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Uniquely Solvable Problems for Abstract Legendre Equation

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Abstract—For loaded abstract Legendre equation we find sufficient conditions of solvability of the Cauchy problem and the boundary control problem. We also consider nonlocal problem that contains fractional integral of a function with respect to another function.

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The research of some physical processes relies on solving of equations with the Laplace operator, which (by means of separation of variables in curvilinear coordinate systems) leads to differential equations containing singularity. If there is a symmetry of some sort, then such equations turn into the Euler–Poisson–Darboux (EPD) equation and the Legendre equation.

Initial problems for the classic and abstract EPD equation were studied in a series of papers and the results are presented in [1] (Chap. 1). The further results in this direction were obtained in author's papers [2, 3].

In this paper we will consider some additional conditions and solvability of corresponding problems for one more abstract singular equation,namely, the Legendre equation.

Let A be a closed operator in Banach space E with domain $D(A)$ which is dense in E. When $k > 0$ let us consider Legendre equation

$$L_k u(t) \equiv u''(t) + k \coth t \, u'(t) + (k/2)^2 u(t) = Au(t), \quad t > 0. \tag{1}$$

Differential operator L_k in the left-hand side of Eq. (1) occurs when solving the Laplace equation in elongated ellipsoid of revolution coordinates ([4], P. 138). If A is a scalar multiplication operator, then the spherical functions considered in [5] (P. 53) satisfies Eq. (1) where $k = 2$. We also note papers [6–11] that study partial differential equations with singular operator of the considered type.

As follows from result in [12], the correct formulation of the initial conditions for abstract Legendre equation (1) consists in setting the initial conditions

$$u(0) = u_0, \quad u'(0) = 0, \tag{2}$$

in the point $t = 0$. Herewith, if $k \geq 1$, then the initial condition $u'(0) = 0$ is removed, which is a characteristic for a number of equations with a singularity in coefficients when $t = 0$.

When $k = 0$, problem (1), (2) is uniformly correct if and only if operator A is a generator of cosine-operator-function (COF) $C(t)$ and we will write down this fact as $A \in G_0$ (see terminology in [13, 14]).

In [12] there are conditions on operator A which provide correct solvability of problem (1), (2). We will denote by G_k the set of operators A with which problem (1), (2) is uniformly correct, the resolving operator of this problem we will denote by $P_k(t)$ and call it the operator Legendre function (OLF).

OLF $P_k(t)$ which was introduced in [12] was used by the author in [15] when establishing the criterion of stabilization of the Cauchy problem for abstract differential first-order equation solution. It can also be used for solution of the weight Cauchy problem for Legendre equation. If $0 < k < 1$, then there is

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a correct statement of the initial conditions which is more general than in (2). Let us consider initial conditions

$$u(0) = u_0, \ \lim_{t\to 0}\left(\frac{\sinh t}{t}\right)^k u'(t) = u_1. \tag{3}$$

When $u_0, u_1 \in D(A)$ and $A \in G_k \subset G_{2-k}$, the unique solution to the Cauchy problem (1), (3) has the form [12]

$$u(t) = P_k(t)u_0 + \frac{1}{1-k}\left(\frac{\sinh t}{t}\right)^{1-k} P_{2-k}(t)u_1.$$

Note that if $A \in G_k$ and $k \geq 1$, then problem (1), (3) is not correct. In what follows we study another statements of additional conditions, which allow us to establish the unique solvability of corresponding problems for the loaded Legendre equation.

1. The Cauchy problem for weakly loaded Legendre equation. Let us consider equation

$$u''(t) + k\coth t\left(u'(t) - \frac{\cosh^{2-k}(t/2)}{\cosh t}u'(0)\right) + \frac{k^2}{4}u(t) = Au(t), \ \ t > 0, \tag{4}$$

which, unlike Eq. (1), contains the value of the derivative of an unknown function in point $t = 0$. We will call Eq. (4) the weakly loaded Legendre equation (see terminology in the introduction to monographs [16, 17]). The growing interest in research of loaded differential equations is caused by an expanding number of their applications and the fact that loaded equations form a distinct class of functional-differential equations with specific problems. The review of papers on the loaded differential equations can be also found in [16, 17].

It is significant that in Eq. (4) there is a load given when $t = 0$ and this fact changes the statement of the initial problem. Unlike weight problem (1), (3), when $k > 0$ we will establish the correctness of Cauchy problem

$$u(0) = u_0, \ \ u'(0) = u_1 \tag{5}$$

for weakly loaded Eq. (4) and specify the resolving operator explicitly.

Further we will assume that $g(t) = \cosh t$. Let us consider the fractional integral of function $f(t)$ with respect to function $g(t) = \cosh t$ ([18], P. 248)

$$I_g^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (\cosh t - \cosh s)^{\alpha-1}\sinh s\ f(s)\,ds.$$

We also assume that $\mu_k = \frac{2^{k/2}\Gamma(k/2+1/2)}{\sqrt{\pi}\,\Gamma(k/2)}$ and use notation $P_k'(t)u_0 = (P_k(t)u_0)'$ for brevity.

Theorem 1 ([12]). *Let operator A be a generator of COF $C(t)$, $u_0 \in D(A)$. Then problem* (1), (2) *is uniformly correct, i.e., $A \in G_k$, and the corresponding OLF can be represented as*

$$P_k(t)u_0 = \mu_k \sinh^{1-k} t\int_0^t (\cosh t - \cosh s)^{k/2-1}C(s)u_0\,ds = \mu_k\,\Gamma(k/2)\ \sinh^{1-k} t\, I_g^{k/2}\left[\frac{C(t)}{\sinh t}\right]u_0, \tag{6}$$

and

$$P_k'(t)u_0 = \frac{\sinh t}{k+1}\,P_{k+2}(t)\left(A - \frac{k^2}{4}I\right)u_0. \tag{7}$$

According to Theorem 1, if $u_1 = 0$, then function $u(t) = P_k(t)u_0$ from (6) is the unique solution to the Cauchy problem (4), (5).

