Research Article

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Pseudo-differential operators on homogeneous spaces of compact and Hausdorff groups

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Abstract: Let G be a compact Hausdorff group and let H be a closed subgroup of G. We introduce pseudo-differential operators with symbols on the homogeneous space G/H. We present a necessary and sufficient condition on symbols for which these operators are in the class of Hilbert–Schmidt operators. We also give a characterization of and a trace formula for the trace class pseudo-differential operators on the homogeneous space G/H.

Keywords: Pseudo-differential operators, Hilbert–Schmidt operators, trace class operators, homogeneous spaces of compact groups

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1 Introduction

The study of pseudo-differential operators is started by Kahn and Nirenberg [17]. Pseudo-differential operators were used by Hörmander to study problems in partial differential operators [11]. Pseudo-differential operators on different classes of groups are extensively studied by several authors [2–4, 19, 22, 23]. Trace class pseudo-differential operators on $\mathbb{S}^1$ are studied by Delgado and Wong [4] and recently by Ghaemi et al. [10]. Molahajloo and Pirhayati [20] gave a characterization of and trace formula for trace class pseudo-differential operators with L^2-symbols. Chen and Wong [1] considered trace class pseudo-differential operators on the unit sphere $\mathbb{S}^{n-1} \cong \mathrm{SO}(n)/\mathrm{SO}(n-1)$ centered in the origin in $\mathbb{R}^n$. In this paper, we try to replace $\mathrm{SO}(n)$ by a compact Hausdorff group G and $\mathrm{SO}(n-1)$ by a closed subgroup H of G. We study the Hilbert–Schmidt and trace class pseudo-differential operators on the homogeneous space G/H. The homogeneous spaces of compact groups play an important role in mathematical physics, geometric analysis, constructive approximation and coherent state transform, see [12–16, 18] and the references therein.

We consider pseudo-differential operators on homogeneous spaces of compact groups with L^2-symbols. Pseudo-differential operators with L^2-symbols are studied by many authors [3, 4, 10]. Recently, Ghaemi et al. [9] characterized nuclear pseudo-differential operators with L^2-symbols on a compact Hausdorff group. By considering the L^p-conditions, $1 \le p < \infty$, on symbols, we can allow singularities, and thus it becomes ideal for applications in several areas of mathematics, ranging from functional analysis to operator algebras or quantization. Our work can be considered a generalization of the corresponding work on $\mathbb{S}^1$ [4], compact Hausdorff groups [20], $\mathbb{S}^{n-1}$ [1] and finite Abelian groups [21]. Our main aim in this paper is to give a charac-

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terization of trace class and Hilbert–Schmidt pseudo-differential operators on the homogeneous space G/H of the compact group G, where H is a closed subgroup of G. We give a trace formula for trace class pseudo-differential operators. The main technique used here for obtaining the trace formula is to obtain a formula for the symbol of the product of two pseudo-differential operators on homogeneous spaces on compact groups.

In Section 2, we give some results from harmonic analysis on homogeneous spaces on compact groups [5–8] and operator theory to make the article self-contained. In Section 3, we define the pseudo-differential operator on the homogeneous space G/H of the compact and Hausdorff group G with the help of the Peter–Weyl theorem for G/H given by Ghani Farashahi [7]. We give a characterization of Hilbert–Schmidt pseudo-differential operators on G/H. We also give a product formula for pseudo-differential operators, a characterization of trace class pseudo-differential operators and then a trace formula for pseudo-differential operators on homogeneous spaces of compact groups.

2 Preliminaries

In this section, we recall some basic and important concepts of harmonic analysis on homogeneous spaces of compact groups, developed by Ghani Farashahi in a series of papers [5–8], and operator theory.

Let G be a compact Hausdorff group with normalized Haar measure dx, and let H be a closed subgroup of G with probability Haar measure dh.

