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ROBUST SOLUTIONS OF MULTIOBJECTIVE LINEAR SEMI-INFINITE PROGRAMS UNDER CONSTRAINT DATA UNCERTAINTY*

M. A. GOBERNA†, V. JEYAKUMAR‡, G. LI‡, AND J. VICENTE-PÉREZ‡

Abstract. The multiobjective optimization model studied in this paper deals with simultaneous minimization of finitely many linear functions subject to an arbitrary number of uncertain linear constraints. We first provide a radius of robust feasibility guaranteeing the feasibility of the robust counterpart under affine data parametrization. We then establish dual characterizations of robust solutions of our model that are immunized against data uncertainty by way of characterizing corresponding solutions of robust counterpart of the model. Consequently, we present robust duality theorems relating the value of the robust model with the corresponding value of its dual problem.

Key words. linear semi-infinite programming, linear multiobjective optimization, robust optimization, duality

AMS subject classifications. 90C29, 90C31, 90C34

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1. Introduction.

Consider the deterministic multiobjective linear semi-infinite program of the form

$$(1.1)\qquad (P)\qquad \begin{array}{ll} \text{V-}\min\limits_{x\in\mathbb{R}^n} & \left(c_1^\top x,\ldots,c_m^\top x\right) \\ \text{such that (s.t.)} & a_t^\top x \geq b_t \quad \forall t\in T, \end{array}$$

where V-min stands for *vector minimization*, $c_i \in \mathbb{R}^n$ for all $i \in I := \{1,\ldots,m\}$, the superscript $^\top$ denotes transpose, $(a_t, b_t) \in \mathbb{R}^n \times \mathbb{R}$ for all $t \in T$, and the *index set* T is arbitrary. When T is finite, (P) becomes an ordinary multiobjective linear optimization problem, whereas, when T is infinite, (P) is a multiobjective linear semi-infinite optimization problem. Some potential applications of these models have been discussed in [9]. In particular, whenever $m = 1$, (P) becomes a single-objective linear semi-infinite program which has been extensively studied in the literature (see [8, 13] and other references therein).

When dealing with real-world optimization problems, the input data associated with a multiobjective linear semi-infinite program are often noisy or uncertain due to *prediction or measurement errors*. For example, a multiobjective optimization problem arising in industry or commerce might involve various costs, financial returns, and future demands that might be unknown at the time of the decision. They have to be predicted and are replaced with their forecasts. They often result in prediction errors. Similarly, some of the data, such as the contents associated with raw materials, might be hard to measure exactly. These input data are subject to measurement errors.

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†Department of Statistics and Operations Research, University of Alicante, 03080 Alicante, Spain (mgoberna@ua.es).

‡Department of Applied Mathematics, University of New South Wales, Sydney 2052, Australia (v.jeyakumar@unsw.edu.au, g.li@unsw.edu.au, jose.vicente@ua.es).

In the single-objective optimization case of constraint data uncertainty, Ben-Tal, El Ghaoui, and Nemirovski [2] provided a highly successful computationally tractable treatment of the robust optimization approach for linear as well as convex optimization problems under data uncertainty. Recently, single-objective linear semi-infinite programming problems under constraint data uncertainty were studied in [10].

In the same vein as in [10], the multiobjective linear problem (P) in the face of *data uncertainty* in the constraints can be captured by a parameterized multiobjective linear problem of the form

$$(1.2)\qquad (P^u)\quad \begin{array}{ll} \operatorname*{V-min}_{x\in\mathbb{R}^n} & \left(c_1^\top x,\ldots,c_m^\top x\right) \\ \text{s.t.} & v_t^\top x \geq w_t \quad \forall t\in T, \end{array}$$

where c_i are given deterministic vectors in $\mathbb{R}^n$ for all $i\in I$, and $u=(v,w):T\to\mathbb{R}^n\times\mathbb{R}$ represents a *selection* of a given *uncertain set-valued mapping* $\mathcal{U}:T\rightrightarrows\mathbb{R}^{n+1}$ (in short, $u\in\mathcal{U}$). Let $\mathcal{U}_t:=\mathcal{U}(t)\subset\mathbb{R}^{n+1}$ for all $t\in T$. Hence, in this robust model, the uncertainty set is the *graph* of $\mathcal{U}$, that is, $\operatorname{gph}\mathcal{U}=\{(t,u_t): u_t\in\mathcal{U}_t, t\in T\}$.

A robust decision maker facing uncertainty in the constraints intends to guarantee the feasibility of her/his decisions, so that the *robust counterpart* of the parametric problem $(P^u)_{u\in\mathcal{U}}$ is the deterministic problem

$$(1.3)\qquad (RP)\quad \begin{array}{ll} \operatorname*{V-min}_{x\in\mathbb{R}^n} & \left(c_1^\top x,\ldots,c_m^\top x\right) \\ \text{s.t.} & v_t^\top x \geq w_t \quad \forall (v_t,w_t)\in\mathcal{U}_t,\ t\in T, \end{array}$$

where the uncertain constraints are enforced for every possible value of the data within the prescribed uncertainty set $\operatorname{gph}\mathcal{U}$. Notice that (RP) is an ordinary multiobjective linear problem whenever $\operatorname{gph}\mathcal{U}$ is finite (unlikely in practice). Otherwise, it is a multiobjective linear semi-infinite optimization problem.

It is worth noting that if the uncertainty also occurs in the objective functions of problem (P), then its corresponding robust counterpart can be rewritten in the form of (RP). For instance, assume that, for each $i\in I$, the vector c_i is an uncertain parameter belonging to the uncertainty set $\mathcal{V}_i$. Then, the *robust counterpart* of the associated parametric problem is given by

$$\begin{array}{ll} \operatorname*{V-min}_{x\in\mathbb{R}^n} & \left(\sup_{c_1\in\mathcal{V}_1} c_1^\top x,\ldots,\sup_{c_m\in\mathcal{V}_m} c_m^\top x\right) \\ \text{s.t.} & v_t^\top x \geq w_t \quad \forall (v_t,w_t)\in\mathcal{U}_t,\ t\in T, \end{array}$$

which is equivalent to

$$(1.4)\qquad \begin{array}{ll} \operatorname*{V-min}_{x\in\mathbb{R}^n, z\in\mathbb{R}^m} & \left(e_1^\top z,\ldots,e_m^\top z\right) \\ \text{s.t.} & e_i^\top z - c_i^\top x \geq 0 \quad \forall c_i\in\mathcal{V}_i,\ i\in I, \\ & v_t^\top x \geq w_t \quad \forall (v_t,w_t)\in\mathcal{U}_t,\ t\in T, \end{array}$$

where $\{e_1,\ldots,e_m\}$ is the canonical basis of $\mathbb{R}^m$.

Following the work of scalar robust optimization (see [1, 2, 14]), some of the key questions of multiobjective optimization under data uncertainty include the following:

I. (*Guaranteeing robust feasibility*) How can one guarantee feasibility for all realizations of an uncertain scenario?

II. (*Identifying robust solutions*) How can one characterize a robust solution that is immunized against data uncertainty?

III. (*Developing robust duality*) How can one relate worst-case (robust counterpart) value with the best-case (optimistic dual) value?

In this paper, we provide some answers to the above questions for the multiobjective linear semi-infinite programming problem with uncertain constraints within the robust optimization framework. In particular, we first establish a radius of robust feasibility guaranteeing the feasibility of the robust counterpart under affine data parametrization. Then, we provide dual characterizations of robust solutions of our uncertain model by way of characterizing corresponding solutions of robust counterpart of the uncertain model. Finally, we present robust duality theorems for our uncertain multiobjective problem.

2. Radius of robust feasibility. In this section, we discuss the feasibility of the robust counterpart of our uncertain multiobjective model under affine data perturbations.

We begin by introducing some notation and preliminary definitions. Given a subset E of a linear space (equipped with a topology not necessarily compatible with the linear structure), $\operatorname{conv} E$, $\operatorname{cone} E$, $\operatorname{int} E$, $\operatorname{cl} E$, and $\operatorname{bd} E$ denote the convex hull, the convex conical hull, the interior, the closure, and the boundary of E, respectively. By 0_n, $\|\cdot\|$, $\mathbb{B}_n$, $\mathbb{R}^n_+$, and $\mathbb{R}^n_{++}$, we denote the zero vector, the Euclidean norm, the Euclidean closed unit ball, the nonnegative orthant, and the positive orthant in $\mathbb{R}^n$, respectively. We also denote by d the Euclidean distance. For a convex cone $K \subset \mathbb{R}^n$, its *positive polar cone* is defined as $K^+ := \{y \in \mathbb{R}^n : x^\top y \geq 0 \, \forall x \in K\}$. Let $\mathbb{R}^{(T)}$ be the linear space of mappings $\mu \in \mathbb{R}^T$ such that $\{t \in T : \mu_t \neq 0\}$ is finite and let us denote by $\mathbb{R}^{(T)}_+$ the positive cone in $\mathbb{R}^{(T)}$. Finally, we recall a useful characterization of feasibility of linear semi-infinite systems which can be found in [13, Theorem 4.4].

LEMMA 2.1. *Let M be an index set and let $(v_t, w_t) \in \mathbb{R}^n \times \mathbb{R}$ for all $t \in M$. Then, $\{x \in \mathbb{R}^n : v_t^\top x \geq w_t, t \in M\} \neq \emptyset$ if and only if $(0_n, 1) \notin \operatorname{cl} \operatorname{cone}\{(v_t, w_t) : t \in M\}$.*

Next, we first discuss the feasibility of the robust counterpart of the uncertain multiobjective model under affine data perturbations under the norm data uncertainty case, where the uncertainty is described as a ball. In other words, let $\alpha \geq 0$ and study the feasibility of the problem

$$(RP_\alpha) \quad \operatorname*{V-min}_{x \in \mathbb{R}^n} \quad \left(c_1^\top x, \ldots, c_m^\top x\right)$$
$$\text{s.t.} \quad v_t^\top x \geq w_t \quad \forall (v_t, w_t) \in (\bar{v}_t, \bar{w}_t) + \alpha \mathbb{B}_{n+1},\, t \in T,$$

where the feasible set $\{x \in \mathbb{R}^n : \bar{v}_t^\top x \geq \bar{w}_t, t \in T\}$ of the unperturbed problem (RP_0) is nonempty. The general case where the uncertainty set is not necessarily a ball will be treated later on.

