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Complete noncompact Spin(7) manifolds from self-dual Einstein 4–orbifolds

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We present an analytic construction of complete noncompact 8–dimensional Ricci-flat manifolds with holonomy Spin(7). The construction relies on the study of the adiabatic limit of metrics with holonomy Spin(7) on principal Seifert circle bundles over asymptotically conical G_2–orbifolds. The metrics we produce have an asymptotic geometry, so-called ALC geometry, that generalises to higher dimensions the geometry of 4–dimensional ALF hyperkähler metrics.

We apply our construction to asymptotically conical G_2–metrics arising from self-dual Einstein 4–orbifolds with positive scalar curvature. As illustrative examples of the power of our construction, we produce complete noncompact Spin(7)–manifolds with arbitrarily large second Betti number and infinitely many distinct families of ALC Spin(7)–metrics on the same smooth 8–manifold.

53C25; 53C10, 53C29, 53C80

1 Introduction

In this paper we provide a new analytic construction of complete noncompact Ricci-flat 8–manifolds with holonomy Spin(7) and nonmaximal volume growth. The starting point of the construction is an asymptotically conical (AC) 7–dimensional orbifold (B, g_0) with holonomy G_2 together with a suitable principal circle orbibundle $\pi\colon M \to B$ with total space M a smooth 8–manifold; we will then say that $\pi\colon M \to B$ is a Seifert bundle. Our method then produces a 1–parameter family $\{g_\epsilon\}_{\epsilon>0}$ of circle-invariant Spin(7)–metrics on M such that (M, g_ϵ) collapses back

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to the orbifold (B, g_0) as $\epsilon \to 0$. The metric g_ϵ has controlled asymptotic geometry, so-called ALC (asymptotically locally conical) geometry: along the (unique) end of M the metric g_ϵ approaches a Riemannian submersion with base a conical (orbifold) metric and circle fibres of fixed finite length.

Theorem A *Let M^8 be a smooth noncompact 8–manifold with an almost-free circle action, ie such that the quotient space $B = M/S^1$ is an orbifold. Assume that B carries an AC orbifold metric g_0 with holonomy G_2 and that the principal circle orbibundle $M \to B$ satisfies the topological condition*

$$c_1^{\mathrm{orb}}(M) \cup [\varphi_0] = 0 \in H^5_{\mathrm{orb}}(B),$$

where φ_0 is the closed and coclosed G_2 3–form on B inducing the G_2–metric g_0.

Then for every $\epsilon > 0$ sufficiently small there exists a circle-invariant ALC $\mathrm{Spin}(7)$–metric g_ϵ on M such that the sequence (M, g_ϵ) collapses to (B, g_0) with bounded curvature as $\epsilon \to 0$.

We refer the reader to Theorem 3.36 later in the paper for a more precise statement.

Motivation and applications In joint work with Haskins and Nordström [32], we developed a similar construction of highly collapsed ALC G_2–holonomy metrics on suitable principal circle bundles over *smooth* AC Calabi–Yau 3–folds. The construction of [32] allowed us to exploit recent progress on the existence of Calabi–Yau cone metrics (see Collins and Székelyhidi [23], Futaki, Ono and Wang [36] and Gauntlett, Martelli, Sparks and Waldram [40]) and AC Calabi–Yau metrics (see Conlon and Hein [24], Goto [43] and van Coevering [22]) to produce infinitely many complete noncompact G_2–manifolds and complete G_2–metrics depending on an arbitrarily large number of parameters. Only a handful of complete noncompact G_2–manifolds was previously known.

The existence of an analogous construction of $\mathrm{Spin}(7)$–metrics from AC G_2–manifolds as in Theorem A is therefore not in itself surprising. The fact that such a construction can be used to produce significant results in $\mathrm{Spin}(7)$–geometry, however, is a priori much less clear: the naive generalisation of [32] to the $\mathrm{Spin}(7)$–setting using only smooth manifolds would be a theorem that currently applies to only one example! The simultaneous extension of [32] to the orbifold setting is the crucial new ingredient that makes Theorem A useful. Indeed, in contrast to the Calabi–Yau case, our current knowledge of smooth AC G_2–manifolds is extremely limited: in 1989 Bryant and

Salamon [17] constructed three (explicit) examples of AC G_2–metrics; only very recently has an infinite family of new simply connected examples been found (Foscolo, Haskins and Nordström [33, Theorem C]). On the other hand, Bryant–Salamon's construction of AC G_2–metrics yields an AC *orbifold* G_2–metric on the total space of the orbibundle of anti-self-dual 2–forms over any self-dual Einstein 4–orbifold with positive scalar curvature. This construction yields a large supply of AC G_2–orbifolds since many self-dual Einstein orbifold metrics can be constructed using the quaternionic Kähler quotient construction of Galicki and Lawson [37]. By using such orbifolds we are able to produce a wealth of new complete noncompact Spin(7)–manifolds.

Theorem B *For every $k \geq 1$ there exists a smooth noncompact 8–manifold that retracts onto $\sharp_k(S^2 \times S^3)$ and carries a family of complete ALC* Spin(7)*–metrics. In particular, there exist complete noncompact* Spin(7)*–manifolds with arbitrarily large second Betti number.*

Only a handful of complete noncompact Spin(7)–metrics was previously known; see Bazaikin [5; 6], Bryant and Salamon [17], Cvetič, Gibbons, Lü and Pope [27; 28], Gukov and Sparks [45] and Kovalev [65]. As a further illustration of the power of our construction, we also find a smooth noncompact 8–manifold that can be described as a circle orbibundle over an AC G_2–orbifold in infinitely many different ways.

