On the convexity of Relativistic Ideal Magnetohydrodynamics

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Abstract. We analyze the influence of the magnetic field in the convexity properties of the relativistic magnetohydrodynamics system of equations. To this purpose we use the approach of Lax, based on the analysis of the linearly degenerate/genuinely non-linear nature of the characteristic fields. Degenerate and non-degenerate states are discussed separately and the non-relativistic, unmagnetized limits are properly recovered. The characteristic fields corresponding to the material and Alfvén waves are linearly degenerate and, then, not affected by the convexity issue. The analysis of the characteristic fields associated with the magnetosonic waves reveals, however, a dependence of the convexity condition on the magnetic field.

The result is expressed in the form of a generalized fundamental derivative written as the sum of two terms. The first one is the generalized fundamental derivative in the case of purely hydrodynamical (relativistic) flow. The second one contains the effects of the magnetic field. The analysis of this term shows that it is always positive leading to the remarkable result that the presence of a magnetic field in the fluid reduces the domain of thermodynamical states for which the EOS is non-convex.

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1. Introduction

There are many astrophysical scenarios governed by relativistic magnetohydrodynamical processes as, e.g., the production of relativistic jets emanating from Active Galactic Nuclei, the structure and dynamics of pulsar wind nebulae, the mechanisms triggering the explosion in core-collapse supernovae, or the production of Gamma Ray Bursts. These scenarios are nowadays the subject of intensive research by means of numerical simulations thanks to recent advances in numerical relativistic magnetohydrodynamics (RMHD) that exploit the fact that the RMHD equations obeying a causal equation of state (EOS) form a hyperbolic system of conservation laws [1].

Matter at densities higher than nuclear matter density can undergo first-order phase transitions to various phases of matter, such as pion condensates [2], hyperonic matter [3] or deconfined quark matter [4, 5]. Several authors [6, 7, 8] have studied, from different points of view, the influence that those exotic states of matter at extreme high densities have on, e.g., the dynamics of stellar core collapse supernovae, the evolution of proto-neutron stars, or the collapse to black hole.

The classical Van der Waals (VdW) EOS is a well known example of EOS displaying a first-order phase transition. Fluids having a thermodynamics governed by a VdW-like EOS exhibit, outside the region of the phase transition, non-classical gasdynamic behaviours in a range of thermodynamic conditions characterized by the negative value of the so-called fundamental derivative, $\mathcal{G}$ [9, 10, 11]

$$\mathcal{G} := -\frac{1}{2} V \frac{\left.\dfrac{\partial^2 p}{\partial V^2}\right|_s}{\left.\dfrac{\partial p}{\partial V}\right|_s} \tag{1}$$

p being the pressure, $V := 1/\rho$ the specific volume (ρ is the rest-mass density) and s the specific entropy. The fundamental derivative measures the convexity of the isentropes in the $p-V$ plane and if $\mathcal{G} > 0$ then the isentropes in the $p-V$ plane are convex, leading to expansive rarefaction waves (and compressive shocks) [12]. In a VdW-like EOS, or in general in a non-convex EOS, rarefaction waves can change to compressive and shock waves to expansive depending on the specific thermodynamical state of the system. These non-classical phenomena have been observed experimentally and their study is, currently, of interest in many engineering applications [13, 14].

Besides this thermodynamical interpretation of convexity, there is an equivalent definition due to Lax [15] that connects with the mathematical properties of the hyperbolic system. According to Lax's approach, a hyperbolic system of conservation laws ‡ is convex if all its characteristic fields are either genuinely non-linear or linearly

‡ The books by LeVeque [16] and Toro [17] are recommendable references for those readers interested in the basic theory of hyperbolic systems of conservation laws. The monograph of [18] on finite-volume methods for hyperbolic problems pays special attention to non-convex flux functions (see their Sects. 13.8.4 -definitions of genuine non-linearity and linear degeneracy, and their relationsphip with convexity-, and 16.1 -devoted entirely to the study of scalar conservation laws with non-convex flux

degenerate. A characteristic field λ is said to be genuinely non-linear or linearly degenerate if, respectively,

$$\mathcal{P} := \vec{\nabla}_{\mathbf{u}}\lambda \cdot \mathbf{r} \neq 0, \tag{2}$$

$$\mathcal{P} := \vec{\nabla}_{\mathbf{u}}\lambda \cdot \mathbf{r} = 0, \tag{3}$$

for all $\mathbf{u}$, where $\vec{\nabla}_{\mathbf{u}}\lambda$ is the gradient of $\lambda(\mathbf{u})$ in the space of conserved variables, $\mathbf{r}$ is the corresponding eigenvector, and the dot stands for the inner product in the space of physical states.

In a non-convex system, non-convexity is associated with those states $\mathbf{u}$ for which the factor $\mathcal{P}$ corresponding to a genuinely non-linear field, Eq. (2), is zero and changes sign in a neighbourhood of $\mathbf{u}$.

A virtue of Lax's approach is that it can be applied to other hyperbolic systems in which the convex or non-convex character of the dynamics is governed by other ingredients beyond the EOS. Among these systems are those of relativistic hydrodynamics (RHD) and classical magnetohydrodynamics (MHD). In these two cases, the convexity of the system has been characterized with the sign of a generalized fundamental derivative that includes an extra term depending of the local speed of sound (in the case of RHD [19]) and the magnetic field (in the case of MHD [20]).

In this work we use the approach of Lax to characterize, from a theoretical point of view, the effects of magnetic fields in the convexity properties of the RMHD system of equations as a previous step to explore its possible impact in the dynamical evolution of different astrophysical scenarios. The result is presented in the form of an extended fundamental derivative whose sign determines the convex/non-convex character of the RMHD system at a given state. Our result recovers the proper non-relativistic and unmagnetized limits.