Let $\mathcal{U}_t := (\bar{v}_t, \bar{w}_t) + \alpha \mathbb{B}_{n+1}$ be the *norm data uncertainty set*. Let $\mathcal{U} : T \rightrightarrows \mathbb{R}^{n+1}$ be defined as $\mathcal{U}(t) := \mathcal{U}_t$ for all $t \in T$. The *radius of feasibility* of the parameterized robust counterpart problem (RP_α) associated with $\mathcal{U}$ is defined to be

$$\rho(\mathcal{U}) := \sup\{\alpha \in \mathbb{R}_+ : \text{ the feasible set of } (RP_\alpha) \text{ is nonempty}\}. \tag{2.1}$$

We observe that $\rho(\mathcal{U})$ is a nonnegative real number as $\{x \in \mathbb{R}^n : \bar{v}_t^\top x \geq \bar{w}_t, t \in T\} \neq \emptyset$. To see that $\rho(\mathcal{U}) < +\infty$, we note that for a given $t \in T$, $(0_n, 1) \in (\bar{v}_t, \bar{w}_t) + \alpha \mathbb{B}_{n+1}$ for a positive large enough α, in which case the corresponding problem (RP_α) is not feasible by Lemma 2.1. This shows that $\rho(\mathcal{U}) < +\infty$.

Moreover, the supremum in the definition of $\rho(\mathcal{U})$ (see (2.1)) may not always be attained, as illustrated in the following simple example, where $\{\mathcal{U}_t, t \in T\}$ is a finite family of closed balls.

Example 2.2. Let $T = \{1, 2\}$, $(\bar{v}_1, \bar{w}_1) = (1, 1, 0)$, $(\bar{v}_2, \bar{w}_2) = (-1, 1, 0)$, and $\alpha = 1$. Let $\mathcal{U}_1 = (1, 1, 0) + \mathbb{B}_3$ and $\mathcal{U}_2 = (-1, 1, 0) + \mathbb{B}_3$. Then, by introducing the index set

$T_R := \bigcup_{t\in T}\mathcal{U}_t$, we get

$$\operatorname{cl}\operatorname{cone} T_R = \operatorname{cl}\left[\left\{x \in \mathbb{R}^3 : x_2 > 0\right\} \cup (\mathbb{R}\times \{0_2\})\right] = \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R},$$

so that $(0_2, 1) \in \operatorname{cl}\operatorname{cone} T_R$ and so Lemma 2.1 implies that (RP_α) is infeasible for $\alpha = 1$. Moreover, it is easy to show that (RP_α) is feasible for any $\alpha < 1$. So, $\rho(\mathcal{U}) = 1$ and the supremum in the definition of $\rho(\mathcal{U})$ is not attained.

Next, we provide a sufficient condition guaranteeing that the supremum in the radius of robust feasibility (2.1) is attained. To do this, recall that for a closed and convex set A, its recession cone A^∞ is defined by

$$A^\infty := \{d : x + \gamma d \in A \ \forall \gamma \geq 0, x \in A\}.$$

Below, we show that, if the recession cone A^∞ of the feasible set A of the unperturbed problem is a subspace, then the supremum in (2.1) is attained. Observe that $A^\infty = \{x \in \mathbb{R}^n : \bar{v}_t^\top x \geq 0, t \in T\} = (\operatorname{cone}\{\bar{v}_t, t \in T\})^+$. So, A^∞ is a subspace if and only if $\operatorname{cl}\operatorname{cone}\{\bar{v}_t, t \in T\}$ is a subspace (a condition in terms of the data). We note that this assumption is satisfied when the corresponding feasible set A can be written as the Minkowski sum of a convex compact set and a subspace.

PROPOSITION 2.3. *Let $A := \{x \in \mathbb{R}^n : \bar{v}_t^\top x \geq \bar{w}_t, t \in T\} \neq \emptyset$. Suppose that A^∞ is a subspace. Then the supremum in* (2.1) *is attained.*

Proof. Let $\rho(\mathcal{U}) \in \mathbb{R}_+$ be the supremum introduced in (2.1). If $\rho(\mathcal{U}) = 0$, then the supremum is automatically attained as $A \neq \emptyset$. So, we assume that $\rho(\mathcal{U}) > 0$ and let $\alpha^k \in (0, \rho(\mathcal{U}))$ be such that $\alpha^k \to \rho(\mathcal{U})$. Then, for each k, there exists $x^k \in \mathbb{R}^n$ such that $v_t^\top x^k - w_t \geq 0$ for all $(v_t, w_t) \in (\bar{v}_t, \bar{w}_t) + \alpha^k \mathbb{B}_{n+1}, t \in T$. This implies that

$$\bar{v}_t^\top x^k - \bar{w}_t + \inf_{(a_t,b_t)\in\mathbb{B}_{n+1}} \alpha^k(a_t^\top x^k - b_t) \geq 0 \quad \forall\, t \in T.$$

So, we have

$$\bar{v}_t^\top x^k - \bar{w}_t - \alpha^k \|(x^k, -1)\| \geq 0 \quad \forall\, t \in T. \tag{2.2}$$

Next we show that $\{x^k\}$ is a bounded sequence. Suppose, on the contrary, that $\|x^k\| \to \infty$. We may assume that $\frac{x^k}{\|x^k\|} \to u \in A^\infty$ with $\|u\| = 1$. Dividing by $\|x^k\|$ on both sides of (2.2) and passing to the limit, we have

$$\bar{v}_t^\top u - \rho(\mathcal{U}) \geq 0 \quad \forall\, t \in T. \tag{2.3}$$

By our assumption A^∞ is a subspace. As $u \in A^\infty$ and $\|u\| = 1$, we see that $-u \in A^\infty$. Take any $x_0 \in A$. Then $x_0 - \gamma u \in A$ for all $\gamma \geq 0$. This implies that $\bar{v}_t^\top u \leq 0$ for all $t \in T$. This contradicts (2.3), and so the claim follows.

Consequently, and by passing to subsequence if necessary, we may assume that $x^k \to \bar{x}$. Passing to the limit in (2.2), we have $\bar{v}_t^\top \bar{x} - \bar{w}_t - \rho(\mathcal{U})\|(\bar{x}, -1)\| \geq 0$ for all $t \in T$. This implies that, for any $(v_t, w_t) \in (\bar{v}_t, \bar{w}_t) + \rho(\mathcal{U})\mathbb{B}_{n+1}, t \in T$, we have

$$\begin{aligned} v_t^\top \bar{x} - w_t &\geq \bar{v}_t^\top \bar{x} - \bar{w}_t + \inf_{(a_t,b_t)\in\mathbb{B}_{n+1}} \rho(\mathcal{U})\{a_t^\top \bar{x} - b_t\} \\ &\geq \bar{v}_t^\top \bar{x} - \bar{w}_t - \rho(\mathcal{U})\|(\bar{x}, -1)\| \geq 0. \end{aligned}$$

Hence, $\bar{x}$ is a feasible point of $(RP_{\rho(\mathcal{U})})$ and so, the supremum in (2.1) is attained. ∎

Example 2.2 violates the condition in Proposition 2.3. Indeed, in this case,

$$\{x \in \mathbb{R}^n : \bar{v}_t^\top x \geq \bar{w}_t, t \in T\}^\infty = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq |x_1|\}$$

is not a subspace.

Below, we provide a formula for computing the radius of robust feasibility. To do this, we first recall some notation. Consider the parameter space $\Theta := (\mathbb{R}^n)^T \times \mathbb{R}^T$. One can endow the parameter space Θ with the extended metric $\tilde{d}$ of the uniform convergence on T; that is,

$$\tilde{d}\,((v,w),(p,q)) := \sup_{t\in T} \|(v_t,w_t)-(p_t,q_t)\| \text{ for } (v,w),(p,q)\in\Theta.$$

Observe that we may have $\tilde{d}\,((v,w),(p,q)) = +\infty$.

Consider the following sets of parameters:

$$\begin{aligned}
\Theta_c &:= \left\{(v,w)\in(\mathbb{R}^n)^T\times\mathbb{R}^T : \exists x\in\mathbb{R}^n, v_t^\top x \geq w_t\,\forall t\in T\right\},\\
\Theta_\infty &:= \left\{(v,w)\in(\mathbb{R}^n)^T\times\mathbb{R}^T : \tilde{d}\,((v,w),\Theta_c) = +\infty\right\}, \quad\text{and}\\
\Theta_s &:= \left\{(v,w)\in(\mathbb{R}^n)^T\times\mathbb{R}^T : \exists \text{ a finite } S\subset T, \{v_t^\top x\geq w_t\,\forall t\in S\} \text{ is not feasible}\right\}.
\end{aligned}$$

Recall also the so-called *hypographical set* [5] of the system $\{v_t^\top x \geq w_t, t\in T\}$ defined as

$$H(v,w) := \operatorname{conv}\{(v_t,w_t), t\in T\} + \mathbb{R}_+\{(0_n,-1)\}. \tag{2.4}$$

The next result provides a formula for the radius of robust feasibility. The proof of this formula relies heavily on the characterization of the elements of Θ_c and some useful results relating the hypographical set and Θ_c, Θ_∞, and Θ_s which are summarized in following lemma.

LEMMA 2.4. *Let $(v,w)\in\Theta$. Then, the following statements hold:*

(i) $0_{n+1}\in\operatorname{int} H(v,w)$ *if and only if* $(0_n,1)\in\operatorname{int}\operatorname{cone}\{(v_t,w_t),t\in T\}$.

(ii) *If $(v,w)\notin\Theta_\infty$, then $(v,w)\in\operatorname{bd}\Theta_c$ if and only if $(v,w)\in\operatorname{bd}\Theta_s$.*

(iii) *If $(v,w)\in\Theta_c$, then $\tilde{d}((v,w),\Theta\backslash\Theta_c) = \tilde{d}\,((v,w),\operatorname{bd}\Theta_c)$.*

(iv) $\tilde{d}\,((v,w),\operatorname{bd}\Theta_s) = d\,(0_{n+1},\operatorname{bd} H(v,w))$.

