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NEW AND UPDATED SEMIDEFINITE PROGRAMMING BOUNDS FOR SUBSPACE CODES

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Abstract. We show that $A_2(7,4) \leq 388$ and, more generally, $A_q(7,4) \leq (q^2-q+1)[7]+q^4-2q^3+3q^2-4q+4$ by semidefinite programming for $q \leq 101$. Furthermore, we extend results by Bachoc et al. on SDP bounds for $A_2(n,d)$, where d is odd and n is small, to $A_q(n,d)$ for small q and small n.

1. Introduction.

By $\mathcal{P}(V)$ we denote the set of all subspaces in a finite dimensional vector space V over a finite field of order q. The set $\mathcal{P}(V)$ forms a metric space with respect to the *subspace metric* $d_s(U,W) = \dim(U+W) - \dim(U \cap W)$. The space $(\mathcal{P}(V), d_s)$ plays an important role in random linear network coding and was introduced by Kötter and Kschischang in [27] to describe error-detecting and -correcting transmission of informations in the subspace channel model. A subset $\mathcal{C}$ of $\mathcal{P}(V)$ is called *subspace code* and its elements are called *codewords*. The subspace distance of $\mathcal{C}$ is given by $d_s(\mathcal{C}) = \min\{d_s(U,W) : U, W \in V \text{ and } U \neq W\}$. We refer the reader to Subsection 2.1 for a more detailed introduction to the used terminology.

The vector $(x_0(\mathcal{C}), \ldots, x_n(\mathcal{C}))$ with $x_k(\mathcal{C})$ as the number of k-subspaces in $\mathcal{C}$ is called the *dimension distribution* of $\mathcal{C}$ and the set $K(\mathcal{C}) = \{\dim(U) : U \in \mathcal{C}\}$ contains the dimensions of all codewords of $\mathcal{C}$. We drop the reference to $\mathcal{C}$ if it is clear by the context. Then $(n, N, d; K)_q$ abbreviates the parameters of $\mathcal{C}$; $\mathcal{C} \subseteq \mathcal{P}(\mathbb{F}_q^n)$, $N = |\mathcal{C}|$, $d \leq d_s(\mathcal{C})$, and $K(\mathcal{C}) \subseteq K$. If $K(\mathcal{C}) = \{k\}$, say, then $\mathcal{C}$ is called *constant-dimension code* (CDC) and is abbreviated as $(n, N, d; k)_q$. In the other extremal case, i.e., $K = \{0, \ldots, n\}$, the parameters of an (unrestricted) subspace code are abbreviated as $(n, N, d)_q$.

The maximum cardinality N of an $(n, N, d; K)_q$ subspace code is denoted as $A_q(n, d; K)$ and the simpler notation $A_q(n, d; k)$ in the constant-dimension case and $A_q(n, d)$ in the unrestricted case applies, too. The determination of $A_q(n, d; K)$,

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or at least suitable bounds, and a classification of all non-isomorphic maximum cardinality codes is known as the *main problem of subspace coding*, since it is the q-analog of the *main problem of classical coding theory*, cf. [29, Page 23].

The smallest undetermined and arguably most interesting constant-dimension code is a maximum cardinality set of planes in $\mathbb{F}_2^7$ mutually intersecting in at most a point. Here the best known result is as follows:

Fact 1.1 ([15, Theorem 2])**.** *We have* $333 \leq A_2(7,4;3) \leq 381$.

The lower bound was derived by finding a $(7,333,4;3)_2$ CDC after modifying *interesting* codes arising in an exhaustive search in the $\mathrm{GL}(\mathbb{F}_2^7)$ for subgroups with the property being subgroup of automorphism groups of *large* $(7,N,4;3)_2$ CDCs. The currently best upper bound is a simple counting argument: There are $\begin{bmatrix}7\\2\end{bmatrix}_2 = 2667$ lines in $\mathbb{F}_2^7$, each plane contains $\begin{bmatrix}3\\2\end{bmatrix}_2 = 7$ of them and no line is incident with two codewords, hence $2667/7 = 381$ upper bounds the size of any $(7,N,4;3)_2$ CDC. Any putative $(7,381,4;3)_2$ CDC is the binary analog of a Fano plane and a lot of previous work tackle its existence question [1, 4–6, 9–11, 13, 14, 22, 24–26, 28, 30, 31, 33, 34].

By omitting the constraint on the dimension of codewords, one arrives at $(7,M,4)_2$ subspace codes. Of course, a $(7,N,4;3)_2$ CDC $\mathcal{C}$ can be extended to $(7,N+1,4)_2$ subspace code $\mathcal{C} \cup \{\mathbb{F}_q^7\}$, providing the best known lower bound $334 \leq A_2(7,4)$. Due to Honold et al. we know the following:

Fact 1.2 ([23, Theorem 4.1])**.** *We have* $A_2(7,4) \leq 407$.

We improve this to:

Theorem 1.1. *We have* $A_2(7,4) \leq 388$.

If equality holds, then the corresponding code consists up to orthogonality of 41 lines and 347 solids (see Lemma 4.1). The correspondence to constant-dimension codes shows in particular that a putative binary Fano plane would imply a $(7,382,4)_2$ subspace code and hence reducing the upper bound to less than 382 would immediately imply the nonexistence of the binary Fano plane – a seemingly very difficult problem.

In the general case, the best bounds are $q^8+q^5+q^4+q^2-q \leq A_q(7,4;3) \leq \begin{bmatrix}7\\2\end{bmatrix}/\begin{bmatrix}3\\2\end{bmatrix} = (q^2-q+1)[7]$; the lower bound is provided by [22, Theorem 4] and the upper bound arises again by counting lines. In the unrestricted case, the augmentation of a CDC by $\mathbb{F}_q^7$ provides again the best known lower bound of $q^8+q^5+q^4+q^2-q+1 \leq A_q(7,4)$. For the upper bound in the unrestricted case, the best previously known method is to relax the minimum distance condition from 4 to 3 and then to apply the integer linear programming argument from [12, Theorem 10].

Define the function $F(q)$ by

$$F(q) = \begin{cases} (q^2-q+1)[7]+q^4-2q^3+3q^2-4q+3 & \text{for } q=2,3, \\ (q^2-q+1)[7]+q^4-2q^3+3q^2-4q+4 & \text{for } q \geq 4. \end{cases}$$

Theorem 1.2. *Let* $2 \leq q \leq 101$ *be a prime power. We have* $A_q(7,4) \leq F(q)$.

This gives $388, 7696, 71157, 410585$ for $q = 2,3,4,5$, while the previous best known bounds were $407, 15802, 144060, 826594$. The bound $q \leq 101$ is chosen rather arbitrarily and we conjecture that it is unnecessary. For general q, we could only show the following.

Theorem 1.3. *Let $2 \le q$ be a prime power. We have $A_q(7,4) \le (q^2-q+1)[7] + 2(q^5+q^3+1)$.*

Previously, Bachoc et al. applied semidefinite programming in [2] to binary subspace codes with odd minimum distance and $n \le 16$. We extend their results in several ways: (1) Since Bachoc et al. computed their bounds, several new upper bounds for small CDC codes were discovered, cf. http://subspacecodes.uni-bayreuth.de/ associated with [16]. Using these new bounds, we provide an update on their bounds (with a slightly differently chosen range of parameters). (2) We provide bounds for d even. (3) We compute bounds for $q > 2$. Our range for all these computations is mostly arbitrary, but chosen in a way that the computations terminate in less than a week on standard hardware at the time of writing.

The paper is organized as follows. In Section 2 we introduce basic definitions and the used theoretical framework of semidefinite programming in coherent configurations, so that we can describe the coherent configuration and semidefinite program which is associated with the symmetry group of the metric space $(\mathcal{P}(V), d_s)$ in Section 3. This culminates in Section 4, in which we investigate $A_q(7,4)$ and show our main results, and Section 5, in which we update the SDP bounds given by Bachoc et al. To conclude this current overview on semidefinite programming for subspace codes, we provide some bounds on quadruples for the binary analog of the Fano plane in Section 6.