The paper is organized as follows. In Sect. 2, the equations of RMHD are introduced as a hyperbolic system of conservation laws. The transformation between primitive and conserved variables are explicitly written. In Sect. 3 the characteristic structure of the RMHD equations is discussed and the analysis of convexity in non-degenerate states presented. In Sect. 4 the analysis of convexity is extended to degenerate states. The non-relativistic, unmagnetized limits are recovered in Sect. 5. Section 6 includes a short summary and presents the conclusions. Finally, there is an Appendix that displays the Jacobian matrices of the RMHD system in quasi-linear form, necessary for the characteristic analysis of Sect. 3.

2. The equations of ideal relativistic magnetohydrodynamics

Let J^μ, $T^{\mu\nu}$ and ${}^*F^{\mu\nu}$§ be the components of the rest-mass current density, the energy–momentum tensor and the Maxwell tensor of an ideal (infinite conductivity) magneto-

functions-).

§ Throughout this paper, Greek indices will run from 0 to 3, while Roman run from 1 to 3, or, respectively, from t to z and from x to z, in Cartesian coordinates.

fluid, respectively

$$J^\mu = \rho u^\mu \tag{4}$$

$$T^{\mu\nu} = \rho h^* u^\mu u^\nu + g^{\mu\nu} p^* - b^\mu b^\nu \tag{5}$$

$$^*F^{\mu\nu} = u^\mu b^\nu - u^\nu b^\mu, \tag{6}$$

where ρ is the proper rest-mass density, $h^* = 1 + \epsilon + p/\rho + b^2/\rho$ is the specific enthalpy including the contribution from the magnetic field (b^2 stands for $b^\mu b_\mu$), ϵ is the specific internal energy, p is the thermal pressure, $p^* = p + b^2/2$ is the total pressure, and $g^{\mu\nu}$ is the metric of the space-time where the fluid evolves. Throughout the paper we use units in which the speed of light is $c = 1$ and the $(4\pi)^{1/2}$ factor is absorbed in the definition of the magnetic field. The four-vectors representing the fluid velocity, u^μ, and the magnetic field measured in the comoving frame, b^μ, satisfy the conditions $u^\mu u_\mu = -1$ and $u^\mu b_\mu = 0$.

The equations of ideal RMHD correspond to the conservation of rest-mass and energy-momentum, and the Maxwell equations. In a flat space-time and Cartesian coordinates, these equations read:

$$J^\mu{}_{,\mu} = 0 \tag{7}$$

$$T^{\mu\nu}{}_{,\mu} = 0 \tag{8}$$

$$^*F^{\mu\nu}{}_{,\mu} = 0, \tag{9}$$

where subscript $({}_{,\mu})$ denotes partial derivative with respect to the corresponding coordinate, (t, x, y, z), and the standard Einstein sum convention is assumed.

The above system can be written as a system of conservation laws as follows

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}^i}{\partial x^i} = 0 \tag{10}$$

where $\mathbf{V} = (\rho, v^i, \epsilon, B^i)^T$ is the set of primitive variables. The state vector (the set of conserved variables) $\mathbf{U}$ and the fluxes, $\mathbf{F}^i$, are, respectively:

$$\mathbf{U} = \begin{pmatrix} D \\ S^i \\ \tau \\ B^i \end{pmatrix}, \tag{11}$$

$$\mathbf{F}^i = \begin{pmatrix} Dv^i \\ S^j v^i + p^* \delta^{ij} - b^j B^i/W \\ \tau v^i + p^* v^i - b^0 B^i/W \\ v^i B^k - v^k B^i \end{pmatrix}. \tag{12}$$

In the preceding equations, D, S^j and τ stand, respectively, for the rest-mass density, the momentum density of the magnetized fluid in the j-direction, and its total energy density, all of them measured in the laboratory (i.e., Eulerian) frame:

$$D = \rho W, \tag{13}$$

$$S^i = \rho h^* W^2 v^i - b^0 b^i, \tag{14}$$

$$\tau = \rho h^* W^2 - p^* - (b^0)^2 - D. \tag{15}$$

The components of the fluid velocity trivector, v^i, as measured in the laboratory frame, are related with the components of the fluid four-velocity according to the following expression: $u^\mu = W(1, v^i)$, where W is the flow Lorentz factor, $W^2 = 1/(1 - v^i v_i)$.

The components of the magnetic field four-vector in the comoving frame and the three vector components B^i measured in the laboratory frame satisfy the relations:

$$b^0 = W v_k B^k, \tag{16}$$

$$b^i = \frac{B^i}{W} + b^0 v^i. \tag{17}$$

Finally, the square of the modulus of the magnetic field can be written as

$$b^2 = \frac{B_k B^k}{W^2} + (v_k B^k)^2. \tag{18}$$

The preceding system must be complemented with the time component of equation (9), that becomes the usual divergence constraint

$$\frac{\partial B^i}{\partial x^i} = 0. \tag{19}$$

An EOS $p = p(\rho, \varepsilon)$ closes the system. Accordingly, the (relativistic) sound speed $a_s := \sqrt{\left.\frac{\partial p}{\partial e}\right|_s}$, e being the mass-energy density of the fluid $e = \rho(1+\epsilon)$, satisfies $h a_s^2 = \chi + \frac{p}{\rho^2}\kappa$, with $\chi := \left.\frac{\partial p}{\partial \rho}\right|_\varepsilon$ and $\kappa := \left.\frac{\partial p}{\partial \varepsilon}\right|_\rho$.

3. Characteristic structure of the RMHD equations and analysis of convexity in non-degenerate states

The characteristic information of the system of RMHD (10) is contained in the set of eigenvalues and right eigenvectors $\{\lambda_\alpha, \mathbf{r}_\alpha\}_{\alpha=1}^8$ of $\zeta_k \mathcal{B}^k$, where $\mathcal{B}^i := \frac{\partial \mathbf{F}^i}{\partial \mathbf{U}}$ are the Jacobian matrices of the vectors of fluxes along the coordinate directions, and ζ_i is an arbitrary unitary 3-vector.