Theorem C *The nontrivial rank-3 real vector bundle over S^5 carries infinitely many families of complete ALC* Spin(7)*–metrics. Different families are distinguished by their (unique) tangent cone at infinity.*

In other words there are infinitely many inequivalent circle actions on the 8–manifold M in question such that the orbit space M/S^1 is the orbibundle of anti-self-dual 2–forms over a self-dual Einstein 4–orbifold with positive scalar curvature.

The analytic framework introduced in this paper to work on AC orbifolds can also be exploited to extend the construction of complete ALC G_2–metrics in [32] to the orbifold setting. In [32] examples of ALC G_2–metrics arose from AC Calabi–Yau metrics on crepant resolutions of Calabi–Yau cones. Often it is natural to consider only *partial* resolutions of Calabi–Yau cones, which replace the singularity of the cone with simpler (albeit nonisolated) orbifold singularities. Combining the techniques of this paper with [32] allows us to construct complete G_2–metrics on suitable circle orbibundles over these orbifold partial resolutions. As an illustration of the possible complete G_2–metrics arising from this construction, Theorem 4.12 establishes an analogue of Theorem C in the G_2 setting by constructing infinitely many distinct families of ALC G_2–metrics on $S^3 \times \mathbb{R}^4$.

Considering sequences of Spin(7)–metrics collapsing to G_2–orbifolds and not only smooth manifolds is also very natural from the point of view of the theory of Riemannian collapse. Indeed, Fukaya [35, Proposition 11.5] has shown that Gromov–Hausdorff limits of Riemannian manifolds that collapse with bounded curvature in codimension 1 must be orbifolds. Thus, besides extending it to the Spin(7) setting, Theorem A extends the construction of [32] to its most general context.

There are three main aspects to the proof of Theorem A and its corollaries, Theorems B and C. The general strategy of the proof of Theorem A relies on the adiabatic limit of Spin(7)–metrics with a circle symmetry. The strategy is partially motivated by known families of cohomogeneity-one ALC Spin(7)–metrics and the duality between M theory and Type IIA string theory in theoretical physics. Successfully implementing this strategy requires a refined knowledge of closed and coclosed forms on AC manifolds and orbifolds. Note that the orbifolds we consider in this paper are noncompact and have a singular set that is allowed to extend all the way to infinity. Describing the analytic framework to work on such orbifolds is an important technical aspect of this paper. Finally, the third aspect of this work is the search for self-dual Einstein 4–orbifolds with positive scalar curvature that give rise to concrete examples of AC G_2–orbifolds to feed into Theorem A. In the rest of this introduction we discuss each of these three aspects.

The adiabatic limit of circle-invariant Spin(7)–metrics Bryant and Salamon [17] constructed the first known example of a complete metric with holonomy Spin(7). The Bryant–Salamon metric is an explicit AC Spin(7)–metric on the spinor bundle $\not{S}(\mathbb{S}^4)$ of the round 4–sphere. The metric is asymptotic at infinity to the Riemannian cone over the squashed Einstein metric on S^7; see Jensen [56].

The Bryant–Salamon metric on $\not{S}(\mathbb{S}^4)$ is invariant under the natural cohomogeneity-one action of $\mathrm{Sp}(2) \simeq \mathrm{Spin}(5)$. Cvetič, Gibbons, Lü and Pope [27] studied the ODE system describing general Sp(2)–invariant Spin(7)–metrics. They found a new explicit example and argued that it moved in a 1–parameter family up to scale, the existence of which was further studied in Cvetič, Gibbons, Lü and Pope [28] and rigorously proved in Bazaikin [6]. The asymptotic behaviour of the metrics in this family, labelled $\mathbb{B}_8$ in the physics literature, is different from the Bryant–Salamon metric: the $\mathbb{B}_8$ metrics have nonmaximal volume growth r^7; at infinity the metric approaches a Riemannian submersion with base metric the Riemannian cone over the nearly Kähler metric on $\mathbb{CP}^3$ and circle fibres with fixed finite length ℓ. In [27] the acronym ALC (asymptotically locally conical) was introduced to describe this asymptotic geometry: the ALC asymptotic geometry is analogous to the asymptotic geometry of ALF (asymptotically locally flat) 4–dimensional manifolds, except that the tangent cone at infinity is not necessarily flat. Up to scale, the asymptotic length ℓ of the circle fibres can be taken as the parameter that distinguishes different members of the $\mathbb{B}_8$ family. In fact, the asymptotic circle action extends to a global symmetry of the $\mathbb{B}_8$ metrics, which are not only invariant under the left action of $\mathrm{Sp}(2)$, but also under the circle acting on the fibres of $\not{S}(\mathbb{S}^4)$ as the Hopf circle action. This circle action is not free since it fixes the zero-section, but the quotient space is still a smooth manifold, $\Lambda^- T^*\mathbb{S}^4$. As $\ell \to 0$ the family of ALC $\mathbb{B}_8$ metrics collapses to the Bryant–Salamon AC G_2–metric on $\Lambda^- T^*\mathbb{S}^4$ [17], which is asymptotic to the cone over the nearly Kähler metric on $\mathbb{CP}^3$. As $\ell \to 0$ the curvature of the $\mathbb{B}_7$ metrics blows up along the zero-section $\mathbb{S}^4$, the fixed locus of the circle action on $\not{S}(\mathbb{S}^4)$.