2. **Preliminaries.**

2.1. **Subspace Codes.**

Let $2 \le q$ be a prime power, $\mathbb{F}_q$ the field with q elements, and $V \cong \mathbb{F}_q^n$ the n-dimensional vector space over $\mathbb{F}_q$. By $\mathcal{P}(V)$ we denote the set of all subspaces in V. For two subspaces $U, W \in \mathcal{P}(V)$ we write $U \le W$ iff U is subspace of W. Recall that $\mathcal{P}(V)$ forms a metric space with respect to the *subspace metric* [27, Section 3.1]

$$d_s(U,W) = \dim(U+W) - \dim(U \cap W).$$

For $k \in \{0,1,\ldots,v\}$, $\begin{bmatrix} V \\ k \end{bmatrix}$ denotes the set of k-dimensional subspaces in V. Its cardinality is given by the *q-binomial coefficient*

$$|\begin{bmatrix} V \\ k \end{bmatrix}| = \begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^{k} \frac{q^{n-k+i}-1}{q^i-1}.$$

As an abbreviation we use the *q-number* $[n]_q = \begin{bmatrix} n \\ 1 \end{bmatrix}_q$ and drop the index q in $[n]_q$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q$ if there is no confusion with $\begin{bmatrix} V \\ k \end{bmatrix}$ and q is clear by the context. Using the *q-factorial* $[n]! = \prod_{i=1}^{n}[i]$, the q-binomial coefficient can then be expressed as $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$. A k-dimensional subspace of V is called simply k-subspace and we refer to 1-subspaces as points, 2-subspaces as lines, 3-subspaces as planes, 4-subspaces as solids, and $(n-1)$-subspaces as hyperplanes.

Let $\mathcal{C}$ be a subspace code. Recall that for $2 \le |\mathcal{C}|$ the subspace distance of $\mathcal{C}$ is given by $d_s(\mathcal{C}) = \min\{d_s(U,W) : U, W \in V \text{ and } U \ne W\}$ and notice that we formally set $d_s(\mathcal{C}) = \infty$ if $|\mathcal{C}| \le 1$.

By $x_i(\mathcal{C})$ we denote the number of i-subspaces in $\mathcal{C}$ and drop the reference to $\mathcal{C}$ if it is clear from the context.

The automorphism group of $(\mathcal{P}(V), d_s)$ for $3 \le n$ was shown to be generated by $\mathrm{P\Gamma L}(V)$ and a polarity $\pi : \mathcal{P}(V) \to \mathcal{P}(V), U \mapsto U^\perp$ (see e.g. [23, Theorem 2.1]). We call $U^\perp$ the *orthogonal space* of U and apply π also to subspace codes $\mathcal{C}$ to obtain their *orthogonal codes* $\mathcal{C}^\perp$. If $\mathcal{C}$ is an $(n, N, d; K)_q$ subspace code with dimension distribution $(x_0(\mathcal{C}), \ldots, x_n(\mathcal{C}))$, then $\mathcal{C}^\perp$ is an $(n, N, d; \{n - i : i \in K\})_q$ subspace code with dimension distribution $(x_n(\mathcal{C}), \ldots, x_0(\mathcal{C}))$, in particular $A_q(n, d; k) = A_q(n, d; n - k)$.

2.2. Coherent Configurations.

We follow the notation and point of view by Hobart and Williford for applying a semidefinite programming bound which is set in the context of coherent configurations and we refer to their work for a general introduction to that topic [17, 18, 20, 21].

Definition 2.1. Let X be a finite set. A coherent configuration is a pair $(X, \mathcal{R})$, where $\mathcal{R} = \{R_0, \ldots, R_l\}$ is a set of binary relations on X with the following properties:

(a) $\mathcal{R}$ is a partition of $X \times X$.
(b) If $R_i \cap \mathrm{diag}(X \times X) \neq \emptyset$, then $R_i \subseteq \mathrm{diag}(X \times X)$.
(c) If $R_i \in \mathcal{R}$, then $R_i^T \in \mathcal{R}$.
(d) For $R_i, R_j, R_k \in \mathcal{R}$ and $x, y \in X$ with $(x, y) \in R_k$, the number of z such that $(x, z) \in R_i$ and $(z, y) \in R_j$ is a constant p_{ij}^k, independent of the choice of x and y.

These p_{ij}^k are commonly called *intersection numbers*. Condition (b) gives a partition of the identity relation into sets X_a called *fibers*. In the group case, i.e., a group G operating on the finite set X, the induced component-wise action of G on $X \times X$ yields a coherent configuration in which the relations are given by the orbits of G on $X \times X$, cf. [19, Pages 212 and 217]. Each relation is contained in some $X_a \times X_{a'}$. If we restrict X to some X_a, then we obtain a (homogeneous) *association scheme*. For each R_i we can define an $|X| \times |X|$ matrix A_i indexed by X with

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in R_i, \\ 0 & \text{otherwise.} \end{cases}$$

The matrices $\{A_0, \ldots, A_l\}$ generate an algebra $\mathcal{A}$ with several useful properties. For the representation theory of $\mathcal{A}$ we follow the notation of [21]. Let $\{\Delta_1, \ldots, \Delta_m\}$ the set of absolutely irreducible representations of $\mathcal{A}$, chosen such that $\Delta_s(A^*) = (\Delta_s(A))^*$. Denote the multiplicity of Δ_s by f_s. Let γ denote the number of fibers of the coherent configuration and E_{ij} the $(\gamma \times \gamma)$-matrix with a 1 at position (i, j) and 0 otherwise. Since $\mathcal{A}$ is semisimple, it decomposes into a direct sum of algebras $\mathcal{E}_s$. There exists a basis $\mathcal{E}_{ij}^s$ for each algebra $\mathcal{E}_s$ satisfying the following equations:

$$\mathcal{E}_{ij}^s \mathcal{E}_{kl}^t = \delta_{st}\delta_{jk}\mathcal{E}_{il}^s, \qquad (\mathcal{E}_{ji}^s)^* = \mathcal{E}_{ij}^s, \text{ and } \qquad \Delta_s(\mathcal{E}_{ij}^t) = \delta_{st} E_{ij}. \tag{1}$$

Let $m_i = |R_i|$. Then

$$A_k = \sum_{i,j,s} (\Delta_s(A_k))_{ij} \mathcal{E}_{ij}^s \qquad \text{and} \qquad \mathcal{E}_{ij}^s = f_s \sum_k \frac{1}{m_k} \overline{(\Delta_s(A_k))_{ij}} A_k. \tag{2}$$

The next lemma shows bounds on subsets of X in terms of the positive semidefiniteness of involved matrices. Bounds arising by this method are commonly called semidefinite programming bound as it is a generalization of Delsarte's linear programming bound [8].

Theorem 2.2 ([20, Theorem 2.2 and 2.3])**.** *Let $(X, \mathcal{R})$ be a coherent configuration, $Y \subseteq X$, and $b_i = |(Y \times Y) \cap R_i|$. Define $D(Y) = \sum_{i=1}^{l} \frac{b_i}{m_i} A_i$. Then the matrices $D(Y)$ and $\Delta_s(D(Y))$ are positive semidefinite for any irreducible representation Δ_s of the coherent configuration satisfying $\Delta_s(A^*) = (\Delta_s(A))^*$.*

If all fibers of a coherent configuration correspond to a commutative association scheme, we can use the the intersection numbers, i.e., the algebra generated by the *intersection matrices* $L_i = (p_{ij}^k)_{kj}$, to first calculate all $\mathcal{E}_{ij}^s$ via the eigenvalues of the association scheme restricted to the fibers (see [7, Chapter 2, Proposition 2.2.2]) and then apply the identities (1) to determine the remaining parameters. In Section 3.3 we provide details for this calculation.

Since each relation is contained in some $X_a \times X_b$ we index the relations, basis matrices, intersection numbers, etc. accordingly: R_{abl}, A_{abl}, $p_{(a,d,i),(d,b,j)}^{(a,b,k)}$, m_{abl}, and b_{abl} such that a, b, d are indices of fibers and l, k, i, j are counters. In particular, all other intersection numbers are zero. The first equation of the identities (2) is hence $A_{abl} = \sum_s (\Delta_s(A_{abl}))_{ab} \mathcal{E}_{ab}^s$.