The left coset space G/H can be seen as a homogeneous space with respect to the action of G on G/H given by left multiplication. The canonical surjection q from G to G/H is given by $q(x) := xH$. Let $\mathcal{C}(\Omega)$ denote the space of continuous functions on a compact Hausdorff space Ω. Define $T_H : \mathcal{C}(G) \to \mathcal{C}(G/H)$ by

$$T_H(f)(xH) = \int_H f(xh)\,dh, \quad xH \in G/H.$$

Then T_H is an onto map. The homogeneous space G/H has a unique normalized G-invariant positive Radon measure μ such that the Weil formula

$$\int_{G/H} T_H(f)(xH)\,d\mu(xH) = \int_G f(x)\,dx$$

holds.

Let $(\pi, \mathcal{H}_\pi)$ be a continuous unitary representation of a compact group G on a Hilbert space $\mathcal{H}_\pi$. It is well known that any irreducible representation $(\pi, \mathcal{H}_\pi)$ is finite-dimensional with the dimension d_π (say). Let $(\pi, \mathcal{H}_\pi)$ be a continuous unitary representation of a compact group G. Consider the operator-valued integral

$$T_H^\pi := \int_H \pi(h)\,dh$$

defined in the weak sense, i.e.,

$$\langle T_H^\pi u, v\rangle = \int_H \langle \pi(h)u, v\rangle\,dh \quad \text{for all } u, v \in \mathcal{H}_\pi.$$

Note that the function $h \mapsto \langle \pi(h)u, v\rangle$ is in $L^1(H)$ for all $u, v \in \mathcal{H}_\pi$. Therefore, the integral $\int_H \langle \pi(h)u, v\rangle\,dh$ is an ordinary integral of an L^1-function. Hence, T_H^π is a bounded linear operator on $\mathcal{H}_\pi$ with norm bounded by one. Further, T_H^π is a partial isometric orthogonal projection, and T_H^π is an identity operator if and only if $\pi(h) = I$ for all $h \in H$. Denote the set of all continuous irreducible unitary representation on G by $\widehat{G}$. Set

$$\mathcal{K}_\pi^H = \{u \in \mathcal{H}_\pi : \pi(h)u = u \text{ for all } h \in H\}.$$

Then $\mathcal{K}_\pi^H$ is a closed subspace of $\mathcal{H}_\pi$. Let $d_{\pi,H}$ be the dimension of $\mathcal{K}_\pi^H$. Then it is evident that $d_{\pi,H} \leq d_\pi$. It can be easily seen that $d_{\pi,H} = d_\pi$ if and only if $[\pi] \in H^\perp := \{[\pi] \in \widehat{G} : \pi(h) = I \text{ for all } h \in H\}$.

Definition 2.1. Let H be a closed subgroup of a compact group G. Then the dual object $\widehat{G/H}$ of G/H is a subset of $\widehat{G}$ and is given by

$$\widehat{G/H} := \{[\pi] \in \widehat{G/H} : T_H^\pi \neq 0\} = \Big\{[\pi] \in \widehat{G} : \int_H \pi(h)\, dh \neq 0\Big\}.$$

Definition 2.2. Let H be a closed subgroup of a compact group G, and let $[\pi] \in \widehat{G/H}$. An ordered orthonormal basis $\mathfrak{B} = \{e_j\}_{j=1}^{d_\pi}$ of the Hilbert space $\mathcal{H}_\pi$ is called H-admissible if it is an extension of an orthonormal basis of $\{e_j\}_{j=1}^{d_{\pi,H}}$ of the closed subspace $\mathcal{K}_\pi^H$.

Now, we state a corollary of the Peter–Weyl theorem for homogeneous spaces of a compact group and some of its consequences.