These first examples led to an explosion of activity in the physics and, later, mathematics literature discussing further (conjectural) families of ALC manifolds with exceptional holonomy. An explicit ALC Spin(7)–metric on $\mathbb{R}^8$ was found in [27] and a new family of Sp(2)–invariant ALC Spin(7)–metrics on the canonical line bundle of $\mathbb{CP}^3$ was studied numerically in [26, Section 2] and later constructed rigorously in Bazaikin [5]. Further work concentrated on the case of cohomogeneity-one $\mathrm{SU}(3)$–invariant Spin(7)–metric with principal orbits the Aloff–Wallach spaces $\mathrm{SU}(3)/U(1)_{k,l}$, where the integers k, l determine the embedding of $U(1)$ in the maximal torus of $\mathrm{SU}(3)$: the discovery of some explicit solutions, numerical investigations of the relevant ODE systems and a rigorous study of local solutions defined in a neighbourhood of the possible singular orbits were carried out by various authors; see Cvetič, Gibbons, Lü and Pope [26], Gukov and Sparks [45], Gukov, Sparks and Tong [46], Kanno and Yasui [60; 61], Reidegeld [80] and Lehmann [69]. Besides [69], in general the existence of complete solutions remains an open question.

From the physics perspective, the interest in ALC metrics with exceptional holonomy arises from the equivalence between M theory and Type IIA string theory in the limit of weak string coupling constant. Kaluza–Klein reduction of supersymmetric M–theory solutions along a circle of small radius proportional to the string coupling constant corresponds geometrically to the study of sequences of manifolds with exceptional holonomy collapsing in codimension 1 along a (degenerate) circle fibration. For instance, the collapse of the $\mathbb{B}_8$ family of ALC Spin(7)–metrics to the Bryant–Salamon AC G_2–metric in the limit $\ell \to 0$ realises the duality between M theory "compactified" on $\cancel{S}(\mathbb{S}^4)$ and Type IIA string theory on $\Lambda^- T^*\mathbb{S}^4$ with a D6–brane wrapping the zero-section. The geometric interpretation of the latter physical jargon is that $\cancel{S}(\mathbb{S}^4) \setminus \mathbb{S}^4$ can be regarded as a principal circle bundle over $\Lambda^- T^*\mathbb{S}^4 \setminus \mathbb{S}^4$ with first Chern class evaluating to one on the 2–sphere linking the zero-section. The collapse with bounded curvature exhibited by the families of ALC Spin(7)–metrics in Theorem A corresponds instead to the physical statement that the weak coupling limit of Type IIA theory on the G_2–orbifold B with Ramond–Ramond 2–form flux representing $c_1^{\mathrm{orb}}(M)$ is equivalent to the low-energy limit of M–theory on the total space M of the circle orbibundle.

The idea of proof of Theorem A is close to this physical interpretation; see for example Cvetič, Gibbons, Lü and Pope [25] and Kaste, Minasian, Petrini and Tomasiello [63]. We consider a Spin(7)–manifold (M, g) with an isometric circle action. Denote by B the orbit space M/S^1 and assume for simplicity it is a smooth manifold. A Spin(7)–metric is uniquely determined by a closed 4–form Φ on M satisfying certain pointwise nonlinear algebraic constraints. In the presence of a Killing field ξ (that we assume preserves also Φ) we can write the metric g on M as $g = h^{\frac{1}{3}} g_B + h^{-1}\theta^2$, where θ is an S^1–invariant 1–form on M dual to ξ, ie a connection 1–form on the principal circle bundle $M \to B$, and h and g_B are a positive function and a Riemannian metric on B. We can then formulate the holonomy reduction of g as a system of nonlinear partial differential equations $\Psi(\theta, h, \varphi) = 0$. Here $\varphi = \xi \lrcorner \Phi$ defines a G_2–structure on B inducing the metric g_B. In dimension 4, the analogous dimensional reduction of hyperkähler metrics along the orbits of a triholomorphic vector field yields the famous Gibbons–Hawking ansatz [41], which reduces the existence of the hyperkähler metric to a *linear* equation on the orbit space $\mathbb{R}^3$. The system of equations $\Psi(\theta, h, \varphi) = 0$ arising from the dimensional reduction of Spin(7)–metrics is nonlinear and in general it is not at all clear how to study existence of solutions.

We then employ the strategy of deforming $\Psi(\theta, h, \varphi) = 0$ to a different equation we can handle better. The most natural geometric degeneration is to consider families of S^1–invariant Spin(7)–metrics with circle orbits of smaller and smaller length. We introduce a small parameter $\epsilon > 0$ and consider a sequence of S^1–invariant metrics $g_\epsilon = h^{\frac{1}{3}} g_B + \epsilon^2 h^{-1}\theta$. The metric g_ϵ has Spin(7)–holonomy if and only if (θ, h, φ) satisfy $\Psi(\epsilon\,\theta, h, \varphi) = 0$. Here $\varphi = \xi \lrcorner \Phi_\epsilon$, where Φ_ϵ is the 4–form inducing g_ϵ. For $\epsilon > 0$ the equation $\Psi(\epsilon\,\theta, h, \varphi) = 0$ is equivalent to $\Psi(\theta, h, \varphi) = 0$ by scaling, but at $\epsilon = 0$ the equation simplifies: solutions are of the form $(0, 1, \varphi_0)$, where φ_0 is a torsion-free G_2–structure on the orbit space B, ie the limiting metric g_B induced by φ_0 has holonomy G_2. It is important to note that the equation $\Psi(\epsilon\,\theta, h, \varphi) = 0$ depends smoothly on ϵ up to and including $\epsilon = 0$. In Theorem A we assume that an AC G_2–metric on B is given and then try to perturb the solution $(0, 1, \varphi_0)$ into a solution of $\Psi(\epsilon\,\theta, h, \varphi) = 0$ for $\epsilon > 0$.