(v) $(v,w)\in\Theta_\infty$ *if and only if* $\sup_{t\in T}\{w_t - v_t^\top x\} \equiv +\infty\,\forall\, x\in\mathbb{R}^n$.

(vi) *If $(v,w)\in\Theta_c$, then $\tilde{d}((v,w),\Theta\backslash\Theta_c) = c(v,w)$ where $c(v,w)$ is the so-called* consistency value *of (v,w) defined by $c(v,w) := \sup_{x\in\mathbb{R}^n}\inf_{t\in T}\frac{v_t^\top x - w_t}{\|(x,-1)\|}$.*

Proof. Statement (i) is from [5, Lemma 5], statement (ii) is from [5, Theorem 5], statement (iii) is from [5, Corollary 1], statement (iv) is from [5, Theorem 6], statement (v) is from [6, Theorem 3], and, finally, statement (vi) follows from [5, Theorem 7]. □

THEOREM 2.5 (radius of robust feasibility). *Let $(\bar{v}_t,\bar{w}_t)\in\mathbb{R}^n\times\mathbb{R}$, $t\in T$, with $\{x : \bar{v}_t^\top x\geq\bar{w}_t, t\in T\}\neq\emptyset$. Let $\mathcal{U}_t := (\bar{v}_t,\bar{w}_t)+\alpha\mathbb{B}_{n+1}$, $t\in T$, and let $\mathcal{U} : T\rightrightarrows\mathbb{R}^{n+1}$ be defined by $\mathcal{U}(t) = \mathcal{U}_t$. Then,*

$$\rho(\mathcal{U}) = d\,(0_{n+1}, H(\bar{v},\bar{w})),$$

where $\rho(\mathcal{U})$ is the radius of robust feasibility given as in (2.1) *and $H(\bar{v},\bar{w})$ is given as in* (2.4).

Proof. We first show that $0_{n+1}\notin\operatorname{int} H(\bar{v},\bar{w})$. Otherwise, Lemma 2.4 (i) gives us

$$(0_n,1)\in\operatorname{int}\operatorname{cone}\{(\bar{v}_t,\bar{w}_t),t\in T\}\subset\operatorname{cl}\operatorname{cone}\{(\bar{v}_t,\bar{w}_t),t\in T\}.$$

Then, Lemma 2.1 implies that $\{x : \bar{v}_t^\top x \geq \bar{w}_t,\, t \in T\} = \emptyset$. This contradicts the feasibility assumption $(\bar{v}, \bar{w}) \in \Theta_c$. Hence, one has $0_{n+1} \notin \operatorname{int} H(\bar{v}, \bar{w})$, and so,

$$d\left(0_{n+1}, \operatorname{bd} H(\bar{v}, \bar{w})\right) = d\left(0_{n+1}, \operatorname{cl} H(\bar{v}, \bar{w})\right) = d\left(0_{n+1}, H(\bar{v}, \bar{w})\right). \tag{2.5}$$

We also note that $(\bar{v}, \bar{w}) \notin \Theta_\infty$. Otherwise, by Lemma 2.4 (v), $\sup_{t\in T}\{\bar{w}_t - \bar{v}_t^\top x\} = +\infty$ for all $x \in \mathbb{R}^n$, but this contradicts the fact that $\{x : \bar{v}_t^\top x \geq \bar{w}_t,\, t \in T\} \neq \emptyset$. Now, it follows from (2.5) and Lemma 2.4 that

$$\begin{aligned} \tilde{d}\left((\bar{v}, \bar{w}), \Theta\backslash\Theta_c\right) &= \tilde{d}\left((\bar{v}, \bar{w}), \operatorname{bd}\Theta_c\right) \\ &= \tilde{d}\left((\bar{v}, \bar{w}), \operatorname{bd}\Theta_s\right) \\ &= d\left(0_{n+1}, \operatorname{bd} H(\bar{v}, \bar{w})\right) = d\left(0_{n+1}, H(\bar{v}, \bar{w})\right), \end{aligned} \tag{2.6}$$

where the first equality is from Lemma 2.4 (iii), the second equality is from Lemma 2.4 (ii), and the third equality follows from Lemma 2.4 (iv).

Let $\alpha \in \mathbb{R}_+$ be such that the feasible set of (RP_α) is nonempty. Then, $(v_t, w_t) \in \mathcal{U}_t$, $t \in T$ (with $\mathcal{U}_t = (\bar{v}_t, \bar{w}_t) + \alpha\mathbb{B}_{n+1}$) implies that $(v, w) \in \Theta_c$. Since $(v_t, w_t) \in \mathcal{U}_t$, $t \in T$ if and only if $\tilde{d}\left((\bar{v}, \bar{w}), (v, w)\right) \leq \alpha$, we can, equivalently, say that $(v, w) \in \Theta\backslash\Theta_c$ implies that $\tilde{d}\left((\bar{v}, \bar{w}), (v, w)\right) > \alpha$. Therefore, (2.6) gives us that $d\left(0_{n+1}, H(\bar{v}, \bar{w})\right) = \tilde{d}\left((\bar{v}, \bar{w}), \Theta\backslash\Theta_c\right) \geq \alpha$ and, as a consequence of (2.1),

$$\rho(\mathcal{U}) \leq d\left(0_{n+1}, H(\bar{v}, \bar{w})\right).$$

We now show that $\rho(\mathcal{U}) = d\left(0_{n+1}, H(\bar{v}, \bar{w})\right)$. To see this, we proceed by the method of contradiction and suppose that there exists $\bar{\alpha} \in \mathbb{R}$ such that $\rho(\mathcal{U}) < \bar{\alpha} < d\left(0_{n+1}, H(\bar{v}, \bar{w})\right)$. Let $\mathcal{U}_t := (\bar{v}_t, \bar{w}_t) + \bar{\alpha}\mathbb{B}_{n+1}$ for all $t \in T$. Then, by the definition of $\rho(\mathcal{U})$,

$$\{x : v_t^\top x \geq w_t \; \forall (v_t, w_t) \in \mathcal{U}_t, t \in T\} = \emptyset. \tag{2.7}$$

Now, recall that

$$c(\bar{v}, \bar{w}) := \sup_{x\in\mathbb{R}^n} \inf_{t\in T} \frac{\bar{v}_t^\top x - \bar{w}_t}{\|(x, -1)\|}$$

is the so-called consistency value of $(\bar{v}, \bar{w})$. Then, from (2.6) and Lemma 2.4 (vi), we have

$$d\left(0_{n+1}, H(\bar{v}, \bar{w})\right) = \tilde{d}\big((\bar{v}, \bar{w}), \Theta\backslash\Theta_c\big) = c(\bar{v}, \bar{w}).$$

It then follows that $c(\bar{v}, \bar{w}) > \bar{\alpha}$, and so, there exists $\bar{x} \in \mathbb{R}^n$ such that $\inf_{t\in T} \frac{\bar{v}_t^\top \bar{x} - \bar{w}_t}{\|(\bar{x}, -1)\|} > \bar{\alpha}$. This implies that $\bar{v}_t^\top \bar{x} - \bar{w}_t > \bar{\alpha}\|(\bar{x}, -1)\|$ for all $t \in T$. Then, for each $(v_t, w_t) \in \mathcal{U}_t$ and for each $t \in T$,

$$\begin{aligned} v_t^\top \bar{x} - w_t &= \bar{v}_t^\top \bar{x} - \bar{w}_t + \langle (v_t, w_t) - (\bar{v}_t, \bar{w}_t), (\bar{x}, -1)\rangle \\ &> \bar{\alpha}\|(\bar{x}, -1)\| - \tilde{d}\big((v, w), (\bar{v}, \bar{w})\big)\, \|(\bar{x}, -1)\| \geq 0. \end{aligned}$$

This contradicts (2.7), and so, the conclusion follows. ∎

Remark 2.6. From the proof of the radius of robust feasibility, we indeed have

$$\rho(\mathcal{U}) = d\left(0_{n+1}, H(\bar{v}, \bar{w})\right) = \sup_{x\in\mathbb{R}^n} \inf_{t\in T} \frac{\bar{v}_t^\top x - \bar{w}_t}{\|(x, -1)\|}.$$

We now provide two examples to illustrate how the radius of robust feasibility can be computed.

Example 2.7. Consider the same example as in Example 2.2; that is, $T = \{1, 2\}$, $(\bar{v}_1, \bar{w}_1) = (1, 1, 0)$, $(\bar{v}_2, \bar{w}_2) = (-1, 1, 0)$, and $\alpha = 1$. As calculated in Example 2.2, $\rho(\mathcal{U}) = 1$. Now, since $H(\bar{v}, \bar{w}) = \operatorname{conv}\{(1, 1, 0), (-1, 1, 0)\} + \mathbb{R}_+ (0, 0, -1)$, the point of $H(\bar{v}, \bar{w})$ closest to 0_3 is $(0, 1, 0)$ and so $d(0_3, H(\bar{v}, \bar{w})) = 1$. This shows that $\rho(\mathcal{U}) = H(\bar{v}, \bar{w}) = 1$.

Example 2.8. Consider the multiobjective problem

$$(RP) \quad \underset{x\in\mathbb{R}^2}{\text{V-min}} \quad (x_1, x_2) \qquad \text{s.t.} \quad v_t^\top x \geq w_t \quad \forall (v_t, w_t) \in \mathcal{U}_t, t \in [0, 1],$$

where $\mathcal{U}_t = (\bar{v}_t, \bar{w}_t) + \alpha\mathbb{B}_{n+1}$, with $(\bar{v}_t, \bar{w}_t) = (t, 1 - t, t - t^2)$ for all $t \in [0, 1]$. The feasible set of $\{\bar{v}_t^\top x \geq \bar{w}_t \, \forall t \in [0, 1]\}$ is $\{x \in \mathbb{R}^2_+ : \sqrt{x_1} + \sqrt{x_2} = 1\} + \mathbb{R}^2_+$ (recall [13, Example 1.1]). Moreover,

$$H(\bar{v}, \bar{w}) := \operatorname{conv}\left\{(t, 1 - t, t - t^2), t \in [0, 1]\right\} + \mathbb{R}_+ \{(0, 0, -1)\}$$

is contained in the hyperplane $\{x \in \mathbb{R}^3 : x_1 + x_2 = 1\}$, so that the point of $H(\bar{v}, \bar{w}) = \operatorname{bd} H(\bar{v}, \bar{w})$ closest to the origin is $\operatorname{proj}_{H(\bar{v},\bar{w})}(0_3) = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$, and so $d(0_3, H(\bar{v}, \bar{w})) = d(0_3, \operatorname{bd} H(\bar{v}, \bar{w})) = \left\|\left(\frac{1}{2}, \frac{1}{2}, 0\right)\right\| = \frac{\sqrt{2}}{2}$.