2.3. **Semidefinite programming.** We abbreviate the term positive semidefinite as psd and for symmetric matrices A and B we write $A \succcurlyeq B$ iff $A - B$ is psd. A *semidefinite program* (SDP) is an optimization problem of the form

$$\begin{aligned} &\min c^T x \\ &\text{subject to } \sum_{i=1}^{m} F_i x_i \succcurlyeq F_0 \\ &x \in \mathbb{R}^m \end{aligned} \tag{3}$$

with $c \in \mathbb{R}^m$ and symmetric $F_i \in \mathbb{R}^{n \times n}$ for $i \in \{0, \ldots, m\}$. The dual problem associated with (3) (which is then called primal) is

$$\begin{aligned} &\max \operatorname{tr}(F_0 Z) \\ &\text{subject to } \operatorname{tr}(F_i Z) = c_i \text{ for all } i \in \{1, \ldots, m\} \\ &Z \succcurlyeq 0 \end{aligned}$$

and, if the primal and dual contain feasible points x and Z, the optimal value of the dual lower bounds the optimal value of the primal. We have equality if the primal or the dual contains strictly feasible points, cf. [35, Page 64 and Theorem 3.1]. Although it can be solved in polynomial time with the ellipsoid method, interior-points methods are often faster in practice cf. [35, Page 52] and [36].

Using the Schur complement, many quadratic inequalities can be modeled as constraints in an SDP: Let $\left(\begin{smallmatrix} A & B \\ B^T & C \end{smallmatrix}\right)$ be symmetric and A be positive definite, then M is psd iff $C - B^T A^{-1} B$ is psd. In particular, using I as an identity matrix of appropriate size, $\left(\begin{smallmatrix} I & Ax-b \\ (Ax-b)^T & c^T x - d \end{smallmatrix}\right)$ is positive semidefinite iff $(Ax - b)^T (Ax - b) \le c^T x - d$.

Unless the complexity classes P and NP coincide, in general quadratic equations are not possible to model in an SDP, e.g. $x \in \{0, 1\}$ is equivalent to $x(x-1) = 0$ and the Schur complement allows to rewrite $x(x-1) \le 0$ as $\left(\begin{smallmatrix} 1 & x \\ x & x \end{smallmatrix}\right) \succcurlyeq 0$ but $x(x-1) \ge 0$ as constraint in an SDP would imply the solvability the NP-complete binary linear programming with polynomial time algorithms of SDPs.

If multiple matrices shall be psd simultaneously, they are commonly arranged as blocks on the main diagonal of the F_i and linear inequalities are commonly embedded as diagonal matrices, hence any linear program can be written as an SDP.

3. The Coherent Configuration of $\mathrm{P\Gamma L}(V)$ operating on $\mathcal{P}(V)$.

3.1. Triples in Vector Spaces. In this section we provide a general formula for counting triples in vector spaces.

Lemma 3.1. *Let A be an a-space and B a b-space with $c = \dim(A \cap B)$ in $\mathbb{F}_q^{a+b-c}$. Then the number of d-spaces D having trivial intersection with A and B is*

$$\psi(a,b,c,d) := \prod_{j=0}^{d-1} \frac{q^{j+c}(q^{a-c-j}-1)(q^{b-c-j}-1)}{q^{d-j}-1}.$$

Proof. We double count $((P_0,\dots,P_{d-1}),D)$, where $(P_0,\dots,P_{d-1})$ is an ordered basis of D. For $P_0,\dots,P_{j-1}$ given, we have

$$[a+b-c]-[a+j]-[b+j]+[c+2j] = \frac{q^{2j+c}(q^{a-c-j}-1)(q^{b-c-j}-1)}{q-1}$$

choices for P_j. Hence, we have $\prod_{j=0}^{d-1} \frac{q^{2j+c}(q^{a-c-j}-1)(q^{b-c-j}-1)}{q-1}$ choices for $(P_0,\dots,P_{d-1})$. Similarly, the number of choices for $(P_0,\dots,P_{d-1})$ with given D is $\prod_{j=0}^{d-1}([d]-[j]) = \prod_{j=0}^{d-1} \frac{q^j(q^{d-j}-1)}{q-1}$, showing the assertion. □

Lemma 3.2. *Let A be an a-space and B a b-space with $c = \dim(A \cap B)$ in $\mathbb{F}_q^{a+b-c}$. Then the number of d-spaces D meeting A in an α-space, B in an β-space and $A \cap B$ in a γ-space is $\varphi(a,b,c,d,\alpha,\beta,\gamma) :=$*

$$\begin{bmatrix} c \\ \gamma \end{bmatrix} q^{(\alpha+\beta-2\gamma)(c-\gamma)} \begin{bmatrix} a-c \\ \alpha-\gamma \end{bmatrix} \begin{bmatrix} b-c \\ \beta-\gamma \end{bmatrix} \psi(a-\alpha, b-\beta, c-\gamma, d-\alpha-\beta+\gamma).$$

Proof. Clearly, there are $\begin{bmatrix} c \\ \gamma \end{bmatrix}$ choices for $A \cap B \cap D$. It is well-known that the remaining choices for $A \cap D$ and $B \cap D$ are

$$q^{(\alpha+\beta-2\gamma)(c-\gamma)} \begin{bmatrix} a-c \\ \alpha-\gamma \end{bmatrix} \begin{bmatrix} b-c \\ \beta-\gamma \end{bmatrix}.$$

In the quotient of $\langle A \cap D, B \cap D \rangle$ we see that we have $\psi(a-\alpha, b-\beta, c-\gamma, d-\alpha-\beta+\gamma)$ choices left to complete D. □

Now we obtain the following:

Lemma 3.3. *Let A be an a-space and B a b-space with $c = \dim(A \cap B)$ in $\mathbb{F}_q^n$. Then the number of d-spaces D meeting A in an α-space, B in an β-space and $A \cap B$ in a γ-space is*

$$\chi(a,b,c,d,n,\alpha,\beta,\gamma) := \sum_{x=\alpha+\beta-\gamma}^{\min\{d,a+b-c\}} q^{(d-x)(a+b-c-x)} \begin{bmatrix} n-a-b+c \\ d-x \end{bmatrix} \varphi(a,b,c,x,\alpha,\beta,\gamma).$$

Hence, we conclude that we can count triples as follows.

Lemma 3.4. *Let A be an a-space and B a b-space which meet in codimension k in $\mathbb{F}_q^n$. Then the number of d-spaces D meeting A in codimension i and B in codimension j is*

$$\sum_{\ell=0}^{\min\{a,b\}-k} \chi(a,b,\min\{a,b\}-k, d, n, \min\{a,d\}-i, \min\{b,d\}-j, \min\{a,b\}-k-\ell).$$

The intersection numbers $p^{(a,b,k)}_{(a,d,i),(d,b,j)}$ are given by the expression in the last lemma and all other intersection numbers vanish.

3.2. **Irreducible Representations.** The coherent configuration in this paper arises by the action of $\mathrm{P\Gamma L}(V)$ on $\mathcal{P}(V) \times \mathcal{P}(V)$. Hence, we have the $n+1$ fibers labeled with $0, 1, \ldots, n$, such that the k-th fiber consists of all k-spaces of V. A pair of subspaces (x, y) is in the relation R_{abc} iff x has dimension a, y has dimension b, and $c = \min\{a, b\} - \dim(x \cap y)$ for all $a, b \in \{0, \ldots, n+1\}$ and $c \in \{0, \ldots, \min\{\min\{a, b\}, n - \min\{a, b\}\}\}$. The benefit of choosing c as the codimension of the intersection is that R_{ii0} corresponds to the identity on the i-th fiber. The fibers of this coherent configuration are obviously symmetric association schemes and hence by [17, Chapter 4] commutative. For $V \cong \mathbb{F}_q^7$, we show in Corollary 4.6 that the 0-space and the 7-space cannot be contained in a large subspace code and hence we restrict ourself in this case to proper subspaces.

Since we investigate the bound on $A_q(7, 4)$ analytically, Table 1 shows the representation explicitly in the style of Hobart and Williford [21]. To improve the notation, we also introduce the abbreviations $\varphi = q^2 + 1$ and $\psi = q^2 - q + 1$. Notice that

$$|X_a| = \begin{bmatrix} 7 \\ a \end{bmatrix}, \qquad \Delta_s(A_{xyc}) = E_{xa}\Delta_s(A_{abc})E_{by}, \text{ and } \qquad m_{xyc} = m_{abc} \tag{4}$$

for $(x, y) \in \{(a, b), (b, a), (7-a, 7-b), (7-b, 7-a)\}$ by orthogonality and symmetry for all a, b, c, and s.