Since the dependence on $\mathbf{U}$ of the fluxes $\mathbf{F}^i$ is implicit, it is useful to write the Jacobian matrices $\mathcal{B}^i$ in terms of matrices involving only explicit derivatives with respect to the primitive variables, $\mathbf{V}$. If we define $\mathcal{A}^0 := \dfrac{\partial \mathbf{U}}{\partial \mathbf{V}}$, and $\mathcal{A}^i := \dfrac{\partial \mathbf{F}^i}{\partial \mathbf{V}}$, then we have that $\mathcal{B}^i = \mathcal{A}^i(\mathcal{A}^0)^{-1}$. Now, the sets of eigenvalues and right eigenvectors of the system in conservation form, $\{\lambda_\alpha, \mathbf{r}_\alpha\}_{\alpha=1}^8$, and of the system in quasi-linear form, $\{\lambda^*_\alpha, \mathbf{r}^*_\alpha\}_{\alpha=1}^8$, satisfying $(\zeta_k \mathcal{A}^k - \lambda^*_\alpha \mathcal{A}^0)\mathbf{r}^*_\alpha = 0$, are related according to $\{\lambda_\alpha, \mathbf{r}_\alpha\}_{\alpha=1}^8 = \{\lambda^*_\alpha, \mathcal{A}^0 \mathbf{r}^*_\alpha\}$. Matrices $\mathcal{A}^0$ and $\zeta_k \mathcal{A}^k$ are displayed in the Appendix.

Once the eigenvalues and eigenvectors are known, we can analyze the convexity of the system studying the expression $\mathcal{P}_\alpha = \vec{\nabla}_{\mathbf{U}} \lambda_\alpha \cdot \mathbf{r}_\alpha$ (see the Introduction). Finally, we can take advantage of the fact that, since $\mathcal{A}^0$ is non-singular, then $\mathcal{P}_\alpha \neq 0$ if, and only if, $\mathcal{P}^*_\alpha := \vec{\nabla}_{\mathbf{V}} \lambda_\alpha \cdot \mathbf{r}^*_\alpha \neq 0$, and perform the analysis of convexity in terms of $\mathcal{P}^*_\alpha$.

The eigenvalues λ_α are the solutions of the following polynomial expression for λ

$$\lambda a \Big(\mathcal{E} a^2 - \mathcal{B}^2 \Big) \Big((b^2 + \rho h a_s^2) a^2 G - W_s^{-2} \rho h a^4 - a_s^2 G \mathcal{B}^2 \Big) = 0, \tag{20}$$

where $\mathcal{E} := \rho h + b^2$, $W_s^{-2} := 1 - a_s^2$ and quantities a, G and $\mathcal{B}$ were defined in ref. [1], $a := \phi_\alpha u^\alpha$, $G := \phi_\alpha \phi^\alpha$, $\mathcal{B} := \phi_\alpha b^\alpha$, being, in our case, $\phi_\alpha := (-\lambda, \zeta_i)$ the normal to the wavefront propagating with speed λ in the spatial direction given by the unit vector ζ_i.

As it is well known, the system of (R)MHD is not-strictly hyperbolic [21]. This means that in some cases, two or more eigenvalues can be equal leading to well studied cases of degeneracy (see refs. [1, 22], for the relativistic case). In Type I degeneracy, the magnetic field is normal to the propagation direction of the wavefront (i.e., $\zeta_k B^k = 0$). In Type II degeneracy, $\zeta_k B^k \neq 0$, but the eigenvalues associated with, at least, one Alfvén wave and one magnetosonic wave are degenerate. Leaving aside the particular cases associated with both degeneracy types, that will be discussed later, the following list compiles the roots of the characteristic equation (20), λ_α $(= \lambda^*_\alpha)$, the right eigenvectors, $\mathbf{r}^*_\alpha$ ‖, and their corresponding scalar products, $\mathcal{P}^*_\alpha$ ¶, in the non-degenerate, general case.

i) $\lambda = \lambda_{\rm null} := 0$. In this case, $\mathcal{P}^*_{\rm null}$ is trivially zero. This eigenvalue is spurious and is associated with the fact that although the RMHD system (10) consists of eight conservation equations, only seven components of the fluxes are non-trivial. Due to the antisymmetric character of the induction equation, the flux of $\zeta_k B^k$ in the ζ^k-direction is identically zero.

ii) $\lambda = \lambda_0 := \zeta_k v^k$ is the eigenvalue associated with the material waves. The corresponding eigenvector is $\mathbf{r}^*_0 = (-\kappa, 0^i, \chi, 0^i)^T$, where κ and χ are thermodynamical derivatives defined at the end of the previous Section, and $0^i = 0$ $(i = 1, 2, 3)$. The scalar product is $\mathcal{P}^*_0 = 0$ and, consequently, the characteristic field defined by λ_0 is linearly degenerate.

‖ The expressions of the eigenvectors have been obtained after tedious algebraic manipulations. They can be verified by direct substituting in the eigenvalue equation, $(\zeta_k \mathcal{A}^k - \lambda^*_\alpha \mathcal{A}^0)\mathbf{r}^*_\alpha = 0$.

¶ For the scalar products $\mathcal{P}^*_{a_\pm}$ and $\mathcal{P}^*_{m_\pm}$, the partial derivatives of the corresponding eigenvalues with respect to the primitive variables, $\mathbf{V}$, have been computed by implicit derivation of the characteristic equations for $\lambda_{a_\pm}$ and $\lambda_{f_\pm}$, respectively, i.e., $\mathcal{A} = 0$ and $\mathcal{N}_4 = 0$ (see below).