On the other hand, direct verification shows that (RP_α) is feasible for $\alpha < \frac{\sqrt{2}}{2}$ and (RP_α) is not feasible for $\alpha = \frac{\sqrt{2}}{2}$ (the constraint corresponding to $t = \frac{1}{2}$ is $0_2^\top x \geq \frac{1}{4}$). So, $\rho(\mathcal{U}) = d(0_3, H(\bar{v}, \bar{w})) = \frac{\sqrt{2}}{2}$, where the supremum is not attained (observe that $\operatorname{cl}\operatorname{cone}\{\bar{v}_t, t \in T\} = \mathbb{R}^2_+$ is not a subspace).

Now we consider a more general case where the uncertain set-valued mapping for affine data perturbations takes the form

$$\mathcal{U}_t := (\bar{v}_t, \bar{w}_t) + \alpha_t Z \quad \forall t \in T, \tag{2.8}$$

with $(\bar{v}, \bar{w}) \in \Theta_c$, $Z \subset \mathbb{R}^{n+1}$ a compact set such that $0_{n+1} \in Z$, and $\alpha \in l^\infty_+(T)$, where $l^\infty_+(T)$ denotes the positive cone of the Banach space $l^\infty(T)$ of all bounded functions of $\mathbb{R}^T$ equipped with the norm $\|\alpha\|_\infty := \sup_{t\in T} |\alpha(t)|$. The next corollary guarantees the feasibility of the robust counterpart (RP) associated with $\mathcal{U}$ as in (2.8) for small enough α.

COROLLARY 2.9 (sufficient feasibility condition). *Let $(\bar{v}_t, \bar{w}_t) \in \mathbb{R}^n \times \mathbb{R}$, $t \in T$, with $\{x : \bar{v}_t^\top x \geq \bar{w}_t, t \in T\} \neq \emptyset$. Let $\mu > 0$, Z be a compact set with $0_{n+1} \in Z \subset \mu\mathbb{B}_{n+1}$, and $\mathcal{U}_t := (\bar{v}_t, \bar{w}_t) + \alpha_t Z$, $t \in T$. Let $\mathcal{U} : T \rightrightarrows \mathbb{R}^{n+1}$ be defined by $\mathcal{U}(t) = \mathcal{U}_t$. Then, the robust counterpart associated with $\mathcal{U}$ as in* (2.8) *is feasible for any $\alpha \in l^\infty_+(T)$ such that*

$$\|\alpha\|_\infty < \frac{d(0_{n+1}, \operatorname{cl} H(\bar{v}, \bar{w}))}{\mu}.$$

Proof. It is an immediate consequence of Theorem 2.5 as $\mu\|\alpha\|_\infty < \varepsilon$ entails that $\alpha_t Z \subset \varepsilon\mathbb{B}_{n+1}$ for all $t \in T$. ∎

The results of this section, including Corollary 2.9, can be easily adapted to multiobjective linear semi-infinite programming with uncertainty in all data by using its reformulation (1.4), where the uncertainty in the objective has been transferred to the constraints.

3. Robust optimality. In this section, we derive conditions characterizing robust solutions of a multiobjective linear semi-infinite programming problem with uncertain constraints.

We first recall different concepts of a solution for a deterministic multiobjective linear semi-infinite program as in (1.1) where the *feasible set* of (P), denoted by X^0, is assumed to be nonempty. A feasible solution $\bar{x} \in X^0$ is said to be *efficient* for (P) if there is no $x \in X^0$ such that $c_i^\top x \le c_i^\top \bar{x}$ for all $i \in I$ and $c_j^\top x < c_j^\top \bar{x}$ for at least one $j \in I$. Analogously, $\bar{x} \in X^0$ is said to be *weakly efficient* if there is no $x \in X^0$ such that $c_i^\top x < c_i^\top \bar{x}$ for all $i \in I$. Moreover, $\bar{x} \in X^0$ is said to be *properly efficient* (in Geoffrion's sense) if there exists $\rho > 0$ such that, for all $i \in I$ and $x \in X^0$ satisfying $c_i^\top x < c_i^\top \bar{x}$, there exists $j \in I$ such that $c_j^\top x > c_j^\top \bar{x}$ and $\frac{c_i^\top \bar{x} - c_i^\top x}{c_j^\top x - c_j^\top \bar{x}} \le \rho$.

Let $\Delta_+^m := \{\lambda \in \mathbb{R}_+^m : \sum_{i=1}^m \lambda_i = 1\}$ and $\Delta_{++}^m := \{\lambda \in \mathbb{R}_{++}^m : \sum_{i=1}^m \lambda_i = 1\}$. As X^0 is a convex set, then it is known (see [7, Chapter 3]) that $\bar{x} \in X^0$ is *weakly efficient* if and only if there exists $\overline{\lambda} \in \Delta_+^m$ such that $\bar{x} \in \operatorname{argmin}_{x \in X^0}(\sum_{i=1}^m \overline{\lambda}_i c_i^\top x)$, and $\bar{x} \in X^0$ is *properly efficient* in the sense of Geoffrion if and only if there exists $\overline{\lambda} \in \Delta_{++}^m$ such that $\bar{x} \in \operatorname{argmin}_{x \in X^0}(\sum_{i=1}^m \overline{\lambda}_i c_i^\top x)$.

3.1. Robust solutions. Recall the robust counterpart introduced in (1.3) of the multiobjective linear semi-infinite program with uncertain constraints. Observe that, by letting $T_R := \bigcup_{t \in T} \mathcal{U}_t$, the program (RP) can be equivalently written as follows:

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{V-min}} & \left(c_1^\top x, \ldots, c_m^\top x\right) \\ \text{s.t.} & v^\top x \ge w \quad \forall (v, w) \in T_R. \end{array}$$

The feasible set of (RP), denoted by X, is said to be the set of *robust feasible solutions* of (P^u).

DEFINITION 3.1 (robust efficient solutions). *A given $\bar{x} \in \mathbb{R}^n$ is said to be a* robust efficient *(*robust weakly efficient, robust properly efficient*) solution of (P^u) whenever x is an efficient (weakly efficient, properly efficient) solution of the robust counterpart (RP). Denote by X_E, X_{pE}, and X_{wE} the sets of robust efficient points, robust properly efficient points, and robust weakly efficient points, respectively.*

Obviously, $X_{pE} \subset X_E \subset X_{wE}$, with $X = X_{wE}$ whenever $c_i = 0_n$ for some $i \in I$, and $X = X_{pE}$ in the trivial case that $c_i = 0_n$ for all $i \in I$.

Let us give an example illustrating the different robust solutions for an uncertain multiobjective linear semi-infinite programming problem.

Example 3.2. Consider the uncertain problem with deterministic objectives

$$\begin{array}{lll} (P^u) & \underset{x \in \mathbb{R}^2}{\text{V-min}} & (x_1, x_2) \\ & \text{s.t.} & v_{(k,t)}^\top x \ge w_{(k,t)} \quad \forall (k,t) \in \mathbb{N} \times ([0,1] \cup \{2\}), \end{array}$$

where $u \in \mathcal{U}$ and $\mathcal{U} : \mathbb{N} \times ([0,1] \cup \{2\}) \rightrightarrows \mathbb{R}^3$ is the uncertain set-valued mapping defined by $\mathcal{U}_{(k,t)} := \left(\bar{a}_{(k,t)}, \bar{b}_{(k,t)}\right) + \alpha \mathbb{B}_3$ for all $(k,t) \in \mathbb{N} \times ([0,1] \cup \{2\})$, with $\alpha > 0$ and

$$\left(\bar{a}_{(k,t)}, \bar{b}_{(k,t)}\right) := \begin{cases} \left(kt, k(1-t), k\left(t - t^2\right) - 1\right) & \text{if } (k,t) \in \mathbb{N} \times [0,1], \\ (-k, -k, -2k-1) & \text{if } (k,t) \in \mathbb{N} \times \{2\}. \end{cases}$$

It can be checked that the systems

$$\left\{\bar{a}_{(k,t)}^\top x \ge \bar{b}_{(k,t)} \; \forall (k,t) \in \mathbb{N} \times ([0,1] \cup \{2\})\right\}$$

and

$$\left\{tx_1 + (1-t)\,x_2 \geq t - t^2, \forall t \in [0,1]\,; -x_1 - x_2 \geq -2\right\}$$

have the same feasible set $F := \text{conv}\,(D \cup \{2e_1, 2e_2\})$, where $D := \{x \in \mathbb{R}^2_+ : \sqrt{x_1} + \sqrt{x_2} = 1\}$ (see [12, Example 1] and [13, Example 1.1] for details). Since $F \subset 3\mathbb{B}_3$, there exists $\delta > 0$ such that the feasible set of any system of the form

$$\left\{v^1_{(k,t)}x_1 + v^2_{(k,t)}x_2 \geq w_{(k,t)}\, \forall (k,t) \in \mathbb{N} \times ([0,1] \cup \{2\})\right\} \tag{3.1}$$

such that

$$\left\|\left(v^1_{(k,t)}, v^2_{(k,t)}, w_{(k,t)}\right) - \left(kt, k(1-t), k(t-t^2) - 1\right)\right\| \leq \delta \quad \forall (k,t) \in \mathbb{N} \times [0,1]\,, \tag{3.2}$$

and

$$\left\|\left(v^1_{(k,2)}, v^2_{(k,2)}, w_{(k,2)}\right) - (-k, -k, -2k-1)\right\| \leq \delta \quad \forall k \in \mathbb{N} \tag{3.3}$$

is also contained in $3\mathbb{B}_3$ (see [13, Corollary 6.2.1] for details). Moreover, if the inequalities (3.2) and (3.3) hold with δ replaced with $\frac{1}{\sqrt{3}} \min\left\{\frac{1}{\sqrt{10}}, \delta\right\}$ then the feasible set of (3.1) is F too (see [12, Example 1]). So, if $\alpha \leq \frac{1}{\sqrt{3}} \min\left\{\frac{1}{\sqrt{10}}, \delta\right\}$, the set of robust feasible solutions of (P^u) is $X = F$, whereas it is easy to see that $X_{pE} = D \setminus \{e_1, e_2\}$, $X_E = D$, and $X_{wE} = D \cup \text{conv}\,\{e_1, 2e_1\} \cup \text{conv}\,\{e_2, 2e_2\}$.