In particular case when operator $A=(\delta+1/2)^2$, $\delta\in\mathbb{R}$ is a multiplication by number operator, OLF $P_k(t)$ can be expressed through attached Legendre function of the first kind $P_\delta^\beta(\cdot)$ ([19], P. 661)

$$P_k(t)=\Gamma(1-\beta)\left(\frac{1}{2}\sinh t\right)^{\beta}P_\delta^\beta(\cosh t),\ \ \beta=\frac{1-k}{2}.$$

Hereinafter we will need the following equations from [12], which was used in the proof of Theorem:

$$L_k\left(\sinh^{1-k}t\,u(t)\right)=\sinh^{1-k}t\,L_{2-k}u(t), \tag{8}$$

$$\left(\frac{1}{\sinh t}\frac{d}{dt}\right)^{-\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(\cosh t-\cosh s)^{\alpha-1}\sinh s\,u(s)\,ds=I_g^\alpha u(t),\ \ \alpha>0; \tag{9}$$

this is a definition of negative fractional exponent of the operator of weight differentiation, which with respect to connection with fractional integral with respect to function $g(t)=\cosh t$ can be extended ([18], P. 248) on all $\alpha\in\mathbb{R}$; if $u(0)=0$, then

$$L_{k+2\alpha}\left(\frac{1}{\sinh t}\frac{d}{dt}\right)^{\alpha}u(t)=\left(\frac{1}{\sinh t}\frac{d}{dt}\right)^{\alpha}L_ku(t), \tag{10}$$

note that when $\alpha\in\mathbb{N}$ condition $u(0)=0$ is not applied.

Note that Eq. (10) means that we can apply to considered in this paper problems the method of transformation operator which is used in [1] (Chap. 2) and is a main method in papers [7, 11].

From (8)–(10) we can obtain

$$\begin{aligned}L_k&\left(\sinh^{1-k}t\left(\frac{1}{\sinh t}\frac{d}{dt}\right)^{-k/2}\frac{u(t)}{\sinh t}\right)=\sinh^{1-k}t\,L_{2-k}\left(\left(\frac{1}{\sinh t}\frac{d}{dt}\right)^{-k/2}\frac{1}{\sinh t}\frac{d}{dt}\int_0^t u(\tau)\,d\tau\right)\\&=\sinh^{1-k}t\,L_{2-k}\left(\left(\frac{1}{\sinh t}\frac{d}{dt}\right)^{1-k/2}\int_0^t u(\tau)\,d\tau\right)=\sinh^{1-k}t\left(\frac{1}{\sinh t}\frac{d}{dt}\right)^{1-k/2}\frac{d^2}{dt^2}\int_0^t u(\tau)\,d\tau\\&=\sinh^{1-k}t\left(\frac{1}{\sinh t}\frac{d}{dt}\right)^{1-k/2}\left(\int_0^t u''(\tau)\,d\tau+u'(0)\right)\\&=\sinh^{1-k}t\left(\frac{1}{\sinh t}\frac{d}{dt}\right)^{-k/2}\frac{u''(t)}{\sinh t}+\sinh^{1-k}t\left(\frac{1}{\sinh t}\frac{d}{dt}\right)^{1-k/2}u'(0).\end{aligned} \tag{11}$$

Let us calculate the term $\left(\frac{1}{\sinh t}\frac{d}{dt}\right)^{1-k/2}u'(0)$ from (11). If $0<k<2$, then by (9) we will get

$$\begin{aligned}\left(\frac{1}{\sinh t}\frac{d}{dt}\right)^{1-k/2}u'(0)&=\frac{1}{\sinh t}\frac{d}{dt}\left(\frac{1}{\sinh t}\frac{d}{dt}\right)^{-k/2}u'(0)\\&=\frac{1}{\sinh t}\frac{d}{dt}\left(\frac{1}{\Gamma(k/2)}\int_0^t(\cosh t-\cosh s)^{k/2-1}\sinh s\,u'(0)\,ds\right)\\&=\frac{1}{\Gamma(k/2+1)\,\sinh t}\frac{d}{dt}\left((\cosh t-1)^{k/2}\right)u'(0)=\frac{1}{\Gamma(k/2)}(\cosh t-1)^{k/2-1}u'(0).\end{aligned}$$

If $k=2$, then $\left(\frac{1}{\sinh t}\frac{d}{dt}\right)^{1-k/2}u'(0)=u'(0)$. Finally, if $k>2$, then by (9)

$$\left(\frac{1}{\sinh t}\frac{d}{dt}\right)^{1-k/2}u'(0)=\frac{1}{\Gamma(k/2)}(\cosh t-1)^{k/2-1}u'(0).$$

Thus, (11) can be rewritten as

$$L_k\left(\sinh^{1-k}t\,I_g^{k/2}\left[\frac{u(t)}{\sinh t}\right]\right)=\sinh^{1-k}t\,I_g^{k/2}\left[\frac{u''(t)}{\sinh t}\right]+\frac{\sinh^{1-k}t}{\Gamma(k/2)}(\cosh t-1)^{k/2-1}u'(0). \tag{12}$$

Note that Eq. (12) is written for function $u(t) = C(t)u_0$, $u''(t) = AC(t)u_0$, $u'(0) = 0$ and was used in the proof of Theorem 1. As we will see further, this equation determines the multiplier in load $u'(0)$ in Eq. (4).

