

On complementable operators in the sense of T. Ando

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AMS Classification: 47A64, 47B65

Keywords: Schur complements, shorted operators, Wiener-Hopf operators, minus order

Abstract

Consider an operator $A : \mathcal{H} \rightarrow \mathcal{K}$ between Hilbert spaces and closed subspaces $\mathcal{S} \subset \mathcal{H}$ and $\mathcal{T} \subset \mathcal{K}$. If there exist projections E on \mathcal{H} and F on \mathcal{K} such that $R(E) = \mathcal{S}$, $R(F) = \mathcal{T}$ and $AE = F^*A$ then A is called $(\mathcal{S}, \mathcal{T})$ -complementable. The origin of this notion comes from the idea of T. Ando of defining Schur complements in terms of operators. In this paper we present some characterizations of these triples $(A, \mathcal{S}, \mathcal{T})$ and applications to bilateral Schur complements and generalized Wiener-Hopf operators.

1 Introduction

Given a square block matrix

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix},$$

where B and E are square and B is invertible, the Schur complement of B in A is $E - DB^{-1}C$. The name, due to E. Haynsworth [20], comes from the formula $\det A = \det B \det(E - DB^{-1}C)$, discovered by I. Schur [30]. By changing the inversion by generalized inverses, the notion is available also for non-square block matrices and milder invertibility hypothesis. On the other side, there exists an extensive theory on the so-called *shorted operators*. M. G. Krein [23], in his studies about self-adjoint extensions of some operators on a Hilbert space defined, for a positive semidefinite operator A on \mathcal{H} and a closed subspace \mathcal{S} of \mathcal{H} , the shorted operator $A_{/\mathcal{S}}$ by means of the existence of the maximum of $\{B \in L(\mathcal{H}) : 0 \leq B \leq A, R(B) \subseteq \mathcal{S}\}$. See also Pekarev-Šmuljan [28] and Arlinskiĭ and E. Tsekanovskiĭ [5]. This notion was rediscovered by Anderson and Trapp [2], who, among many other results, proved that $A_{/\mathcal{S}}$ coincides with the Schur complement. It is impossible to record the different situations, disciplines and applications which rely on these "complements". We refer the reader to the excellent surveys by R. W. Cottle [13] and D. Carlson [8] and the more recent book edited by F. Zhang [33]. We do mention a note by C. A. Butler and T. D. Morley [7] which compares six different extensions and generalizations of Schur complements. One among them, due to T. Ando [3] is the origin of this note. Given an $n \times n$ complex matrix A and a subspace \mathcal{S} of \mathbb{C}^n , A is called \mathcal{S} -complementable if there exist matrices M_r, M_l such that $P_{\mathcal{S}}M_r = M_r$, $M_lP_{\mathcal{S}} = M_l$, $P_{\mathcal{S}}AM_r = P_{\mathcal{S}}A$ and $M_lAP_{\mathcal{S}} = AP_{\mathcal{S}}$; here $P_{\mathcal{S}}$ denotes the orthogonal projection onto \mathcal{S} . The matrices $A_{\mathcal{S}} := AM_r$ and $A_{/\mathcal{S}} := A - A_{\mathcal{S}}$ are called the Schur compression of A to \mathcal{S} and the generalized Schur complement of A with respect to \mathcal{S} . Ando proved that if

*M. L. Arias was partially supported by CONICET (PIP 11220120100426), FONCYT (PICT 2017-0883 and 2014-1776), G. Corach by CONICET (PIP 11220120100426), FONCYT (PICT 2014-1776) and A. Maestriperi by CONICET (PIP 11220130100168)

$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$, $\mathbb{C}^n = \mathbb{C}^{n_1} \oplus \mathbb{C}^{n_2}$ and the $n_1 \times n_1$ matrix B is invertible then $M_r = \begin{pmatrix} I & B^{-1}C \\ 0 & 0 \end{pmatrix}$, and $M_l = \begin{pmatrix} I & 0 \\ DB^{-1} & 0 \end{pmatrix}$, show that A is $\mathbb{C}^{n_1} \times \{0\}$ -complementable and $A_{\mathcal{S}} = \begin{pmatrix} B & C \\ D & DB^{-1}C \end{pmatrix}$, $A_{/\mathcal{S}} = \begin{pmatrix} 0 & 0 \\ 0 & E - DB^{-1}C \end{pmatrix}$, (here $\mathcal{S} = \mathbb{C}^{n_1} \times \{0\}$, of course). Ando's definition have many advantages. We only mention two. Firstly, it has a neat extension to Hilbert space operators (see [11]); secondly, it can be easily extended, at least for matrices, to linear transformations between different vector spaces (i.e., non square matrices) with a fixed subspace on each. This extension is due to Mitra and Puri [26] and, independently, to Carlson and Haynsworth [9]. Much later, Antezana et al. [4] extended this idea to Hilbert space operators: for a bounded linear operator $A : \mathcal{H} \rightarrow \mathcal{K}$ between Hilbert spaces and closed subspaces $\mathcal{S} \subset \mathcal{H}$, $\mathcal{T} \subset \mathcal{K}$, they gave appropriate definitions of complementability and a weaker notion called weak complementability which allows the definition of a generalized bi-Schur complement (or, bilateral shorted operator) $A_{/\mathcal{S}, \mathcal{T}}$. Antezana et al. [4] applied this construction to the extension of parallel sums of bounded linear operators. This study started with Anderson and Duffin [1]; see also Filmore-Williams [18]. The present paper continues their research in different directions. On the one hand, we explore in more detail several features of the complementability. On the other hand, we use them to get results on the operators $T_{\mathcal{T}, \mathcal{S}}(A) = P_{\mathcal{T}}A|_{\mathcal{S}} : \mathcal{S} \rightarrow \mathcal{T}$ which Devinatz and Shinbrot [15] called general Wiener-Hopf operators.

It should be mentioned that the theory of compatibility between subspaces and selfadjoint operators plays a significant role in this paper, mainly through its methods. This theory, first introduced by Hassi and Nordström [19], was later developed in a large series of papers (among them, [10], [11], [12]). It should be noticed that a selfadjoint operator A and a closed subspace \mathcal{S} are compatible if and only if A is \mathcal{S} -complementable (see [11]).

We briefly describe the contents of the paper. In Section 2 we study complementable and weakly complementable operators. The main results of this section are Propositions 2.4 and 2.15, where we prove that the notion of complementable operators (resp. weakly complementable operators) introduced by Antezana et al. in [4] is equivalent to the existence of certain bounded projections (resp. densely defined projections). In addition, we fully describe those projections. In Section 3 we study the generalized invertibility (or, invertibility) of general Wiener-Hopf operators through the notions of complementable and weakly complementable operators. For instance, under the condition of weakly complementability, we obtain a formula of the Moore-Penrose inverse of this kind of operators by means of the projections associated to this condition. The main contributions of this section are Theorems 3.2, 3.6 and 3.7. Finally, in Section 4 we study bilateral shorted operators. As we mentioned before, this class of operators was introduced in [4] under the hypothesis of weakly complementability. Here, we present a formula of this operator by means of the projections associated to the weakly complementability, see Theorem 4.2. Moreover, in Theorem 4.6 we provide a variational characterization of the bilateral shorted operator which extends a result given in [4].

2 Complementable and weakly-complementable operators

Throughout this paper $\mathcal{G}, \mathcal{H}, \mathcal{K}$ denote complex Hilbert spaces. $L(\mathcal{H}, \mathcal{K})$ is the space of all bounded linear operators from \mathcal{H} to \mathcal{K} and the algebra $L(\mathcal{H}, \mathcal{H})$ is abbreviated by $L(\mathcal{H})$. Given $T \in L(\mathcal{H}, \mathcal{K})$ its range is denoted by $R(T)$, its nullspace by $N(T)$ and its adjoint by T^* . We denote by T^\dagger the Moore-Penrose inverse of T . Along this article we shall deal with not necessarily bounded projections. A densely defined operator E , with domain $\mathcal{D}(E)$, is called *projection* if $R(E) \subseteq \mathcal{D}(E)$ and $E(Ex) = Ex$ for all $x \in \mathcal{D}(E)$. In this case, $\mathcal{D}(E) = R(E) \dot{+} N(E)$, see [27]. The set of bounded (oblique) projections in $L(\mathcal{H})$ is denoted by $\mathcal{Q}(\mathcal{H})$ and the set of orthogonal projections in $L(\mathcal{H})$ by

$\mathcal{P}(\mathcal{H})$, i.e., $\mathcal{Q}(\mathcal{H}) = \{E \in L(\mathcal{H}) : E^2 = E\}$ and $\mathcal{P}(\mathcal{H}) = \{P \in \mathcal{Q}(\mathcal{H}) : P = P^*\}$. To simplify the notation, we will often omit the Hilbert space and just write \mathcal{Q}, \mathcal{P} when no confusion arises. For two fixed closed subspace \mathcal{S}, \mathcal{T} of \mathcal{H} such that $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{T}$, denote by $Q_{\mathcal{S} // \mathcal{T}}$ the element in \mathcal{Q} with $R(Q_{\mathcal{S} // \mathcal{T}}) = \mathcal{S}$ and $N(Q_{\mathcal{S} // \mathcal{T}}) = \mathcal{T}$ and by $P_{\mathcal{S}}$ the element in \mathcal{P} with $R(P_{\mathcal{S}}) = \mathcal{S}$. The next result will be frequently applied along this note, see [14, Theorem 22] for its proof.

Theorem 2.1. *If $A, B \in L(\mathcal{H})$ have closed ranges, then AB has closed range if and only if $R(B) + N(A)$ is closed.*

Along this article, $A \in L(\mathcal{H}, \mathcal{K})$ and $\mathcal{S} \subseteq \mathcal{H}$ and $\mathcal{T} \subseteq \mathcal{K}$ are closed subspaces.

2.1 Complementability

In what follows we study complementable operators. This notion was introduced by T. Ando [3] in finite dimensional spaces with the aim of studying the Schur compression and Schur complement of matrices. Later this concept was generalized for two subspaces by Mitra and Puri [26] and Carlson and Haynsworth in [9], and extended to infinite dimensional Hilbert spaces by Antezana et al. in [4]. Although our next definition of complementability differs from the one considered in [4], we show in Proposition 2.4 that both definitions coincide.