3.2. Characterizations of robust efficient solutions. We now provide some simple characterizations for the robust weakly efficient solutions and robust properly efficient solutions. These characterizations involve the so-called *active cone* at $\bar{x} \in X$,

$$A\,(\bar{x}) := \text{cone}\left\{v : (v,w) \in T_R \text{ and } v^\top \bar{x} = w\right\} \subset \mathbb{R}^n,$$

defined in terms of the data of the problem (RP), which is closely related to the cone of *feasible directions* at $\bar{x} \in X$, given by

$$D\,(X; \bar{x}) := \{d \in \mathbb{R}^n : \exists\, \mu > 0 \text{ such that } \bar{x} + \mu d \in X\}\,.$$

On the other hand, the program (RP) (or its constraints system) is said to satisfy the *Farkas–Minkowski* constraint qualification (FMCQ) when $X \neq \emptyset$ and any linear consequence of $\left\{v^\top x \geq w, (v,w) \in T_R\right\}$ is also consequence of some finite subsystem. FMCQ holds if and only if cone $\{T_R \cup (0_n, -1)\}$ is closed. Moreover, FMCQ holds whenever $\{\mathcal{U}_t, t \in T\}$ is a finite family of finite sets. We will say that (RP) satisfies the *local Farkas–Minkowski* constraint qualification (LFMCQ) at $\bar{x} \in X$ when $D\,(X; \bar{x})^+ = A\,(\bar{x})$ or, equivalently, when any consequence of $\left\{v^\top x \geq w, (v,w) \in T_R\right\}$ determining a supporting hyperplane to X at $\bar{x}$ is also consequence of some finite subsystem. Obviously, if (RP) satisfies the FMCQ, then it also satisfies the LFMCQ at any $\bar{x} \in X$. These constraint qualifications allow us to replace $D\,(X; \bar{x})^+$ by $A\,(\bar{x})$, with $A\,(\bar{x})$ being expressed in terms of the data of the problem. Given a feasible solution $\overline{x}$ of a scalar linear semi-infinite program $\min\left\{c^\top x : x \in X\right\}$, the KKT condition $c \in A\,(\bar{x})$ guarantees the optimality of $\overline{x}$, and it is also necessary whenever the LFMCQ holds at $\bar{x}$.

Below, we present characterizations of robust solutions under the constraint qualifications LFMCQ. Recall that $\mathbb{R}^{(T)}_+$ denotes the positive cone of the space $\mathbb{R}^{(T)}$ of functions from T to $\mathbb{R}$ which have a finite support.

THEOREM 3.3 (characterization of robust solutions w.r.t. weak efficiency). *Let X be the feasible set of problem (RP). Suppose that the LFMCQ at $\bar{x} \in X$ holds and*

$\mathcal{U}_t$ is convex for all $t \in T$. Then, $\bar{x}$ is a robust weakly efficient solution of (P^u) if and only if there exists $\bar{\lambda} \in \Delta_+^m, \bar{y}^i \in \mathbb{R}^{(T)}$, $(\bar{v}_t, \bar{w}_t) \in \mathcal{U}_t$, $t \in T$, and $\bar{r}_i \in \mathbb{R}$ such that

$$\left\{ \begin{array}{l} \sum_{i=1}^m \bar{\lambda}_i \big(c_i - \sum_{t\in T} \bar{y}_t^i \bar{v}_t\big) = 0_n, \\ \sum_{i=1}^m \bar{\lambda}_i \bar{y}^i \in \mathbb{R}_+^{(T)}, \\ \sum_{i=1}^m \bar{\lambda}_i \bar{r}_i = 0, \\ c_i^\top \bar{x} = \sum_{t\in T} \bar{w}_t \bar{y}_t^i + \bar{r}_i, \ \ i = 1, \dots, m. \end{array} \right. \tag{3.4}$$

Proof. [$\Rightarrow$] Let $\bar{x}$ be a robust weakly efficient solution. Let $\overline{\lambda} \in \Delta_+^m$ be such that $\bar{x} \in \operatorname{argmin}_{x\in X}(\sum_{i=1}^m \bar{\lambda}_i c_i^\top x)$. As the LFMCQ holds at $\bar{x}$, $\sum_{i=1}^m \bar{\lambda}_i c_i \in A(\bar{x})$. Thus, we can write

$$\sum_{i=1}^m \bar{\lambda}_i c_i = \sum_{t\in T'} \sum_{l=1}^{m_t} \mu_t^l v_t^l$$

for some $m_t \in \mathbb{N}$, $\mu_t^l > 0$, and $(v_t^l, w_t^l) \in \mathcal{U}_t$ with $(v_t^l)^\top \bar{x} = w_t^l$, for all $l = 1, \dots, m_t$, $t \in T'$, with $T' \subset T$ being a finite set. By defining $\mu_t := 0$ and $(\bar{v}_t, \bar{w}_t)$ any point in $\mathcal{U}_t$, for those $t \in T \backslash T'$, and $\mu_t := \sum_{l=1}^{m_t} \mu_t^l > 0$ and

$$(\bar{v}_t, \bar{w}_t) := \sum_{l=1}^{m_t} \frac{\mu_t^l}{\mu_t} (v_t^l, w_t^l) \in \mathcal{U}_t,$$

for those $t \in T'$, then we get that $\mu \in \mathbb{R}_+^{(T)}$ and $(\bar{v}_t, \bar{w}_t) \in \mathcal{U}_t$, $t \in T$. Moreover, $\sum_{i=1}^m \bar{\lambda}_i c_i = \sum_{t\in T} \mu_t \bar{v}_t$ with $\bar{v}_t^\top \bar{x} = \bar{w}_t$ for those $t \in T$ such that $\mu_t \neq 0$. Letting $\bar{y}^i := \mu$ for all $i = 1, \dots, m$, we have

$$\sum_{i=1}^m \bar{\lambda}_i \left(c_i - \sum_{t\in T} \bar{y}_t^i \bar{v}_t \right) = \sum_{i=1}^m \bar{\lambda}_i c_i - \sum_{t\in T} \mu_t \bar{v}_t = 0_n.$$

Furthermore,

$$\sum_{i=1}^m \bar{\lambda}_i c_i^\top \bar{x} = \sum_{t\in T} \mu_t \bar{v}_t^\top \bar{x} = \sum_{t\in T} \mu_t \bar{w}_t = \sum_{t\in T} \bar{w}_t \bar{y}_t^i \quad \forall i = 1, \dots, m.$$

Now, for each $i = 1, \dots, m$, let $\bar{r}_i := c_i^\top \bar{x} - \sum_{t\in T} \bar{w}_t \bar{y}_t^i$. Then, $c_i^\top \bar{x} = \sum_{t\in T} \bar{w}_t \bar{y}_t^i + \bar{r}_i$ for all $i = 1, \dots, m$ and

$$\sum_{i=1}^m \bar{\lambda}_i \bar{r}_i = \sum_{i=1}^m \bar{\lambda}_i (c_i^\top \bar{x} - \sum_{t\in T} \bar{w}_t \bar{y}_t^i) = \sum_{i=1}^m \bar{\lambda}_i c_i^\top \bar{x} - \sum_{t\in T} \bar{w}_t \mu_t = 0.$$

[$\Leftarrow$] Suppose that there exist $\bar{\lambda} \in \Delta_+^m, \bar{y}^i \in \mathbb{R}^{(T)}$ and $\bar{r}_i \in \mathbb{R}$ such that (3.4) holds. Take any feasible point x of (RP). Then, we have $v_t^\top x \geq w_t \ \forall (v_t, w_t) \in \mathcal{U}_t$, $t \in T$. It then follows that

$$\begin{aligned}\sum_{i=1}^{m}\bar{\lambda}_i(c_i^\top x - c_i^\top \bar{x}) &= \sum_{i=1}^{m}\bar{\lambda}_i c_i^\top x - \sum_{i=1}^{m}\bar{\lambda}_i c_i^\top \bar{x}\\ &= \sum_{i=1}^{m}\sum_{t\in T}\bar{\lambda}_i \bar{y}_t^i v_t^\top x - \sum_{i=1}^{m}\bar{\lambda}_i c_i^\top \bar{x}\\ &\geq \sum_{i=1}^{m}\sum_{t\in T}\bar{\lambda}_i \bar{y}_t^i w_t - \sum_{i=1}^{m}\bar{\lambda}_i\left(\sum_{t\in T}\bar{w}_t\bar{y}_t^i + \bar{r}_i\right)\\ &= -\sum_{i=1}^{m}\bar{\lambda}_i\bar{r}_i = 0.\end{aligned}$$

This shows that for any feasible solution x of (RP), $c_i^\top x < c_i^\top \bar{x}$, $i = 1, \ldots, m$, cannot happen simultaneously. So, $\bar{x}$ is a robust weakly efficient solution. ▯

Remark 3.4. We note that, in the special case when T is finite, the above robust weakly efficient solution characterization was obtained in [11]. In fact, [11] established a characterization for robust weakly efficient solution for multiobjective linear programming problems where the data uncertainty occurs in both the objective function and the constraints.

Next, in the case where $\mathcal{U}_t$ is the *scenario uncertainty set* (the polytope defined, in a parametric way, in (3.5)) and T is the unit ball in $\mathbb{R}^q$, $q \in \mathbb{N}$, we show that, whether a robust feasible point $\bar{x}$ is a robust weakly efficient solution or not can be verified by solving a *second-order cone programming problem*. To do this, we recall that the second-order cone SOC_r, $r \in \mathbb{N}\cup\{0\}$, is given by

$$\mathrm{SOC}_r := \{(x_0, x_1, \ldots, x_r) \in \mathbb{R}^{r+1} : x_0 \geq \|(x_1, \ldots, x_r)\|\}$$

(in particular, $\mathrm{SOC}_0 = \mathbb{R}_+$). It is known that a second-order cone programming problem can be efficiently solved (for example, by interior point method; see [16]).