iii) $\lambda = \lambda_{a\pm}$ are the roots of the second-order polynomial in λ, $\mathcal{A}$,

$$\mathcal{A} := \mathcal{E}a^2 - \mathcal{B}^2. \tag{21}$$

They define the Alfvén waves. Since $\zeta_k B^k \neq 0$, then $a \neq 0$ and the corresponding eigenvectors are

$$\mathbf{r}^*_{a\pm} = (0, r_2^i, 0, r_4^i)^T, \tag{22}$$

where $r_2^i = a_1 B^i + a_2 v^i + a_3 \zeta^i$, $r_4^i = W a^{-1}(r_2^i \zeta_k B^k - B^i \zeta_k r_2^k)$. The coefficients $a_p\,(p = 1, 2, 3)$ are such that $v_k r_2^k = 1$, $\zeta_k r_2^k = -Wa$, and $B_k r_2^k = -v_k B^k W^2$. The scalar products are

$$\mathcal{P}^*_{a\pm} = \left(\frac{\partial \lambda_{a\pm}}{\partial v^i}\right) r_2^i + \left(\frac{\partial \lambda_{a\pm}}{\partial B^i}\right) r_4^i \propto \Big(\zeta_k r_2^k + W\,a\,(v_k r_2^k)\Big) = 0, \tag{23}$$

in agreement with the linearly degenerate character of the Alfvén waves.

iv) The four eigenvalues $\lambda_{f\pm}$, $\lambda_{s\pm}$, are the roots of the fourth-order polynomial in λ, $\mathcal{N}_4$,

$$\mathcal{N}_4 := (b^2 + \rho h a_s^2) a^2 G - W_s^{-2} \rho h a^4 - a_s^2 G \mathcal{B}^2, \tag{24}$$

associated with the fast and slow magnetosonic wavespeeds, respectively. Since $\zeta_k B^k \neq 0$, then $a \neq 0$ and the corresponding eigenvectors are

$$\mathbf{r}^*_{m\pm} = (r_1, r_2^i, r_3, r_4^i)^T, \tag{25}$$

$(m = f, s)$, where

$$\begin{aligned}
r_1 &= \rho W^3 \Big(\rho h a (G + a^2) - G\mathcal{B}^2/a\Big),\\
r_2^i &= W\Big(G\mathcal{B}B^i + \rho h W a^2(\lambda_{m\pm} v^i - \zeta^i)\Big),\\
r_3 &= r_1 p/\rho^2,\\
r_4^i &= \rho h W^3 a\Big((\lambda_{m\pm}\, v^i - \zeta^i)\zeta_k B^k - B^i(\lambda_{m\pm} a W^{-1} - G)\Big).
\end{aligned} \tag{26}$$

The scalar products are

$$\begin{aligned}
\mathcal{P}^*_{m\pm} &= \left(\frac{\partial \lambda_{m\pm}}{\partial \rho}\right) r_1 + \left(\frac{\partial \lambda_{m\pm}}{\partial v^i}\right) r_2^i + \left(\frac{\partial \lambda_{m\pm}}{\partial \epsilon}\right) r_3 + \left(\frac{\partial \lambda_{m\pm}}{\partial B^i}\right) r_4^i\\
&= \frac{W^3 a^4 G^2}{2 a_s^2 d} \mathcal{P}_1^*\,\mathcal{P}_2^*,
\end{aligned} \tag{27}$$

where d, the derivative of $\mathcal{N}_4$ with respect to λ at $\lambda = \lambda_{m,\pm}$ $(m = f, s)$, $\mathcal{N}_4'(\lambda_{m,\pm})$, is

$$d = a_s^2 G^2 \mathcal{B}(\zeta_k B^k) - (G - \lambda_{m\pm} a W^{-1}) \rho h W W_s^{-2} a^4, \tag{28}$$

and

$$\mathcal{P}_1^* = b^2 G - \rho h a^2, \tag{29}$$

$$\mathcal{P}_2^* = \left(\rho \left.\frac{\partial a_s^2}{\partial \rho}\right|_\epsilon + \frac{p}{\rho} \left.\frac{\partial a_s^2}{\partial \epsilon}\right|_\rho \right) W_s^2 \left(\frac{\mathcal{B}^2}{a^2} - \mathcal{E} \right) - b^2(3 - a_s^2) - 2\rho h a_s^2 + \frac{a_s^2(5 - 3a_s^2)\mathcal{B}^2}{a^2}. \tag{30}$$

It is interesting to note that d can only be zero in degenerate states, since it is only in these states where both $\mathcal{N}_4(\lambda) = 0$ and $\mathcal{N}_4'(\lambda) = 0$ are satisfied simultaneously. Let us now discuss the conditions under which the remaining factors in Eq. (27) can become zero. Quantity a is non-zero as far as $\zeta_k B^k \neq 0$. On the other hand, it can be proven by simple algebraic manipulation of Eqs. $\mathcal{A}(\lambda) = 0$ and $\mathcal{N}_4(\lambda) = 0$ that $\mathcal{P}_1^* = 0$ if and only if the corresponding magnetosonic eigenvalue is also an Alfvén eigenvalue (i.e., Type II degeneracy). Since we are avoiding degenerate states, and G is always non-zero, we shall concentrate on the changes of sign of $\mathcal{P}_2^*$, in order to analyze the possible loss of convexity associated with the magnetosonic waves.

Since in the case of zero magnetic field, the purely relativistic result has to be recovered, we shall try now to rewrite expression (30) in terms of the relativistic fundamental derivative

$$\tilde{\mathcal{G}} = 1 + \frac{\rho}{2a_s^2} \left.\frac{\partial a_s^2}{\partial \rho}\right|_s - a_s^2 \tag{31}$$

derived in ref. [19]. The sought expression is

$$\mathcal{P}_2^* = -2a_s^2 W_s^2 \mathcal{E}(1 - R)\, \tilde{\mathcal{G}}_{\rm M}, \tag{32}$$

with $\tilde{\mathcal{G}}_{\rm M}$, the fundamental derivative for relativistic, magnetized fluids, being

$$\tilde{\mathcal{G}}_{\rm M} := \tilde{\mathcal{G}} + F, \tag{33}$$

where

$$F := \frac{3}{2} W_s^{-4} \left(\frac{c_a^2/a_s^2 - R}{1 - R} \right). \tag{34}$$

In the previous expressions, $R := \dfrac{\mathcal{B}^2}{\mathcal{E}a^2}$, and $c_a^2 := \dfrac{b^2}{\mathcal{E}}$ stands for the square of the Alfvén velocity. Moreover, in deriving expression (32) from (30) we have used the following relation among thermodynamical derivatives $\left.\dfrac{\partial}{\partial \rho}\right|_s = \left.\dfrac{\partial}{\partial \rho}\right|_\epsilon + \dfrac{p}{\rho^2} \left.\dfrac{\partial}{\partial \epsilon}\right|_\rho$.