Theorem 2.3 ([7, Corollary 4.1]). *Let H be a closed subgroup of a compact group G, and let μ be the normalized G-invariant measure on G/H. For each $[\pi] \in \widehat{G/H}$, let $\mathfrak{B}_\pi = \{e_{\ell,\pi} : 1 \leq \ell \leq d_\pi\}$ be an H-admissible basis for the representation space $\mathcal{H}_\pi$. Then we have the following statements:*

(i) *The set $\mathfrak{B}(G/H) = \{\sqrt{d_\pi}\pi_{ij} : \pi \in \widehat{G/H},\ 1 \leq i \leq d_\pi,\ 1 \leq j \leq d_{\pi,H}\}$ constitutes an orthonormal basis for the Hilbert space $L^2(G/H)$.*

(ii) *Each $f \in L^2(G/H)$ decomposes as the following:*

$$f = \sum_{[\pi]\in\widehat{G/H}} d_\pi \sum_{j=1}^{d_{\pi,H}} \sum_{i=1}^{d_\pi} \langle f, \pi_{ij}\rangle_{L^2(G/H)} \pi_{ij},$$

where the series converge in $L^2(G/H)$.

Using the above decomposition of $f \in L^2(G/H)$ and the orthogonality relation

$$\langle \pi_{ij}, \pi'_{kl}\rangle_{L^2(G/H)} = d_\pi^{-1} \delta_{\pi\pi'} \delta_{ik} \delta_{jl},$$

we get the following Plancherel's theorem (see [5]).

Theorem 2.4. *For $f \in L^2(G/H)$, we have*

$$\|f\|_{L^2(G/H)}^2 = \sum_{[\pi]\in\widehat{G/H}} d_\pi \sum_{j=1}^{d_{\pi,H}} \sum_{i=1}^{d_\pi} |\langle f, \pi_{ij}\rangle_{L^2(G/H)}|^2.$$

Let $\mathcal{H}$ be a complex and separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|_{\mathcal{H}}$. Denote by $\|\cdot\|_{\mathcal{B}(\mathcal{H})}$ the norm in the C^*-algebra of all bounded linear operators on $\mathcal{H}$.

An operator $T \in \mathcal{B}(\mathcal{H})$ is a Hilbert–Schmidt operator if for any (hence all) orthonormal basis $\{e_j\}_{j=1}^\infty$ of $\mathcal{H}$, we have $\sum_j \|Te_j\|_{\mathcal{H}} < \infty$. The set of Hilbert–Schmidt operators, denoted by S_2, is a two-sided ideal of $\mathcal{B}(\mathcal{H})$. The Hilbert–Schmidt norm on S_2 is given by

$$\|T\|_{HS} = \Big(\sum_{j=1}^\infty \|Te_j\|_{\mathcal{H}}^2\Big)^{\frac{1}{2}}.$$

An operator $T \in \mathcal{B}(\mathcal{H})$ is a trace class operator if for any (hence all) orthonormal basis $\{e_j\}_{j=1}^\infty$ of $\mathcal{H}$, we have $\operatorname{tr}(T) = \sum_{j=1}^\infty \langle Te_j, e_j\rangle < 0$. The set of all trace class operators on $\mathcal{H}$ is denoted by S_1.

The following well-known theorem describes a relation between a trace class operator and Hilbert–Schmidt operators.

Theorem 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is a trace class operator if and only if there exist two Hilbert–Schmidt operators U and V on $\mathcal{H}$ such that $T = UV$.*

3 Pseudo-differential operators on homogeneous space of compact groups

Throughout this section, we always assume that G is a compact Hausdorff group, and H is a closed subgroup of G.

In this section, we assume that the homogeneous space G/H has a unique G-invariant positive Radon measure μ, and $\widehat{G/H}$ is the abstract dual of G/H. The inner product on $L^2(G/H)$ will be denoted by $\langle\cdot,\cdot\rangle$. We will freely use the notation and concepts explained in the previous section. Now, we start with the definition of pseudo-differential operators.