Definition 2.2. *An operator $A \in L(\mathcal{H}, \mathcal{K})$ is $(\mathcal{S}, \mathcal{T})$ -complementable if there exist two projections $E \in \mathcal{Q}(\mathcal{H}), F \in \mathcal{Q}(\mathcal{K})$ with $R(E) = \mathcal{S}, R(F) = \mathcal{T}$ such that $AE = F^*A$.*

In other words, $A \in L(\mathcal{H}, \mathcal{K})$ is $(\mathcal{S}, \mathcal{T})$ -complementable if there exist two bounded projections E, F onto \mathcal{S}, \mathcal{T} respectively, such that F is an A -adjoint of E , i.e., $\langle Ex, y \rangle_A = \langle x, Fy \rangle_A$, for all $x \in \mathcal{H}, y \in \mathcal{K}$ where $\langle x, y \rangle_A := \langle Ax, y \rangle$. When $\mathcal{H} = \mathcal{K}$ and $\mathcal{S} = \mathcal{T}$ we say that $A \in L(\mathcal{H})$ is \mathcal{S} -complementable when A is $(\mathcal{S}, \mathcal{S})$ -complementable. We denote by $\mathcal{P}(A, \mathcal{S}, \mathcal{T})$ the set of projections arising from 2.2, i.e.,

$$\mathcal{P}(A, \mathcal{S}, \mathcal{T}) := \{(E, F) \in \mathcal{Q} \times \mathcal{Q} : R(E) = \mathcal{S}, R(F) = \mathcal{T}, AE = F^*A\}.$$

Clearly, the set $\mathcal{P}(A, \mathcal{S}, \mathcal{T})$ is not empty if and only if A is $(\mathcal{S}, \mathcal{T})$ -complementable.

Remarks 2.3. The next properties are straightforward:

1. A is $(\mathcal{S}, \mathcal{T})$ -complementable if and only if A^* is $(\mathcal{T}, \mathcal{S})$ -complementable. In this case, $(E, F) \in \mathcal{P}(A, \mathcal{S}, \mathcal{T})$ if and only if $(F, E) \in \mathcal{P}(A^*, \mathcal{T}, \mathcal{S})$.
2. A selfadjoint operator $A \in L(\mathcal{H})$ is \mathcal{S} -complementable if and only if there exists a projection E with $R(E) = \mathcal{S}$ such that $AE = E^*A$. In fact, if A is \mathcal{S} -complementable, there exist two projections $E, F \in \mathcal{Q}$ with $R(E) = R(F) = \mathcal{S}$ such that $AE = F^*A$ then $E^*AE = E^*F^*A = E^*A$ is selfadjoint and so $AE = E^*A$ as desired. The converse is straightforward. Therefore, if A is a selfadjoint operator then A is \mathcal{S} -complementable if and only if (A, \mathcal{S}) is compatible in the sense of [10].

Let us start by proving that our definition of $(\mathcal{S}, \mathcal{T})$ -complementability is equivalent to the definition given by Carlson and Haynsworth in [9].

Proposition 2.4. *The following conditions are equivalent:*

1. A is $(\mathcal{S}, \mathcal{T})$ -complementable.

2. Given $Q_s \in \mathcal{Q}(\mathcal{H})$ and $Q_t \in \mathcal{Q}(\mathcal{K})$ with $R(Q_s) = \mathcal{S}$ and $N(Q_t) = \mathcal{T}^\perp$ there exist operators $M_r \in L(\mathcal{H}), M_l \in L(\mathcal{K})$ such that $M_l Q_t = M_l, Q_s M_r = M_r, M_l A Q_s = A Q_s$ and $Q_t A M_r = Q_t A$.

Proof. $1 \Rightarrow 2$. Assume that A is $(\mathcal{S}, \mathcal{T})$ -complementable and let E, F be two projections with $R(E) = \mathcal{S}$ and $R(F) = \mathcal{T}$ such that $AE = F^*A$. Consider two projections Q_s, Q_t with $R(Q_s) = \mathcal{S}$ and $N(Q_t) = \mathcal{T}^\perp$. Then, $F^*Q_t = F^*, Q_s E = E, F^*A Q_s = AE Q_s = A Q_s$ and $Q_t A E = Q_t F^*A = Q_t A$. Hence, item 2 holds, with $M_r = E$ and $M_l = F^*$.

$2 \Rightarrow 1$. Let Q_s, Q_t be two projections with $R(Q_s) = \mathcal{S}$ and $N(Q_t) = \mathcal{T}^\perp$ and M_r, M_l be two operators such that $M_l Q_t = M_l, Q_s M_r = M_r, M_l A Q_s = A Q_s$ and $Q_t A M_r = Q_t A$. Define $E := Q_s + M_r(I - Q_s)$ and $F := Q_t^* + M_l^*(I - Q_t)^*$. Thus, from $Q_s M_r = M_r$ and $M_l Q_t = M_l$, we obtain that $E^2 = E, F^2 = F$ and $R(E) = R(Q_s) = \mathcal{S}$ and $R(F) = R(Q_t^*) = \mathcal{T}$. We claim that $AE = F^*A$. In fact, $F^*AE = (Q_t + (I - Q_t)M_l)AE = Q_t AE + (I - Q_t)M_l A Q_s E = Q_t AE + (I - Q_t)AE = AE$, where we use that $M_l A Q_s = A Q_s$ and $Q_s E = E$. In a similar way $F^*AE = F^*A$. \square

In the next result we present equivalent conditions to the notion of complementability in terms of ranges of operators and Hilbert spaces decompositions. In the sequel, we will frequently use that $(AS)^\perp = (A^*)^{-1}(\mathcal{S}^\perp)$.

Proposition 2.5. *The following conditions are equivalent:*

1. A is $(\mathcal{S}, \mathcal{T})$ -complementable.
2. $R(P_{\mathcal{S}}A^*P_{\mathcal{T}}) = R(P_{\mathcal{S}}A^*)$ and $R(P_{\mathcal{T}}AP_{\mathcal{S}}) = R(P_{\mathcal{T}}A)$.
3. $\mathcal{H} = \mathcal{S} + (A^*\mathcal{T})^\perp$ and $\mathcal{K} = \mathcal{T} + (AS)^\perp$.

Proof. $1 \Rightarrow 2$. Let E, F be two bounded projections with $R(E) = \mathcal{S}$ and $R(F) = \mathcal{T}$ such that $AE = F^*A$. To prove item 2 it suffices to prove that $R(P_{\mathcal{S}}A^*) \subseteq R(P_{\mathcal{S}}A^*P_{\mathcal{T}})$ and $R(P_{\mathcal{T}}A) \subseteq R(P_{\mathcal{T}}AP_{\mathcal{S}})$. Let us see the second inclusion: $P_{\mathcal{T}}A = P_{\mathcal{T}}AE + P_{\mathcal{T}}A(I - E) = P_{\mathcal{T}}AP_{\mathcal{S}}E + P_{\mathcal{T}}(I - F^*)A = P_{\mathcal{T}}AP_{\mathcal{S}}E$ because $P_{\mathcal{T}}F^* = P_{\mathcal{T}}$. Then $R(P_{\mathcal{T}}A) \subseteq R(P_{\mathcal{T}}AP_{\mathcal{S}})$. Similarly, $R(P_{\mathcal{S}}A^*) \subseteq R(P_{\mathcal{S}}A^*P_{\mathcal{T}})$.

$2 \Rightarrow 3$. Let $y = Ax \in R(A)$. Then, $Ax = P_{\mathcal{T}}Ax + P_{\mathcal{T}^\perp}Ax = P_{\mathcal{T}}As + P_{\mathcal{T}^\perp}Ax$, for some $s \in \mathcal{S}$ because $R(P_{\mathcal{T}}AP_{\mathcal{S}}) = R(P_{\mathcal{T}}A)$. Thus, $Ax = As - P_{\mathcal{T}^\perp}As + P_{\mathcal{T}^\perp}Ax \in AS + \mathcal{T}^\perp$. Therefore, $R(A) \subseteq AS + \mathcal{T}^\perp$. Then $\mathcal{H} = A^{-1}(AS + \mathcal{T}^\perp) = \mathcal{S} + A^{-1}(\mathcal{T}^\perp)$. The proof of the second equality is similar.

$3 \Rightarrow 1$. If $\mathcal{H} = \mathcal{S} + (A^*\mathcal{T})^\perp$ and $\mathcal{K} = \mathcal{T} + (AS)^\perp$ then there exist bounded projections E and F with $R(E) = \mathcal{S}$, $N(E) \subseteq (A^*\mathcal{T})^\perp$, $R(F) = \mathcal{T}$ and $N(F) \subseteq (AS)^\perp$. We claim that $AE = F^*A$. In fact, $F^*As = As = AEs$ for all $s \in \mathcal{S}$ and $F^*Ax = 0 = AEx$ for all $x \in N(E)$. \square

The following is a necessary condition for complementability.

Lemma 2.6. *If A is $(\mathcal{S}, \mathcal{T})$ -complementable then $(AS)^\perp \cap \mathcal{T} = N(A^*) \cap \mathcal{T}$ and $(A^*\mathcal{T})^\perp \cap \mathcal{S} = N(A) \cap \mathcal{S}$.*

Proof. Assume that A is $(\mathcal{S}, \mathcal{T})$ -complementable and let E, F be projections onto \mathcal{S}, \mathcal{T} respectively, such that $AE = F^*A$. Thus, if $x \in (AS)^\perp \cap \mathcal{T}$ then, using that $(AS)^\perp = (A^*)^{-1}(\mathcal{S}^\perp)$, $A^*x \in \mathcal{S}^\perp$ and $x \in \mathcal{T}$, and so $\langle A^*x, y \rangle = \langle A^*Fx, y \rangle = \langle E^*A^*x, y \rangle = \langle A^*x, Ey \rangle = 0$ for all $y \in \mathcal{H}$. Therefore, $A^*x = 0$ and so $x \in N(A^*) \cap \mathcal{T}$. Since $(AS)^\perp \cap \mathcal{T} \supseteq N(A^*) \cap \mathcal{T}$ always holds, the equality is proved. Similarly, it can be proved that $(A^*\mathcal{T})^\perp \cap \mathcal{S} = N(A) \cap \mathcal{S}$. \square

Corollary 2.7. *A is $(\mathcal{S}, \mathcal{T})$ -complementable if and only if $R(A) = \mathcal{AS} \dot{+} \mathcal{T}^\perp \cap R(A)$ and $R(A^*) = A^*\mathcal{T} \dot{+} \mathcal{S}^\perp \cap R(A^*)$.*

Proof. For any $A \in L(\mathcal{H}, \mathcal{K})$, the equality $(A^*\mathcal{T})^\perp \cap \mathcal{S} = N(A) \cap \mathcal{S}$ holds if and only if $\mathcal{AS} \cap \mathcal{T}^\perp = \{0\}$. In fact, suppose that $\mathcal{AS} \cap \mathcal{T}^\perp = \{0\}$ and let $x \in (A^*\mathcal{T})^\perp \cap \mathcal{S}$ then $Ax \in \mathcal{AS} \cap \mathcal{T}^\perp = \{0\}$. Then $x \in N(A)$. The converse is similar.