Now let us consider the Cauchy problem (4), (5) in the case when $u_0 = 0$. Let $\nu_k = k2^{k/2-1}$. We introduce the sine-operator-function (SOF) $S(t) = \int\limits_0^t C(s)\,ds$.

Theorem 2. *If $u_0 = 0$, $u_1 \in D(A)$ and operator A is a generator of COF $C(t)$, then function $u(t) = Q_k(t)u_1$, where*

$$Q_k(t)u_1 = \nu_k \sinh^{1-k} t \int_0^t (\cosh t - \cosh\tau)^{k/2-1} S(\tau)u_1\,d\tau = \nu_k\,\Gamma(k/2)\,\sinh^{1-k} t\, I_g^{k/2}\left[\frac{S(t)}{\sinh t}\right]u_1 \tag{13}$$

is a solution to problem (4), (5), and

$$Q_k'(t)u_1 = \frac{\sinh t}{k+2} Q_{k+2}(t)\left(A - \frac{k^2}{4}I\right)u_1 + \frac{u_1}{\cosh^k(t/2)}. \tag{14}$$

Proof. Let us verify that function $Q_k(t)u_1$ satisfies Eq. (4). In order to do this let us substitute function $u(t) = S(t)u_1$, $u''(t) = AS(t)u_1$, $u'(0) = u_1$ in (11). After some elementary transformations we will get

$$L_kQ_k(t)u_1 = L_k\left(\nu_k\,\Gamma(k/2)\,\sinh^{1-k} t\, I_g^{k/2}\left[\frac{S(t)}{\sinh t}\right]u_1\right)$$
$$= \nu_k\,\Gamma(k/2)\left(\sinh^{1-k} t\, I_g^{k/2}\left[\frac{AS(t)}{\sinh t}\right]u_1 + \frac{\sinh^{1-k} t}{\Gamma(k/2)}(\cosh t - 1)^{k/2-1}u_1\right)$$
$$= AQ_k(t)u_1 + \frac{k\,2^k\,(2\sinh^2(t/2))^{k/2-1}}{\sinh t\big(2\sinh(t/2)\,\cosh(t/2)\big)^{k-2}}u_1 = AQ_k(t)u_1 + k\coth t\,\frac{\cosh^{2-k}(t/2)}{\cosh t}u_1$$

and, consequently, function $Q_k(t)u_1$ satisfies Eq. (4).

Let us check that if the function satisfies initial conditions (5) when $u_0 = 0$. Since when t is small, there is the inequality

$$\|S(t)\| \le M\sinh t,$$

then, with respect to (13), when $t \to 0$ we get

$$\|Q_k(t)\| \le M\nu_k\,\sinh^{1-k} t \int_0^t (\cosh t - \cosh\tau)^{k/2-1}\sinh\tau\,d\tau = M\nu_k\,\sinh^{1-k} t \int_1^{\cosh t}(\cosh t - s)^{k/2-1}\,ds$$
$$= M\,2^k\,\sinh^{1-k} t\,(\cosh t - 1)^{k/2} = M\,2^{3k/2}\,\sinh^{1-k} t\,\sinh^k(t/2) \le M_1 t \to 0,$$

That is why function $Q_k(t)u_1$ satisfies the first condition from (5).

To check whether the function $Q_k(t)u_1$ satisfies the second condition from (5), we will derive formula (14) for its derivative. In order to do this we will rewrite (4) as

$$\frac{1}{\sinh^k t}(\sinh^k t\,u'(t))' + \frac{k^2}{4}u(t) = Au(t) + \frac{k\,\cosh^{2-k}(t/2)}{\sinh t}u'(0).$$

Now let us substitute function $Q_k(t)u_1$ in this equation and integrate it after multiplying by $\sinh^k t$:

$$\sinh^k t\,Q_k'(t)u_1 = \int_0^t \sinh^k s\left(A - \frac{k^2}{4}I\right)Q_k(t)u_1\,ds + k\int_0^t \sinh^{k-1} s\,\cosh^{2-k}(s/2)\,ds\,u_1$$
$$= \int_0^t \sinh^k s\left(A - \frac{k^2}{4}I\right)Q_k(t)u_1\,ds + k2^{k-1}\int_0^t \sinh^{k-1}(s/2)\,\cosh(s/2)\,ds\,u_1$$

$$= \int_0^t \sinh^k s\Big(A-\frac{k^2}{4}I\Big)Q_k(t)u_1\,ds + 2^k\ \sinh^k(t/2)\,u_1.$$

Taking into account (13) and changing the order of integration, we will obtain

$$\sinh^k t\,Q_k'(t)u_1 = \nu_k\int_0^t \sinh\tau\int_0^\tau(\cosh\tau-\cosh y)^{k/2-1}S(y)\Big(A-\frac{k^2}{4}I\Big)u_1\,dy\,d\tau + 2^k\ \sinh^k(t/2)\,u_1$$

$$= \nu_k\int_0^t S(y)\Big(A-\frac{k^2}{4}I\Big)u_1\int_y^t \sinh\tau\,(\cosh\tau-\cosh y)^{k/2-1}d\tau\,dy + 2^k\ \sinh^k(t/2)\,u_1$$

$$= \frac{2\nu_k}{k}\int_0^t(\cosh\tau-\cosh y)^{k/2}S(y)\Big(A-\frac{k^2}{4}I\Big)u_1\,dy + 2^k\ \sinh^k(t/2)\,u_1$$

$$= \frac{2\nu_k}{k\nu_{k+2}}\ \sinh^{k+1}t\,Q_{k+2}(t)\Big(A-\frac{k^2}{4}I\Big)u_1 + 2^k\ \sinh^k(t/2)\,u_1.$$

That is why the derivative of function $Q_k(t)u_1$ has the form (14), and function $Q_k(t)u_1$ satisfies the second condition from (5). □

Theorem 3. *Let $u_0, u_1 \in D(A)$ and let operator A be a generator of COF $C(t)$. Then function $u(t) = P_k(t)u_0 + Q_k(t)u_1$ is the unique solution to the Cauchy problem* (4), (5).