Let σ be a measurable function on $G/H \times \widehat{G/H} \times \mathbb{N} \times \mathbb{N}$. Define the *pseudo-differential operator* T_σ *corresponding to the symbol* σ as follows. For any measurable function f on G/H, define $T_\sigma f$ formally on G/H by

$$(T_\sigma f)(gH) = \sum_{[\pi]\in G/H} d_\pi \sum_{i=1}^{d_{\pi,H}} \sum_{j=1}^{d_\pi} \sigma(gH,\pi,i,j)\langle f,\pi_{ij}\rangle \pi_{ij}(gH) \quad \text{for almost all } gH \in G/H,$$

where $\langle\cdot,\cdot\rangle$ denotes the L^2-inner product.

Let $L^2(G/H \times \widehat{G/H} \times \mathbb{N} \times \mathbb{N})$ denote the space of all measurable functions σ on $G/H \times \widehat{G/H} \times \mathbb{N} \times \mathbb{N}$ such that

$$\|\sigma\|_{L^2(G/H\times\widehat{G/H}\times\mathbb{N}\times\mathbb{N})} := \Big(\sum_{[\pi]\in G/H} d_\pi \sum_{i=1}^{d_{\pi,H}} \sum_{j=1}^{d_\pi} \int_{G/H} |\sigma(gH,\pi,i,j)\pi_{ij}(gH)|^2 \, d\mu(gH)\Big)^{\frac{1}{2}} < \infty.$$

The following theorem gives the boundedness of pseudo-differential operators on G/H with corresponding symbols in L^2-space.

Theorem 3.1. *Let G be a compact group, and let H be a closed subgroup of G. Let* $\sigma \in L^2(G/H \times \widehat{G/H} \times \mathbb{N} \times \mathbb{N})$. *Then the pseudo-differential operator* $T_\sigma : L^2(G/H) \to L^2(G/H)$ *is bounded and*

$$\|T_\sigma\|_{\mathcal{B}(L^2(G/H))} \le \|\sigma\|_{L^2(G/H\times\widehat{G/H}\times\mathbb{N}\times\mathbb{N})}.$$

Proof. Let $f \in L^2(G/H)$. Then, by Minkowski's inequality, we have

$$\begin{aligned}\|T_\sigma f\|_{L^2(G/H)} &= \Big(\int_{G/H} |(T_\sigma f)(gH)|^2 \, d\mu(gH)\Big)^{\frac{1}{2}} \\ &= \Big(\int_{G/H} \Big|\sum_{[\pi]\in G/H} d_\pi \sum_{i=1}^{d_{\pi,H}} \sum_{j=1}^{d_\pi} \sigma(gH,\pi,i,j)\langle f,\pi_{ij}\rangle \pi_{ij}(gH)\Big|^2 \, d\mu(gH)\Big)^{\frac{1}{2}} \\ &\le \sum_{[\pi]\in G/H} \Big(\int_{G/H} d_\pi^2 \Big|\sum_{i=1}^{d_{\pi,H}} \sum_{j=1}^{d_\pi} \sigma(gH,\pi,i,j)\langle f,\pi_{ij}\rangle \pi_{ij}(gH)\Big|^2 \, d\mu(gH)\Big)^{\frac{1}{2}}.\end{aligned}$$