It is important to note that $R = 1$ if and only if the eigenvalue corresponds to an Alfvén wavespeed (i.e., it satisfies equation $\mathcal{A}(\lambda) = 0$). Since we are not considering degeneracies, we conclude that $R \neq 1$ for magnetosonic waves and, consequently, 1) the denominator in the second term of $\tilde{\mathcal{G}}_{\rm M}$ is well defined, and 2) $\mathcal{P}_2^* = 0$ if and only if $\tilde{\mathcal{G}}_{\rm M} = 0$.

The price to pay for using primitive (or conserved) variables in our analysis of convexity is the loss of covariance and a dependence of the fundamental derivative $\tilde{\mathcal{G}}_{\rm M}$ on kinematics through quantity R. For fast and slow magnetosonic fields, let us carry out the analysis of the magnetic correction to the purely hydrodynamic (relativistic) fundamental derivative (Eq. (34)) in the comoving frame (CF,

$u^\mu = \delta^\mu_0$), which we will name $F_{\mathrm{CF},m}$ $(m = f, s)$ henceforth. A simple algebraic calculation leads to

$$F_{\mathrm{CF},m} = \frac{3}{2} W_\omega^{-2} \left(\frac{c_m^2 - a_s^2}{c_m^2 - c_a^2} \right), \tag{35}$$

where c_m^2 are the solutions of the quadratic equation in λ^2, $\mathcal{N}_{4,\mathrm{CF}}(\lambda) = 0$, namely

$$c_m^2 = \frac{1}{2} \left((\omega^2 + a_s^2\, c_A^2) \pm \left((\omega^2 + a_s^2\, c_A^2)^2 - 4a_s^2\, c_A^2 \right)^{1/2} \right), \tag{36}$$

with $c_A^2 = \dfrac{(\zeta_k B^k)^2}{\mathcal{E}}$ and $W_\omega^{-2} := 1 - \omega^2$, $\omega^2 = a_s^2 + c_a^2 - a_s^2 c_a^2$.
Taking into account that, for non-degenerate states, $a_s^2, c_a^2 \in (c_s^2, c_f^2)^+$, we have that $F_{\mathrm{CF},m} > 0$ $(m = f, s)$. Now, the transformation of R as a scalar ensures that $F_m > 0$ $(m = f, s)$ in any reference frame, with important consequences for the influence of the magnetic field on the convexity of the system.

4. Analysis of convexity in degenerate states

4.1. Type I degeneracy

This degeneracy appears in states in which $\zeta_k B^k = 0$. Now, the roots of the characteristic equation (20), the right eigenvectors, and the corresponding scalar products have the following properties:

i) $\lambda = \lambda_{\mathrm{null}} := 0$. It is again the spurious eigenvalue analyzed in the previous Section associated with the null flux component. $\mathcal{P}^*_{\mathrm{null}}$ is trivially zero.

ii) The eigenvalue $\lambda = \lambda_0 := \zeta_k v^k$ has multiplicity 5. The corresponding eigenvectors are of the form $\mathbf{r}_0^* = (r_1, a_1 B^i + a_2 \zeta_\perp^i, r_3, a_3 B^i + a_4 \zeta_\perp^i)^T$, where $\zeta_\perp^i$ is an arbitrary vector orthogonal to ζ^i and B^i, and r_1, r_3 and a_p, $(p = 1, 2, 3, 4)$ are functions of the primitive variables. Since only the derivative $\partial\lambda/\partial v^k$ $(= \zeta_k)$ is different from zero, the scalar product is

$$\mathcal{P}^*_0 = \zeta_i (a_1 B^i + a_2 \zeta_\perp^i) = 0. \tag{37}$$

Hence, the characteristic fields defined by λ_0 are linearly degenerate.

iii) $\lambda_{f,\pm}$ are the solutions of the quadratic equation in λ

$$\left(b^2 + \rho h a_s^2 - a_s^2 (v_k B^k)^2 \right) G - W_s^{-2} \rho h a^2 = 0, \tag{38}$$

and are associated with the fast magnetosonic wavespeeds. The explicit expression of these eigenvalues when $\zeta_k = (1, 0, 0)$ can be found in ref. [23].
The corresponding eigenvectors can be obtained from those of the fast magnetosonic eigenvalues in the general case (see Eq. (25)) making $\zeta_k B^k = 0$, i.e., $\mathcal{B} = a(v_k B^k)$.

$^+$ In the CF it can be easily proven that $\mathcal{N}_{4,\mathrm{CF}}(a_s) < 0$ and $\mathcal{N}_{4,\mathrm{CF}}(c_a) < 0$, implying that both a_s^2 and c_a^2 are between the roots of $\mathcal{N}_{4,\mathrm{CF}}(\lambda) = 0$, namely c_s^2, c_f^2.

The scalar products are *

$$\mathcal{P}^*_{f\pm} = \frac{W_s^2 G^2}{2\rho h} \mathcal{P}^*_1 \mathcal{P}^*_2, \quad (39)$$

where

$$\mathcal{P}^*_1 = \frac{\mathcal{E} - (v_k B^k)^2}{1 - \zeta_k v^k} \quad (40)$$

$$\mathcal{P}^*_2 = \left(\rho \left. \frac{\partial a_s^2}{\partial \rho} \right|_\epsilon + \frac{p}{\rho} \left. \frac{\partial a_s^2}{\partial \epsilon} \right|_\rho \right) W_s^2 \left((v_k B^k)^2 - \mathcal{E} \right) - b^2(3 - a_s^2) - 2\rho h a_s^2 + a_s^2(5 - 3a_s^2)(v_k B^k)^2. \quad (41)$$