Proof. The fact that function $u(t) = P_k(t)u_0 + Q_k(t)u_1$ is a solution to problem (4), (5) is established in Theorems 1 and 2. We will prove the uniqueness of solution to problem (4), (5) by contradiction. Let $u_1(t)$ and $u_2(t)$ be the two solutions to problem (4), (5). Then function $v(t) = u_1(t) - u_2(t)$ satisfies Eq. (4) and conditions (5). By means of Theorem 1 $v(t) \equiv 0$, and this proves the uniqueness of the solution and the theorem. □

As it is established in [12], when $u_0 \in E$ uniformly with respect to $t \in [0, t_0]$, $t_0 > 0$,

$$\lim_{k\to 0} P_k(t)u_0 = C(t)u_0.$$

Analogously, for $u_1 \in E$ we can obtain equation $\lim_{k\to 0} Q_k(t)u_1 = S(t)u_1$. Consequently, there occurs the "coupling" of the resulting solution in Theorem 3 and the well-known solution to the Cauchy problem for abstract wave equation

$$\lim_{k\to 0}\big(P_k(t)u_0 + Q_k(t)u_1\big) = C(t)u_0 + S(t)u_1.$$

2. The boundary control problem for weakly loaded Legendre equation. We will seek solution $u(t) \in C^2([0,1], E) \cap C((0,1], D(A))$ to Eq. (4), satisfying two final conditions, given for the sake of convenience in point $t = 1$

$$u(1) = u_2,\ \ u'(1) = u_3. \tag{15}$$

As it follows from Theorem 3, it suffices to define unknown initial elements u_0, u_1 in conditions (5) over final conditions (15) to justify the solvability of problem (4), (15). Applying conditions (15) to function $u(t) = P_k(t)u_0 + Q_k(t)u_1$ and using Eqs. (6), (7), (13), (14) to find elements u_0, u_1 we will get the system

$$P_k(1)u_0 + Q_k(1)u_1 = u_2, \tag{16}$$

$$\frac{\sinh 1}{k+1}P_{k+2}(1)\Big(A-\frac{k^2}{4}I\Big)u_0 + \frac{\sinh 1}{k+2}Q_{k+2}(1)\Big(A-\frac{k^2}{4}I\Big)u_1 + \frac{1}{\cosh^k 1/2}u_1 = u_3. \tag{17}$$

It is convenient to rewrite Eqs. (16), (17) as a matrix equation

$$Bv = w,\ \ B : D(A)\times D(A) \longrightarrow E\times E, \tag{18}$$

where

$$v = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \quad w = \begin{pmatrix} u_2 \\ u_3 \end{pmatrix},$$
$$B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} = \begin{pmatrix} P_k(1) & Q_k(1) \\ \frac{\sinh 1}{k+1} P_{k+2}(1)\left(A - \frac{k^2}{4}I\right) & \frac{\sinh 1}{k+2} Q_{k+2}(1)\left(A - \frac{k^2}{4}I\right) + \frac{1}{\cosh^k 1/2} I \end{pmatrix}, \tag{19}$$

wherein all operators B_1, B_2, B_3, B_4 commute on $D(A)$.

Thus the unique solvability of problem (4), (15) reduces to the problem of existence of inverse operator matrix of operator matrix $B : D(A) \times D(A) \to E \times E$, given by (19), defined on some subset from $E \times E$. As in scalar case, an important role is played by the determinant of operator matrix B, which we will denote by $\Delta = B_1 B_4 - B_2 B_3$.

Let $x \in D(A)$, with respect to (6) and (13), after elementary transformations we will get

$$\begin{aligned}
\Delta x &= P_k(1)\left(\frac{\sinh 1}{k+2} Q_{k+2}(1)\left(A - \frac{k^2}{4}I\right)x + \frac{1}{\cosh^k 1/2}x\right) - \frac{\sinh 1}{k+1} Q_k(1) P_{k+2}(1)\left(A - \frac{k^2}{4}I\right)x \\
&= \frac{1}{\sinh^k 1} P_k(1) \int_0^1 \sinh^k s\left(A - \frac{k^2}{4}I\right) Q_k(s)x\,ds + \frac{1}{\cosh^k 1/2} P_k(1)x \\
&\quad - \frac{1}{\sinh^k 1} Q_k(1) \int_0^1 \sinh^k s\left(A - \frac{k^2}{4}I\right) P_k(s)x\,ds \\
&= \frac{1}{\sinh^k 1} P_k(1) \int_0^1 \sinh^k s\left(Q_k''(s) + k\coth s\, Q_k'(s) - k\frac{\cosh^{2-k}(s/2)}{\sinh s}\right)x + \frac{1}{\cosh^k 1/2} P_k(1)x \\
&\quad - \frac{1}{\sinh^k 1} Q_k(1) \int_0^1 \sinh^k s\left(P_k''(s) + k\coth s\, P_k'(s)\right)x \\
&= P_k(1)Q_k'(1)x - \frac{k}{\sinh^k 1} P_k(1)x \int_0^1 \sinh^{k-1} s\, \cosh^{2-k}(s/2)\,ds + \frac{1}{\cosh^k 1/2} P_k(1)x - Q_k(1)P_k'(1)x \\
&= P_k(1)Q_k'(1)x - Q_k(1)P_k'(1)x.
\end{aligned} \tag{20}$$

Let us introduce

$$W_k(t)x = \begin{vmatrix} P_k(t) & Q_k(t) \\ P_k'(t) & Q_k'(t) \end{vmatrix} x = P_k(t)Q_k'(t)x - P_k'(t)Q_k(t)x,$$

the Wronski operator determinant built over operator functions $P_k(t)$ and $Q_k(t)$.

