quad I^{1-\beta}_{y,b-}\tau(b) = 0.$$

Proof. **Existence of the solution.** Since the system of eigenfunctions $\{\tau_k(y)\}_{k=\mathbb{N}}$ of the fractional problem (2.10)–(2.11) forms an orthonormal basis in $L^2(a,b)$, then for u we obtain the representation

$$u(x,y) = \sum_{k=1}^{\infty} \nu_k(x)\tau_k(y), \quad (x,y) \in \Omega, \tag{5.5}$$

where $\nu_k(x)$ are unknown functions.

By using the representation (5.5), from (5.1)–(5.2) for the unknown functions $\nu_k(x)$ we get the problem

$$\mathcal{D}^{\alpha}_{x,0+}\mathcal{D}^{\alpha}_{x,0+}\nu_k(x) - \lambda_k\nu_k(x) = f_k(x), \quad 0 < x < 1, \tag{5.6}$$

$$\nu_k(0) = 0, \quad \nu_k(1) = 0. \tag{5.7}$$

Applying the method in [1], it is not difficult to show that the general solution of Equation (5.6) has the form

$$\begin{aligned} \nu_k(x) &= C_1E_{\alpha,1}(\sqrt{\lambda_k}x^{\alpha}) + C_2E_{\alpha,1}(-\sqrt{\lambda_k}x^{\alpha}) \\ &\quad + \int_0^x (x-s)^{2\alpha-1}C_k(\lambda_k(x-s))f_k(s)ds. \end{aligned} \tag{5.8}$$

Using the boundary conditions (5.7), we obtain the unique solution of the problem (5.6)-(5.7)

$$\begin{aligned} \nu_k(x) &= \int_0^x (x-s)^{2\alpha-1}C_k(\lambda_k(x-s))f_k(s)ds \\ &\quad - S(\lambda_k x)\int_0^1 (1-s)^{2\alpha-1}C_k(\lambda_k(1-s))f_k(s)ds, \end{aligned}$$

where $S(\lambda_k x)$ is defined by (2.5). Furthermore, according to Lemma 2.7, the following inequality holds

$$0 \le S(\lambda_k x), \quad C(\lambda_k x) \le 1, \quad x \in [0,1].$$

Now, By Lemma 2.7, ν_k satisfies

$$\begin{aligned} &|\nu_k(x)| \\ &\le \int_0^x (x-s)^{2\alpha-1}C_k(\lambda_k(x-s))|f_k(s)|ds + \int_0^1 (1-s)^{2\alpha-1}C_k(\lambda_k(1-s))|f_k(s)|ds \\ &\le \max_x |f_k|(x^{2\alpha}C_k(\lambda_k x) + C_k(\lambda_k)) \\ &\le C\frac{\max_x |f_k|}{1+\lambda_k}, \end{aligned}$$

where C is a constant. Then the series (5.4) converges uniformly in the domain $\bar{\Omega}$ and therefore $u(x,y) \in C(\bar{\Omega})$. Further, using the estimate

$$S_k(\lambda_k x) = O(\mathrm{e}^{\lambda_k^{1/\alpha}(x-1)}),$$

we can prove that $\mathcal{D}^{\beta}_{y,a+}D^{\beta}_{y,b-}u(x,y), \mathcal{D}^{\alpha}_{x,0+}\mathcal{D}^{\alpha}_{x,0+}u(x,y) \in C(\Omega)$.

Uniqueness of the solution of the problem (5.1)-(5.3) follows from the uniqueness of the solution of Problem 1.1. □

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Niyaz Tokmagambetov
Al-Farabi Kazakh National University, 050040, Al-Farabi ave., 71, Almaty, Kazakhstan.
Institute of Mathematics and Mathematical Modeling, 050010, Pushkin st., 125, Almaty, Kazakhstan
E-mail address: `tokmagambetov@math.kz`

Berikbol T. Torebek
Institute of Mathematics and Mathematical Modeling, 050010, Pushkin st., 125, Almaty, Kazakhstan.
Al-Farabi Kazakh National University, 050040, Al-Farabi ave., 71, Almaty, Kazakhstan
E-mail address: `torebek@math.kz`