If A is $(\mathcal{S}, \mathcal{T})$ -complementable then, by Proposition 2.5, $\mathcal{K} = \mathcal{T} + (\mathcal{AS})^\perp$ and, by Lemma 2.6, $(\mathcal{AS})^\perp \cap \mathcal{T} = N(A^*) \cap \mathcal{T}$. Thus, $R(A^*) = A^*\mathcal{T} + \mathcal{S}^\perp \cap R(A^*)$ and $A^*\mathcal{T} \cap \mathcal{S}^\perp = \{0\}$ by the previous comment. The proof of the second equality is similar.

For the converse, suppose that $R(A) = \mathcal{AS} \dot{+} \mathcal{T}^\perp \cap R(A)$. Hence, $\mathcal{H} = A^{-1}(\mathcal{AS} \dot{+} \mathcal{T}^\perp \cap R(A)) = \mathcal{S} + N(A) + A^{-1}(\mathcal{T}^\perp) = \mathcal{S} + A^{-1}(\mathcal{T}^\perp)$. Analogously, from $R(A^*) = A^*\mathcal{T} \dot{+} (\mathcal{S}^\perp \cap R(A^*))$, we get that $\mathcal{K} = \mathcal{T} + (\mathcal{AS})^\perp$, and the proof is complete. \square

By Proposition 2.5, if $A = I \in L(\mathcal{H})$ and \mathcal{S}, \mathcal{T} are two closed subspaces of \mathcal{H} then A is $(\mathcal{S}, \mathcal{T})$ -complementable if and only if $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{T}^\perp$. By [6, Theorem 1], $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{T}^\perp$ if and only if $\|P_{\mathcal{S}} - P_{\mathcal{T}}\| < 1$. More generally, if A has closed range or A is invertible, the complementability can be characterized as follows:

Proposition 2.8. *Given $A \in L(\mathcal{H}, \mathcal{K})$ with closed range, the following conditions are equivalent:*

1. *A is $(\mathcal{S}, \mathcal{T})$ -complementable.*
2. *$\mathcal{H} = \mathcal{S} + (A^*\mathcal{T})^\perp$, $A^*(\mathcal{T})$ is closed and $(A^*\mathcal{T})^\perp \cap \mathcal{S} = N(A) \cap \mathcal{S}$.*
3. *$\mathcal{K} = \mathcal{T} + (\mathcal{AS})^\perp$, $A(\mathcal{S})$ is closed and $(\mathcal{AS})^\perp \cap \mathcal{T} = N(A^*) \cap \mathcal{T}$.*

Proof. $1 \Leftrightarrow 2$. Assume that A is $(\mathcal{S}, \mathcal{T})$ -complementable. We only need to prove that $A^*(\mathcal{T})$ is closed. Now, by Corollary 2.7, $R(A^*) = A^*(\mathcal{T}) \dot{+} \mathcal{S}^\perp \cap R(A^*)$ and, since $R(A^*)$ is closed, we get that $A^*(\mathcal{T})$ is closed because of [32, Theorem 5.10].

Conversely, suppose that $\mathcal{H} = \mathcal{S} + (A^*\mathcal{T})^\perp$, $A^*(\mathcal{T})$ is closed and $(A^*\mathcal{T})^\perp \cap \mathcal{S} = N(A) \cap \mathcal{S} =: \mathcal{N}$. We shall prove that $\mathcal{K} = \mathcal{T} + (\mathcal{AS})^\perp$. Now, $\mathcal{H} = (\mathcal{S} \ominus \mathcal{N}) \dot{+} (A^*\mathcal{T})^\perp$ or, equivalently, $\mathcal{H} = (\mathcal{S} \ominus \mathcal{N})^\perp \dot{+} A^*\mathcal{T}$. Therefore, if $y \in R(A^*)$, then $y = s + A^*t$ for some $s \in (\mathcal{S} \ominus \mathcal{N})^\perp$ and $t \in \mathcal{T}$; and so $s \in (\mathcal{S} \ominus \mathcal{N})^\perp \cap R(A^*)$. Hence, $R(A^*) = (\mathcal{S} \ominus \mathcal{N})^\perp \cap R(A^*) \dot{+} A^*\mathcal{T}$. From this, $\mathcal{K} = (A^*)^{-1}((\mathcal{S} \ominus \mathcal{N})^\perp) + \mathcal{T} = (A(\mathcal{S} \ominus \mathcal{N}))^\perp + \mathcal{T} = (\mathcal{AS})^\perp + \mathcal{T}$. Therefore, A is $(\mathcal{S}, \mathcal{T})$ -complementable.

The equivalence $1 \Leftrightarrow 3$ can be proved similarly. \square

Corollary 2.9. *Let $A \in L(\mathcal{H}, \mathcal{K})$ be an invertible operator. The next conditions are equivalent:*

1. *A is $(\mathcal{S}, \mathcal{T})$ -complementable*
2. *$\mathcal{H} = \mathcal{S} \dot{+} (A^*\mathcal{T})^\perp$*
3. *$\mathcal{K} = \mathcal{T} \dot{+} (\mathcal{AS})^\perp$.*
4. *$\|P_{\mathcal{S}} - P_{(A^*\mathcal{T})^\perp}\| < 1$.*
5. *$\|P_{\mathcal{T}} - P_{(\mathcal{AS})^\perp}\| < 1$.*

The next result shows some equivalent conditions for complementability when $A = P_{\mathcal{M}}$, where \mathcal{M} is a closed subspace of \mathcal{H} .

Proposition 2.10. *Let $\mathcal{M} \subseteq \mathcal{H}$ be a closed subspace. The next conditions are equivalent:*

1. *$P_{\mathcal{M}}$ is $(\mathcal{S}, \mathcal{T})$ -complementable*

2. $\mathcal{M} = (\mathcal{S} + \mathcal{M}^\perp) \cap \mathcal{M} \dot{+} \mathcal{T}^\perp \cap \mathcal{M}$, and $\mathcal{T} + \mathcal{M}^\perp$ is closed.
3. $\mathcal{H} = \mathcal{S} + \mathcal{M}^\perp + \mathcal{T}^\perp \cap \mathcal{M} = \mathcal{T} + \mathcal{M}^\perp + \mathcal{S}^\perp \cap \mathcal{M}$.
4. The subspaces $P_{\mathcal{M}}\mathcal{S}$ and $P_{\mathcal{M}}\mathcal{T}$ are closed and the Grammian operator $G : P_{\mathcal{M}}\mathcal{S} \rightarrow P_{\mathcal{M}}\mathcal{T}$, $G := P_{P_{\mathcal{M}}\mathcal{T}}|_{P_{\mathcal{M}}\mathcal{S}}$ is invertible.

Proof. It always holds that $P_{\mathcal{M}}\mathcal{S} = (\mathcal{S} + \mathcal{M}^\perp) \cap \mathcal{M}$: if $x \in P_{\mathcal{M}}\mathcal{S}$, there exists $s \in \mathcal{S}$ such that $x = P_{\mathcal{M}}s$. Then $s = x + P_{\mathcal{M}^\perp}s$, or $x = s - P_{\mathcal{M}^\perp}s \in \mathcal{S} + \mathcal{M}^\perp$. Then $x \in (\mathcal{S} + \mathcal{M}^\perp) \cap \mathcal{M}$. Conversely, if $x \in (\mathcal{S} + \mathcal{M}^\perp) \cap \mathcal{M}$ then $x = s + m^\perp$, for some $s \in \mathcal{S}, m^\perp \in \mathcal{M}^\perp$. Then $x = P_{\mathcal{M}}x = P_{\mathcal{M}}s \in P_{\mathcal{M}}\mathcal{S}$.

1 \Rightarrow 2. If $P_{\mathcal{M}}$ is $(\mathcal{S}, \mathcal{T})$ -complementable then, by Corollary 2.7 and [32, Theorem 5.10], $P_{\mathcal{M}}\mathcal{S}$ and $P_{\mathcal{M}}\mathcal{T}$ are closed and $\mathcal{M} = P_{\mathcal{M}}\mathcal{S} \dot{+} \mathcal{T}^\perp \cap \mathcal{M}$. Then, $\mathcal{M} = (\mathcal{S} + \mathcal{M}^\perp) \cap \mathcal{M} \dot{+} \mathcal{T}^\perp \cap \mathcal{M}$ and $\mathcal{T} + \mathcal{M}^\perp$ is closed because of Theorem 2.1.

2 \Rightarrow 3. If $\mathcal{M} = (\mathcal{S} + \mathcal{M}^\perp) \cap \mathcal{M} \dot{+} \mathcal{T}^\perp \cap \mathcal{M}$ then $\mathcal{H} = (\mathcal{S} + \mathcal{M}^\perp) \cap \mathcal{M} \dot{+} \mathcal{T}^\perp \cap \mathcal{M} + \mathcal{M}^\perp \subseteq \mathcal{S} + \mathcal{M}^\perp + \mathcal{T}^\perp \cap \mathcal{M}$, as desired. To prove the second equality it suffices to check that by taking orthogonal complement in the Hilbert space \mathcal{M} in the equality $\mathcal{M} = P_{\mathcal{M}}\mathcal{S} \dot{+} \mathcal{T}^\perp \cap \mathcal{M}$, and using that $\mathcal{T} + \mathcal{M}^\perp$ is closed, we get that $\mathcal{M} = (P_{\mathcal{M}}\mathcal{S})^\perp \cap \mathcal{M} \dot{+} (\mathcal{T} + \mathcal{M}^\perp) \cap \mathcal{M}$. But $(P_{\mathcal{M}}\mathcal{S})^\perp = N(P_{\mathcal{S}}P_{\mathcal{M}}) = (\mathcal{M} \cap \mathcal{S}^\perp) \oplus \mathcal{M}^\perp$. Therefore, $\mathcal{M} = \mathcal{S}^\perp \cap \mathcal{M} \dot{+} (\mathcal{T} + \mathcal{M}^\perp) \cap \mathcal{M}$. Proceeding as before, we get that $\mathcal{H} = \mathcal{T} + \mathcal{M}^\perp + \mathcal{S}^\perp \cap \mathcal{M}$.