3.3. **Calculating the Irreducible Representation.** Let us outline how to calculate Δ_s. Since our fibers are commutative, we can use standard techniques for commutative association schemes, see [7, Prop. 2.2.2], to calculate

$$A_{iik} = \sum_s (\Delta_s(A_{iik}))_{ii}\mathcal{E}^s_{ii}.$$

This yields the entries of $\mathcal{E}^s_{ii}$ for all i and s. Notice that $M_{xy} = M_{x'y'}$ for all matrices $M \in \mathcal{A}$ if (x, y) and (x', y') are in the same relation, in particular we write $M_{(i,j,k)}$ for M_{xy} with some $(x, y) \in R_{ijk}$. Now let $i \neq j$. By Equation (1), we know that

$$\mathcal{E}^s_{ii}A_{ijk} = \mathcal{E}^s_{ii}\left(\sum_{s'}(\Delta_{s'}(A_{ijk}))_{ij}\mathcal{E}^{s'}_{ij}\right) = (\Delta_s(A_{ijk}))_{ij}\mathcal{E}^s_{ij}.$$

Note that $\mathcal{E}^s_{ii}A_{ijk} = ((A_{ijk})^T(\mathcal{E}^s_{ii})^T)^T = (A_{jik}\mathcal{E}^s_{ii})^T$ since $\mathcal{E}^s_{ii}$ is symmetrical. Hence, using the triple intersection numbers, we can derive $(\Delta_s(A_{ijk}))_{ij}\mathcal{E}^s_{ij}$. To be more precise, using the previous two equalities we have

$$\begin{aligned}((\Delta_s(A_{ijk}))_{ij}\mathcal{E}^s_{ij})_{xy} = (\mathcal{E}^s_{ii}A_{ijk})_{xy} = (A_{jik}\mathcal{E}^s_{ii})_{yx} &= \sum_z (A_{jik})_{yz}(\mathcal{E}^s_{ii})_{zx} \\ &= \sum_{(y,z)\in R_{jik}} (\mathcal{E}^s_{ii})_{zx} = \sum_\ell p^{(j,i,m)}_{(j,i,k),(i,i,l)}(\mathcal{E}^s_{ii})_{(i,i,l)},\end{aligned}$$

in which m is defined by $(y, x) \in R_{jim}$. As $\mathcal{E}^s_{ij}\mathcal{E}^s_{ji} = \mathcal{E}^s_{ii}$, this is sufficient to calculate $\mathcal{E}^s_{ij}$. Notice that this is not unique as we can replace $\mathcal{E}^s_{ij}$ by $-\mathcal{E}^s_{ij}$ and all conditions on the $\mathcal{E}^s_{ij}$ such as $\mathcal{E}^s_{ij}\mathcal{E}^s_{jk} = \mathcal{E}^s_{ik}$ are still satisfied. After we have chosen $\mathcal{E}^s_{ij}$, we can determine $(\Delta(A_{ijk}))_{ij}$ by solving Equation (2).

A_{abc}	$m_{abc}/\lvert X_a\rvert$	$\Delta_0(A_{abc})$	$\Delta_1(A_{abc})$	$\Delta_2(A_{abc})$	$\Delta_3(A_{abc})$
A_{110}	1	E_{11}	E_{11}		
A_{111}	$q[6]$	$q[6]E_{11}$	$-E_{11}$		
A_{120}	$[6]$	$[2]\sqrt{\psi[3]}E_{12}$	$\sqrt{q[5]}E_{12}$		
A_{121}	$q^2\psi[3][5]$	$q^2[5]\sqrt{\psi[3]}E_{12}$	$-\sqrt{q[5]}E_{12}$		
A_{130}	$\genfrac{[}{]}{0pt}{}{6}{2}$	$[3]\sqrt{\psi[5]}E_{13}$	$q\sqrt{\varphi[5]}E_{13}$		
A_{131}	$q^3(q^3+1)\genfrac{[}{]}{0pt}{}{5}{2}$	$q^3[4]\sqrt{\psi[5]}E_{13}$	$-q\sqrt{\varphi[5]}E_{13}$		
A_{140}	$\genfrac{[}{]}{0pt}{}{6}{3}$	$[4]\sqrt{\psi[5]}E_{14}$	$q\sqrt{q\varphi[5]}E_{14}$		
A_{141}	$q^4\psi[3][5]$	$q^4[3]\sqrt{\psi[5]}E_{14}$	$-q\sqrt{q\varphi[5]}E_{14}$		
A_{150}	$\genfrac{[}{]}{0pt}{}{6}{4}$	$[5]\sqrt{\psi[3]}E_{15}$	$q^2\sqrt{[5]}E_{15}$		
A_{151}	$q^5[6]$	$q^5[2]\sqrt{\psi[3]}E_{15}$	$-q^2\sqrt{[5]}E_{15}$		
A_{160}	$\genfrac{[}{]}{0pt}{}{6}{5}$	$[6]E_{16}$	$q^{5/2}E_{16}$		
A_{161}	q^6	q^6E_{16}	$-q^{5/2}E_{16}$		
A_{220}	1	E_{22}	E_{22}	E_{22}	
A_{221}	$q[2][5]$	$q[2][5]E_{22}$	$(q^2[4]-1)E_{22}$	$-[2]E_{22}$	
A_{222}	$q^4\varphi[5]$	$q^4\varphi[5]E_{22}$	$-q^2[4]E_{22}$	qE_{22}	
A_{230}	$[5]$	$\sqrt{[3][5]}E_{23}$	$[2]\sqrt{q\varphi}E_{23}$	$q\sqrt{[3]}E_{23}$	
A_{231}	$q^2[4][5]$	$q^2[4]\sqrt{[3][5]}E_{23}$	$(q^3[3]-[2])\sqrt{q\varphi}E_{23}$	$-q[2]\sqrt{[3]}E_{23}$	
A_{232}	$q^6\varphi[5]$	$q^6\varphi\sqrt{[3][5]}E_{23}$	$-q^3[3]\sqrt{q\varphi}E_{23}$	$q^2\sqrt{[3]}E_{23}$	
A_{240}	$\varphi[5]$	$\varphi\sqrt{[3][5]}E_{24}$	$q[3]\sqrt{\varphi}E_{24}$	$q^2\sqrt{[3]}E_{24}$	
A_{241}	$q^3[4][5]$	$q^3[4]\sqrt{[3][5]}E_{24}$	$q(q^4[2]-[3])\sqrt{\varphi}E_{24}$	$-q^2[2]\sqrt{[3]}E_{24}$	
A_{242}	$q^8[5]$	$q^8\sqrt{[3][5]}E_{24}$	$-q^5[2]\sqrt{\varphi}E_{24}$	$q^3\sqrt{[3]}E_{24}$	
A_{250}	$\varphi[5]$	$\varphi[5]E_{25}$	$q^{3/2}[4]E_{25}$	q^3E_{25}	
A_{251}	$q^4[2][5]$	$q^4[2][5]E_{25}$	$q^{3/2}(q^5-[4])E_{25}$	$-[2]q^3E_{25}$	
A_{252}	q^{10}	$q^{10}E_{25}$	$-q^{13/2}E_{25}$	q^4E_{25}	
A_{330}	1	E_{33}	E_{33}	E_{33}	E_{33}
A_{331}	$q[3][4]$	$q[3][4]E_{33}$	$(q^2[2][3]-1)E_{33}$	$(q^2-1)[3]E_{33}$	$-[3]E_{33}$
A_{332}	$q^4\varphi[3]^2$	$q^4\varphi[3]^2E_{33}$	$q^2[3](q^4-q-1)E_{33}$	$-q[3](q^2+q-1)E_{33}$	$q[3]E_{33}$
A_{333}	$q^9[4]$	$q^9[4]E_{33}$	$-q^6[3]E_{33}$	$q^4[2]E_{33}$	$-q^3E_{33}$
A_{340}	$[4]$	$[4]E_{34}$	$[3]\sqrt{q}E_{34}$	$q[2]E_{34}$	$\sqrt{q^3}E_{34}$
A_{341}	$q^2\varphi[3]^2$	$q^2\varphi[3]^2E_{34}$	$[3](q^3[2]-1)\sqrt{q}E_{34}$	$q[3](q^2-q-1)E_{34}$	$-[3]\sqrt{q^3}E_{34}$
A_{342}	$q^6[3][4]$	$q^6[3][4]E_{34}$	$q^3(q^5-[2][3])\sqrt{q}E_{34}$	$-q^2(q^3-1)[2]E_{34}$	$q[3]\sqrt{q^3}E_{34}$
A_{343}	q^{12}	$q^{12}E_{34}$	$-q^8\sqrt{q}E_{34}$	q^6E_{34}	$-q^3\sqrt{q^3}E_{34}$
f_s		1	$[7]-1$	$\genfrac{[}{]}{0pt}{}{7}{2}-[7]$	$\genfrac{[}{]}{0pt}{}{7}{3}-\genfrac{[}{]}{0pt}{}{7}{2}$

TABLE 1. Here $\varphi = q^2+1$ and $\psi = q^2-q+1$.