Corollary 3.5 (tractable characterization w.r.t. weak efficiency: scenario uncertainty). *Let $p, q \in \mathbb{N}$. For problem (P^u), suppose that*

$$\mathcal{U}_t := \mathrm{conv}\left\{(v_1^0, w_1^0) + \sum_{j=1}^{q} t_j(v_1^j, w_1^j), \ldots, (v_p^0, w_p^0) + \sum_{j=1}^{q} t_j(v_p^j, w_p^j)\right\}, \tag{3.5}$$

where $(v_k^j, w_k^j) \in \mathbb{R}^n \times \mathbb{R}$, $k = 1, \ldots, p$, $j = 1, \ldots, q$, and $T = \mathbb{B}_q$. Let X be the feasible set of its associated robust counterpart problem (RP). Suppose that the LFMCQ at $\bar{x} \in X$ holds. Then, the following statements are equivalent:

(i) *$\bar{x}$ is a robust weakly efficient solution of (P^u).*

(ii) *The following second-order cone system has a solution:*

$$A(\lambda,\mu) = 0_{n+2} \text{ and } (\lambda,\mu) \in \left(\prod_{i=1}^{m}\mathrm{SOC}_0\right)\times\left(\prod_{k=1}^{p}\mathrm{SOC}_q\right), \tag{3.6}$$

where $A : \mathbb{R}^m \times \mathbb{R}^{p(q+1)} \to \mathbb{R}^{n+2}$ is an affine mapping given by

$$A(\lambda,\mu) = \begin{pmatrix} \sum_{i=1}^{m}\lambda_i c_i - \sum_{k=1}^{p}(\mu_k^0 v_k^0 + \sum_{j=1}^{q}\mu_k^j v_k^j) \\ \sum_{k=1}^{p}\left(\mu_k^0((v_k^0)^\top\bar{x} - w_k^0) + \sum_{j=1}^{q}\mu_k^j((v_k^j)^\top\bar{x} - w_k^j)\right) \\ \sum_{i=1}^{m}\lambda_i - 1\end{pmatrix}.$$

(iii) *The following second-order cone programming problem has a solution:*

$$\min_{(\lambda,\mu)\in\mathbb{R}^m\times\mathbb{R}^{p(q+1)}}\left\{0 : A(\lambda,\mu)=0_{n+2}, (\lambda,\mu)\in\left(\prod_{i=1}^m \mathrm{SOC}_0\right)\times\left(\prod_{k=1}^p \mathrm{SOC}_q\right)\right\}.$$

Proof. [(i) $\Rightarrow$ (ii)] Let $\bar{x}$ be a robust weakly efficient solution of (P^u). As the LFMCQ at $\bar{x}\in X$ holds, the preceding theorem implies that there exist $\bar{\lambda}\in\Delta_+^m, \bar{y}^i\in\mathbb{R}^{(T)}$, $(\bar{v}_t,\bar{w}_t)\in\mathcal{U}_t$, $t\in\mathbb{B}_q$, and $\bar{r}_i\in\mathbb{R}$ such that

$$(3.7)\qquad \begin{cases} \sum_{i=1}^m \bar{\lambda}_i\big(c_i-\sum_{t\in T}\bar{y}_t^i\bar{v}_t\big)=0_n,\\ \sum_{i=1}^m \bar{\lambda}_i\bar{y}^i\in\mathbb{R}_+^{(T)},\\ \sum_{i=1}^m \bar{\lambda}_i\bar{r}_i=0,\\ c_i^\top\bar{x}=\sum_{t\in T}\bar{w}_t\bar{y}_t^i+\bar{r}_i,\ \ i=1,\ldots,m. \end{cases}$$

For each $t\in\mathbb{B}_q$, as $(\bar{v}_t,\bar{w}_t)\in\mathcal{U}_t$, there exist $\gamma_k\geq 0$, $k=1,\ldots,p$, with $\sum_{k=1}^p\gamma_k=1$ such that

$$(\bar{v}_t,\bar{w}_t)=\sum_{k=1}^p\gamma_k\left((v_k^0,w_k^0)+\sum_{j=1}^q t_j(v_k^j,w_k^j)\right).$$

For each $k=1,\ldots,p$, let $\bar{\mu}_k^0=\sum_{i=1}^m\sum_{t\in T}\bar{\lambda}_i\bar{y}_t^i\gamma_k$ and $\bar{\mu}_k^j=\sum_{i=1}^m\sum_{t\in T}\bar{\lambda}_i\bar{y}_t^i\gamma_k t_j$, $j=1,\ldots,q$. Then, the second relation of (3.7) and $\gamma_k\geq 0$ imply that $\bar{\mu}_k^0=\sum_{t\in T}(\sum_{i=1}^m\bar{\lambda}_i\bar{y}_t^i)\gamma_k\geq 0$. Moreover, $t\in\mathbb{B}_q$ gives us that $\bar{\mu}_k^0\geq\|(\bar{\mu}_k^1,\ldots,\bar{\mu}_k^q)\|$. Note that

$$\sum_{i=1}^m\sum_{t\in T}\bar{\lambda}_i\bar{y}_t^i\bar{v}_t=\sum_{i=1}^m\sum_{t\in T}\sum_{k=1}^p\bar{\lambda}_i\bar{y}_t^i\gamma_k\left(v_k^0+\sum_{j=1}^q t_j v_k^j\right)=\sum_{k=1}^p\left(\bar{\mu}_k^0 v_k^0+\sum_{j=1}^q\bar{\mu}_k^j v_k^j\right).$$

Then, the first relation of (3.7) implies that $\sum_{i=1}^m\bar{\lambda}_i c_i-\sum_{k=1}^p(\bar{\mu}_k^0 v_k^0+\sum_{j=1}^q\bar{\mu}_k^j v_k^j)=0_n$. To see the last assertion, we first note that

$$\sum_{i=1}^m\sum_{t\in T}\bar{\lambda}_i\bar{y}_t^i\bar{w}_t=\sum_{k=1}^p\left(\bar{\mu}_k^0 w_k^0+\sum_{j=1}^q\bar{\mu}_k^j w_k^j\right).$$

It then follows that

$$\begin{aligned}\sum_{t\in T}\sum_{i=1}^m(\bar{\lambda}_i\bar{y}_t^i)(\bar{v}_t^\top\bar{x}-\bar{w}_t)&=\sum_{i=1}^m\sum_{t\in T}\bar{\lambda}_i\bar{y}_t^i\bar{v}_t^\top\bar{x}-\sum_{i=1}^m\bar{\lambda}_i\sum_{t\in T}\bar{y}_t^i\bar{w}_t\\ &=\sum_{i=1}^m\bar{\lambda}_i c_i^\top\bar{x}-\sum_{i=1}^m\bar{\lambda}_i(c_i^\top\bar{x}-\bar{r}_i)\ =\ 0,\end{aligned}$$

where the second equality follows from the first and the last relations in (3.7), and the third equality follows from the third relation in (3.7). So, we have

$$\begin{aligned}\sum_{k=1}^p\left(\bar{\mu}_k^0((v_k^0)^\top\bar{x}-w_k^0)+\sum_{j=1}^q\bar{\mu}_k^j((v_k^j)^\top\bar{x}-w_k^j)\right)&=\sum_{i=1}^m\sum_{t\in T}\bar{\lambda}_i\bar{y}_t^i(\bar{v}_t^\top\bar{x}-\bar{w}_t)\\ &=\sum_{t\in T}\sum_{i=1}^m(\bar{\lambda}_i\bar{y}_t^i)(\bar{v}_t^\top\bar{x}-\bar{w}_t)\ =\ 0.\end{aligned}$$

Therefore, we see that there exist $\bar{\lambda} \in \Delta_+^m$ and $\bar{\mu}_k^j \in \mathbb{R}$, $k = 1, \ldots, p$, $j = 0, 1, \ldots, q$ such that

$$\begin{cases} \sum_{i=1}^m \bar{\lambda}_i c_i - \sum_{k=1}^p (\bar{\mu}_k^0 v_k^0 + \sum_{j=1}^q \bar{\mu}_k^j v_k^j) = 0_n, \\ \overline{\mu}_k^0 \geq \|(\overline{\mu}_k^1, \ldots, \overline{\mu}_k^q)\|, \ k = 1, \ldots, p, \\ \sum_{k=1}^p \left(\bar{\mu}_k^0((v_k^0)^\top \bar{x} - w_k^0) + \sum_{j=1}^q \bar{\mu}_k^j((v_k^j)^\top \bar{x} - w_k^j)\right) = 0. \end{cases}$$

This implies that the second-order cone system (3.6) has a solution.