3 \Rightarrow 1. Assume that $\mathcal{H} = \mathcal{S} + \mathcal{M}^\perp + \mathcal{T}^\perp \cap \mathcal{M}$. Now, $\mathcal{M}^\perp + \mathcal{T}^\perp \cap \mathcal{M} = N(P_{\mathcal{T}}P_{\mathcal{M}}) = R(P_{\mathcal{M}}P_{\mathcal{T}})^\perp = (P_{\mathcal{M}}\mathcal{T})^\perp$. Thus, $\mathcal{H} = \mathcal{S} + (P_{\mathcal{M}}\mathcal{T})^\perp$. The equality $\mathcal{H} = \mathcal{T} + (P_{\mathcal{M}}\mathcal{S})^\perp$ follows in the same way from $\mathcal{H} = \mathcal{T} + \mathcal{M}^\perp + \mathcal{S}^\perp \cap \mathcal{M}$.

2 \Leftrightarrow 4. Suppose that $\mathcal{M} = (\mathcal{S} + \mathcal{M}^\perp) \cap \mathcal{M} \dot{+} \mathcal{T}^\perp \cap \mathcal{M}$, and $\mathcal{T} + \mathcal{M}^\perp$ is closed. From this, $P_{\mathcal{M}}\mathcal{S}$ and $P_{\mathcal{M}}\mathcal{T}$ are closed subspaces. Let us prove that G is invertible. Consider $y \in P_{\mathcal{M}}\mathcal{T}$. Then, $y \in \mathcal{M}$ and so $y = x_1 + x_2$ for some $x_1 \in (\mathcal{S} + \mathcal{M}^\perp) \cap \mathcal{M} = P_{\mathcal{M}}\mathcal{S}$ and $x_2 \in \mathcal{T}^\perp \cap \mathcal{M}$. Hence, since $\mathcal{T}^\perp \cap \mathcal{M} \subseteq (P_{\mathcal{M}}\mathcal{T})^\perp$ we get that $Gx_1 = P_{P_{\mathcal{M}}\mathcal{T}}x_1 = P_{P_{\mathcal{M}}\mathcal{T}}y = y$. Thus, G is surjective. On the other side, let $x \in P_{\mathcal{M}}\mathcal{S}$ such that $Gx = 0$. Then, $x \in (P_{\mathcal{M}}\mathcal{T})^\perp \cap P_{\mathcal{M}}\mathcal{S} = (\mathcal{T}^\perp \cap \mathcal{M}) \cap P_{\mathcal{M}}\mathcal{S} = \{0\}$. Therefore G is injective, so that G is invertible.

Conversely, assume that $P_{\mathcal{M}}\mathcal{S}$ and $P_{\mathcal{M}}\mathcal{T}$ are closed subspaces and $G := P_{P_{\mathcal{M}}\mathcal{T}}|_{P_{\mathcal{M}}\mathcal{S}}$ is invertible. Then, by [24, Proposition 2.1], $(P_{\mathcal{M}}\mathcal{T})^\perp \dot{+} P_{\mathcal{M}}\mathcal{S} = \mathcal{H}$. Hence, $\mathcal{M} = P_{\mathcal{M}}((P_{\mathcal{M}}\mathcal{T})^\perp) \dot{+} P_{\mathcal{M}}\mathcal{S}$. Now, as $P_{\mathcal{M}}\mathcal{T} = (\mathcal{T} + \mathcal{M}^\perp) \cap \mathcal{M}$ a simple computation shows that $P_{\mathcal{M}}((P_{\mathcal{M}}\mathcal{T})^\perp) = \mathcal{T}^\perp \cap \mathcal{M}$. The proof is complete. \square

By the proof of Proposition 2.5, the set $\mathcal{P}(A, \mathcal{S}, \mathcal{T})$ can be characterized in the following way:

Proposition 2.11. *Let A be $(\mathcal{S}, \mathcal{T})$ -complementable. Then*

$$\mathcal{P}(A, \mathcal{S}, \mathcal{T}) = \{(E, F) \in \mathcal{Q} \times \mathcal{Q} : R(E) = \mathcal{S}, N(E) \subseteq (A^*\mathcal{T})^\perp, R(F) = \mathcal{T} \text{ and } N(F) \subseteq (A\mathcal{S})^\perp\}.$$

If A is $(\mathcal{S}, \mathcal{T})$ -complementable, consider $\mathcal{N} = \mathcal{S} \cap N(A)$ and $\mathcal{N}^* = \mathcal{T} \cap N(A^*)$. By Lemma 2.6, $\mathcal{H} = \mathcal{S} \dot{+} (A^*\mathcal{T})^\perp \ominus \mathcal{N}$ and $\mathcal{K} = \mathcal{T} \dot{+} (A\mathcal{S})^\perp \ominus \mathcal{N}^*$ and the corresponding projections

$$\tilde{E} = Q_{\mathcal{S} // (A^*\mathcal{T})^\perp \ominus \mathcal{N}} \text{ and } \tilde{F} = Q_{\mathcal{T} // (A\mathcal{S})^\perp \ominus \mathcal{N}^*} \quad (1)$$

are such that $(\tilde{E}, \tilde{F}) \in \mathcal{P}(A, \mathcal{S}, \mathcal{T})$. Moreover, the set $\mathcal{P}(A, \mathcal{S}, \mathcal{T})$ is an affine manifold as the next corollary shows:

Corollary 2.12. *Let A be $(\mathcal{S}, \mathcal{T})$ -complementable. Then*

$$\mathcal{P}(A, \mathcal{S}, \mathcal{T}) = (\tilde{E}, \tilde{F}) + L(\mathcal{S}^\perp, \mathcal{N}) \times L(\mathcal{T}^\perp, \mathcal{N}^*),$$

where \tilde{E} and \tilde{F} are the projections given by (1) and $L(\mathcal{S}^\perp, \mathcal{N}), L(\mathcal{T}^\perp, \mathcal{N}^*)$ are viewed as subspaces of $L(\mathcal{H}), L(\mathcal{K})$, respectively. In particular, $\mathcal{P}(A, \mathcal{S}, \mathcal{T}) = \{(\tilde{E}, \tilde{F})\}$ if and only if $\mathcal{S} \cap N(A) = \{0\}$ and $\mathcal{T} \cap N(A^*) = \{0\}$.

Given two closed subspaces \mathcal{S}, \mathcal{T} of \mathcal{H}, \mathcal{K} , respectively, we identify each $A \in L(\mathcal{H}, \mathcal{K})$ with the matrix

$$A = \begin{array}{c} \mathcal{T} \\ \mathcal{T}^\perp \end{array} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{array}{c} \mathcal{S} \\ \mathcal{S}^\perp \end{array} \quad (2)$$

where $A_{11} = P_{\mathcal{T}}A|_{\mathcal{S}}, A_{12} = P_{\mathcal{T}}A|_{\mathcal{S}^\perp}, A_{21} = P_{\mathcal{T}^\perp}A|_{\mathcal{S}}, A_{22} = P_{\mathcal{T}^\perp}A|_{\mathcal{S}^\perp}$. The next result provides a characterization of the $(\mathcal{S}, \mathcal{T})$ -complementability in terms of the matrix representation of A and A^* . The first part of the following result can be also found in [4, Proposition 3.2]

Corollary 2.13. *Consider the matrix representation (2). Then, A is $(\mathcal{S}, \mathcal{T})$ -complementable if and only if $R(A_{12}) \subseteq R(A_{11})$ and $R(A_{21}^*) \subseteq R(A_{11}^*)$. In this case,*

$$\mathcal{P}(A, \mathcal{S}, \mathcal{T}) = \{(E, F) : E = \begin{pmatrix} 1 & X \\ 0 & 0 \end{pmatrix} \begin{array}{c} \mathcal{S} \\ \mathcal{S}^\perp \end{array} \text{ and } F = \begin{pmatrix} 1 & Y \\ 0 & 0 \end{pmatrix} \begin{array}{c} \mathcal{T} \\ \mathcal{T}^\perp \end{array} \text{ with } A_{11}X = A_{12}, A_{11}^*Y = A_{21}^*\}.$$

Proof. From Proposition 2.5, A is $(\mathcal{S}, \mathcal{T})$ -complementable if and only if $R(P_{\mathcal{S}}A^*P_{\mathcal{T}}) = R(P_{\mathcal{S}}A^*)$ and $R(P_{\mathcal{T}}AP_{\mathcal{S}}) = R(P_{\mathcal{T}}A)$. But these last equalities hold if and only if $R(P_{\mathcal{S}}A^*P_{\mathcal{T}^\perp}) \subseteq R(P_{\mathcal{S}}A^*P_{\mathcal{T}})$ and $R(P_{\mathcal{T}}AP_{\mathcal{S}^\perp}) \subseteq R(P_{\mathcal{T}}AP_{\mathcal{S}})$, which proves the assertion. In this case, by Douglas' inclusion operator range theorem [17, Theorem 1], there exist $X \in L(\mathcal{S}^\perp, \mathcal{S})$ and $Y \in L(\mathcal{T}^\perp, \mathcal{T})$ such that $A_{11}X = A_{12}$ and $A_{11}^*Y = A_{21}^*$. Define $E = \begin{pmatrix} 1 & X \\ 0 & 0 \end{pmatrix}$ and $F = \begin{pmatrix} 1 & Y \\ 0 & 0 \end{pmatrix}$; clearly, E, F are projections onto \mathcal{S}, \mathcal{T} , respectively and it is straightforward that $AE = F^*A$. Therefore, $(E, F) \in \mathcal{P}(A, \mathcal{S}, \mathcal{T})$. The other inclusion follows in a similar way. \square

2.2 Weak complementability

By considering the adjoints of E and F in Definition 2.2, the definition of complementability can be rewritten in the following way: $A \in L(\mathcal{H}, \mathcal{K})$ is $(\mathcal{S}, \mathcal{T})$ -complementable if there exist two bounded projections E, F with $N(E) = \mathcal{S}^\perp$ and $N(F) = \mathcal{T}^\perp$ such that $EA^* = (FA)^*$. Motivated by this fact, we introduce now a weaker notion:

Definition 2.14. *An operator $A \in L(\mathcal{H}, \mathcal{K})$ is called $(\mathcal{S}, \mathcal{T})$ -weakly complementable if there exist two densely defined projections E, F with $N(E) = \mathcal{S}^\perp$ and $N(F) = \mathcal{T}^\perp$ such that:*

- a) $E|P_{\mathcal{T}}AP_{\mathcal{S}}|^{1/2}$ and $F|P_{\mathcal{S}}A^*P_{\mathcal{T}}|^{1/2}$ are bounded operators;
- b) $FA \in L(\mathcal{H}, \mathcal{K}), EA^* \in L(\mathcal{K}, \mathcal{H})$ and $EA^* = (FA)^*$.