3.4. **Semidefinite programming.** We apply Theorem 2.2 for $(n, \lvert\mathcal{C}\rvert, d)_q$ subspace codes $\mathcal{C} \subseteq \mathcal{P}(V)$. Then $b_{ijl} = \lvert(\mathcal{C}\times\mathcal{C})\cap R_{ijl}\rvert$ is the number of pairs (U, W) of codewords in $\mathcal{C}$ such that $\dim(U) = i$, $\dim(W) = j$, and $\min\{i,j\} - \dim(U\cap W) = l$. The minimum subspace distance of d implies that $b_{ijl} = 0$ for triples i, j, l satisfying $i \neq j$ or $1 \le l$ if $l < \min\{i,j\} + (d-i-j)/2$. In particular, the number of i-subspaces in $\mathcal{C}$ is given by $x_i = b_{ii0}$ and they fulfill

$$b_{ijl} = b_{jil}, \qquad b_{ii0}^2 = \sum_l b_{iil}, \text{ and } \qquad b_{ii0}b_{jj0} = \sum_l b_{ijl}. \tag{5}$$

Since the last two conditions of Equations (5) cannot be expressed as constraints in an SDP, we implement only two inequalities: First, $b_{ii0}^2 \le \sum_l b_{iil}$ corresponds via the Schur complement to $\begin{pmatrix} 1 & b_{ii0} \\ b_{ii0} & \sum_l b_{iil} \end{pmatrix} \succcurlyeq 0$. Second, $b_{ii0}b_{jj0} \ge \sum_l b_{ijl}$ is equivalent to $b_{ii0}^2 b_{jj0}^2 \geq (\sum_l b_{ijl})^2$ and using Equations (5) this is again equivalent to $\begin{pmatrix} \sum_l b_{iil} & \sum_l b_{ijl} \\ \sum_l b_{ijl} & \sum_l b_{jjl} \end{pmatrix} \succcurlyeq 0$. But this constraint is redundant as it is implied by $\sum_{il} \frac{b_{iil}}{m_{iil}} \Delta_0(A_{iil}) + \sum_{i<j,l} \frac{b_{ijl}}{m_{ijl}} (\Delta_0(A_{ijl}) + \Delta_0(A_{jil})) \succcurlyeq 0$.

Since $|\mathcal{C}| = \sum_i b_{ii0}$ and $|\mathcal{C}|^2 = \sum_{ijl} b_{ijl}$, the inequality $\sum_{ijl} b_{ijl} \geq (\sum_i b_{ii0})^2$ is valid and, using again the Schur complement, can be expressed as $\begin{pmatrix} 1 & \sum_i b_{ii0} \\ \sum_i b_{ii0} & \sum_{ijl} b_{ijl} \end{pmatrix} \succcurlyeq 0$. This constraint can be sharpened by considering pairs of fibers. On the one hand, we have $x_i + x_j = b_{ii0} + b_{jj0}$. On the other hand, we have $(x_i + x_j)^2 = x_i^2 + 2x_i x_j + x_j^2 = \sum_l b_{iil} + 2\sum_l b_{ijl} + \sum_l b_{jjl}$. The Schur complement shows then that $\begin{pmatrix} 1 & b_{ii0}+b_{jj0} \\ b_{ii0}+b_{jj0} & \sum_l b_{iil} + 2\sum_l b_{ijl} + \sum_l b_{jjl} \end{pmatrix} \succcurlyeq 0$ is equivalent to $\sum_l b_{iil} + 2\sum_l b_{ijl} + \sum_l b_{jjl} \geq (b_{ii0} + b_{jj0})^2$.

Using Equations (4) and (5), we have

$$\frac{b_{ijl}}{m_{ijl}} \Delta_s(A_{ijl}) + \frac{b_{jil}}{m_{jil}} \Delta_s(A_{jil}) = \frac{b_{ijl}}{m_{ijl}} \left(\Delta_s(A_{ijl}) + \Delta_s(A_{jil}) \right)$$

for $i \neq j$, which is a symmetric matrix. Hence, using only b_{ijl} for $i \leq j$ fulfills the condition of SDPs to consist of symmetric matrices.

The complete SDP is given by the general conditions

$$\max \sum_i b_{ii0} \text{ subject to}$$

$$\sum_{il} \frac{b_{iil}}{m_{iil}} \Delta_s(A_{iil}) + \sum_{i<j,l} \frac{b_{ijl}}{m_{ijl}} (\Delta_s(A_{ijl}) + \Delta_s(A_{jil})) \succcurlyeq 0 \text{ for all } s$$

$$\begin{pmatrix} 1 & b_{ii0} \\ b_{ii0} & \sum_l b_{iil} \end{pmatrix} \succcurlyeq 0 \text{ for all } i$$

$$\begin{pmatrix} 1 & b_{ii0}+b_{jj0} \\ b_{ii0}+b_{jj0} & \sum_l b_{iil} + 2\sum_l b_{ijl} + \sum_l b_{jjl} \end{pmatrix} \succcurlyeq 0 \text{ for all } i < j$$

$$b_{ijl} \in \mathbb{R} \text{ for all } i \leq j, l$$

and the problem specific conditions are given by

$$0 \leq b_{ijl} \leq A_q(n, 2\lceil d/2 \rceil; i) \cdot A_q(n, 2\lceil d/2 \rceil; j) \text{ for all } i \leq j, l \text{ with } i \neq j \text{ or } 1 \leq l$$

$$0 \leq b_{ii0} \leq A_q(n, 2\lceil d/2 \rceil; i) \text{ for all } i$$

$$b_{ijl} = 0 \text{ for all } i \leq j, l \text{ satisfying } i \neq j \text{ or } 1 \leq l \text{ if } l < \min\{i, j\} + (d - i - j)/2.$$

This SDP is bounded and the assignment $b_{ijl} = 0$ for all $i \leq j, l$ is a feasible solution. Although $A_q(n, 2\lceil d/2 \rceil; k)$ is often not known explicitly, it can be replaced by a suitable upper bound, cf. http://subspacecodes.uni-bayreuth.de/ associated with [16].

The restriction of the variables in the SDP to a subset of the fibers implies the following

Lemma 3.5. *Let K be a subset of $\{0, \ldots, n\}$. If i and j in the SDP above are restricted to values in K, then the optimal value of this SDP is an upper bound for $A_q(n, d; K)$.*

4. **Theorem 1.2 and Related Results.** Throughout this section let $\mathcal{C}$ be a subspace code of $\mathbb{F}_q^7$ with minimum distance 4. We denote the number of elements of $\mathcal{C}$ in the i-th fiber (so of dimension i) by x_i. By Theorem 2.2 and Table 1, we obtain a

semidefinite program. Optimizing this program with the SDP solver SDPA-GMP, we verified Theorem 1.2. The purpose of this section is to motivate Theorem 1.2 and provide some partial results which might show Theorem 1.2 for all q.

First let us note the following result for the inner distributions of $\mathcal{C}$ in the binary case:

Lemma 4.1. *Let $\mathcal{C}$ be a subspace code of $\mathbb{F}_2^7$ with $384 \le |\mathcal{C}| \le 388$ and minimum distance 4, then one of the following occurs (up to orthogonality):*

$$
\begin{aligned}
&|\mathcal{C}| = 388 \text{ and } x_2 = 41,\ x_4 = 347,\\
&|\mathcal{C}| = 387 \text{ and } x_2 = 41-\alpha,\ x_4 = 346+\alpha \text{ for } \alpha \in \{0,1,\dots,5\},\\
&|\mathcal{C}| = 386 \text{ and } x_2 = 41-\alpha,\ x_4 = 345+\alpha \text{ for } \alpha \in \{0,1,\dots,12\},\\
&|\mathcal{C}| = 385 \text{ and } x_2 = 41-\alpha,\ x_4 = 344+\alpha \text{ for } \alpha \in \{0,1,\dots,18\},\\
&|\mathcal{C}| = 384 \text{ and } x_2 = 41-\alpha,\ x_4 = 343+\alpha \text{ for } \alpha \in \{0,1,\dots,23\} \text{ or}\\
&|\mathcal{C}| = 384 \text{ and } x_2 = 38-\alpha,\ x_4 = 345+\alpha,\ x_6 = 1 \text{ for } \alpha \in \{0,1,2\}.
\end{aligned}
$$