Using Cauchy–Schwarz inequality, we get

$$\begin{aligned}\|T_\sigma f\|_{L^2(G/H)} &\le \sum_{[\pi]\in G/H} \Big(\int_{G/H} d_\pi^2 \Big(\sum_{i=1}^{d_{\pi,H}} \sum_{j=1}^{d_\pi} |\langle f,\pi_{ij}\rangle|^2\Big)\Big(\sum_{i=1}^{d_{\pi,H}} \sum_{j=1}^{d_\pi} |\sigma(gH,\pi,i,j)\pi_{ij}(gH)|^2\Big) \, d\mu(gH)\Big)^{\frac{1}{2}} \\ &= \sum_{[\pi]\in G/H} \Big(d_\pi \sum_{i=1}^{d_{\pi,H}} \sum_{j=1}^{d_\pi} |\langle f,\pi_{ij}\rangle|^2\Big)^{\frac{1}{2}} \Big(\int_{G/H} d_\pi \sum_{i=1}^{d_{\pi,H}} \sum_{j=1}^{d_\pi} |\sigma(gH,\pi,i,j)\pi_{ij}(gH)|^2 \, d\mu(gH)\Big)^{\frac{1}{2}} \\ &\le \Big(\sum_{[\pi]\in G/H} d_\pi \sum_{i=1}^{d_{\pi,H}} \sum_{j=1}^{d_\pi} |\langle f,\pi_{ij}\rangle|^2\Big)^{\frac{1}{2}} \Big(\sum_{[\pi]\in G/H} d_\pi \int_{G/H} \sum_{i=1}^{d_{\pi,H}} \sum_{j=1}^{d_\pi} |\sigma(gH,\pi,i,j)\pi_{ij}(gH)|^2 \, d\mu(gH)\Big)^{\frac{1}{2}}.\end{aligned}$$

Therefore, Plancherel's theorem gives

$$\|T_\sigma f\|_{L^2(G/H)} \le \|f\|_{L^2(G/H)} \|\sigma\|_{L^2(G/H\times\widehat{G/H})\times\mathbb{N}\times\mathbb{N}}.$$

Hence, T_σ is a bounded operator on $L^2(G/H)$ and $\|T_\sigma\|_{\mathcal{B}(L^2(G/H))} \le \|\sigma\|_{L^2(G/H\times\widehat{G/H})\times\mathbb{N}\times\mathbb{N}}$. □

Our next result provides a characterization of Hilbert–Schmidt pseudo-differential operators on G/H.

Theorem 3.2. *Let G be a compact group, and let H be a closed subgroup of G. Let σ be a measurable function on $G/H \times \widehat{G/H} \times \mathbb{N} \times \mathbb{N}$. Then the corresponding pseudo-differential operator $T_\sigma : L^2(G/H) \to L^2(G/H)$ is a Hilbert–Schmidt operator if and only if $\sigma \in L^2(G/H \times \widehat{G/H} \times \mathbb{N} \times \mathbb{N})$. Furthermore, we have*

$$\|T_\sigma\|_{HS} = \|\sigma\|_{L^2(G/H\times\widehat{G/H}\times\mathbb{N}\times\mathbb{N})}.$$

Proof. Let $\pi' \in \widehat{G/H}$ and $1 \le i_0 \le d_\pi$, $1 \le j_0 \le d_{\pi,H}$. Then, by the relation $\langle \pi'_{lk}, \pi_{l'k'}\rangle_{L^2(G/H)} = d_\pi^{-1}\delta_{ll'}\delta_{kk'}\delta_{\pi\pi'}$, we get

$$\begin{aligned}(T_\sigma \pi'_{i_0 j_o})(gH) &= \sum_{[\pi]\in\widehat{G/H}} d_\pi \sum_{i=1}^{d_\pi}\sum_{i=1}^{d_{\pi,H}} \sigma(gH,\pi,i,j)\langle \pi'_{i_0 j_0}, \pi_{i,j}\rangle \pi_{ij}(gH)\\ &= \sigma(gH,\pi',i_0,j_0)\pi_{i_0 j_0}(gH), \quad gH \in G/H.\end{aligned}$$