The first step is to understand elements in the kernel of the linearisation $\mathcal{L}$ of Ψ at $(0, 1, \varphi_0)$, since they correspond to formal tangent vectors to curves of solutions to $\Psi(\epsilon\,\theta, h, \varphi) = 0$ for $\epsilon \in [0, \epsilon_0)$. A dichotomy arises at this stage: it is geometrically meaningful to consider bounded solutions to the linearised problem as well as unbounded solutions with prescribed singularities in codimension 4. In this paper we only consider the former case, which corresponds to sequences of Spin(7)–metrics collapsing with bounded curvature; the case of codimension-4 singularities, related to collapse with unbounded curvature along the fibres of a circle fibration which degenerates in codimension 4, is more involved and will be treated elsewhere. It turns out that bounded solutions (θ_0, h_0, ρ_0) to the linearised problem $\mathcal{L}(\theta_0, h_0, \rho_0) = 0$ are completely determined by the choice of a principal circle bundle $M \to B$ with $c_1(M) = [d\theta_0]$. The topological constraint $c_1(M) \cup [\varphi_0] = 0 \in H^5(B)$ arises at this stage as the necessary and sufficient condition for solving the linearised problem.

We can now imagine reconstructing a curve of solutions to $\Psi(\epsilon\,\theta, h, \varphi) = 0$ for $\epsilon \geq 0$ sufficiently small by deforming away from the initial solution $(0, 1, \varphi_0)$ in the direction of (θ_0, h_0, ρ_0) via an application of the implicit function theorem. The key step is the study of the mapping properties of the linear operator $\mathcal{L}$. Now, $\mathcal{L}$ is not obviously elliptic as it involves a combination of differential, codifferential and decomposition of differential forms into different types induced by the representation theory of G_2 (analogous to the (p, q)–type decomposition on complex manifolds). It is therefore not immediately obvious how to identify the cokernel of $\mathcal{L}$. In the construction of ALC G_2–metrics from AC Calabi–Yau 3–folds in [32] the linearised problem was complicated enough that we were only able to prove existence of solutions by solving the analogue of the equation $\Psi(\epsilon\,\theta, h, \varphi) = 0$ as a power series in ϵ, exploiting special cancellations that were only evident by solving the equation order-by-order in ϵ. In this paper we are instead able to set up a direct argument using the implicit function theorem. The key difference with respect to [32] is that the space of G_2–structures on $\mathbb{R}^7$ is an open set in a linear space, while the space of SU(3)–structures on $\mathbb{R}^6$ is cut out by nonlinear constraints and therefore a further choice of "exponential map" is necessary. In order to understand the mapping properties of the linearised operator $\mathcal{L}$ and therefore prove Theorem A, we need to exploit the interplay between the Laplacian, the Dirac operator and type decomposition of differential forms on AC G_2–orbifolds and, crucially, the fact that we restrict to variations of the G_2–structure φ in the same cohomology class as φ_0.

Analysis on AC orbifolds The discussion so far has been formal. In order to implement the strategy we have just outlined we need to develop analytic tools to work on AC orbifolds. There are two main issues to take into account: the fact that we work on noncompact spaces, and the orbifold singularities. The analysis of linear elliptic operators on *smooth* AC manifolds using weighted Sobolev and Hölder spaces is well established. Analysis on *compact* orbifolds has also been used in many geometric applications. However, the orbifolds we consider in this paper are noncompact and in view of the applications we have in mind we cannot insist that the singular set be compact. To the author's knowledge the simultaneous presence of an AC end and orbifold singularities allowed to extend to infinity seems not to have been considered before in the literature. We therefore felt it was necessary to include a self-contained exposition of the geometric and analytic tools we need. Since the orbifolds we consider arise as global quotients of smooth noncompact manifolds by a circle action, we develop the theory in a way that makes crucial use of this assumption. Instead of working on the AC orbifold B itself we work on the smooth total space M of the circle orbibundle over B: elliptic operators acting on differential forms on B are replaced with transversally elliptic operators acting on basic forms on M. Note that since every Riemannian orbifold arises as the quotient of a smooth manifold by the action of a compact Lie group (the orthogonal frame orbibundle of an orbifold is always a smooth manifold, see [1, Corollary 1.24]), a similar strategy can be (and has been) applied more generally. The case of *Seifert circle bundles* (ie principal circle orbibundles with smooth total space) allows us to give a particularly clean exposition.

The central object in our exposition is the so-called *adapted connection* ∇ of a Riemannian foliation, see [8, Definition 1.7] and [81, Definition 3.13]: a certain metric connection with torsion on TM that preserves the splitting of the tangent bundle of M into vertical and horizontal subbundles. We use ∇ instead of the Levi-Civita connection of M to define natural elliptic operators acting on basic sections of appropriate vector bundles. For instance, the exterior differential and codifferential acting on differential forms are replaced by the *covariant* differential and codifferential induced by ∇. Restricting these "adapted" operators to basic forms allows us to develop the linear theory of elliptic operators acting on weighted Banach spaces on AC orbifolds exactly as in the case of smooth AC manifolds. Once the right language has been developed, the only new analytic and geometric ingredient is Parker's equivariant Sobolev inequality [79].