The concept of weakly complementability was first defined by Antezana et al. in [4] with the purpose of extending the notion of shorted operator to bounded linear operators between two different Hilbert spaces. Their definition relies on the matricial form of A in terms of \mathcal{S} and \mathcal{T} as in (2). Next, we prove that our definition coincides with that of [4].

Proposition 2.15. *If A has the matrix representation given by (2), then the following conditions are equivalent:*

- 1. A is $(\mathcal{S}, \mathcal{T})$ -weakly complementable.

2. $R(A_{12}) \subseteq R(|A_{11}^*|^{1/2})$ and $R(A_{21}^*) \subseteq R(|A_{11}|^{1/2})$.

Proof. $1 \Rightarrow 2$. Suppose that A is $(\mathcal{S}, \mathcal{T})$ -weakly complementable and let E, F two densely defined projections with $N(E) = \mathcal{S}^\perp$ and $N(F) = \mathcal{T}^\perp$ such that conditions a) and b) of Definition 2.14 hold. Hence, if $E = \begin{pmatrix} I & 0 \\ X & 0 \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^\perp \end{matrix}$ and $F = \begin{pmatrix} I & 0 \\ Y & 0 \end{pmatrix} \begin{matrix} \mathcal{T} \\ \mathcal{T}^\perp \end{matrix}$, then from condition a) the operators $X|A_{11}|^{1/2}$ and $Y|A_{11}^*|^{1/2}$ are bounded. On the other hand, by condition b), we obtain that $XA_{11}^* = A_{12}^*$ and $(YA_{11})^* = A_{21}^*$. Let $A_{11}^* = U^*|A_{11}^*|$ be the polar decomposition of A_{11}^* where U is the partial isometry with $N(U) = N(A_{11})$. Then, $A_{12}^* = XA_{11}^* = X|A_{11}|^{1/2}U^*|A_{11}^*|^{1/2} \in L(\mathcal{T}, \mathcal{S}^\perp)$ and since $Z := X|A_{11}|^{1/2}U^* \in L(\mathcal{S}, \mathcal{T}^\perp)$, we have that $A_{12} = |A_{11}^*|^{1/2}Z^*$ and so $R(A_{12}) \subseteq R(|A_{11}^*|^{1/2})$. In a similar way, from $A_{21} = YA_{11} = Y|A_{11}^*|^{1/2}U|A_{11}|^{1/2} \in L(\mathcal{S}, \mathcal{T}^\perp)$ we get that $R(A_{21}^*) \subseteq R(|A_{11}|^{1/2})$ as desired.

$2 \Rightarrow 1$. Assume that $R(A_{12}) \subseteq R(|A_{11}^*|^{1/2})$ and $R(A_{21}^*) \subseteq R(|A_{11}|^{1/2})$. Then, by Douglas' theorem [17, Theorem 1], there exist operators B, C such that $A_{12} = |A_{11}^*|^{1/2}B$, $R(B) \subseteq \overline{R(A_{11})}$ and $A_{21}^* = |A_{11}|^{1/2}C$ and $R(C) \subseteq \overline{R(A_{11}^*)}$. Therefore,

$$A = \begin{matrix} \mathcal{T} \\ \mathcal{T}^\perp \end{matrix} \begin{pmatrix} |A_{11}^*|U & |A_{11}^*|^{1/2}B \\ C^*|A_{11}|^{1/2} & A_{22} \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^\perp \end{matrix},$$

where $A_{11}^* = U^*|A_{11}^*| = |A_{11}|U^*$ is the polar decomposition of A_{11}^* .

Define: $E_0 = \begin{matrix} \mathcal{S} \\ \mathcal{S}^\perp \end{matrix} \begin{pmatrix} I & 0 \\ B^*U(|A_{11}|^{1/2})^\dagger & 0 \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^\perp \end{matrix}$ and $F_0 = \begin{matrix} \mathcal{T} \\ \mathcal{T}^\perp \end{matrix} \begin{pmatrix} I & 0 \\ C^*U^*(|A_{11}^*|^{1/2})^\dagger & 0 \end{pmatrix} \begin{matrix} \mathcal{T} \\ \mathcal{T}^\perp \end{matrix}$. Then, E_0, F_0 are two densely defined projections, $\mathcal{D}(E_0) = \mathcal{D}((|A_{11}|^{1/2})^\dagger) + \mathcal{S}^\perp$ and $\mathcal{D}(F_0) = \mathcal{D}((|A_{11}^*|^{1/2})^\dagger) + \mathcal{T}^\perp$, with $N(E_0) = \mathcal{S}^\perp$ and $N(F_0) = \mathcal{T}^\perp$. Thus, since $|P_{\mathcal{T}}AP_{\mathcal{S}}| = |A_{11}|$ and $|P_{\mathcal{S}}A^*P_{\mathcal{T}}| = |A_{11}^*|$ then $E_0|P_{\mathcal{T}}AP_{\mathcal{S}}|^{1/2} = \begin{pmatrix} |A_{11}|^{1/2} & 0 \\ B^*UP_{\overline{R(|A_{11}|^{1/2})}} & 0 \end{pmatrix} \in L(\mathcal{H})$, $F_0|P_{\mathcal{S}}A^*P_{\mathcal{T}}|^{1/2} = \begin{pmatrix} |A_{11}^*|^{1/2} & 0 \\ C^*U^*P_{\overline{R(|A_{11}^*|^{1/2})}} & 0 \end{pmatrix} \in L(\mathcal{K})$ and $E_0A^* = \begin{pmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & B^*UC \end{pmatrix} = (F_0A)^* \in L(\mathcal{H}, \mathcal{K})$, where we used that $U^*|A_{11}^*|^{1/2} = |A_{11}|^{1/2}U^*$. Therefore, A is $(\mathcal{S}, \mathcal{T})$ -weakly complementable. \square

Notice that every positive operator in $L(\mathcal{H})$ is \mathcal{S} -weakly complementable for every closed subspace \mathcal{S} of \mathcal{H} : in fact, if $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^\perp \end{matrix}$ is the matrix representation of A then $A_{11} \geq 0$ and $R(A_{12})$ is always contained in $R(A_{11}^{1/2})$.

Remark 2.16. Taking into account the proof of the above result we observe that if $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ is $(\mathcal{S}, \mathcal{T})$ -weakly complementable then

$$A = \begin{matrix} \mathcal{T} \\ \mathcal{T}^\perp \end{matrix} \begin{pmatrix} |A_{11}^*|U & |A_{11}^*|^{1/2}B \\ C^*|A_{11}|^{1/2} & A_{22} \end{pmatrix}, \quad (3)$$

where $A_{11}^* = U^*|A_{11}^*|$ is the polar decomposition of A_{11}^* , $B = (|A_{11}^*|^{1/2})^\dagger A_{12}$ and $C = (|A_{11}|^{1/2})^\dagger A_{21}^*$. Moreover,

$$E_0 = \begin{matrix} \mathcal{S} \\ \mathcal{S}^\perp \end{matrix} \begin{pmatrix} I & 0 \\ B^*U(|A_{11}|^{1/2})^\dagger & 0 \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^\perp \end{matrix} \quad (4)$$

and

$$F_0 = \begin{matrix} \mathcal{T} \\ \mathcal{T}^\perp \end{matrix} \begin{pmatrix} I & 0 \\ C^*U^*(|A_{11}^*|^{1/2})^\dagger & 0 \end{pmatrix} \begin{matrix} \mathcal{T} \\ \mathcal{T}^\perp \end{matrix}, \quad (5)$$

are two densely defined projections with $\mathcal{D}(E_0) = \mathcal{D}(|A_{11}|^{1/2})^\dagger + \mathcal{S}^\perp$ and $\mathcal{D}(F_0) = \mathcal{D}(|A_{11}^*|^{1/2})^\dagger + \mathcal{T}^\perp$, that verify the conditions of Definition 2.14. We remark that these projections are not closed, in general, but they are semiclosed operators in the sense given by Kaufman in [22]. He defined that an operator C is semiclosed if there exists a bounded operator B on \mathcal{H} , with range the domain of C , such that CB is bounded. In our case, if we consider $\Gamma := |A_{11}|^{1/2}P_{\mathcal{S}} + P_{N(|A_{11}|)} + P_{\mathcal{S}^\perp}$ then $\Gamma \in L(\mathcal{H})$, $R(\Gamma) = \mathcal{D}(E_0)$ and it is easy to check, using the formula for E_0 , that $E_0\Gamma \in L(\mathcal{H})$, i.e., E_0 is semiclosed. Similarly, it can be proved that F_0 is semiclosed.

Finally, we highlight that if A is $(\mathcal{S}, \mathcal{T})$ -complementable then E_0^* and F_0^* , are the projections defined by (1).

3 Complementability and general Wiener-Hopf operators

If $A \in L(\mathcal{H})$ and \mathcal{S} is closed subspace of \mathcal{H} then the operator $P_{\mathcal{S}}A|_{\mathcal{S}}$, has been called by Devinatz and Shinbrot [15] a general Wiener-Hopf operator. It must be pointed out that this operator emerges as a generalization of the classical integral operator $g(x) = \int_0^\infty K(x-y)f(y)dy$ studied by Wiener, Hopf and others. Here, we shall consider the following more general definition:

Definition 3.1. *Given two closed subspaces \mathcal{S}, \mathcal{T} of \mathcal{H}, \mathcal{K} respectively, the general Wiener-Hopf operator of $A \in L(\mathcal{H}, \mathcal{K})$, $T_{\mathcal{T}, \mathcal{S}}(A) : \mathcal{S} \rightarrow \mathcal{T}$ is defined as*

$$T_{\mathcal{T}, \mathcal{S}}(A) = P_{\mathcal{T}}A|_{\mathcal{S}}.$$

Using the matrix representation (2), it is clear that $T_{\mathcal{T}, \mathcal{S}}(A) = A_{11}$. There is an extensive literature concerning to the invertibility of the general Wiener-Hopf operator $P_{\mathcal{S}}A|_{\mathcal{S}}$ see [15], [21],[29], [31], among others. Our aim in this section is to apply the previous results about complementability and weakly complementability to the study of the generalized invertibility (or, invertibility) of $T_{\mathcal{T}, \mathcal{S}}(A)$ and to obtain a formula for $T_{\mathcal{T}, \mathcal{S}}(A)^\dagger$ (or, $T_{\mathcal{T}, \mathcal{S}}(A)^{-1}$) in terms of the projections associated to these notions. Recall that for $A \in L(\mathcal{H}, \mathcal{K})$, the Moore-Penrose inverse of A is the (non necessarily bounded) operator $A^\dagger : R(A) \oplus R(A)^\perp \rightarrow \mathcal{H}$, defined as $A^\dagger(Ax + y) = x$ where $y \in R(A)^\perp$ and $x \in N(A)^\perp$.