Since $\{\sqrt{d_\pi}\pi_{ij} : 1 \le d_\pi,\ 1 \le j \le d_{\pi,H}\}$ forms an orthonormal basis for $L^2(G/H)$, we have

$$\begin{aligned}\|T_\sigma\|_{HS}^2 &= \sum_{[\pi]\in\widehat{G/H}} \sum_{i=1}^{d_\pi}\sum_{j=1}^{d_{\pi,H}} \|T_\sigma(\sqrt{d_\pi}\pi_{ij})\|_{L^2(G/H)}^2\\ &= \sum_{[\pi]\in\widehat{G/H}} d_\pi \sum_{i=1}^{d_\pi}\sum_{j=1}^{d_{\pi,H}} \int_{G/H} |(T_\sigma \pi_{ij})(gH)|^2\, d\mu(gH)\\ &= \sum_{[\pi]\in\widehat{G/H}} d_\pi \sum_{i=1}^{d_\pi}\sum_{j=1}^{d_{\pi,H}} \int_{G/H} |\sigma(gH,\pi,i,j)\pi_{ij}(gH)|^2\, d\mu(gH)\\ &= \|\sigma\|^2_{L^2(G/H\times\widehat{G/H}\times\mathbb{N}\times\mathbb{N})}.\end{aligned}$$

Therefore, T_σ is Hilbert–Schmidt operator if and only if $\sigma \in L^2(G/H \times \widehat{G/H} \times \mathbb{N} \times \mathbb{N})$. ☐

Let σ and τ be two measurable functions on $\widehat{G/H} \times G/H \times \mathbb{N} \times \mathbb{N}$. Define $\sigma \circledast \tau$ by

$$\begin{aligned}&(\sigma \circledast \tau)(gH,\xi,k,l)\\ &\quad = \int_{G/H} \tau(wH,\xi,k,l)\xi_{kl}(wH) \sum_{[\pi]\in\widehat{G/H}} d_\pi \sum_{i=1}^{d_\pi}\sum_{j=1}^{d_{\pi,H}} \sigma(gH,\pi,i,j)\overline{\pi_{ij}(wH)}\pi_{ij}(gH)\, d\mu(wH)(\xi_{kl}(gH))^{-1}\end{aligned}$$

for all $gH \in G/H$, $[\xi] \in \widehat{G/H}$, $1 \le k \le d_\pi$ and $1 \le l \le d_{\pi,H}$.

In the following theorem, we prove that the product of two pseudo-differential operators on G/H is again a pseudo-differential operator on G/H.

Theorem 3.3. *Let G be a compact group, and let H be a closed subgroup of G. Let σ and τ be measurable functions on $G/H \times \widehat{G/H} \times \mathbb{N} \times \mathbb{N}$. Then*

$$T_\sigma T_\tau = T_\lambda,$$

where $\lambda = \sigma \circledast \tau$.

Proof. For $f \in L^2(G/H)$ and $gH \in G/H$, we have

$$\begin{aligned}(T_\sigma T_\tau f)(gH) &= \sum_{[\pi]\in\widehat{G/H}} d_\pi \sum_{i=1}^{d_\pi}\sum_{j=1}^{d_{\pi,H}} \sigma(gH,\pi,i,j)\langle T_\tau f, \pi_{ij}\rangle \pi_{ij}(gH)\\ &= \sum_{[\pi]\in\widehat{G/H}} d_\pi \sum_{i=1}^{d_\pi}\sum_{j=1}^{d_{\pi,H}} \sigma(gH,\pi,i,j)\Big(\int_{G/H} (T_\tau f)(wH)\overline{\pi_{ij}(wH)}\, d\mu(wH)\Big)\pi_{ij}(gH)\\ &= \sum_{[\pi]\in\widehat{G/H}} d_\pi \sum_{i=1}^{d_\pi}\sum_{j=1}^{d_{\pi,H}} \sigma(gH,\pi,i,j)\\ &\qquad \times \Big(\int_{G/H} \sum_{[\xi]\in\widehat{G/H}} d_\xi \sum_{k=1}^{d_\pi}\sum_{l=1}^{d_{\xi,H}} \tau(wH,\xi,k,l)\langle f, \xi_{kl}\rangle \xi_{kl}(wH)\overline{\pi_{ij}(wH)}\, d\mu(wH)\Big)\pi_{ij}(gH)\end{aligned}$$