From Eq. (18), $b^2 - (v_k B^k)^2 \geq 0$ and then $\mathcal{P}^*_1$ is always positive. Hence the possible changes of sign of $\mathcal{P}^*_{f\pm}$ coincide with those of $\mathcal{P}^*_2$. Let us note that the expression for $\mathcal{P}^*_2$ coincides with that of the general case (Eq. 30) making $\mathcal{B} = a(v_k B^k)$. Then, proceeding in exactly the same way as in the general case we conclude that the fundamental derivative for relativistic, magnetized fluids for Type I degenerate states is

$$\tilde{\mathcal{G}}_{\mathrm{M,deg\,I}} = \tilde{\mathcal{G}} + \frac{3}{2} W_s^{-4} \left(\frac{c_a^2/a_s^2 - R_{\mathrm{deg\,I}}}{1 - R_{\mathrm{deg\,I}}} \right), \quad (42)$$

where now, $R_{\mathrm{deg\,I}} = \dfrac{(v_k B^k)^2}{\mathcal{E}}$. As discussed in the non-degenerate case, $R_{\mathrm{deg\,I}} \neq 1$, and the corresponding factor is $F_{\mathrm{deg\,I}} > 0$.

The special case when $v_k B^k = 0$ is obtained by making $R_{\mathrm{deg\,I}} = 0$ in the previous expression. The same result for this case is obtained through a purely hydrodynamical approach (see Appendix in ref. [24]) by building up a thermodynamically consistent EOS incorporating the effects of the magnetic field.

4.2. Type II degeneracy

Now, $\zeta_k B^k \neq 0$ and, at least, one eigenvalue associated with an Alfvén wave and an eigenvalue associated with a magnetosonic wave are degenerated. Three cases are distinguished. In cases 1 ($c_a > a_s$) and 2 ($c_a < a_s$) one fast or slow magnetosonic eigenvalue, respectively, and an Alfvén eigenvalue are degenerated. In these cases, as discussed in the previous Section, the quantity $\mathcal{P}^*_1$ defined in Eq. (29) is zero for the degenerate eigenvalues and, hence, the corresponding characteristic fields are linearly degenerate. When $c_a = a_s$ (case 3), an Alfvén eigenvalue is degenerated with a pair (slow and fast) of magnetosonic eigenvalues. Now, quantity d defined in Eq. (28) is also 0, and we have an indetermination in $\mathcal{P}^*_{m\pm}$ (Eq. 27). In this case, we have checked that the dot product of the magnetosonic eigenvectors associated with the degenerated fields and the gradient of the Alfvén eigenvalue is zero, which means that the degenerate characteristic field is again linearly degenerate.

* As in the non-degenerate case, for the scalar products $\mathcal{P}^*_{f\pm}$, the partial derivatives of the corresponding eigenvalues with respect to the primitive variables, $\mathbf{V}$, have been computed by implicit derivation of the characteristic equation (38).

5. Purely hydrodynamical and classical limits

The purely (relativistic) hydrodynamical limit can be obtained as a particular case of the Type I degeneracy, in which besides having $\zeta_k B^k = 0$ and $v_k B^k = 0$, we make $b^2 = 0$. Hence, from Eq. (33), and making $R = 0$ and $c_a = 0$, we have $\tilde{\mathcal{G}}_{\mathrm{M},b^2=0} = \tilde{\mathcal{G}}$.

We now discuss the classical (magnetized) limits for both degenerate and non-degenerate states. These limits are obtained by expanding all the quantities in the definition of the fundamental derivative in powers of $1/c^2$ (c is the speed of light) and keeping the leading term. On one hand, the relativistic (non-magnetized) fundamental derivative is $\tilde{\mathcal{G}} = \mathcal{G} + \mathcal{O}(1/c^2)$, where $\mathcal{G}$ is the classical (non-magnetized) counterpart [19]. On the other hand, $R = (\zeta_k B^k)^2/(\rho c_{m,\mathrm{cl}}^2) + \mathcal{O}(1/c^2)$, where $c_{m,\mathrm{cl}}$ $(m = f, s)$ is $c_{m,\mathrm{cl}} = \dfrac{1}{\sqrt{2}}\left(a_{s,\mathrm{cl}}^2 + B^2/\rho \pm \sqrt{(a_{s,\mathrm{cl}}^2 + B^2/\rho)^2 - 4a_{s,\mathrm{cl}}^2(\zeta_k B^k)^2/\rho}\right)^{1/2}$, and $a_{s,\mathrm{cl}}$ stands for the classical definition of the sound speed. Hence, we get from Eq. (33)

$$\tilde{\mathcal{G}}_{\mathrm{M,cl}} := \mathcal{G} + \frac{3}{2}\left(\frac{c_{a,\mathrm{cl}}^2/a_{s,\mathrm{cl}}^2 - (\zeta_k B^k)^2/(\rho c_m^2)}{1 - (\zeta_k B^k)^2/(\rho c_m^2)}\right). \tag{43}$$

In the previous expression, $c_{a,\mathrm{cl}}$ stand for the classical definition of the Alfvén speed, $\sqrt{B^2/\rho}$.

It can be shown that, taking $\zeta_k = (1, 0, 0)$, the resulting expression of $\tilde{\mathcal{G}}_{\mathrm{M,cl}}$ is proportional to the non-linearity factor for the non-linear fields of the (classical) MHD system obtained in ref. [20] (see their equation (17)).

For Type I degenerate states, since $R = \mathcal{O}(1/c^2)$,

$$\tilde{\mathcal{G}}_{\mathrm{M,deg\,I,cl}} = \mathcal{G} + \frac{3}{2}\left(\frac{c_{a,\mathrm{cl}}^2}{a_{s,\mathrm{cl}}^2}\right), \tag{44}$$

proportional to the corresponding result obtained in ref. [20] (see their table I).

Finally for Type II degenerate states, the eigenvalues that are degenerated lead to characteristic fields which are linearly degenerate, whereas the (hypothetical) non-degenerate magnetosonic field (subcases 1 and 2) is genuinely non-linear and its properties in relation with convexity are governed by the fundamental derivative in Eq. (43), with $c_{m,\mathrm{cl}} = c_{s,\mathrm{cl}}$ (subcase 1), and $c_{m,\mathrm{cl}} = c_{f,\mathrm{cl}}$ (subcase 2).