As nonexperts in the theory of Riemannian foliations, we are unable to evaluate the originality in our treatment and how much our clean exposition depends on the restriction to the simple case of foliations with totally geodesic 1–dimensional leaves. For example other authors use different "adapted" connections for different purposes instead of our uniform approach using ∇; see [86]. An original contribution of this paper is a calculation of all the topological contributions to the weighted L^2–cohomology of AC manifolds and orbifolds. The L^2–cohomology of smooth AC manifolds is well known; see [48, Theorem 1.A; 70, Example 0.15]. For geometric applications, however, it is often important to work with differential forms that are not necessarily square-integrable. For instance, there are many examples of higher-dimensional AC manifolds with special holonomy that are asymptotic to their tangent cone at infinity with a non-L^2 rate of decay. In Theorem 2.31 we apply the Fredholm theory we develop in Section 2 to give a complete description of the topological contributions to the weighted L^2–cohomology of AC manifolds. Our elementary proof immediately generalises to the case of AC orbifolds. Special cases of our result have been derived by other authors (see for example [62, Section 4.5]), but as far as we are aware a proof in arbitrary dimension is not currently available in the literature.

Self-dual Einstein 4–orbifolds and special holonomy The analytic tools we develop in Section 2, including our results about weighted L^2–cohomology of AC orbifolds, allow us to implement the adiabatic-limit strategy and prove our main abstract existence result, Theorem A. The final part of the paper is devoted to the study of concrete examples produced by this construction. All the examples we consider in the paper are obtained by applying Theorem A to Bryant–Salamon's AC G_2–metrics arising from suitable self-dual Einstein 4–orbifolds with positive scalar curvature.

It is well known that self-dual Einstein metrics with positive scalar curvature in dimension 4 generate many different geometries related to special holonomy; see [11, Section 13.4]. If Q is a self-dual Einstein 4–manifold (or orbifold) with positive scalar curvature then its twistor space Z, the unit-sphere bundle in the (orbi)bundle of anti-self-dual 2–forms, carries two Einstein metrics with positive scalar curvature: a Kähler–Einstein metric [82] and a nearly Kähler metric [30]. The Konishi bundle S, the principal SU(2)– or SO(3)–bundle associated with Z, also has two Einstein metrics: a 3–Sasaki metric [64; 85] and a (strict) nearly parallel G_2–metric [39; 34]. Except for the Kähler–Einstein metric, these higher-dimensional compact Einstein spaces carry real Killing spinors and are therefore related to special holonomy via the cone construction [3]: the cone over the nearly Kähler metric on Z has holonomy G_2, the cone over the 3–Sasaki metric on S is hyperkähler and the one over the nearly parallel G_2–metric on S has holonomy $\mathrm{Spin}(7)$. Furthermore, vector-bundle constructions of Ricci-flat metrics on (orbi)bundles over Q can be used to produce noncompact spaces with special holonomy (partially) desingularising these cones. For example, a well known seminal construction by Calabi [18] yields an AC Calabi–Yau metric on the canonical line bundle over the Kähler–Einstein 3–fold Z; this metric is asymptotic to the cone over a finite quotient of S by a cyclic group that only preserves one Sasaki structure in the 3–Sasaki structure. Bryant–Salamon's construction [17] of a (unique up to scale) AC G_2–metric on the (orbi)bundle of anti-self-dual 2–forms on Q plays a distinguished role in this paper.

There are also known constructions of $\mathrm{Spin}(7)$–holonomy metrics from self-dual Einstein 4–manifolds: in [17] Bryant–Salamon construct an AC $\mathrm{Spin}(7)$–metric on the spinor bundle of a spin self-dual Einstein 4–manifold with positive scalar curvature; in [6] Bazaĭkin shows that the Bryant–Salamon AC metric is in fact the limit of a 1–parameter family of ALC $\mathrm{Spin}(7)$–metrics on the same 8–manifold; Bazaĭkin [5] also constructs families of ALC $\mathrm{Spin}(7)$–metrics arising as deformations of Calabi's AC Calabi–Yau metrics on K_Z. By a result of Hitchin [51] the only smooth self-dual Einstein 4–manifolds with positive scalar curvature are $\mathbb{S}^4$ and $\mathbb{CP}^2$. As a consequence, there are only three smooth $\mathrm{Spin}(7)$–manifolds produced by these constructions. The constructions of Bryant–Salamon and Bazaĭkin immediately generalise to self-dual Einstein 4–orbifolds Q to produce many *singular* $\mathrm{Spin}(7)$–metrics. The reasons these metrics are never complete is that the self-dual Einstein 4–orbifold Q or its twistor space Z (which is always singular if Q is) are always embedded in the resulting spaces. If one considers principal orbibundles instead of vector bundles, however, it is instead often possible to obtain smooth manifolds. For example, Boyer and Galicki and their collaborators constructed infinitely many smooth 3–Sasaki manifolds using self-dual Einstein 4–orbifolds [14]. Similarly, many smooth Sasaki–Einstein manifolds arise as circle orbibundles over Kähler–Einstein Fano orbifolds; see [11, Chapter 11]. Theorem A allows us to obtain smooth $\mathrm{Spin}(7)$–manifolds from self-dual Einstein 4–orbifolds in an analogous way.