Thus, with respect to (20), the question of existence of inverse operator to $\Delta = B_1B_4 - B_2B_3$ reduces to existence of operator which is inverse to Wronski operator determinant $W_k(1)$.

Lemma. *Let $k > 0$, $x \in D(A)$ and let operator A be a generator of COF $C(t)$. Then the Wronski operator determinant built over defined correspondingly by Eqs.* (6), (13) *operator functions $P_k(t)$ and $Q_k(t)$, equals*

$$W_k(t)x = \frac{k}{\sinh^k t} \int_0^t \frac{\sinh^{k-1} \tau}{\cosh^{k-2} \tau/2} P_k(\tau)x\,d\tau. \tag{21}$$

Proof. Let us show that function $W_k(t)x$ satisfies the equation

$$W_k'(t)x + k\coth t\, W_k(t)x = \frac{k\cosh^{2-k} t/2}{\sinh t} P_k(t)x \tag{22}$$

and initial condition

$$\lim_{t\to 0} W_k(t)x = x. \tag{23}$$

Indeed,

$$\begin{aligned} W_k'(t)x &= P_k'(t)Q_k'(t)x + P_k(t)Q_k''(t)x - P_k'(t)Q_k'(t)x - P_k''(t)Q_k(t)x \\ &= P_k(t)\left(Q_k''(t) + k\coth t\, Q_k'(t) - \frac{k\cosh^{2-k} t/2}{\sinh t} I + \frac{k^2}{4}Q_k(t)\right)x \\ &\quad - P_k(t)\left(k\coth t\, Q_k'(t) - \frac{k\cosh^{2-k} t/2}{\sinh t} I + \frac{k^2}{4}Q_k(t)\right)x \\ &\quad - Q_k(t)\left(P_k''(t) + k\coth t\, P_k'(t) + \frac{k^2}{4}P_k(t)\right)x + Q_k(t)\left(k\coth t\, P_k'(t) + \frac{k^2}{4}P_k(t)\right)x \\ &= P_k(t)AQ_k(t)x - Q_k(t)AP_k(t)x - k\coth t\, W_k(t)x + \frac{k\cosh^{2-k} t/2}{\sinh t}\, P_k(t)x = \\ &= -k\coth t\, W_k(t)x + \frac{k\cosh^{2-k} t/2}{\sinh t}\, P_k(t)x, \end{aligned}$$

and therefore the function $W_k(t)x$ satisfies Eq. (22).

When $P_k(0)x = Q_k'(0)x = x$, $P_k'(0)x = Q_k(0)x = 0$, function $W_k(t)x$ also satisfies initial condition (23), and the unique solution to problem (22), (23) is a function defined by equality (21). □

According to the lemma, we need to study the invertibility of bounded operator

$$W_k(1)x = \frac{k}{\sinh^k 1}\int_0^1 \frac{\sinh^{k-1}\tau}{\cosh^{k-2}\tau/2} P_k(\tau)x\, d\tau. \tag{24}$$

Note that if $k = 0$, then $W_0(t)x = C(t)S'(t)x - C'(t)S(t)x = C^2(t)x - AS^2(t)x = x$ and operator $W_0(t) = I$ is always invertible, but in general case $k > 0$ it is not true, and the question of invertibility of operator $W_k(t)$ is very difficult. Hereinafter entire function

$$\cosh i_k(\lambda) = \frac{k\mu_k}{\sinh^k 1}\sum_{j=0}^{\infty}\frac{a_j(k)}{(2j)!}\lambda^j = \frac{k\mu_k}{\sinh^k 1}\int_0^1 \cosh s\sqrt{\lambda}\int_s^1 \cosh^{2-k}\tau/2\,(\cosh\tau - \cosh s)^{k/2-1}\, d\tau\, ds, \tag{25}$$

where $a_j(k) = \int\limits_0^1 s^{2j}\int\limits_s^1 \cosh^{2-k}\tau/2\,(\cosh\tau - \cosh s)^{k/2-1}\, d\tau\, ds$ will play an important role.

Theorem 4. *Let A be a bounded operator. For operator $W_k(1)$ defined by Eq. (24) to be invertible it is necessary and sufficient that on spectrum $\sigma(A)$ of operator A condition*

$$\cosh i_k(\lambda) \neq 0, \quad \lambda \in \sigma(A) \tag{26}$$

be satisfied.

Proof. We substitute (6) in (24), after elementary transformations we get

$$W_k(1) = \frac{k\mu_k}{\sinh^k 1}\sum_{j=0}^{\infty}\frac{1}{(2j)!}A^j\int_0^1 \cosh^{2-k}\tau/2\int_0^\tau s^{2j}\,(\cosh\tau - \cosh s)^{k/2-1}\, ds\, d\tau = \cosh i_k(A). \tag{27}$$

Let Ω be an open set on complex plane containing spectrum $\sigma(A)$ of bounded operator A, boundary of which $\partial\Omega$ consists of finite number of rectifiable Jordan curves oriented in positive direction. Then by representing the operator from right-hand side of (27) through resolvent $R(\lambda)$ of operator A, we will get

$$W_k(1) = \frac{1}{2\pi i}\int_{\partial\Omega} \cosh i_k(\lambda) R(\lambda)\, d\lambda. \tag{28}$$