6. Summary and conclusions

In this paper we have analyzed the influence of the magnetic field in the convexity properties of the RMHD equations. To this purpose we have used the approach of Lax, based on the analysis of the linearly degenerate/genuinely non-linear nature of the characteristic fields. Degenerate and non-degenerate states have been discussed separately and the non-relativistic, unmagnetized limits are properly recovered. The characteristic fields corresponding to the material and Alfvén waves are linearly degenerate and, then, not affected by the convexity issue. The analysis of the characteristic fields associated with the magnetosonic waves reveals, however, a dependence of the convexity condition on the magnetic field.

The result is expressed in the form of a generalized fundamental derivative, Eq. (33), written as the sum of two terms. The first one is the generalized fundamental derivative in the case of purely hydrodynamical (relativistic) flow already obtained in ref. [19]. The second one contains the effects of the magnetic field. The analysis of this term in the comoving frame (extendable to any other reference system given the scalar nature of the term) shows that it is always positive leading to the remarkable result that the presence of a magnetic field in the fluid reduces the domain of thermodynamical states for which the EOS is non-convex, as it happens in the non-relativistic MHD limit [20].

We speculate with the possibility that our findings can be relevant in the context of massive stellar core collapse. Depending mostly on the pre-collapse stellar magnetic field and on the gradient of the rotational velocity, dynamically relevant magnetic fields may develop after the core bounce (see, e.g., [25, 26, 27]). Should these magnetic fields become as large as the existing numerical models point out, then our results indicate that the loss of convexity would be rather limited, if existing at all. However, it is still a matter of debate what is the actual level of magnetic field saturation due to the action of the Magneto Rotational Instability (MRI; see, e.g., [28, 29]), and hence, whether or not the MRI-amplified magnetic field may have the sufficient strength as to impede the development of non-convex regions in the collapsed core. It is very likely that under the most common conditions (namely, non-rotating or slowly rotating cores), the magnetic field will not play central dynamical role in the post-collapse evolution, though it may set the time scale for supernova explosions (e.g., [30]). In such cases, we foresee that there might exist a range of physical conditions in which a non-convex EOS may render a convexity loss in the post-collapse core that cannot be compensated by the growth of pre-collapse magnetic fields, e.g., in slowly rotating (including non-rotating) massive stellar cores. Addressing this issue by means of numerical simulations is beyond the scope of the present work, and will be considered elsewhere.

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Appendix. Jacobian matrices of the RMHD system in quasi-linear form

Matrices $\mathcal{A}^0$ and $\zeta_k\mathcal{A}^k$ associated with the system (10) in quasilinear form are:

$$\mathcal{A}^0 = \begin{pmatrix} W & \rho W^3 v_j & 0 & 0_j \\ (\mathcal{A}^0)^{S^i}_{\rho} & (\mathcal{A}^0)^{S^i}_{v^j} & (\mathcal{A}^0)^{S^i}_{\epsilon} & (\mathcal{A}^0)^{S^i}_{B^j} \\ (\mathcal{A}^0)^{\tau}_{\rho} & (\mathcal{A}^0)^{\tau}_{v^j} & (\mathcal{A}^0)^{\tau}_{\epsilon} & (\mathcal{A}^0)^{\tau}_{B^j} \\ 0^i & 0^i_j & 0^i & \delta^i_j \end{pmatrix},$$

where

$$\begin{aligned}
(\mathcal{A}^0)^{S^i}_{\rho} &= (1+\epsilon+\chi)W^2 v^i, \\
(\mathcal{A}^0)^{S^i}_{v^j} &= B^i B_j + B^2\delta^i_j + hW^2(\delta^i_j + 2W^2 v^i v_j), \\
(\mathcal{A}^0)^{S^i}_{\epsilon} &= (\rho+\kappa)W^2 v^i, \\
(\mathcal{A}^0)^{S^i}_{B^j} &= -\delta^i_j v_k B^k - B^i v_j + 2v^i B_j, \\
(\mathcal{A}^0)^{\tau}_{\rho} &= (1+\epsilon)W^2 - W + \chi(W^2-1), \\
(\mathcal{A}^0)^{\tau}_{v^j} &= -B_j v_k B^k + v_j[B^2 + \rho W^3(2hW-1)], \\
(\mathcal{A}^0)^{\tau}_{\epsilon} &= \rho W^2 + \kappa(W^2-1), \\
(\mathcal{A}^0)^{\tau}_{B^j} &= -v_j v_k B^k + B_j(2-1/W^2).
\end{aligned}$$

$$\zeta_k\mathcal{A}^k = \begin{pmatrix} W\zeta_k v^k & (\zeta_k\mathcal{A}^k)^{D}_{v^j} & 0 & 0_j \\ (\zeta_k\mathcal{A}^k)^{S^i}_{\rho} & (\zeta_k\mathcal{A}^k)^{S^i}_{v^j} & (\zeta_k\mathcal{A}^k)^{S^i}_{\epsilon} & (\zeta_k\mathcal{A}^k)^{S^i}_{B^j} \\ (\zeta_k\mathcal{A}^k)^{\tau}_{\rho} & (\zeta_k\mathcal{A}^k)^{\tau}_{v^j} & (\zeta_k\mathcal{A}^k)^{\tau}_{\epsilon} & (\zeta_k\mathcal{A}^k)^{\tau}_{B^j} \\ 0^i & B^i\zeta_j - \delta^i_j\zeta_k B^k & 0^i & \delta^i_j\zeta_k v^k - v^i\zeta_j \end{pmatrix},$$