Now, a self-dual Einstein 4–orbifold Q with positive scalar curvature yields complete $\mathrm{Spin}(7)$–metrics via Theorem A if and only if there exists a smooth 8–manifold M

arising as a circle orbibundle over $B=\Lambda^-T^*Q$. Indeed, note that $H^5_{\mathrm{orb}}(B)=0$ since B retracts onto Q and therefore the necessary topological condition in Theorem A is vacuous. We prove in Lemma 4.2 that B is the circle quotient of a smooth 8–manifold if and only if Q itself is the circle quotient of a smooth 5–manifold; whenever this happens we say that Q is Spin(7)*–admissible*. Theorem A is useful only if we can find a large supply of Spin(7)–admissible self-dual Einstein 4–orbifolds with positive scalar curvature.

Infinitely many self-dual Einstein 4–orbifolds with positive scalar curvature are known thanks to the quaternionic Kähler quotient construction of Galicki and Lawson [37]. For example, infinitely many self-dual Einstein 4–orbifolds with positive scalar curvature arise as quaternionic Kähler reductions of quaternionic projective space $\mathbb{HP}^n$ by a subgroup of $\mathrm{Sp}(n+1)$. In [37] Galicki and Lawson illustrate their quotient construction by considering self-dual Einstein metrics on weighted complex projective planes $\mathbb{WCP}^2[q_1,q_2,q_3]$ arising as quotients of $\mathbb{HP}^2$ by a circle. All these orbifolds are clearly Spin(7)–admissible, since weighted projective planes are all circle quotients of S^5. Theorem C follows from applying our main existence result, Theorem A, to these Galicki–Lawson examples.

It is likely that many more examples of self-dual Einstein 4–orbifolds are Spin(7)–admissible. For example, all toric self-dual Einstein 4–orbifolds, ie 4–orbifolds with a T^2–symmetry, must arise as quaternionic Kähler quotients of $\mathbb{HP}^n$ by an $(n-1)$–dimensional torus [19]. The geometry of these toric orbifolds is then completely encoded in the combinatorics of the embedding of the Lie algebra of T^{n-1} into the Lie algebra of the maximal torus of $\mathrm{Sp}(n+1)$. It is likely that combinatorial conditions characterising Spin(7)–admissibility can be given in the same way that clear combinatorial criteria characterise the existence of a smooth 3–Sasaki Konishi bundle [14, Theorem 2.14]. Instead of pursuing such a systematic combinatorial approach, however, in this paper we construct by hand an explicit family of examples with unbounded second orbifold Betti number. In the proof of Theorem B we use an infinite list of self-dual Einstein 4–orbifolds with positive scalar curvature arising from ALE gravitational instantons of type A_n via the hyperkähler/quaternionic Kähler correspondence. This correspondence associates to each hyperkähler metric with a circle action that preserves only one complex structure in the twistor sphere an S^1–invariant quaternionic Kähler space of the same dimension. The examples of self-dual Einstein 4–orbifolds we consider (originally considered by Galicki and Nitta [38] without any reference to the hyperkähler/quaternionic Kähler correspondence) give rise to 8–manifolds that are rank-3 real vector bundles over $\sharp_k(S^2\times S^3)$ for any $k\geq 1$. Joyce's analytic constructions of compact Spin(7)–manifolds [57; 58] can also be adapted to produce complete noncompact Spin(7)–metrics by desingularising orbifold quotient singularities of noncompact flat orbifolds [59, Sections 13.1 and 15.1] or asymptotically cylindrical Calabi–Yau 4–folds [65]. However, the variety of examples produced by Theorem B is new.

ALC G_2–manifolds from AC Calabi–Yau orbifolds The geometric and analytic framework to work on AC orbifolds we introduce in this paper allows us to extend the construction of ALC G_2–manifolds from AC Calabi–Yau 3–folds in [32] to the orbifold case. While [32] already yields infinitely many examples of complete noncompact G_2–manifolds, as a simple application of our orbifold extension we produce infinitely many distinct families of ALC G_2–metrics on $S^3 \times \mathbb{R}^4$; see Theorem 4.12. As in Theorem C, the families are distinguished by their tangent cones at infinity. In order to prove Theorem 4.12 we show that there are infinitely many ways of realising $S^3 \times \mathbb{R}^4$ as a circle orbibundle over a small orbifold partial resolution of a Gorenstein toric Kähler cone. For example, there is an infinite list of S^1–actions on $S^3 \times \mathbb{R}^4$ labelled by two coprime positive integers p and q such that $B = S^3 \times \mathbb{R}^4 / S^1_{p,q}$ is a small partial resolution of the Calabi–Yau cone over the so-called $Y^{p,q}$ Sasaki–Einstein 5–manifold [40]. AC Calabi–Yau metrics on B are constructed by Martelli and Sparks [74] using the formalism of Hamiltonian 2–forms. The construction of [32], suitably extended to the orbifold setting using the analysis of Section 2 in this paper, then immediately yields families of highly collapsed ALC G_2–metrics on $S^3 \times \mathbb{R}^4$.

Plan The rest of the paper is organised in three main sections corresponding to the three different aspects of the proof of Theorem A and its applications. Section 2 develops the necessary geometric and analytic framework to work on AC orbifolds and includes the proof of Theorem 2.31 about weighted L^2–cohomology of AC orbifolds. Section 3 contains the proof of Theorem A implementing the adiabatic-limit strategy we have outlined. Finally, Section 4 presents the concrete examples of Theorems B and C.