If $|\mathcal{C}| = 388$, then (b_{241}, b_{442}) is one of the following:

$$
\begin{aligned}
&(5026, 44058),\\
&(5027, 44054+x) \text{ for } x \in \{0,\dots,3\},\\
&(5028, 44051+x) \text{ for } x \in \{0,\dots,4\},\\
&(5029, 44047+x) \text{ for } x \in \{0,\dots,6\},\\
&(5030, 44044+x) \text{ for } x \in \{0,\dots,7\},\\
&(5031, 44042+x) \text{ for } x \in \{0,\dots,7\},\\
&(5032, 44039+x) \text{ for } x \in \{0,\dots,9\},\\
&(5033, 44037+x) \text{ for } x \in \{0,\dots,9\},\\
&(5034, 44035+x) \text{ for } x \in \{0,\dots,9\},\\
&(5035, 44033+x) \text{ for } x \in \{0,\dots,9\},\\
&(5036, 44032+x) \text{ for } x \in \{0,\dots,8\},\\
&(5037, 44031+x) \text{ for } x \in \{0,\dots,8\},\\
&(5038, 44030+x) \text{ for } x \in \{0,\dots,7\},\\
&(5039, 44029+x) \text{ for } x \in \{0,\dots,6\},\\
&(5040, 44029+x) \text{ for } x \in \{0,\dots,4\},\\
&(5041, 44029+x) \text{ for } x \in \{0,1,2\},\\
&(5042, 44029+x) \text{ for } x \in \{0,1\}.
\end{aligned}
$$

To obtain this result, we solve the SDP as described in Subsection 3.4 and added additional constraints which forced certain distributions for the x_i. For $|\mathcal{C}| = 388$ we additionally determined all possible distributions of the b_{ijk}'s using the same idea. This ruled out $x_2 = 40$ and $x_4 = 248$ (which is otherwise feasible).

We use $x_2 \le A_q(7,4;2) = q^5 + q^3 + 1$ and $x_5 \le A_q(7,4;5) = A_q(7,4;2) = q^5 + q^3 + 1$. This is implied by the following lemma due to Beutelspacher and orthogonality.

Lemma 4.2 ([3])**.** *$A_q(n,2k;k) = \frac{q^n-q}{q^k-1} - q + 1$ if k divides $n-1$.*

The following lemma generalizes $x_3 + x_4 \le 381$ in the binary case from [23, Lemma 4.2.ii].

Lemma 4.3. *We have $x_3 + x_4 \le (q^2 - q + 1)[7]$ with equality only if $x_3 = 0$ or $x_4 = 0$.*

Proof. We write $b = x_3$ and $c = x_4$ to avoid indices. The only allowed relations are (up to transposition and orthogonality) $R_{330}, R_{332}, R_{333}, R_{342}, R_{343}$. Let $x_3\beta$ denote the number of pairs in relation R_{332}, δ the number of pairs in relation R_{342}, $x_4\gamma$ the number of pairs in relation R_{442}. From $\Delta_1(A_{abc})$ and, respectively, $\Delta_2(A_{abc})$ and Theorem 2.2 we obtain the following positive semidefinite matrices (after some simplifications and multiplying by $q^3\sqrt{q}\psi[3][4][5][7]$):

$$N_1 = \begin{pmatrix} bq^3([3][7] - [3]^2 b + \beta[7]) & q^{5/2}([7]\delta - bc[3][4]) \\ q^{5/2}([7]\delta - bc[3][4]) & cq^3([3][7] - [3]^2 c + \gamma[7]) \end{pmatrix}$$

$$N_2 = \begin{pmatrix} bq[2]([3](q^7 + q^5 + b - 1) - \beta[2]^2\psi) & -[2][3]((q^3+1)\delta - \varphi bc) \\ -[2][3]((q^3+1)\delta - \varphi bc) & cq[2]([3](q^7 + q^5 + c - 1) - \gamma[2]^2\psi) \end{pmatrix}$$

For an $m \times m$ matrix M and a set I, let M_I denote the $m \times m$ with $(M_I)_{xy} = M_{xy}$ if $x, y \in I$ and $(M_I)_{xy} = 0$ otherwise. We set $N_t = N_1 + t_1 N_2 + t_2((N_2)_{\{1\}} + (N_2)_{\{2\}})$, where

$$t_1 = \frac{q^{5/2}[7]}{[2][6]}, \qquad t_2 = \frac{q^2[5][7]}{[2]^2[6](q^2 + q^{3/2} + q + q^{1/2} + 1)}$$

For $q \ge 2$ the factors t_1, t_2 are positive, so N_t is a positive semidefinite matrix. Hence, $\det(N_t) \ge 0$. Rearranging for b yields

$$0 \le b \le ((q^2 - q + 1)[7] - c)\frac{1}{1 + \frac{c}{q\psi[3]^2}}.$$

This implies the assertion. $\square$

This can be improved to:

Corollary 4.4. *We have $x_1 + x_3 + x_4 \le (q^2 - q + 1)[7]$ with equality only if $x_3 = 0$ or $x_4 = 0$.*

Proof. The minimum distance implies $x_1 \le 1$. If $x_1 = 0$, then Lemma 4.3 shows the claim. Hence, we assume $x_1 = 1$.

The only allowed relations are (up to transposition and orthogonality) R_{110}, R_{131}, R_{141}, R_{333}, R_{332}, R_{330}, R_{343}, and R_{342}. Let $(x_3^2 - x_3)a_{332}$ denote the number of pairs in relation R_{332}, $(x_4^2 - x_4)a_{442}$ the number of pairs in relation R_{442}, and $x_3x_4a_{342}$ the number of pairs in relation R_{342}. From $\Delta_1(A_{abc})$ and, respectively, $\Delta_2(A_{abc})$ and Theorem 2.2 we obtain the following positive semidefinite matrices (after some simplifications and multiplying by [7]):

$$N_1 = \begin{pmatrix} 1 & -\frac{x_3}{\sqrt{[5]\varphi}(q^5+q^2)} & -\frac{x_4\sqrt{\varphi}}{\sqrt{[5]}q^{5/2}[3]\psi} \\ -\frac{x_3}{\sqrt{[5]\varphi}(q^5+q^2)} & \frac{x_3([7][3]-a_{332}[7]+x_3(a_{332}[7]-q^2-[4]-[5]+1))}{[5]q^3(q^4+q^2+1)\varphi[2]} & \frac{x_3x_4(a_{342}[7]+[2]-[4]-[5]-[6]+1)}{q^{7/2}[5][3]\varphi\psi[2]} \\ -\frac{x_4\sqrt{\varphi}}{\sqrt{[5]}q^{5/2}[3]\psi} & \frac{x_3x_4(a_{342}[7]+[2]-[4]-[5]-[6]+1)}{q^{7/2}[5][3]\varphi\psi[2]} & \frac{x_4([7][3]-a_{442}[7]+x_4(a_{442}[7]-q^2-[4]-[5]+1))}{[5]q^3(q^4+q^2+1)\varphi[2]} \end{pmatrix}$$

$$N_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{x_3(a_{332}(x_3-1)(q^2-[5])+[7](q^3+q-1)+[3]x_3-[5]+1)}{[5]q^5\varphi(q^4+q^2+1)} & -\frac{x_3x_4(a_{342}(q^3+1)-\varphi)}{[5]q^6\psi\varphi} \\ 0 & -\frac{x_3x_4(a_{342}(q^3+1)-\varphi)}{[5]q^6\psi\varphi} & \frac{x_4(a_{442}(x_4-1)(q^2-[5])+[7](q^3+q-1)+x_4[3]-[5]+1)}{[5]q^5\varphi(q^4+q^2+1)} \end{pmatrix}$$

We set $N_t = N_1 + t_1 N_2 + t_2((N_2)_{\{2\}} + (N_2)_{\{3\}})$, where

$$t_1 = \frac{q^{5/2}[7]}{[3]\psi[2]^2}, \qquad t_2 = \frac{[7]q^2([3]-\sqrt{q}[2])}{[3]\psi[2]^3}.$$

For $q \geq 2$ the factors t_1, t_2 are positive, so N_t is a positive semidefinite matrix. Hence, $\det(N_t) \geq 0$ and solving this inequality for x_3 yields an upper bound for x_3, say $u(q, x_4)$. Then, the objective function is upper bounded by $1 + u(q, x_4) + x_4$, which has its maximum on $0 \leq x_4 \leq (q^2-q+1)[7]$ at $\sqrt{q[4]^2(q^4+q^2+1)^2} - q([7] + q^2\varphi)$ with the value $2\sqrt{q}(q([7]+q[4]) - \sqrt{q} - q^{3/2} - 5/2q^{5/2} - q^{7/2} - 2q^{9/2} - q^{11/2} - q^{13/2} + 1)$, which is at most $(q^2-q+1)[7]$.