$$= \sum_{[\pi]\in\widehat{G/H}} d_\pi \sum_{i=1}^{d_\pi} \sum_{j=1}^{d_{\pi,H}} \sigma(gH,\pi,i,j)$$
$$\times \left(\int_{G/H} \sum_{[\xi]\in\widehat{G/H}} d_\xi \sum_{k=1}^{d_\pi} \sum_{l=1}^{d_{\xi,H}} \tau(wH,\xi,k,l)\langle f,\xi_{kl}\rangle \xi_{kl}(wH)\overline{\pi_{ij}(wH)}\,d\mu(wH)\right)\pi_{ij}(gH)$$
$$= \sum_{[\xi]\in\widehat{G/H}} d_\xi \sum_{k=1}^{d_\xi} \sum_{l=1}^{d_{\xi,H}} \int_{G/H} \tau(wH,\xi,k,l)\xi_{kl}(wH) \sum_{[\pi]\in\widehat{G/H}} d_\pi \sum_{i=1}^{d_\pi} \sum_{j=1}^{d_{\pi,H}} \sigma(gH,\pi,i,j)$$
$$\times \overline{\pi_{ij}(wH)}\pi_{ij}(gH)\,d\mu(wH)(\xi_{kl}(gH))^{-1}\langle f,\xi_{kl}\rangle \xi_{kl}(gH)$$
$$= \sum_{[\xi]\in\widehat{G/H}} d_\xi \sum_{k=1}^{d_\xi} \sum_{l=1}^{d_{\xi,H}} \left\{ \int_{G/H} \tau(wH,\xi,k,l)\xi_{kl}(wH) \sum_{[\pi]\in\widehat{G/H}} d_\pi \sum_{i=1}^{d_\pi} \sum_{j=1}^{d_{\pi,H}} \sigma(gH,\pi,i,j)\right.$$
$$\left. \times \overline{\pi_{ij}(wH)}\pi_{ij}(gH)\,d\mu(wH)(\xi_{kl}(gH))^{-1}\right\}\langle f,\xi_{kl}\rangle \xi_{kl}(gH)$$
$$= \sum_{[\xi]\in\widehat{G/H}} d_\xi \sum_{k=1}^{d_\xi} \sum_{l=1}^{d_{\xi,H}} \lambda(gH,\xi,k,l)\langle f,\xi_{kl}\rangle \xi_{kl}(gH)$$
$$= (T_\lambda f)(gH),$$

where $\lambda(gH,\xi,k,l) = (\sigma \circledast \tau)(gH,\xi,k,l)$ defined as above.

Hence, $T_\sigma T_\tau = T_\lambda$. □

Finally, we present the trace formula for a pseudo-differential operator on G/H.

Theorem 3.4. *Let G be a compact group, and let H be a closed subgroup of G. Let λ be a measurable function on $G/H \times \widehat{G/H} \times \mathbb{N} \times \mathbb{N}$. Then T_λ is a trace class operator if and only if there exist two measurable functions σ and τ in $L^2(G/H \times \widehat{G/H} \times \mathbb{N} \times \mathbb{N})$ such that $\lambda = \sigma \circledast \tau$. Furthermore,*

$$\operatorname{tr}(T_\lambda) = \sum_{[\pi]\in\widehat{G/H}} d_\pi \sum_{i=1}^{d_\pi} \sum_{j=1}^{d_{\pi,H}} \int_{G/H} \lambda(gH,\pi,i,j)|\pi_{ij}(gH)|^2\,d\mu(gH)$$
$$= \sum_{[\pi]\in\widehat{G/H}} d_\pi \sum_{k=1}^{d_\pi} \sum_{l=1}^{d_{\pi,H}} \sum_{[\xi]\in\widehat{G/H}} d_\xi \sum_{i=1}^{d_\xi} \sum_{j=1}^{d_{\xi,H}} \langle \tau(\cdot,\xi,i,j)\xi_{ij},\pi_{kl}\rangle\langle \tau(\cdot,\pi,k,l)\pi_{kl},\xi_{ij}\rangle.$$