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2 Asymptotically conical orbifolds

In this section we develop the necessary geometric and analytic framework to work on AC orbifolds. For the geometric applications of the paper it will suffice to consider orbifolds arising as quotients of smooth manifolds by a circle action. While we will restrict to this situation for ease of exposition, note that every (effective) orbifold arises as the quotient of a smooth manifold by an effective *almost-free* (ie with finite stabilisers) action of a compact Lie group [1, Corollary 1.24]. In Section 2.1 we collect preliminary materials on orbifolds and foliations and introduce the language we are going to use in the rest of the paper. In Section 2.2 we develop a good Fredholm theory for linear elliptic operators on AC orbifolds. We apply this theory in Section 2.3 to provide a computation of the weighted L^2–cohomology of AC orbifolds.

2.1 Orbifolds and principal Seifert circle bundles

As a preliminary, we collect the facts about orbifolds and foliations that we are going to use throughout the section. We use the traditional notion of an orbifold, ie an *effective* orbifold in the sense of [1, Definitions 1.1 and 1.2], and avoid almost completely the language of groupoids. Indeed, the orbifolds we will consider all arise as quotients of an effective *almost-free* (ie with finite stabilisers) circle action on a smooth manifold. Our exposition uses this fact in an essential way.

Let ξ be a nowhere-vanishing vector field on a smooth manifold M of dimension $n+1$. Assume that the orbits of ξ are all closed, ie (possibly after an appropriate normalisation) ξ generates an effective almost-free circle action on M. The orbit space $B=M/S^1$ has a natural *orbifold* structure and $\pi: M\to B$ is a principal circle *orbibundle*. We refer to [1, Chapters 1 and 2] and [11, Chapter 4] for basics on orbifolds and orbibundles. Since its total space is smooth, $\pi: M\to B$ is a *Seifert fibration* in the sense of [42, Definition 1.2]. An alternative viewpoint is that the vector field ξ defines a foliation on M. We are going to use various notions from the theory of foliations; see [75; 81; 86].

We fix a Riemannian metric g on M such that (a) ξ has unit length, and (b) the orbits of ξ are geodesics. By [86, Proposition 6.7] such a metric exists if and only if there exists a 1–form θ on M such that $\theta(\xi)=1$ and $\mathcal{L}_\xi\theta=0$. Denote by $\mathcal{H}$ the *horizontal* bundle $\ker\theta=\xi^\perp$. Observe that $\mathcal{H}$ can be identified with the pullback to M of the orbifold tangent bundle of B. The restriction of g to $\mathcal{H}$ will be denoted by g_B since it defines a Riemannian metric on the orbifold B. We will refer to the data (M,π,θ,g_B) as a *Riemannian principal Seifert* (*circle*) *bundle*.

We will always assume that M is oriented with volume form $\mathrm{dv}_g = \theta \wedge \mathrm{dv}_B$, where $\mathrm{dv}_B = \xi \lrcorner \mathrm{dv}_g$ is a nowhere-vanishing section of $\Lambda^n \mathcal{H}^*$ satisfying $\mathcal{L}_\xi \mathrm{dv}_B = 0$. We will denote by $*_M$ the Hodge-star operator of (M, g, dv_g).

2.1.1 Projectable bundles and connections Let $\pi\colon M \to B$ be a Riemannian principal Seifert circle bundle and let $P \to M$ be a principal G–bundle, where G is a compact Lie group. We say that P is *projectable* if the circle action on M lifts to a circle action on P commuting with the G–action. Projectable principal bundles on M are in one-to-one correspondence with principal orbibundles on the orbifold B. In the theory of foliations there is a weaker notion of a foliated bundle [75, Section 2.6], where one only assumes that the vector field ξ lifts to a vector field $\widetilde{\xi}$ on P. The restriction of a foliated bundle P to an orbit $\mathcal{O} \simeq S^1$ of ξ is a trivial principal G–bundle endowed with a flat connection. If P is projectable then this flat connection has trivial holonomy.

Let V be a G–representation. Consider the associated vector bundle $E = P \times_G V \to M$. If P is foliated with lift $\widetilde{\xi}$ of ξ, we say that a section $s\colon M \to E$ of E is *basic* if $\mathcal{L}_{\widetilde{\xi}} \widetilde{s} = 0$, where $\widetilde{s}\colon P \to V$ is the G–equivariant function corresponding to s. If P is projectable we can interpret basic sections as sections of the orbibundle $E/S^1 \to B$. For this reason we will denote the space of basic smooth sections of E by $C^\infty(B; E)$. When the circle action on M is free and B is a manifold, then $C^\infty(B; E)$ coincides with the space of smooth sections of the bundle $E/S^1 \to B$. Spaces of basic sections with lower regularity (for example L^2 sections) are defined in a similar way.

A connection A on P, thought of as a Lie algebra-valued 1–form on P, is *projectable* if $\widetilde{\xi} \lrcorner A = 0 = \widetilde{\xi} \lrcorner F_A$. Doing analysis on M with projectable connections acting on basic sections is a replacement for doing analysis on the orbifold B without worrying about its singularities.