Necessary and sufficient condition of invertibility of operator $W_k(1)$ is absence of point $\lambda = 0$ in spectrum $\sigma(W_k(1))$ of operator $W_k(1)$. Eq. (28) means that operator $W_k(1)$ is an analytic function of operator A, $W_k(1) = \cosh i_k(A)$. By the theorem on the mapping of the spectrum of a bounded operator $\sigma(W_k(1)) = \cosh i_k(\sigma(A))$. Thus value $\lambda = 0$ is not a point of the spectrum of operator $W_k(1)$ only when function $\cosh i_k(\lambda)$ does not vanishes on spectrum $\sigma(A)$ or, what is the same, condition (26) is satisfied

$$W_k^{-1}(1) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{1}{\cosh i_k(\lambda)} R(\lambda)\,d\lambda. \qquad (29) \quad \square$$

From Theorem 4 it follows that the location of zeros of function $\cosh i_k(\lambda)$ defines the invertibility of operator $W_k(1)$ in the case of bounded operator A. In the case of unbounded operator A condition of the form (26) will not be sufficient condition of invertibility, despite the zeros location will still play an important role.

Let us consider the case when in Eq. (4) parameter $k = 2$. In this case

$$\cosh i_2(\lambda) = \frac{2}{\sinh^2 1}\frac{\cosh\lambda - 1}{\lambda^2}, \qquad (30)$$

zeros of function $\cosh i_2(\lambda)$ can be calculated using the formula

$$\lambda_j = j\frac{\pi i}{2}, \ \ j \in \mathbb{Z}\setminus\{0\}, \qquad (31)$$

let us indicate the sufficient condition of invertibility of operator $W_k(1)$ in the case of unbounded operator A.

We denote by Υ_0 the contour on complex plane consisting of straight line $\operatorname{Re} z = \sigma_0 > \omega$ (we pass it bottom-up), ω is a $C(t)$ COF growth exponent, Υ_0^2 is a parabola, image Υ_0 under mapping $w = z^2$ ($z \in \Upsilon_0$, $w \in \Upsilon_0^2$).

There can be only finite number of zeros λ_j to the left of parabola Υ_0^2, we will denote their set by Λ, $\operatorname{card}(\Lambda) < \infty$.

Condition 1. Let $k = 2$ and each zero λ_j of entire function $\cosh i_2(\lambda)$, defined by Eq. (31), which lies to the left of parabola Υ_0^2, belong to resolvent set $\rho(A)$ of operator A and there exists $d > 0$ such that $\max\limits_{j\in\Lambda}\|R(\lambda_j)\| \le d$.

We assume that Condition 1 is satisfied. Since each zero $\lambda_j \in \Lambda$ belongs to $\rho(A)$, it belongs to $\rho(A)$ with circular neighborhood Ω_j of radius $1/d$, boundary of which (we pass it clockwise) we denote by γ_j and let

$$\Xi = \Upsilon_0^2 \bigcup \bigcup_{j\in\Lambda}\gamma_j.$$

Our problem reduces to a problem of existence of an operator defined on some subset of $D(A)$ which is inverse operator to the bounded operator given by Eq. (28) when $k = 2$ and continuously extended on E. In order to do it when $x \in E$, $\lambda_0 \in \mathbb{C}$ we introduce bounded operator

$$Hx = \frac{1}{2\pi i}\int_{\Xi} \frac{R(z)x\,dz}{\cosh i_2(z)(z-\lambda_0)^3}, \ \ H : E \to E. \qquad (32)$$

Let us show that the integral in (32) converges absolutely when Condition 1 is satisfied. Indeed, by virtue of the choice of contour Υ_0^2, inequality [13]

$$\|\lambda\, R(\lambda^2)\| \le \frac{M}{\operatorname{Re}\lambda - \omega}, \ \ \operatorname{Re}\lambda > \omega,$$

and the boundedness of function $\big(\cosh\lambda^2 - 1\big)^{-1}$ integral

$$\int_{\Upsilon_0^2} \frac{R(z)\,dz}{\cosh i_2(z)\,(z-\lambda_0)^3} = 2\int_{\Upsilon_0} \frac{\lambda\, R(\lambda^2)\,d\lambda}{\cosh i_2(\lambda^2)\,\big(\lambda^2-\lambda_0\big)^3} = 2\int_{\Upsilon_0} \frac{\lambda^5\, R(\lambda^2)\,d\lambda}{\big(\cosh\lambda^2 - 1\big)\,\big(\lambda^2-\lambda_0\big)^3}$$

converges absolutely.

Theorem 5. *Let operator A be a generator of COF $C(t)$, $x \in D(A^4)$ and Condition 1 is satisfied. Then operator $W_k(1)$ has inverse operator $W_k^{-1}(1) : D(A^3) \to E$.*

Proof. Let $x \in D\big(A\big)$, $\sigma_0 < \sigma < \operatorname{Re}\xi$. Then by substituting operator $W_k(1)$ defined by Eq. (28) in (32) and applying the Hilbert identity

$$R(z)R(\xi^2) = \frac{R(z) - R(\xi^2)}{\xi^2 - z},$$

we get

$$\begin{aligned} HW_2(1)x &= \frac{1}{2\pi i}\int_{\Xi}\frac{R\big(z\big)}{\cosh i_2(z)(z-\lambda_0)^3}\,\frac{1}{i\pi}\int_{\sigma-i\infty}^{\sigma+i\infty}\cosh i_2\big(\xi^2\big)\xi R(\xi^2)x\,d\xi dz \\ &= -\frac{1}{2\pi^2}\int_{\Xi}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{\xi\cosh i_2\big(\xi^2\big)R\big(z\big)x}{\cosh i_2\big(z\big)(z-\lambda_0)^3(\xi^2-z)} - \frac{\xi\cosh i_2\big(\xi^2\big)R\big(\xi^2\big)x}{\cosh i_2\big(z\big)(z-\lambda_0)^3(\xi^2-z)}\right)d\xi\,dz. \quad (33) \end{aligned}$$