Let us begin with the following result:

Theorem 3.2. *If A is $(\mathcal{S}, \mathcal{T})$ -weakly complementable, then*

$$T_{\mathcal{T}, \mathcal{S}}(A)^\dagger = (AP_{\mathcal{S}})^\dagger F_0 P_{\overline{R(T_{\mathcal{T}, \mathcal{S}}(A))}}|_{\mathcal{D}(T_{\mathcal{T}, \mathcal{S}}(A)^\dagger)}, \quad (6)$$

where F_0 is the densely defined projection given by (5).

Proof. Assume that A is $(\mathcal{S}, \mathcal{T})$ -weakly complementable and let F_0 be the densely defined projection given by (5). Observe that, in this case, $T_{\mathcal{T}, \mathcal{S}}(A)^\dagger$ is not necessarily a bounded operator. Let us prove formula (6). For this, consider the matrix representations of A and F_0 given in Remark 2.16 and $T := P_{\mathcal{T}}AP_{\mathcal{S}} = \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix}$. Therefore, a simple matrix computation shows that $AP_{\mathcal{S}} = F_0AP_{\mathcal{S}} = F_0P_{\mathcal{T}}AP_{\mathcal{S}} = F_0T$, where F_0T is well defined since $R(A_{11}) \subseteq \mathcal{D}(|A_{11}^*|^{1/2})^\dagger$. From this and noting that $R(T^\dagger) = N(P_{\mathcal{T}}AP_{\mathcal{S}})^\perp \subseteq N(AP_{\mathcal{S}})^\perp$, it follows that $T^\dagger = P_{N(AP_{\mathcal{S}})^\perp}T^\dagger = (AP_{\mathcal{S}})^\dagger AP_{\mathcal{S}}T^\dagger = (AP_{\mathcal{S}})^\dagger F_0TT^\dagger = (AP_{\mathcal{S}})^\dagger F_0 P_{\overline{R(T)}}$. Finally, notice that $T^\dagger|_{\mathcal{T}} = T_{\mathcal{T}, \mathcal{S}}(A)^\dagger$, that $\overline{R(T)} = \overline{R(T_{\mathcal{T}, \mathcal{S}}(A))}$ and $\mathcal{D}(T^\dagger) \cap \mathcal{T} = \mathcal{D}(T_{\mathcal{T}, \mathcal{S}}(A)^\dagger)$. The result is proved. \square

Given $T \in L(\mathcal{H}, \mathcal{K})$, $T^\dagger \in L(\mathcal{K}, \mathcal{H})$ if and only if $R(T)$ is closed. Hence, in what follows we provide conditions in terms of complementability that guarantee that $R(T_{\mathcal{T}, \mathcal{S}}(A))$ is closed. For this purpose, we start with the next result.

Lemma 3.3. $N(T_{\mathcal{T},\mathcal{S}}(A)) = \mathcal{S} \cap N(A)$ if and only if $A(\mathcal{S}) \cap \mathcal{T}^\perp = \{0\}$.

Proof. The result follows noticing that $N(T_{\mathcal{T},\mathcal{S}}(A)) = \mathcal{S} \cap A^{-1}(\mathcal{T}^\perp)$ and applying the remark at the beginning of the proof of Corollary 2.7. \square

The following result follows by Theorem 2.1:

Proposition 3.4. *If $T_{\mathcal{T},\mathcal{S}}(A)$ has closed range then $A\mathcal{S} + \mathcal{T}^\perp$ is closed. If $A\mathcal{S}$ is closed, the converse holds.*

Corollary 3.5. *$T_{\mathcal{T},\mathcal{S}}(A)$ has closed range and is injective if and only if $A(\mathcal{S}) \dot{+} \mathcal{T}^\perp$ is closed and $\mathcal{S} \cap N(A) = \{0\}$.*

Proof. The nullspace of $T_{\mathcal{T},\mathcal{S}}(A)$ is $N(T_{\mathcal{T},\mathcal{S}}(A)) = \mathcal{S} \cap A^{-1}(\mathcal{T}^\perp)$.

Suppose that $T_{\mathcal{T},\mathcal{S}}(A)$ has closed range and is injective. Then, by Proposition 3.4, $A\mathcal{S} + \mathcal{T}^\perp$ is closed. Since $\mathcal{S} \cap N(A) \subseteq \mathcal{S} \cap A^{-1}(\mathcal{T}^\perp) = \{0\}$, then $\mathcal{S} \cap N(A) = \mathcal{S} \cap A^{-1}(\mathcal{T}^\perp) = \{0\}$. Also, by the proof of Corollary 2.7, this implies that $A\mathcal{S} \cap \mathcal{T}^\perp = \{0\}$ so the sum is direct.

Conversely, suppose that $A(\mathcal{S}) \dot{+} \mathcal{T}^\perp$ is closed and $\mathcal{S} \cap N(A) = \{0\}$. Then, by [32, Theorem 5.10], $A(\mathcal{S})$ is closed. Applying Proposition 3.4, the range of $T_{\mathcal{T},\mathcal{S}}(A)$ is closed. Since $A(\mathcal{S}) \cap \mathcal{T}^\perp = \{0\}$ then, by Lemma 3.3, $T_{\mathcal{T},\mathcal{S}}(A)$ is injective. \square

Theorem 3.6. *The following conditions are equivalent:*

1. A is $(\mathcal{S}, \mathcal{T})$ -complementable and $A(\mathcal{S}), A^*(\mathcal{T})$ are closed subspaces.
2. $T_{\mathcal{T},\mathcal{S}}(A)$ has closed range, $N(T_{\mathcal{T},\mathcal{S}}(A)) = \mathcal{S} \cap N(A)$ and $N(T_{\mathcal{S},\mathcal{T}}(A^*)) = \mathcal{T} \cap N(A^*)$.

Proof. Assume that A is $(\mathcal{S}, \mathcal{T})$ -complementable and $A(\mathcal{S}), A^*(\mathcal{T})$ are closed subspaces. Thus, $\mathcal{H} = \mathcal{S} + (A^*\mathcal{T})^\perp$ and $\mathcal{K} = \mathcal{T} + (A\mathcal{S})^\perp$ or, equivalently, $\mathcal{S}^\perp \dot{+} A^*\mathcal{T}$ and $\mathcal{T}^\perp \dot{+} A\mathcal{S}$ are closed subspaces. Hence, by Proposition 3.4 and Lemma 3.3 we obtain item 2.

Conversely, suppose that item 2 holds. Then as $T_{\mathcal{T},\mathcal{S}}(A)$ has closed range and $N(T_{\mathcal{T},\mathcal{S}}(A)) = \mathcal{S} \cap N(A)$ we get that $A(\mathcal{S}) \dot{+} \mathcal{T}^\perp$ is closed. Then $A\mathcal{S}$ is closed and $\mathcal{H} = \mathcal{T} + (A\mathcal{S})^\perp$. Similarly, but considering that $T_{\mathcal{S},\mathcal{T}}(A^*)$ has also closed range and $N(T_{\mathcal{S},\mathcal{T}}(A^*)) = \mathcal{T} \cap N(A^*)$, we get that $A^*\mathcal{T}$ is closed and $\mathcal{K} = \mathcal{S} + (A^*\mathcal{T})^\perp$. Therefore, A is $(\mathcal{S}, \mathcal{T})$ -complementable. \square

We devote the remain of this section to study the relationship between the invertibility of $T_{\mathcal{T},\mathcal{S}}(A)$ and the notion of complementability. For that, in the following result we collect some equivalent conditions to the invertibility of $T_{\mathcal{T},\mathcal{S}}(A)$.

Proposition 3.7. *The next conditions are equivalent:*

1. $T_{\mathcal{T},\mathcal{S}}(A)$ is invertible
2. $\mathcal{H} = A(\mathcal{S}) \dot{+} \mathcal{T}^\perp$ and $N(A) \cap \mathcal{S} = \{0\}$.
3. $\mathcal{H} = (A\mathcal{S})^\perp \dot{+} \mathcal{T}$, $N(A) \cap \mathcal{S} = \{0\}$ and $A(\mathcal{S})$ is closed.

Proof. $1 \Rightarrow 2$. Clearly, if $T_{\mathcal{T},\mathcal{S}}(A)$ is invertible then $R(P_{\mathcal{T}}AP_{\mathcal{S}}) = \mathcal{T}$. Thus, $\mathcal{H} = P_{\mathcal{T}}^{-1}(R(P_{\mathcal{T}}AP_{\mathcal{S}})) = P_{\mathcal{T}}^{-1}(P_{\mathcal{T}}(A(\mathcal{S}))) = A(\mathcal{S}) + \mathcal{T}^\perp$. Then, the result follows by Corollary 3.5.

$2 \Rightarrow 1$. If $\mathcal{H} = A(\mathcal{S}) \dot{+} \mathcal{T}^\perp$ then $\mathcal{T} = P_{\mathcal{T}}(\mathcal{H}) = P_{\mathcal{T}}(A(\mathcal{S})) = R(P_{\mathcal{T}}AP_{\mathcal{S}})$, i.e., $T_{\mathcal{T},\mathcal{S}}(A)$ is surjective. The proof is complete by considering Corollary 3.5.

The equivalence $3 \Leftrightarrow 1$ follows from the fact that $\mathcal{H} = A\mathcal{S} \dot{+} \mathcal{T}^\perp$ if and only if $\mathcal{H} = (A\mathcal{S})^\perp \dot{+} \mathcal{T}$ and $A\mathcal{S}$ is closed, see [14, Theorem 15]. \square

We refer the reader to [15, Theorem 1 and Lemma 3] and [21, Theorems 1 and 2] for some other equivalent conditions to the invertibility of $T_{\mathcal{T},\mathcal{S}}(A)$.

Corollary 3.8. $T_{\mathcal{T},\mathcal{S}}(A)$ is invertible if and only if $T_{\mathcal{S},\mathcal{T}}(A^*)$ is invertible.