[(ii) ⇒ (i)] Suppose that the second-order cone system (3.6) has a solution. Then, there exist $\bar{\lambda} \in \Delta_+^m$ and $\bar{\mu}_k^j \in \mathbb{R}$, $k = 1, \ldots, p$, $j = 0, 1, \ldots, q$, such that

$$\begin{cases} \sum_{i=1}^m \bar{\lambda}_i c_i - \sum_{k=1}^p (\bar{\mu}_k^0 v_k^0 + \sum_{j=1}^q \bar{\mu}_k^j v_k^j) = 0_n, \\ \overline{\mu}_k^0 \geq \|(\overline{\mu}_k^1, \ldots, \overline{\mu}_k^q)\|, \ k = 1, \ldots, p, \\ \sum_{k=1}^p \left(\bar{\mu}_k^0((v_k^0)^\top \bar{x} - w_k^0) + \sum_{j=1}^q \bar{\mu}_k^j((v_k^j)^\top \bar{x} - w_k^j)\right) = 0. \end{cases}$$

Let x be a feasible point of (RP). Then, we have $v_t^\top x \geq w_t \ \forall (v_t, w_t) \in \mathcal{U}_t, \ t \in T$. It then follows that, for each $k = 1, \ldots, p$ and $t \in \mathbb{B}_q$,

$$\left(v_k^0 + \sum_{j=1}^q t_j v_k^j\right)^\top x - \left(w_k^0 + \sum_{j=1}^q t_j w_k^j\right) \geq 0.$$

This implies that for each $k = 1, \ldots, p$,

$$(3.8) \quad (v_k^0)^\top x - w_k^0 \geq \sup_{t \in \mathbb{B}_q} \sum_{j=1}^q t_j \left(w_k^j - (v_k^j)^\top x\right) = \left\|\left((v_k^1)^\top x - w_k^1, \ldots, (v_k^q)^\top x - w_k^q\right)\right\|.$$

Hence,

$$\begin{aligned} &\sum_{i=1}^m \bar{\lambda}_i (c_i^\top x - c_i^\top \bar{x}) \\ &= \sum_{i=1}^m \bar{\lambda}_i \left(\sum_{k=1}^p \left(\bar{\mu}_k^0 v_k^0 + \sum_{j=1}^q \bar{\mu}_k^j v_k^j \right)^\top x \right) - \sum_{i=1}^m \bar{\lambda}_i \left(\sum_{k=1}^p \left(\bar{\mu}_k^0 v_k^0 + \sum_{j=1}^q \bar{\mu}_k^j v_k^j \right)^\top \bar{x} \right) \\ &= \sum_{i=1}^m \bar{\lambda}_i \left(\sum_{k=1}^p \left(\bar{\mu}_k^0 v_k^0 + \sum_{j=1}^q \bar{\mu}_k^j v_k^j \right)^\top x \right) - \sum_{i=1}^m \bar{\lambda}_i \sum_{k=1}^p \left(\bar{\mu}_k^0 w_k^0 + \sum_{j=1}^q \bar{\mu}_k^j w_k^j \right) \\ &= \sum_{i=1}^m \bar{\lambda}_i \sum_{k=1}^p \left(\bar{\mu}_k^0 \left((v_k^0)^\top x - w_k^0 \right) + \sum_{j=1}^q \bar{\mu}_k^j \left((v_k^j)^\top x - w_k^j \right) \right) \ \geq \ 0, \end{aligned}$$

where the last inequality follows from the fact that $\overline{\mu}_k^0 \geq \|(\overline{\mu}_k^1, \ldots, \overline{\mu}_k^q)\|, \ k = 1, \ldots, p$, and (3.8). Thus, we see that $\bar{x}$ is a robust weakly efficient solution for (P^u).

[(ii) ⇔ (iii)] This equivalence is immediate. ∎

THEOREM 3.6 (characterization of robust solutions w.r.t. proper efficiency). *Let X be the feasible set of problem (RP). Suppose that the LFMCQ at $\bar{x} \in X$ holds and $\mathcal{U}_t$ is convex for all $t \in T$. Then, $\bar{x}$ is a robust properly efficient solution of (P^u) if* *and only if there exists* $\bar{\lambda} \in \Delta^m_{++}, \bar{y}^i \in \mathbb{R}^{(T)}$, $(\bar{v}_t, \bar{w}_t) \in \mathcal{U}_t$, $t \in T$, *and* $\bar{r}_i \in \mathbb{R}$ *such that*

$$(3.9) \qquad \begin{cases} \sum_{i=1}^m \bar{\lambda}_i \big(c_i - \sum_{t\in T} \bar{y}^i_t \bar{v}_t\big) = 0_n, \\ \sum_{i=1}^m \bar{\lambda}_i \bar{y}^i \in \mathbb{R}^{(T)}_+, \\ \sum_{i=1}^m \bar{\lambda}_i \bar{r}_i = 0, \\ c_i^\top \bar{x} = \sum_{t\in T} \bar{w}_t \bar{y}^i_t + \bar{r}_i, \ i = 1, \ldots, m. \end{cases}$$

Proof. [$\Rightarrow$] Let $\bar{x}$ be a properly efficient solution of (RP). Since the LFMCQ at $\bar{x}$ holds, there exists $\bar{\lambda} \in \Delta^m_{++}$ such that $\bar{x} \in \operatorname{argmin}_{x\in X}(\sum_{i=1}^m \bar{\lambda}_i c_i^\top x)$. Then, following similar arguments as in the proof of Theorem 3.3, we see that (3.9) holds.

[$\Leftarrow$] Suppose that there exists $\bar{\lambda} \in \Delta^m_{++}, \bar{y}^i \in \mathbb{R}^{(T)}$, and $\bar{r}_i \in \mathbb{R}$ such that (3.9) holds. Take any feasible point x of (RP). Then, we have $v_t^\top x \geq w_t \ \forall (v_t, w_t) \in \mathcal{U}_t$, $t \in T$. Following similar arguments as in the proof of Theorem 3.3, we see that $\sum_{i=1}^m \bar{\lambda}_i (c_i^\top x - c_i^\top \bar{x}) \geq 0$. Thus, the conclusion follows. □

Similarly to Corollary 3.5, in the case where $\mathcal{U}_t$ is the scenario uncertainty set and T is the unit ball in $\mathbb{R}^q$, $q \in \mathbb{N}$, we obtain the following numerically checkable robust optimality condition for verifying whether a robust feasible point is robust properly efficient or not.

COROLLARY 3.7 (tractable sufficient robust optimality condition w.r.t. proper efficiency: scenario uncertainty). *Let* $p, q \in \mathbb{N}$. *For problem* (P^u), *suppose that*

$$\mathcal{U}_t := \operatorname{conv}\left\{(v_1^0, w_1^0) + \sum_{j=1}^q t_j (v_1^j, w_1^j), \ldots, (v_p^0, w_p^0) + \sum_{j=1}^q t_j (v_p^j, w_p^j)\right\},$$

where $(v_k^j, w_k^j) \in \mathbb{R}^n \times \mathbb{R}$, $k = 1, \ldots, p$, $j = 1, \ldots, q$, *and* $T = \mathbb{B}_q$. *Let* X *be the feasible set of its associated robust counterpart problem* (RP). *Suppose that the LFMCQ at* $\bar{x} \in X$ *holds. Consider the second-order cone system* (3.6) *as in Corollary* 3.5. *If this second-order cone system has a solution* $(\bar{\lambda}, \bar{\mu})$ *with* $\bar{\lambda} \in \mathbb{R}^m_{++}$, *then* $\bar{x}$ *is a robust properly efficient solution of* (P^u).

Proof. Let $(\bar{\lambda}, \bar{\mu})$ be a solution of the second-order cone system (3.6) with $\bar{\lambda} \in \mathbb{R}^m_{++}$. This implies that $\bar{\lambda} \in \Delta^m_{++}, \bar{\mu}_k^j \in \mathbb{R}$, $k = 1, \ldots, p$, $j = 0, 1, \ldots, q$, and

$$\begin{cases} \sum_{i=1}^m \bar{\lambda}_i c_i - \sum_{k=1}^p (\bar{\mu}_k^0 v_k^0 + \sum_{j=1}^q \bar{\mu}_k^j v_k^j) = 0_n, \\ \overline{\mu}_k^0 \geq \|(\overline{\mu}_k^1, \ldots, \overline{\mu}_k^q)\|, \ k = 1, \ldots, p, \\ \sum_{k=1}^p \big(\bar{\mu}_k^0((v_k^0)^\top \bar{x} - w_k^0) + \sum_{j=1}^q \bar{\mu}_k^j ((v_k^j)^\top \bar{x} - w_k^j)\big) = 0. \end{cases}$$

Using a similar method of proof as in Corollary 3.5, we see that $\bar{x}$ is a robust properly efficient solution of (P^u). □

4. Robust duality. In this section, we now develop a suitable robust duality framework for the multiobjective linear semi-infinite programming problem with uncertain constraints. Related details for scalar optimization problems can be found in [1, 4, 10, 15, 14].

As stated in section 1, the multiobjective linear problem (P) in the face of data uncertainty in the constraints can be captured by the parameterized problem (P^u), for each fixed selection $u = (v, w) \in \mathcal{U}$, introduced in (1.2). The robust counterpart of problem (P^u) is obtained by finding the "worst" value over all possible scenario $u \in \mathcal{U}$,

$$(RP) \quad \underset{x\in\mathbb{R}^n}{\text{V-min}} \quad \big(c_1^\top x, \ldots, c_m^\top x\big) \quad \text{s.t.} \quad v_t^\top x \geq w_t \quad \forall (v_t, w_t) \in \mathcal{U}_t, t \in T.$$

For each fixed selection $u = (v, w) \in \mathcal{U}$, we associate to (P^u) the dual problem (D^u) as follows:

$$(D^u) \quad \underset{\substack{\lambda \in \Delta_+^m \\ y^i \in \mathbb{R}^{(T)},\, r_i \in \mathbb{R}}}{\text{V-max}} \quad \left(w^\top y^1 + r_1, \dots, w^\top y^m + r_m\right)$$
$$\text{s.t.} \quad \sum_{i=1}^m \lambda_i \left(c_i - \sum_{t \in T} y_t^i v_t\right) = 0_n, \quad \sum_{i=1}^m \lambda_i y^i \in \mathbb{R}_+^{(T)}, \quad \sum_{i=1}^m \lambda_i r_i = 0,$$

where $w^\top y^i := \sum_{t \in T} w_t y_t^i \,\forall i \in I$. In short, we will write $y := \left(y^1, \dots, y^m\right)$ and $r := (r_1, \dots, r_m)$. We note that, for each fixed parameter u, the dual problem (D^u) is nothing but the standard Fenchel–Lagrange type dual of the primal problem which was extensively studied in the literature. For a comprehensive survey, see [3].

We now define a deterministic problem by looking at the optimistic counterpart of (D^u),

$$(OD) \quad \underset{\substack{\lambda \in \Delta_+^m,\, (v,w) \in \mathcal{U} \\ y^i \in \mathbb{R}^{(T)},\, r_i \in \mathbb{R}}}{\text{V-max}} \quad \left(w^\top y^1 + r_1, \dots, w^\top y^m + r_m\right)$$
$$\text{s.t.} \quad \sum_{i=1}^m \lambda_i \left(c_i - \sum_{t \in T} y_t^i v_t\right) = 0_n, \quad \sum_{i=1}^m \lambda_i y^i \in \mathbb{R}_+^{(T)}, \quad \sum_{i=1}^m \lambda_i r_i = 0.$$

The optimal value of (OD) is the "best" value over all possible scenarios $u = (v, w) \in \mathcal{U}$ of (D^u). In the special case when $m = 1$ (that is, the scalar value case), the problem (OD) reduces to

$$\max_{y \in \mathbb{R}_+^{(T)},\, (v,w) \in \mathcal{U}} \quad w^\top y \qquad \text{s.t.} \quad c - \sum_{t \in T} y_t v_t = 0_n,$$

which is the optimistic counterpart of the usual Lagrangian dual of a semi-infinite programming problem introduced in [10].