The integral in (33) converges absolutely. By changing the order of integration we will obtain

$$\begin{aligned} HW_2(1)x = &-\frac{1}{2\pi^2}\int_{\Xi}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\xi\cosh i_2\big(\xi^2\big)R\big(z\big)x\,d\xi dz}{\cosh i_2\big(z\big)(z-\lambda_0)^3(\xi^2-z)} \\ &+\frac{1}{2\pi^2}\int_{\sigma-i\infty}^{\sigma+i\infty}\xi\cosh i_2\big(\xi^2\big)R\big(\xi^2\big)x\int_{\Xi}\frac{dz}{\cosh i_2\big(z\big)(z-\lambda_0)^3\,(\xi^2-z)}\,d\xi. \quad (34) \end{aligned}$$

If we close integration contour Υ_0^2 to the left without intersection with $\bigcup\limits_{j\in\Lambda}\gamma_j$, then the inner integral in the second summand in (34) vanishes by virtue of choice of contour Ξ and the Cauchy theorem for multiply connected domains. For calculations of the integrals in the first summand in (34) we will use the Cauchy integral formula. Thus we get

$$\begin{aligned} HW_2(1)x &= -\frac{1}{2\pi^2}\int_{\Xi}\int_{\Upsilon}\frac{\xi\cosh i_2\big(\xi^2\big)R\big(z\big)x\,d\xi\,dz}{\cosh i_2\big(z\big)(z-\lambda_0)^3(\xi^2-z)} \\ &= -\frac{1}{4\pi^2}\int_{\Xi}\int_{\Upsilon^2}\frac{\cosh i_2\big(\lambda\big)R(z)x\,d\lambda dz}{\cosh i_2\big(z\big)(z-\lambda_0)^3(\lambda-z)} = \frac{1}{2\pi i}\int_{\Xi}\frac{R(z)x\,dz}{(z-\lambda_0)^3} = \frac{1}{2\pi i}\int_{\Upsilon_0^2}\frac{R(z)x\,dz}{(z-\lambda_0)^3} \\ &= -\frac{1}{2}R''(\lambda_0)x = -R^3(\lambda_0)x. \end{aligned}$$

Commuting operators H, $W_2(1)$, $R^3(\lambda_0)$ are bounded and domain $D(A)$ is dense in E, that is why equation $HW_2(1)x = -R^3(\lambda_0)x$ is fair for $x \in E$, herewith $HW_2(1) : E \to D\big(A^3\big)$. Consequently, the operator

$$W_2^{-1}(1)x = -(\lambda_0 I\ -\ A)^3 Hx,\ \ W_2^{-1}(1) : D\big(A^3\big) \to E, \quad (35)$$

is an inverse operator for $W_k(1)$. Indeed,

$$\begin{aligned} W_2(1)W_2^{-1}(1)x &= -W_2(1)(\lambda_0 I - A)^3 Hx = -W_2(1)H(\lambda_0 I - A)^3 x \\ &= R^3(\lambda_0)(\lambda_0 I - A)^3 x = x,\ \ x \in D\big(A^3\big), \end{aligned}$$

$$W_2^{-1}(1)W_2(1)x = -(\lambda_0 I - A)^3 HW_2(1)x = (\lambda_0 I - A)^3 R^3(\lambda_0)x = x,\ \ x \in E. \qquad \square$$

In Theorems 4 and 5 we use the set on which operator $W_k(1)$ has inverse operator $W_k^{-1}(1)$ which has the form (29) in the case of bounded operator A and $k > 0$ and the form (35) in the case of unbounded operator A and $k = 2$. Thus by virtue of (20) we proved the existence of the inverse operator to Δ. By solving matrix Eq. (18), like in the scalar case we will get

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \Delta^{-1} \begin{pmatrix} B_4 & -B_3 \\ -B_2 & B_1 \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}.$$

Thereby, by virtue of Theorems 4 and 5 we can obtain the following statements on solvability of the boundary control problem considered in this Item in which function $\cosh i_k(\lambda)$ from (25) plays a great role.

Theorem 6. *Let $u_2, u_3 \in E$, A be a bounded operator and condition $\cosh i_k(\lambda) \neq 0$, $\lambda \in \sigma(A)$ be satisfied on spectrum $\sigma(A)$ of operator A. Then problem* (4), (15) *has unique solution $u(t) = P_k(t)u_0 + Q_k(t)u_1$, where*

$$u_0 = W_k^{-1}(1)\left(\frac{\sinh 1}{k+2} Q_{k+2}(1)\left(A - \frac{k^2}{4}I\right)u_2 + \frac{u_2}{\cosh^k 1/2} - \frac{\sinh 1}{k+1} P_{k+2}(1)\left(A - \frac{k^2}{4}I\right)u_3\right), \quad (36)$$

$$u_1 = W_k^{-1}(1)\big(-Q_k(1)u_2 + P_k(1)u_3\big), \quad (37)$$

and operator $W_k^{-1}(1)$ has the form (29).