Proof. The first part of the theorem follows from Theorem 3.2 and the fact that the product of two Hilbert–Schmidt operators is a trace class operator (see Theorem 2.5). Now, the absolute convergence of the series

$$\sum_{[\pi]\in\widehat{G/H}} d_\pi \sum_{k=1}^{d_\pi} \sum_{l=1}^{d_{\pi,H}} \sum_{[\xi]\in\widehat{G/H}} d_\xi \sum_{i=1}^{d_\pi} \sum_{j=1}^{d_{\pi,H}} \langle \tau(\cdot,\xi,i,j)\xi_{ij},\pi_{kl}\rangle\langle \tau(\cdot,\pi,k,l)\pi_{kl},\xi_{ij}\rangle$$

follows from Plancherel's theorem and Cauchy–Schwarz inequality.

Since the set $\{\sqrt{d_\pi}\pi_{ij} : \pi \in \widehat{G/H},\ 1 \le i \le d_\pi,\ 1 \le j \le d_{\pi,H}\}$ forms an orthonormal basis for $L^2(G/H)$, we have

$$\operatorname{tr}(T_\lambda) = \sum_{[\xi]\in\widehat{G/H}} d_\xi \sum_{i=1}^{d_\pi} \sum_{j=1}^{d_{\pi,H}} \langle T_\lambda \xi_{ij},\xi_{ij}\rangle$$
$$= \sum_{[\xi]\in\widehat{G/H}} d_\xi \sum_{i=1}^{d_\xi} \sum_{j=1}^{d_{\xi,H}} \int_{G/H} \lambda(gH,\xi,i,j)\xi_{ij}(gH)\overline{\xi_{ij}(gH)}\,d\mu(gH)$$
$$= \sum_{[\xi]\in\widehat{G/H}} d_\xi \sum_{i=1}^{d_\xi} \sum_{j=1}^{d_{\xi,H}} \int_{G/H} \left\{ \int_{G/H} \tau(wH,\xi,i,j)\xi_{ij}(wH) \sum_{[\pi]\in\widehat{G/H}} d_\pi \sum_{k=1}^{d_\pi} \sum_{l=1}^{d_{\pi,H}} \sigma(gH,\pi,k,l)\right.$$
$$\left. \times \overline{\pi_{kl}(wH)}\pi_{kl}(gH)\,d\mu(wH)\right\}\overline{\xi_{ij}(gH)}\,d\mu(gH)$$

$$= \sum_{[\pi]\in\widehat{G/H}} d_\pi \sum_{k=1}^{d_\pi}\sum_{l=1}^{d_{\pi,H}} \sum_{[\xi]\in\widehat{G/H}} d_\xi \sum_{i=1}^{d_\xi}\sum_{j=1}^{d_{\xi,H}} \int_{G/H} \tau(wH,\xi,i,j)\xi_{ij}(wH)\overline{\pi_{jk}(wH)}\,d\mu(wH)$$
$$\times \int_{G/H} \sigma(gH,\pi,k,l)\pi_{kl}(gH)\overline{\xi_{gH}}\,d\mu(gH)$$
$$= \sum_{[\pi]\in\widehat{G/H}} d_\pi \sum_{k=1}^{d_\pi}\sum_{l=1}^{d_{\pi,H}} \sum_{[\xi]\in\widehat{G/H}} d_\xi \sum_{i=1}^{d_\xi}\sum_{j=1}^{d_{\xi,H}} \langle\tau(\cdot,\xi,i,j)\xi_{ij},\pi_{kl}\rangle\langle\tau(\cdot,\pi,k,l)\pi_{kl},\xi_{ij}\rangle. \qquad \square$$

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