DEFINITION 4.1. *A feasible point* $(\bar\lambda, \bar y, \bar r, (\bar v, \bar w))$ *of the optimistic counterpart* (OD) *is said to be a* weakly efficient solution *of* (OD) *if there is no feasible point* $(\lambda, y, r, (v, w))$ *of* (OD) *such that*

$$\bar w^\top \bar y^i + \bar r_i < w^\top y^i + r_i \quad \forall i \in I.$$

THEOREM 4.2 (robust duality with respect to weak efficiency). *Let* $\bar x$ *be a robust weakly efficient solution of* (P^u). *Suppose that the LFMCQ at* $\bar x \in X$ *holds and* $\mathcal{U}_t$ *is convex for all* $t \in T$. *Then, there exists a weakly efficient solution* $(\bar\lambda, \bar y, \bar r, (\bar v, \bar w))$ *for* (OD) *such that* $c_i^\top \bar x = \bar w^\top \bar y^i + \bar r_i \,\forall i \in I$.

Proof. We first show that weak duality holds between the robust counterpart (RP) and the optimistic counterpart (OD); that is, there is no $x \in X$ and no $(\lambda, y, r, (v, w))$ feasible for (OD) such that $c_i^\top x < \sum_{t \in T} w_t y_t^i + r_i \,\forall i \in I$. We proceed by the method of contradiction and assume that there exists $x \in X$ and $(\lambda, y, r, (v, w))$ feasible for (OD) such that $c_i^\top x < \sum_{t \in T} w_t y_t^i + r_i \,\forall i \in I$. As $x \in X$ and $(\lambda, y, r, (v, w))$ is feasible for problem (OD), we have $v_t^\top x \geq w_t\ \forall t \in T$, and

$$\begin{cases} \sum_{i=1}^m \lambda_i \left(c_i - \sum_{t\in T} y_t^i v_t\right) = 0_n, \\ \sum_{i=1}^m \lambda_i y^i \in \mathbb{R}_+^{(T)}, \\ \sum_{i=1}^m \lambda_i r_i = 0, \\ \lambda \in \Delta_+^m, y^i \in \mathbb{R}^{(T)}, r_i \in \mathbb{R}, (v,w) \in \mathcal{U}. \end{cases}$$

This implies that

$$\begin{aligned} \sum_{i=1}^m \lambda_i \left(c_i^\top x - \sum_{t\in T} w_t y_t^i - r_i \right) &= \sum_{i=1}^m \lambda_i \left(\sum_{t\in T} y_t^i (v_t^\top x - w_t) \right) \\ &= \sum_{t\in T} \left(\sum_{i=1}^m \lambda_i y_t^i \right) (v_t^\top x - w_t) \ \geq \ 0. \end{aligned}$$

On the other hand, as $c_i^\top x < \sum_{t\in T} w_t y_t^i + r_i\, \forall i \in I$ and $\lambda \in \Delta_+^m$, we see that $\sum_{i=1}^m \lambda_i \left(c_i^\top x - \sum_{t\in T} w_t y_t^i - r^i\right) < 0$. This makes a contradiction, and so, the conclusion follows.

In virtue of Theorem 3.3, there exists $\bar{\lambda} \in \Delta_+^m, \bar{y}^i \in \mathbb{R}^{(T)}$, $(\bar{v}_t, \bar{w}_t) \in \mathcal{U}_t$, $t \in T$, and $\bar{r}_i \in \mathbb{R}$ such that (3.4) holds. In particular, $(\bar{\lambda}, \bar{y}, \bar{r}, (\bar{v}, \bar{w}))$ is feasible for (OD). To see the conclusion, it suffices to show that $(\bar{\lambda}, \bar{y}, \bar{r}, (\bar{v}, \bar{w}))$ is a weakly efficient solution for (OD). To see this, we proceed by the method of contradiction and assume that there exists $(\lambda, y, r, (v, w))$ feasible for (OD) such that, for all $i \in I$, $\sum_{t\in T} \bar{w}_t \bar{y}_t^i + \bar{r}_i < \sum_{t\in T} w_t y_t^i + r_i$. This, together with the last relation in (3.4), implies that $c_i^\top \bar{x} < \sum_{t\in T} w_t y_t^i + r_i\, \forall i \in I$. Since $\bar{x} \in X$, this contradicts the weak duality statement, and so, the conclusion follows. □

As a corollary, we obtain a version of the robust duality theorem which was given in [10] for a single-objective linear semi-infinite programming problem under data uncertainty using a local constraint qualification.

COROLLARY 4.3. *Consider the programs (RP) and (OD) with $m = 1$, and let $\bar{x}$ be a robust solution of (P^u). Suppose that the LFMCQ at $\bar{x} \in X$ holds and $\mathcal{U}_t$ is convex for all $t \in T$. Then, there exists a solution $(\bar{y}, (\bar{v}, \bar{w}))$ for (OD) such that $c^\top \bar{x} = \bar{w}^\top \bar{y}$.*

Proof. Note that the programs (RP) and (OD) with $m = 1$ collapse to

$$(RP_1) \quad \min_{x\in\mathbb{R}^n} \ c^\top x \quad \text{s.t.} \quad v_t^\top x \geq w_t \quad \forall (v_t, w_t) \in \mathcal{U}_t, t \in T,$$

and

$$(OD_1) \quad \max_{y\in\mathbb{R}_+^{(T)}, (v,w)\in\mathcal{U}} \ w^\top y \quad \text{s.t.} \quad c - \textstyle\sum_{t\in T} y_t v_t = 0_n.$$

Thus, the conclusion follows from the preceding robust duality theorem. □

Similarly, one can obtain duality theorems with respect to properly efficient solutions. For that purpose, consider the following modified optimistic dual problem:

$$(MOD) \quad \underset{\substack{\lambda\in\Delta^m_{++},\,(v,w)\in\mathcal{U}\\ y^i\in\mathbb{R}^{(T)},\,r_i\in\mathbb{R}}}{\text{V-max}} \quad \left(w^\top y^1 + r_1, \ldots, w^\top y^m + r_m\right)$$
$$\text{s.t.} \quad \begin{array}{l} \sum_{i=1}^m \lambda_i \left(c_i - \sum_{t\in T} y^i_t v_t\right) = 0_n, \\ \sum_{i=1}^m \lambda_i y^i \in \mathbb{R}^{(T)}_+, \\ \sum_{i=1}^m \lambda_i r_i = 0, \end{array}$$

where we replace $\lambda \in \Delta^m_+$ by $\lambda \in \Delta^m_{++}$ in (OD).

DEFINITION 4.4. *A feasible point* $(\bar\lambda, \bar y, \bar r, (\bar v, \bar w))$ *of the problem* (MOD) *is said to be an* efficient solution *of* (MOD) *if there is no feasible point* $(\lambda, y, r, (v, w))$ *of* (MOD) *such that*

$$\left\{\begin{array}{ll} \bar w^\top \bar y^i + \bar r_i \le w^\top y^i + r_i & \forall\, i \in I, \textit{and} \\ \bar w^\top \bar y^j + \bar r_j < w^\top y^j + r_j & \textit{for at least one } j \in I. \end{array}\right.$$

Analogously to Theorem 4.2, we prove the following robust duality theorem with respect to properly efficient solutions.

THEOREM 4.5 (robust duality with respect to proper efficiency). *Let* $\bar x$ *be a robust properly efficient solution of* (P^u). *Suppose that the LFMCQ holds at* $\bar x \in X$ *and* $\mathcal{U}_t$ *is convex for all* $t \in T$. *Then, there exists an efficient solution* $(\bar\lambda, \bar y, \bar r, (\bar v, \bar w))$ *for* (MOD) *such that* $c_i^\top \bar x = \bar w^\top \bar y^i + \bar r_i \,\forall\, i \in I$.

Proof. Using a similar method of proof as in Theorem 4.2, one can show that weak duality holds; that is, there is no $x \in X$ and no $(\lambda, y, (v, w))$ feasible for (MOD) such that

$$\left\{\begin{array}{ll} c_i^\top x \le w^\top y^i + r_i & \forall\, i \in I, \text{and} \\ c_j^\top x < w^\top y^j + r_j & \text{for at least one } j \in I. \end{array}\right.$$

As $\bar x$ is a properly efficient solution of (P^u) and the LFMCQ at $\bar x$ holds, from Theorem 3.6, there exists $\bar\lambda \in \Delta^m_{++}, \bar y^i \in \mathbb{R}^{(T)}$, $(\bar v_t, \bar w_t) \in \mathcal{U}_t$, $t \in T$, and $\bar r_i \in \mathbb{R}$ such that (3.9) holds. In particular, we have $(\bar\lambda, \bar y, \bar r, (\bar v, \bar w))$ is feasible for (MOD). The conclusion will follow if we show that $(\bar\lambda, \bar y, \bar r, (\bar v, \bar w))$ is an efficient solution of (MOD). To see this, we proceed by the method of contradiction and assume that there exists $(\lambda, y, r, (v, w))$ feasible for (MOD) such that

$$\left\{\begin{array}{ll} \bar w^\top \bar y^i + \bar r_i \le w^\top y^i + r_i & \forall\, i \in I, \text{and} \\ \bar w^\top \bar y^j + \bar r_j < w^\top y^j + r_j & \text{for at least one } j \in I. \end{array}\right.$$

This together with the last relation in (3.9) implies that $c_i^\top \bar x \le w^\top y^i + r_i \,\forall\, i \in I$ and $c_j^\top \bar x < w^\top y^j + r_j$ for at least one $j \in I$. Since $\bar x \in X$, this contradicts the weak duality statement, and so, the conclusion follows. ∎

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