Proof. By Proposition 3.7, $T_{\mathcal{T},\mathcal{S}}(A)$ is invertible if and only if $R(P_{\mathcal{T}}AP_{\mathcal{S}}) = R(P_{\mathcal{T}})$ and $N(P_{\mathcal{T}}AP_{\mathcal{S}}) = N(P_{\mathcal{S}})$ or, equivalently, $N(P_{\mathcal{S}}A^*P_{\mathcal{T}}) = N(P_{\mathcal{T}})$ and $R(P_{\mathcal{S}}A^*P_{\mathcal{T}}) = R(P_{\mathcal{S}})$, i.e., applying again Proposition 3.7, $T_{\mathcal{S},\mathcal{T}}(A^*)$ is invertible. \square

Corollary 3.9. If $T_{\mathcal{T},\mathcal{S}}(A)$ is invertible then A is $(\mathcal{S}, \mathcal{T})$ -complementable.

Proof. Assume that $T_{\mathcal{T},\mathcal{S}}(A)$ is invertible. Then, by Proposition 3.7, $N(A) \cap \mathcal{S} = N(T_{\mathcal{T},\mathcal{S}}(A)) = \{0\}$ and $N(T_{\mathcal{S},\mathcal{T}}(A^*)) = \mathcal{T} \cap N(A^*) = \{0\}$. Then, by Theorem 3.6, A is $(\mathcal{S}, \mathcal{T})$ -complementable. \square

Theorem 3.10. The next conditions are equivalent:

1. $T_{\mathcal{T},\mathcal{S}}(A)$ is invertible.
2. $\mathcal{P}(A, \mathcal{S}, \mathcal{T})$ has a unique element and $A(\mathcal{S})$ is closed.

Moreover, if the above conditions hold then

$$T_{\mathcal{T},\mathcal{S}}(A)^{-1} = (AP)^{\dagger}Q_{AS//\mathcal{T}^{\perp}|\mathcal{T}}. \quad (7)$$

Proof. $1 \Rightarrow 2$. Suppose that $T_{\mathcal{T},\mathcal{S}}(A)$ is invertible. By Corollary 3.9, A is $(\mathcal{S}, \mathcal{T})$ -complementable or, equivalently, $\mathcal{P}(A, \mathcal{S}, \mathcal{T})$ is not empty. Moreover, by Proposition 3.7, $A\mathcal{S}$ is closed and $(A^*)^{-1}(\mathcal{S}^{\perp}) \cap \mathcal{T} = \{0\}$. Applying again Proposition 3.7 to $T_{\mathcal{S},\mathcal{T}}(A^*)$ (which is also invertible because of Corollary 3.8) we obtain that $A^{-1}(\mathcal{T}^{\perp}) \cap \mathcal{S} = \{0\}$. Therefore, by Corollary 2.12, $\mathcal{P}(A, \mathcal{S}, \mathcal{T})$ has a unique element.

$2 \Rightarrow 1$. Suppose that $\mathcal{P}(A, \mathcal{S}, \mathcal{T})$ has a unique element and $A\mathcal{S}$ is closed. Then, by Corollary 2.12, $\mathcal{H} = (A^*)^{-1}(\mathcal{S}^{\perp}) + \mathcal{T}$ and $\mathcal{S} \cap A^{-1}(\mathcal{T}^{\perp}) = \{0\}$. Now, $N(A) \cap \mathcal{S} \subseteq \mathcal{S} \cap A^{-1}(\mathcal{T}^{\perp})$, i.e., $N(A) \cap \mathcal{S} = \{0\}$. Therefore, by Proposition 3.7, $T_{\mathcal{T},\mathcal{S}}(A)$ is invertible. Finally, formula (7) follows by Theorem 3.2, noticing that if $\mathcal{P}(A, \mathcal{S}, \mathcal{T}) = \{(E, F)\}$ then $F = Q_{\mathcal{T} // (A\mathcal{S})^{\perp}}$ or, equivalently, $Q_{AS//\mathcal{T}^{\perp}} = F^*$. \square

4 Bilateral shorted operator

As mentioned in the introduction, the bilateral shorted operator defined in [4] extends to Hilbert spaces the matricial analogue introduced by Mitra and Puri [26] and Carlson and Haynsworth [9]. Next, we recall the definition given in [4], where we have interchanged \mathcal{S}, \mathcal{T} and $\mathcal{S}^{\perp}, \mathcal{T}^{\perp}$ to maintain notational consistency with the previous concepts. In what follows, we consider the matrix form of an operator $A \in L(\mathcal{H}, \mathcal{K})$ given in (2).

Definition 4.1. If A is $(\mathcal{S}, \mathcal{T})$ -weakly complementable, the bilateral shorted operator of A to the subspaces \mathcal{S}, \mathcal{T} is the operator

$$A_{/\mathcal{S},\mathcal{T}} := \begin{pmatrix} 0 & 0 \\ 0 & A_{22} - C^*U^*B \end{pmatrix} \quad (8)$$

where $B = (|A_{11}^*|^{1/2})^{\dagger}A_{12}$, $C = (|A_{11}|^{1/2})^{\dagger}A_{21}^*$, and $A_{11}^* = U^*|A_{11}^*|$ is the polar decomposition of A_{11}^* .

The aim of this section is to provide new characterizations of $A_{/\mathcal{S},\mathcal{T}}$ by means of the densely defined projections given by the weakly-complementability of A . In this direction, our first result is a formula for $A_{/\mathcal{S},\mathcal{T}}$ in terms of the semiclosed projections E_0, F_0 given by (4) and (5):

Theorem 4.2. *If A is $(\mathcal{S}, \mathcal{T})$ -weakly complementable then,*

$$A_{/\mathcal{S},\mathcal{T}} = (I - F_0)A \text{ and } A_{/\mathcal{T},\mathcal{S}}^* = (I - E_0)A^*,$$

where E_0, F_0 are the densely defined projections given by (4) and (5), respectively.

Proof. Let A be $(\mathcal{S}, \mathcal{T})$ -weakly complementable. Then, by Remark 2.16,

$$A = \begin{array}{c} \mathcal{T} \\ \mathcal{T}^\perp \end{array} \left(\begin{array}{cc} |A_{11}^*|U & |A_{11}^*|^{1/2}B \\ C^*|A_{11}|^{1/2} & A_{22} \end{array} \right) \begin{array}{c} \mathcal{S} \\ \mathcal{S}^\perp \end{array}, \quad (9)$$

where $A_{11}^* = U^*|A_{11}^*|$ is the polar decomposition of A_{11}^* , $B = (|A_{11}^*|^{1/2})^\dagger A_{12}$, and $C = (|A_{11}|^{1/2})^\dagger A_{21}^*$.

Then, a simple computation shows that $(I - F_0)A = \begin{pmatrix} 0 & 0 \\ 0 & A_{22} - C^*U^*B \end{pmatrix} = A_{/\mathcal{S},\mathcal{T}}$ and $(I - E_0)A^* =$

$$\begin{pmatrix} 0 & 0 \\ 0 & A_{22}^* - B^*UC \end{pmatrix} = A_{/\mathcal{T},\mathcal{S}}^*, \text{ where } E_0 \text{ and } F_0 \text{ are given by (4) and (5), respectively.}$$

□

For a positive operator $A \in L(\mathcal{H})$, Krein [23] proved that the shorted operator $A_{/\mathcal{S}}$ turns out to be the maximum of all positive operators which are smaller than A with respect to the Lowner order (i.e., $A \leq B$ if $\langle Ax, x \rangle \leq \langle Bx, x \rangle \forall x \in \mathcal{H}$) and which have range lying in the fixed subspace \mathcal{S}^\perp . See also Anderson and Trapp [2], Pekarev-Šmuljan [28] and Arlinskii and E. Tsekanovskii [5]. For $A \in L(\mathcal{H})$ non positive, Mitra [25] proved (for matrices in $\mathbb{C}^{m \times n}$) that a similar result can be obtained if the minus order is considered. More precisely, he proved that

$$A_{/\mathcal{S},\mathcal{T}} = \max_{\leq} \{C \in \mathbb{C}^{m \times n} : C \leq A, R(C) \subseteq \mathcal{T}^\perp \text{ and } R(C^*) \subseteq \mathcal{S}^\perp\},$$

where $A \leq B$ if $R(A) \cap R(B - A) = \{0\}$ and $R(A^*) \cap R(A^* - B^*) = \{0\}$. Later, in order to extend this result for bounded operators on infinite dimensional Hilbert spaces, Antezana et al. [4] introduced the next generalization of the minus order: given $A, B \in L(\mathcal{H}, \mathcal{K})$, $A \leq B$ if $\overline{R(A)} \dot{+} \overline{R(B - A)}$ and $\overline{R(A^*)} \dot{+} \overline{R(B^* - A^*)}$ are closed subspaces, or equivalently, there are bounded (oblique) projections P and Q such that $A = PB$ and $A^* = QB^*$, see [4, Proposition 4.13]. See also [16], where it was proved that $A \leq B$ if and only if $R(B) = R(A) \dot{+} R(B - A)$ and $R(B^*) = R(A^*) \dot{+} R(B^* - A^*)$. If A is $(\mathcal{S}, \mathcal{T})$ -complementable then

$$A_{/\mathcal{S},\mathcal{T}} := \max_{\leq} \{C \in L(\mathcal{H}, \mathcal{K}) : C \leq A, R(C) \subseteq \mathcal{T}^\perp, R(C^*) \subseteq \mathcal{S}^\perp\}, \quad (10)$$

see [4, Theorem 4.15]. This result extends Mitra's result under the hypothesis of complementability.

Our goal now is to extend, under the condition of weakly-complementability, the variational characterization (10) of the bilateral shorted operator. For this, we define a partial order in $L(\mathcal{H}, \mathcal{K})$ that extends the minus order:

Definition 4.3. *Given two operators $A, B \in L(\mathcal{H}, \mathcal{K})$, we shall write $A \prec B$ if there exist two densely defined projections Q, P with closed ranges such that $A = QB$ and $A^* = PB^*$.*

Lemma 4.4. *Let $A, B \in L(\mathcal{H}, \mathcal{K})$. The next conditions are equivalent:*

1. $A \prec B$.

2. $\overline{R(A)} \cap R(B - A) = \{0\}$ and $\overline{R(A^*)} \cap R(B^* - A^*) = \{0\}$.