where

$$\begin{aligned}
(\zeta_k\mathcal{A}^k)^{D}_{v^j} &= \rho W(W^2 v_j\zeta_k v^k + \zeta_j), \\
(\zeta_k\mathcal{A}^k)^{S^i}_{\rho} &= (1+\epsilon+\chi)W^2 v^i\zeta_k v^k + \chi\zeta^i, \\
(\zeta_k\mathcal{A}^k)^{S^i}_{v^j} &= (\zeta_i B_j - \delta^i_j\zeta_l B^l)v_k B^k + B^2(\delta^i_j\zeta_k v^k - \zeta^i v_j + v^i\zeta_j) \\
&\quad - B^i(\zeta_j v_k B^k - 2v_j\zeta_k B^k + B_j\zeta_k v^k) - v^i B_j\zeta_k B^k \\
&\quad + \rho h W^2(\delta^i_j\zeta_k v^k + v^i\zeta_j + 2W^2 v^i v_j\zeta_k v^k), \\
(\zeta_k\mathcal{A}^k)^{S^i}_{\epsilon} &= v^i(\rho+\kappa)W^2\zeta_k v^k + \zeta^i\kappa, \\
(\zeta_k\mathcal{A}^k)^{S^i}_{B^j} &= \zeta^i v_j v_k B^k - \delta^i_j v_k B^k\zeta_l v^l - B^i v_j\zeta_k v^k \\
&\quad - v^i(\zeta_j v_k B^k + v_j\zeta_k B^k - 2B_j\zeta_k v^k) - W^{-2}(B^i\zeta_j - \zeta^i B_j + \delta^i_j\zeta_k B^k), \\
(\zeta_k\mathcal{A}^k)^{\tau}_{\rho} &= (1+\epsilon+\chi)W^2\zeta_k v^k - W\zeta_k v^k, \\
(\zeta_k\mathcal{A}^k)^{\tau}_{v^j} &= -B_j\zeta_k B^k + B^2\zeta_j + \rho W[\zeta_j(hW-1) + v_j\zeta_k v^k W^2(2hW-1)], \\
(\zeta_k\mathcal{A}^k)^{\tau}_{\epsilon} &= (\rho+\kappa)W^2\zeta_k v^k, \\
(\zeta_k\mathcal{A}^k)^{\tau}_{B^j} &= 2B_j\zeta_k v^k - v_j\zeta_k B^k - \zeta_j v_k B^k.
\end{aligned}$$

All the quantities appearing in the definition of the matrices are defined in the body of the paper and $0^i = (0,0,0)^T$, $0_j = (0,0,0)$ and 0^i_j is the null 3×3 matrix.

References

[1] Anile M A 1989 *Relativistic fluids and magneto-fluids* (Cambridge, England: Cambridge University Press)
[2] Kostyuk A, Gorenstein M I, Stöcker H and Greiner W 2001 *Phys. Lett. B* **500** 273
[3] Gulminelli F, Raduta Ad R, Oertel M and Margueron J 2013 *Phys. Rev. C* **87** 055809
[4] Gorenstein M I, Gaździcki M and Greiner W 2005 *Phys. Rev. C* **72** 024909
[5] Gorenstein M I 2012 *Phys. Rev. C* **86** 044907
[6] Dimmelmeier H, Bejger M, Haensel P and Zdunik J L 2009 *Monthly Notices of the Royal Astronomical Society* **396** 2269
[7] Bombaci I, Logoteta D, Providencia C and Vidaã I 2012 2nd Iberian Nuclear Astrophysics Meeting on Compact Stars IOP Publishing *Journal of Physics: Conference Series* **342** 012001
[8] Peres B, Oertel M and Novak J 2013 *Phys. Rev. D* **87** 043006
[9] Thompson P A 1971 *Phys. Fluids* **14** 1843
[10] Menikoff R and Plohr B J 1989 *Reviews of Modern Physics* **61** 75
[11] Guardone A, Zamfirescu C and Colonna P 2010 *J. Fluid Mech.* **642** 127
[12] Rezzolla L and Zanotti O 2013 *Relativistic hydrodynamics* (Oxford, England: Oxford University Press)
[13] Cinnella P and Congedo P M 2007 *J. Fluid Mech.* **580** 179
[14] Cinnella P, Congedo P M, Pediroda V and Parussini L 2011 *Phys. Fluids* **23** 116101
[15] Lax P D 1957 *Comm. Pure Appl. Math.* **10** 537
[16] LeVeque R J 1992 *Numerical methods for conservation laws* (Basel, Switzerland: Birkhäuser)
[17] Toro E F 2009 *Riemann solvers and numerical methods for fluid dynamics: a practical introduction* (Berlin, Germany: Springer, 3rd. edition)
[18] LeVeque R J 2002 *Finite-Volume Methods for Hyperbolic Problems* (Cambridge, United Kingdom: Cambridge University Press)
[19] Ibáñez J M, Cordero-Carrión I, Martí J M and Miralles, J A 2013 *Classical and Quantum Gravity* **30** 057002
[20] Serna S and Marquina A 2014 *Phys. Fluids* **26** 016101
[21] Brio M and Wu C C 1988 *J.Comput.Phys.* **75** 400
[22] Antón L, Miralles J A, Martí, J M, Ibáñez, J M, Aloy, M A, Mimica, P 2010 *Astroph. J. Suppl.* **188** 1
[23] Leismann T, Antón L, Aloy M A, Müller E, Martí J M, Miralles J A and Ibáñez J M 2005 *Astron. Astrophys.* **436** 503
[24] Romero R , Martí J M, Pons J A, Ibáñez J M, and Miralles J A 2005 *Journal of Fluid Mechanics* **544** 323
[25] Akiyama S, Wheeler, J C, Meier, D L, Lichtenstadt, I 2003, *Astroph. J.* **584** 954
[26] Obergaulinger M, Aloy M A, Dimmelmeier H, Müller E 2006 *Astron. Astrophys.* **457** 209
[27] Sawai H., Yamada S., Suzuki H. 2013 *Astroph. J.* **770** L19
[28] Obergaulinger M, Cerdá-Durán P, Müller E, Aloy M A, 2009 *Astron. Astrophys.* **498** 241
[29] Pessah M E 2010 *Astroph. J.* **716** 1012
[30] Obergaulinger M, Janka H T, Aloy M A 2014 *Monthly Notices of the Royal Astronomical Society* **445** 3169