Theorem 7. *Let $u_2, u_3 \in D(A^4)$ and Condition* 1 *be satisfied. Then problem* (4), (15) *has unique solution $u(t) = P_2(t)u_0 + Q_2(t)u_1$, where initial elements u_0, u_1 are defined in* (36), (37) *when $k = 2$, and*

$$W_2^{-1}(1)x = -\frac{1}{2\pi i}\int_{\Xi} \frac{R(z)(\lambda_0 I - A)^3 x\, dz}{\cosh i_2(z)\,(z-\lambda_0)^3}, \quad x \in D(A^3).$$

3. Nonlocal problem for the Legendre equation. We will seek solution $u(t) \in C^2([0,1], E) \cap C((0,1], D(A))$ to Eq. (1) that satisfies the nonlocal integral condition with fractional integral I_g^{β}, $\beta > 0$ with respect to function $g(t) = \cosh t$,

$$\lim_{t\to 1} I_g^{\beta}\big(\sinh^{k-1} t\, u(t)\big) = u_4, \quad (38)$$

and the condition

$$u'(0) = 0. \quad (39)$$

Problem (1), (38), (39) with nonlocal conditions (38), (39), broadly speaking, is not correct. Let us specify the conditions on operator A and element $u_4 \in E$ providing its unique solvability.

Among publications on solvability of nonlocal problems with integral condition for abstract differential first-order equations we will note papers [20] and [21]. The criterion of the uniqueness of solution is established in [22]. The nonlocal problem for Euler–Poisson–Darboux equation is studied in [23].

The research on solvability of nonlocal problem (1), (38), (39) are based on locating of initial element u_0 in condition (2) using nonlocal condition (38).

We apply the fractional integral with respect to function $g(t) = \cosh t$ to function $u(t) = P_k(t)u_0$, multiplied by $\sinh^{k-1} t$, where $P_k(t)$ is defined by Eq. (6). Taking into account the semigroup property of fractional integration and condition (38), we get

$$\lim_{t\to 1} I_g^{\beta}\big(\sinh^{k-1} t\, u(t)\big) = \mu_k\,\Gamma(k/2)\, \lim_{t\to 1} I_g^{k/2+\beta}\left[\frac{C(t)}{\sinh t}\right]u_0$$
$$= \frac{\mu_k\,\Gamma(k/2)}{\Gamma(k/2+\beta)}\int_0^1 (\cosh 1 - \cosh s)^{k/2+\beta-1} C(s)u_0\, ds = u_4.$$

As before, when establishing the solvability of nonlocal problem (1), (38), (39) we will use entire function

$$\psi_{k,\beta}(\lambda) = \frac{\mu_k\,\Gamma(k/2)}{\Gamma(k/2+\beta)}\sum_{j=0}^{\infty}\frac{b_j(k,\beta)}{(2j)!}\lambda^j = \frac{\mu_k\,\Gamma(k/2)}{\Gamma(k/2+\beta)}\int_0^1 \cosh s\sqrt{\lambda}\,(\cosh 1-\cosh s)^{k/2+\beta-1}\,ds,$$

where

$$b_j(k,\beta) = \int_0^1 s^{2j}(\cosh 1-\cosh s)^{k/2+\beta-1}\,ds.$$

Theorem 8. *Let A be a bounded operator and $u_4 \in E$. In order for problem* (1), (38), (39) *to have unique solution it is necessary and sufficient that condition $\psi_{k,\beta}(\lambda) \neq 0$, $\lambda \in \sigma(A)$ be satisfied on spectrum $\sigma(A)$ of operator A. Herewith $u(t) = P_k(t)u_0$, where*

$$u_0 = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{1}{\psi_{k,\beta}(\lambda)}R(\lambda)u_4\,d\lambda.$$

Proof is analogous to the proof of theorem 4.

Let us consider the case in which $\beta = 1-k/2$, $0 < k \leq 2$:

$$\psi_k(\lambda) = \frac{\mu_k\,\Gamma(k/2)\,\sinh\sqrt{\lambda}}{\sqrt{\lambda}},$$

zeros λ_j of function $\psi_k(\lambda)$ can be calculated explicitly:

$$\lambda_j = -\pi^2 j^2,\ \ j \in \mathbb{N}.$$

Let us specify the sufficient solvability condition of nonlocal problem (1), (38), (39) in the case of unbounded operator A.

Condition 2. Let each zero $\lambda_j = -\pi^2 j^2$, $j \in \mathbb{N}$ of a function $\psi_k(\lambda)$ belong to resolvent set $\rho(A)$ and there exists $d > 0$ such that $\sup\limits_{j=1,2,\dots}\|R(\lambda_j)\| \leq d$.

Since each zero λ_j, $j = 1,2,\dots$ of function $\cosh i_k(\lambda)$ belongs to $\rho(A)$, it belongs to $\rho(A)$ with a circular neighborhood Ω_j of radius $\frac{1}{d}$, boundary of which (we pass it clockwise) we denote by γ_j. Let Υ_0 be a contour on complex plane consisting of straight line (we pass it bottom-up) $\operatorname{Re} z = \sigma_0 > \omega$, Υ_0^2 be a parabola, image Υ_0 under mapping $w = z^2$ ($z \in \Upsilon_0$, $w \in \Upsilon_0^2$), and $\Xi = \Upsilon_0^2 \bigcup\limits_{j=1,2,\dots}\gamma_j$.

We put $\lambda_0 \in \rho(A)$, $\operatorname{Re}\lambda_0 > \sigma > \sigma_0$ and introduce bounded operator

$$Hv = \frac{1}{2\pi i}\int_{\Xi}\frac{R(z)v\,dz}{\psi_k(z)(z-\lambda_0)^2},\ \ H : E \to E. \tag{40}$$

As in the proof of Theorem 5 we establish that the integral in (40) under Condition 2 converges absolutely and there is valid

Theorem 9. *Let operator A be a generator of COF $C(t)$, $x \in D(A^3)$ and Condition 2 be satisfied. Then problem* (1), (38), (39) *is uniquely solvable and the solution has he form $u(t) = P_k(t)u_0$, where $u_0 = (\lambda_0 I - A)^2 Hu_4$, and operator H is defined in* (40).

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