Proof. Assume that $A \prec B$, and let Q, P be two densely defined projections with closed ranges such that $A = \overline{QB}$ and $A^* = \overline{PB^*}$. Without loss of generality, we can assume that $R(Q) = \overline{R(A)}$ and $R(P) = \overline{R(A^*)}$. Hence, $Q(B - A) = 0$, i.e., $R(B - A) \subseteq N(Q)$ and so $\overline{R(A)} \cap R(B - A) = \{0\}$. Similarly, $\overline{R(A^*)} \cap R(B^* - A^*) = \{0\}$. Conversely, suppose that $\overline{R(A)} \cap R(B - A) = \{0\}$ and $\overline{R(A^*)} \cap R(B^* - A^*) = \{0\}$. Hence, $Q_{A,B} := Q_{\overline{R(A)}/R(B-A) + (\overline{R(A)} + R(B-A))^\perp}$ and $Q_{A^*,B^*} = Q_{\overline{R(A^*)}/R(B^*-A^*) + (\overline{R(A^*)} + R(B^*-A^*))^\perp}$ are two densely defined projections with closed ranges and a simple computation shows that $A = Q_{A,B}B$ and $A^* = Q_{A^*,B^*}B^*$. Therefore, $A \prec B$. \square

From now on, given $A, B \in L(\mathcal{H}, \mathcal{K})$ we denote by

$$\prec_{A,B} := \{Q : Q \text{ is a densely defined projection with closed range s.t. } A = QB\}.$$

Thus, $A \prec B$ if and only if $\prec_{A,B} \neq \emptyset$ and $\prec_{A^*,B^*} \neq \emptyset$. Moreover, if $A, B \in L(\mathcal{H}, \mathcal{K})$ verify that $\overline{R(A)} \cap R(B - A) = \{0\}$ then we shall denote by $Q_{A,B} := Q_{\overline{R(A)}/R(B-A) + (\overline{R(A)} + R(B-A))^\perp}$. Therefore, taking into account the proof of Lemma 4.4, if $A \prec B$ then $Q_{A,B} \in \prec_{A,B}$ and $Q_{A^*,B^*} \in \prec_{A^*,B^*}$.

Let us prove that \prec defines an order in $L(\mathcal{H}, \mathcal{K})$:

Lemma 4.5. *The relation \prec is a partial order in $L(\mathcal{H}, \mathcal{K})$.*

Proof. In fact, the relation \prec is:

1. Reflexive: for all $A \in L(\mathcal{H}, \mathcal{K})$, $A = P_{\overline{R(A)}}A$, and $A^* = P_{\overline{R(A^*)}}A^*$ i.e., $A \prec A$.

2. Antisymmetric: If $A \prec B$ and $B \prec A$ then $A^* = Q_{A^*,B^*}B^*$, so $N(B^*) \subseteq N(A^*)$, and $B^* = Q_{B^*,A^*}A^*$, so $N(A^*) \subseteq N(B^*)$. Thus, $N(A^*) = N(B^*)$ or, equivalently, $\overline{R(A)} = \overline{R(B)}$. Hence, $A = Q_{A,B}B = B$.

3. Transitive: Let $A \prec B$ and $B \prec C$. Then, $A = Q_{AB}B = Q_{A,B}Q_{B,C}C$. Let us prove that $Q_{A,B}Q_{B,C}$ is a densely defined projection with closed range. In fact, $N(Q_{B,C}) \subseteq \mathcal{D}(Q_{A,B}Q_{B,C})$ and $R(B) \subseteq \mathcal{D}(Q_{A,B}Q_{B,C})$. Thus $N(Q_{B,C}) + R(B) \subseteq \mathcal{D}(Q_{A,B}Q_{B,C})$, and so $Q_{A,B}Q_{B,C}$ is densely defined. To see that $Q_{A,B}Q_{B,C}$ is a projection, $R(Q_{A,B}Q_{B,C}) \subseteq R(Q_{A,B}) = \overline{R(A)} \subseteq \overline{R(B)} \subseteq \mathcal{D}(Q_{B,C})$. Hence, $Q_{B,C}Q_{A,B}Q_{B,C} = Q_{A,B}Q_{B,C}$ in $\mathcal{D}(Q_{A,B}Q_{B,C})$ and so $Q_{A,B}Q_{B,C} = (Q_{A,B}Q_{B,C})^2$. Finally, $R(Q_{A,B}Q_{B,C}) \subseteq \overline{R(A)}$. On the other hand, as $\overline{R(A)} \subseteq \overline{R(B)}$, then $Q_{B,C}Q_{A,B} = Q_{A,B}$. Thus, $Q_{A,B}Q_{B,C}Q_{A,B} = Q_{A,B}$ and so $\overline{R(A)} = R(Q_{A,B}) \subseteq R(Q_{A,B}Q_{B,C})$. Therefore, $R(Q_{A,B}Q_{B,C}) = \overline{R(A)}$ is closed.

Similarly, $A^* = Q_{A^*,B^*}B^* = Q_{A^*,B^*}Q_{B^*,C^*}C^*$ and $Q_{A^*,B^*}Q_{B^*,C^*}$ is a densely defined projection with range $\overline{R(A^*)}$. Then, $A \prec C$. \square

Clearly, the partial order \prec extends the minus order, in the sense that if $A \leq B$ then $A \prec B$.

Now we are in position to prove the last result of this paper. If A is $(\mathcal{S}, \mathcal{T})$ -weakly complementable, consider the factorization given in (3) and let $\Gamma_1 := \begin{pmatrix} |A_{11}^*|^{1/2} & 0 \\ 0 & I \end{pmatrix}$ and $\Gamma_2 := \begin{pmatrix} |A_{11}|^{1/2} & 0 \\ 0 & I \end{pmatrix}$. Notice that $A = \Gamma_1 A_0 \Gamma_2$ where $A_0 = \begin{pmatrix} U & B \\ C^* & A_{22} \end{pmatrix}$. Hence, define $\mathcal{M}^\prec(A, \mathcal{S}, \mathcal{T}) = \{C \in L(\mathcal{H}, \mathcal{K}) : C \prec A, R(C) \subseteq \mathcal{T}^\perp, R(C^*) \subseteq \mathcal{S}^\perp, \exists Q \in \prec_{C,A}, \exists P \in \prec_{C^*,A^*} \text{ s.t. } Q\Gamma_1 \in L(\mathcal{K}), P\Gamma_2 \in L(\mathcal{H})\}$.

Theorem 4.6. *Let A be $(\mathcal{S}, \mathcal{T})$ -weakly complementable. Then,*

$$A_{/\mathcal{S}, \mathcal{T}} = \max_{\prec} \mathcal{M}^{\prec}(A, \mathcal{S}, \mathcal{T}).$$

Proof. Let us start by showing that $A_{/\mathcal{S}, \mathcal{T}} \in \mathcal{M}^{\prec}(A, \mathcal{S}, \mathcal{T})$. By Theorem 4.2, $A_{/\mathcal{S}, \mathcal{T}} = (I - F_0)A$ and $(A_{/\mathcal{S}, \mathcal{T}})^* = (I - E_0)A^*$, where E_0, F_0 are given by (4) and (5), respectively. Thus, since $I - E_0$ and $I - F_0$ are two densely defined projections with closed ranges ($R(I - F_0) = N(F_0) = \mathcal{T}^{\perp}$ and $R(I - E_0) = N(E_0) = \mathcal{S}^{\perp}$), we get that $A_{/\mathcal{S}, \mathcal{T}} \prec A$, $R(A_{/\mathcal{S}, \mathcal{T}}) \subseteq \mathcal{T}^{\perp}$ and $R(A_{/\mathcal{S}, \mathcal{T}}^*) \subseteq \mathcal{S}^{\perp}$. Moreover, $(I - F_0)\Gamma_1 = (I - F_0)(|P_{\mathcal{S}}A^*P_{\mathcal{T}}|^{1/2} + P_{\mathcal{T}^{\perp}}) \in L(\mathcal{K})$ and $(I - E_0)\Gamma_2 = (I - E_0)(|P_{\mathcal{T}}AP_{\mathcal{S}}|^{1/2} + P_{\mathcal{S}^{\perp}}) \in L(\mathcal{H})$ because E_0 and F_0 verify the conditions of Definition 2.14. Then, $A_{/\mathcal{S}, \mathcal{T}} \in \mathcal{M}^{\prec}(A, \mathcal{S}, \mathcal{T})$.

Now, consider $C \in \mathcal{M}^{\prec}(A, \mathcal{S}, \mathcal{T})$. Then, $C = QA = Q\Gamma_1A_0\Gamma_2$. Therefore, as $Q\Gamma_1 \in L(\mathcal{K})$, we have that $C^* = \Gamma_2A_0^*(Q\Gamma_1)^* = (I - E_0)\Gamma_2A_0^*(Q\Gamma_1)^*$, because $R(C^*) \subseteq \mathcal{S}^{\perp} = R(I - E_0)$. Moreover, $(I - E_0)\Gamma_2 \in L(\mathcal{H})$. Hence, $C = Q\Gamma_1A_0((I - E_0)\Gamma_2)^*$. We claim that $A_{/\mathcal{S}, \mathcal{T}} = \Gamma_1A_0((I - E_0)\Gamma_2)^*$. In fact, $A_{/\mathcal{S}, \mathcal{T}} = ((I - E_0)A^*)^* = ((I - E_0)\Gamma_2A_0^*\Gamma_1)^* = \Gamma_1A_0((I - E_0)\Gamma_2)^*$, as desired. Then, $C = QA_{/\mathcal{S}, \mathcal{T}}$.

On the other hand, $C^* = PA^* = P\Gamma_2A_0^*\Gamma_1$. Hence, as $P\Gamma_2 \in L(\mathcal{H})$, we get that $C = \Gamma_1A_0(P\Gamma_2)^* = (I - F_0)\Gamma_1A_0(P\Gamma_2)^*$, because $R(C) \subseteq \mathcal{T}^{\perp} = R(I - F_0)$. Then, $C^* = P\Gamma_2A_0^*((I - F_0)\Gamma_1)^*$. Now, $(A_{/\mathcal{S}, \mathcal{T}})^* = \Gamma_2A_0^*((I - F_0)\Gamma_1)^*$ because $A_{/\mathcal{S}, \mathcal{T}} = (I - F_0)A = (I - F_0)\Gamma_1A_0\Gamma_2$ and $(I - F_0)\Gamma_1 \in L(\mathcal{K})$. Summarizing, $C = QA_{/\mathcal{S}, \mathcal{T}}$ and $C^* = P(A_{/\mathcal{S}, \mathcal{T}})^*$, i.e., $C \prec A_{/\mathcal{S}, \mathcal{T}}$. The proof is complete. \square

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