□

Lemma 4.5. *We have $x_2 + x_3 \leq (q^2-q+1)[7]$ with equality only if $x_2 = 0$.*

Proof. We write $a = x_2$ and $b = x_3$ to avoid indices. The only allowed relations are (up to transposition and orthogonality) $R_{220}, R_{222}, R_{232}, R_{330}, R_{332}, R_{333}$. Let $x_3\beta$ denote the number of pairs in relation R_{332}. From $\Delta_1(A_{abc})$ and, respectively, $\Delta_2(A_{abc})$ and Theorem 2.2 we obtain the following positive semidefinite matrices:

$$N_1 = \begin{pmatrix} a[4]([7]-[2]a) & -abq^{7/2}[2][3]\sqrt{\varphi} \\ -abq^{7/2}[2][3]\sqrt{\varphi} & bq^3([3][7]-[3]^2b+\beta[7]) \end{pmatrix}$$

$$N_2 = \begin{pmatrix} aq^3[2]((\psi[3](q^2[4]-1)+a) & abq^2[2]\sqrt{[3]} \\ abq^2[2]\sqrt{[3]} & bq[2]([3](q^7+q^5+b-1)-\beta[2]^2\psi) \end{pmatrix}$$

Set $N_t = N_1 + t_1 N_2$, where $t_1 = \frac{q^2[7]}{[2]^2\psi}$. As $t_1 \geq 0$, N_t is positive semidefinite, so $\det(N_t) \geq 0$. Rearranging this for b yields

$$b \leq ((q^2-q+1)[7]-a)\frac{1}{1+a\frac{[2]^2C}{q[5]^3}},$$

where $C = 2[2]\sqrt{q[3]\psi} - (q^4+3q^3+3q^2+3q+1)$. The assertion follows. □

This also shows that only proper subspaces are of interest.

Corollary 4.6. *If $(q^2-q+1)[7]+3 \leq |\mathcal{C}|$, then $x_0 = x_7 = 0$ and $x_1 + x_6 \leq 1$.*

Proof. By the minimum distance, we have $0 \leq x_i \leq 1$ for $i \in \{0,1,6,7\}$. If $x_0 = x_7 = 1$ then the minimum distance shows $\mathcal{C} \subseteq \{\{0\}, \mathbb{F}_q^7\}$. If $x_0 + x_7 = 1$ then by orthogonality we can assume without loss of generality that $x_0 = 0$ and $x_7 = 1$ and in particular $|\mathcal{C}| = x_1 + x_2 + x_3 + 1$. If $x_1 = 1$ then $x_2 = 0$ and $|\mathcal{C}| \leq A_q(7,4;3) + 2 \leq (q^2-q+1)[7]+2$ contradicting the claim. Hence, we have $|\mathcal{C}| = x_2 + x_3 + 1 \leq (q^2-q+1)[7]+1$ using the inequality from Lemma 4.5.

Assume now that $x_0 = x_7 = 0$ and $x_1 = x_6 = 1$. Then $x_2 = x_5 = 0$ by the minimum distance and $|\mathcal{C}| = x_3 + x_4 + 2 \leq (q^2-q+1)[7]+2$ using the inequality from Lemma 4.3 and completing the proof. □

We finish with the motivation for the bound in Theorem 1.2.

Lemma 4.7. *We have $x_2 + x_4 \leq F(q)$.*

Proof. We write $a = x_2$ and $c = x_4$ to avoid indices. The only allowed relations are (up to transposition and orthogonality) $R_{220}, R_{222}, R_{241}, R_{242}, R_{440}, R_{442}, R_{443}$. Let α denote the number of pairs in relation R_{241}, and $x_4\gamma$ the number of pairs in relation R_{442}. From $\Delta_1(A_{abc})$ and, respectively, $\Delta_2(A_{abc})$ and Theorem 2.2 we obtain the following positive semidefinite matrices:

$$N_1 = \begin{pmatrix} a[4]([7]-[2]a) & [2]\varphi([7]\alpha - acq^3[2][4]) \\ [2]\varphi([7]\alpha - acq^3[2][4]) & bq^3([3][7]-[3]^2b+\beta[7]) \end{pmatrix}$$

$$N_2 = \begin{pmatrix} aq^3[2]((\psi[3](q^2[4]-1)+a) & q[2]\sqrt{[3]}(ac\varphi - \alpha\psi[3]) \\ q[2]\sqrt{[3]}(ac\varphi - \alpha\psi[3]) & bq[2]([3](q^7+q^5+b-1)-\beta[2]^2\psi) \end{pmatrix}$$

Set $N_t = N_1 + t_1 N_2 + t_2 (N_1)_{22}$, where

$$t_1 = \frac{q^2\sqrt{\varphi}[7]}{[6]\sqrt{[3]}}, \qquad t_2 = \frac{[2]^2\sqrt{\varphi}}{\sqrt{[3]^3}} - 1.$$

As $t_1, t_2 \geq 0$, N_t is positive semidefinite, so $\det(N_t) \geq 0$. Solving this inequality for c gives an upper bound on c in terms of a, say $c(a)$. Then $a + c \leq \lfloor a + c(a) \rfloor$. The function $F(q)$ is defined such that $F(q) = \max_{0 \leq a \leq q^5+q^3+1} \lfloor a + c(a) \rfloor$ for q a prime power. Here we use Lemma 4.2. $\square$

Combining Lemma 4.5, Lemma 4.7, and Lemma 4.3 shows Theorem 1.3.

We applied also the strategy of [23, Section 4.1] in the binary case with functions $x_3 \leq f'(x_4)$, $x_3 \leq g'(x_2)$, and $x_3 \leq h'(x_5)$ defined by

$$f'(x) = \left\lfloor \frac{294(381-x)}{294+x} \right\rfloor, \qquad g'(x) = \left\lfloor \frac{62(6\sqrt{70}+59)(381-x)}{372\sqrt{70}+3658+9x} \right\rfloor, \text{ and}$$

$$h'(x) = \left\lfloor \frac{(13209651 - 28575x)\sqrt{35} + 73499853 - 192913x}{192913 + 34671\sqrt{35} - 98x} \right\rfloor,$$

as implied by the same reasoning as in Lemmata 4.3, 4.5, and 4.7. Denote the previous upper bounds f^{HKK}, g^{HKK}, and h^{HKK} from [23, Lemma 4.2], [23, Lemma 4.3], and [23, Lemma 4.4], respectively. The bounds f' and h' are stronger than f^{HKK} and h^{HKK}, respectively, for large arguments while $g^{\mathrm{HKK}}(x) \leq g'(x)$ for all $0 \leq x \leq 41$. Assuming $x_4 \leq x_3$, we have $x_4 \leq 151$ by f', improving $x_4 \leq 190$ from [23, Lemma 4.2.i]. Then, as shown in [23, Section 4.1], if $x_4 \leq x_3$ we have the bound

$$x_2 + x_3 + x_4 + x_5 \leq \max_{\substack{0 \leq x_2 \leq 41 \\ 0 \leq x_5 \leq 41}} x_2 + F(\min\{g(x_2), h(x_5)\}, \min\{g(x_5), h(x_2)\}) + x_5 \text{ with}$$

$$F(u_3, u_4) = \max_{0 \leq x_4 \leq \min\{u_3, u_4, 151\}} \min\{u_3, f(x_4)\} + x_4$$

in which we fixed an error with the max in F from [23, Section 4.1]. Using only the functions implied by the SDP arguments, i.e., $f = f'$, $g = g'$, and $h = h'$, an exhaustive computer calculation determines the right hand side as 432. By taking $f = \min\{f', f^{\mathrm{HKK}}\}$, $g = g^{\mathrm{HKK}}$, and $h = \min\{h', h^{\mathrm{HKK}}\}$, the right hand side of the maximization problem is 393 which improves the 406 from [23, Section 4.1] but is inferior to Theorem 1.2. Nevertheless, this calculation involved only integer computations and is resilient against numerical errors. Then Corollary 4.6 shows $A_2(7,4) \leq 394$.

5. **New and Updated SDP Bounds.** Bachoc et al. [2] provided bounds for network codes with odd distances, but not for even distances or $q > 2$. With the general formulas for triple intersection numbers described in Section 3.1, we can calculate the corresponding coherent configuration with standard techniques and let a semidefinite programming solver (here SDPA-GMP[1]) find a bound on the corresponding problem. The following tables list bounds on $A_q(n,d)$ for small q and small n, complementing and, for $q = 2$ and odd d, improving the work by Bachoc et al. New best bounds are **bold**. If $q = 2$ and d is odd, the new SDP bound is better than the old or there was no previous SDP bound in literature, then the entry is in *italics*.

$d \setminus n$	8	9	10	11	12	13	14
3	***9191***	**107419**	***2531873***	***57201557***	***2685948795***	***119527379616***	***11215665059647***
4	**6479**	**53710**	**1705394**	**28600778**	**1816165540**	**59763689822**	***7496516673358***
5	*327*	*2458*	*48255*	***660265***	***26309023***	***688127334***	***54724534275***
6	260	**1240**	38455	**330133**	**21362773**	**344063682**	***43890879895***
7			*1219*	*8844*	***314104***	***4678401***	***330331546***
8			1090	4480	**279476**	**2343888**	***292988615***
9					***4483***	*34058*	***2298622***
10					4226	**17133**	***2164452***
11						259	***17155***
12							*16642*

TABLE 2. SDP bounds on $A_2(n,d)$.

$d \setminus n$	6	7	8	9	10	11	12
3	967	**15394**	760254	**34143770**	**5026344026**	**675225312722**	**298950313257852**
4	**788**	**7696**	**627384**	**17071886**	**4112061519**	**337612656529**	**244829520433920**
5		166	7222	123535	**16008007**	**818518696**	**320387589445**
6			6727	**61962**	**14893814**	**409259348**	**298571221318**
7				490	61002	1076052	**400831735**
8					59539	**539351**	**391178436**
9						1462	537278
10							532903

TABLE 3. SDP bounds on $A_3(n,d)$.

$d \setminus n$	6	7	8	9	10	11
3	4772	**142313**	20482322	**2341621613**	**1343547758223**	**614496020025690**
4	**4231**	**71156**	**18245203**	**1170810807**	**1194101275238**	**307248010015067**
5		516	68117	**2132181**	**1122729102**	**140323867490**
6			66054	**1067796**	**1088550221**	**70161933745**
7				2052	1058831	33669242
8					1050630	16847095
9						8196

TABLE 4. SDP bounds on $A_4(n,d)$.

[1]Some numbers require a higher precision output than what SDPA offers. See `https://github.com/ferihr/sdpa-gmp` for a version where the constants `P_FORMAT_obj` and `P_FORMAT_gap` in `sdpa_io.h` adjust the output length.

$d \setminus n$	6	7	8	9	10
3	17179	**821170**	277100135	**64262978412**	**108238287449582**
4	**15883**	**410585**	**256754528**	**32131489207**	**100215014898311**
5		1254	398154	**19675409**	**31196584033**
6			391883	**9847885**	**30703887393**
7				6254	9803150
8					9771883

TABLE 5. SDP bounds on $A_5(n, d)$.

$d \setminus n$	6	7	8	9	10
3	123239	**11807778**	14753449680	**9728400942608**	**85039309360944189**
4	**118347**	**5903889**	**14176726504**	**4864200471305**	**81703574152063079**
5		4806	5803270	**566262547**	**4784663914039**
6			5769615	**283240686**	**4756893963688**
7				33618	282744208
8					282508875

TABLE 6. SDP bounds on $A_7(n, d)$.

We added these bounds and will continuously add data on the SDP bound for larger numbers on http://subspacecodes.uni-bayreuth.de/, cf. [16].

6. **Quadruple Conditions for the 2-Fano plane.** Famously, Schrijver used semidefinite programming to improve the bounds on constant weight codes [32] and considered the centralizer algebra of a vertex, i.e., a codeword. In principle the same method is feasible for any (sufficiently symmetric) graph. In vector spaces this corresponds to constant-dimension codes. One way of obtaining the necessary structural information is to calculate the triples (so the p_{ij}^k) in relationship to one fixed vertex. Let π be a plane in $\mathbb{F}_q^7$. We can now define a coherent configuration on planes in $\mathbb{F}_q^7$ in the following way: Our a-th fiber consists of all planes τ with $\operatorname{codim}(\pi \cap \tau) = a$. Clearly, $a \in \{0, 1, 2, 3\}$. The relations between elements are characterized as follows: two planes x and y are in relation $R_{a,b;\alpha,\beta,\gamma}$ if

$$(\operatorname{codim}(x \cap \pi), \operatorname{codim}(y \cap \pi); \operatorname{codim}(x \cap y), \operatorname{codim}(x \cap y \cap \pi), \operatorname{codim}(\langle x, y\rangle \cap \pi)) = (a, b; \alpha, \beta, \gamma).$$

It can be easily verified that feasible parameter sets up to transposition are as follows:

$$\begin{aligned}
&(0,0;0,0,0),(0,1;1,1,0),(0,1;2,2,0),(0,1;3,3,0),\\
&(1,1;0,1,1),(1,1;1,1,0),(1,1;1,1,1),(1,1;1,2,0),(1,1;2,2,0),\\
&(1,2;1,2,0),(1,2;1,2,1),(1,2;2,2,0),(1,2;2,2,1),(1,2;2,3,0),(1,2,3,3,0),\\
&(1,3;2,3,1),(1,3;3,3,0),(1,3;3,3,1),\\
&(2,2;0,2,2),(2,2;1,2,2),(2,2;1,2,1),(2,2;2,2,0),(2,2;2,2,1),(2,2;2,2,2),\\
&\quad(2,2;1,3,1),(2,2;2,3,1),(2,2;2,3,0),(2,2;3,3,0),(2,2;3,3,1),\\
&(2,3;1,3,2),(2,3;2,3,2),(2,3;2,3,1),(2,3;3,3,0),(2,3;3,3,1),\\
&(3,3;0,3,3),(3,3;1,3,3),(3,3;1,3,2),(3,3;2,3,2),(3,3;2,3,1),(3,3;3,3,0),\\
&\quad(3,3;3,3,1).
\end{aligned}$$

Notice that these relations also characterize the relations for the centralizer algebra of k-spaces in $\mathbb{F}_q^n$ in general, but it is non-trivial to count triple intersection numbers here. Hence, we limit ourselves to the one open case where the p_{ij}^k's can be counted with the computer explicitly, that is $(n,k,q)=(7,3,2)$.

For the q-Fano plane upper and lower bounds on pairs of planes in certain relations are well-known. Using the same techniques as before, we obtain the following upper and lower bounds on the number of quadruples occurring for the 2-Fano plane. We assume that π is in the q-Fano plane. We leave pairs, which are always 0, out. The notation $ab\alpha^*$ refers to the maximal sum of pairs in a relation of type $(a,b;\alpha,\beta,\gamma)$. The numbers $ab\alpha^*$ are known for general q. We mostly provide them for completeness.

Rel	00000	02220	03330				
$\# =$	1	140	240				
Rel	22220	22221	22222	22231	22230	22330	22331
$\# \leq$	420	1260	2240	5040	420	5040	7560
$\# \geq$	0	0	1400	4620	0	4200	6720
Rel	23232	23231	23330	23331			
$\# \leq$	7560	5040	2520	20160			
$\# \geq$	6720	4200	1680	19320			
Rel	33232	33231	33330	33331			
$\# \leq$	20160	2520	1920	34440			
$\# \geq$	19320	1680	1080	33600			
Rel	222^*	223^*	232^*	233^*	332^*	333^*	
$\# =$	7700	11760	11760	21840	21840	35520	

TABLE 7. Upper and lower bounds on the number of pairs in relation $ab\alpha\beta\gamma=(a,b;\alpha,\beta,\gamma)$ for π in the 2-Fano plane.

7. **Future Work.** An obvious open problem is to show the bound of Theorem 1.2 for general q. This might be of larger interest as it is usually very hard to optimize SDP problems with parameters except for certain special cases. For all bounds an interesting question is if we can find constructions which match them.

In [32] Schrijver successfully improved the best known bounds for constant weight codes with semidefinite programming. If one can calculate a version of Lemma 3.4 for the relations of Section 6, then it is surely feasible to improve the known bounds on constant-dimension codes.

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