The oriented orthonormal frame bundle of M is projectable but the Levi-Civita connection ∇^{LC} of g is not. Following [81, Definition 3.13] (see also [8, Definition 1.7] in the case where B is smooth and $M \to B$ is an arbitrary fibration) we will introduce an adapted connection ∇, which is better suited to the Seifert fibration structure than the Levi-Civita connection. Let ξ be the vertical vector field generating the circle action on M and let θ be its dual 1–form. Let X and Y denote vectors in $\mathcal{H}$. The Levi-Civita connection ∇^{LC} of g is

$$\begin{aligned} \nabla^{\mathrm{LC}}_\xi \xi &= 0, & \nabla^{\mathrm{LC}}_X \xi &= \tfrac{1}{2}(X \lrcorner d\theta)^\sharp, \\ \nabla^{\mathrm{LC}}_\xi Y &= \tfrac{1}{2}(Y \lrcorner d\theta)^\sharp + [\xi, Y], & \nabla^{\mathrm{LC}}_X Y &= -\tfrac{1}{2} d\theta(X, Y)\, \xi + (\nabla^{\mathrm{LC}}_X Y)_{\mathcal{H}}. \end{aligned}$$

Note that since $\theta([\xi, X]) = -d\theta(\xi, X) = 0$ we have $[\xi, \mathcal{H}] \subset \mathcal{H}$. We now define

$$\nabla_\xi \xi = 0, \quad \nabla_X \xi = 0, \quad \nabla_\xi Y = [\xi, Y], \quad \nabla_X Y = (\nabla^{\mathrm{LC}}_X Y)_{\mathcal{H}}. \tag{2.1}$$

We will refer to ∇ as the *adapted connection* of the Seifert bundle $\pi\colon M \to B$. The adapted connection is a projectable metric connection, but it has nonvanishing torsion $T(U, V) = d\theta(U, V)\,\xi$. A vector field X on M is called *basic* if it is a basic section of $\mathcal{H}$, ie $X \in \mathcal{H}$ and $[\xi, X] = 0$. Note that $\nabla_\xi X = 0$ for every basic vector field. Basic vector fields are identified with vector fields on the orbifold B and under this identification the adapted connection ∇ corresponds to the Levi-Civita connection of g_B.

2.1.2 Transverse elliptic operators Let $E \to M$ be a projectable metric bundle endowed with a projectable metric connection A. Combining A with the adapted connection ∇ on M we obtain a projectable connection, still denoted by ∇, on any tensor bundle with values in E. We can then use ∇ to define differential operators on M acting on E–valued tensors.

A *basic tensor* is an S^1–invariant section of $\bigotimes^r \mathcal{H} \otimes \bigotimes^s \mathcal{H}^*$. Since $\mathcal{L}_\xi$ coincides with ∇_ξ, the adapted connection preserves basic tensors. Hence if P is a differential operator defined using the adapted connection and acting on sections of (a subbundle of) $\bigotimes^r TM \otimes \bigotimes^s T^*M \otimes E$, then the restriction of P to basic E–valued tensors is well-defined. We will refer to the restriction of P to basic tensors as a *transverse* (or basic) operator. A basic operator is *elliptic* if its extension as an operator acting on arbitrary E–valued tensors is elliptic.

We are particularly interested in "basic versions" of $d+d^*$, the rough Laplacian and the Dirac operator D acting on differential forms and spinors on M with values in E. Fix an orthonormal frame $e_1, \dots, e_{n+1}$ for (M, g). We will assume that $\{e_1, \dots, e_{n+1}\}$ is an *adapted frame*, ie $e_1, \dots, e_n$ are basic vector fields and $e_{n+1} = \xi$. We then define

$$d_\nabla = \sum_{i=1}^{n+1} e_i \wedge \nabla_{e_i}, \quad d_\nabla^* = -\sum_{i=1}^{n+1} e_i \lrcorner \nabla_{e_i}, \quad \nabla^*\nabla = -\sum_{i=1}^{n+1} \nabla_{e_i}\nabla_{e_i} \tag{2.2}$$

acting on E–valued differential forms and arbitrary E–valued tensors, respectively.

As the notation suggests, d_∇^* is the formal L^2–adjoint of d_∇, where the L^2–inner product on forms is defined using the metric g and the volume form $\theta \wedge \mathrm{dv}_B$. We want to understand the restriction of d_∇ and d_∇^* to basic forms. According to our definition of basic tensors, a differential form γ on M is basic if and only if $\xi \lrcorner \gamma = 0 = \mathcal{L}_\xi \gamma$. Let

$\Omega^\bullet(B)$ denote the space of smooth basic forms. We now define a transverse Hodge-star operator $*$ by

$$*\gamma = *_M(\theta \wedge \gamma), \tag{2.3}$$

for every basic form γ. Note that we also have $*_M\gamma = (-1)^k\,\theta \wedge *\gamma$ if $\gamma \in \Omega^k(B)$. The following lemma follows from straightforward manipulations of (2.2) using the relations between $*_M$ and $*$ and between ∇^{LC} and the adapted connection ∇.

Lemma 2.4 *For every $\gamma \in \Omega^k(M)$ we have*

$$d_\nabla\gamma = d\gamma - d\theta \wedge (\xi \lrcorner \gamma) \quad \text{and} \quad d_\nabla^*\gamma = d_M^*\gamma - (-1)^{k(n+1-k)}\,\theta \wedge *_M(d\theta \wedge *_M\gamma).$$

Here d_M^* denotes the codifferential on (M, g, dv_g). In particular, if $\gamma \in \Omega^k(B)$ is basic then

$$d_\nabla\gamma = d\gamma \quad \text{and} \quad d_\nabla^*\gamma = (-1)^{n(k-1)+1} * d * \gamma.$$

Proof The formulas for d_∇ and its restriction to basic forms are immediate. The formula for d_∇^* is deduced from the formula for d_∇ using the fact that d_∇^* is the formal L^2–adjoint of d_∇. The description of the restriction of d_∇^* to basic forms uses the fact that

$$d_M^*\gamma = (-1)^{n(k-1)+1} * d * \gamma + (-1)^{k(n+1)}\theta \wedge *(d\theta \wedge *\gamma),$$
$$*_M(d\theta \wedge *_M\gamma) = (-1)^k * (d\theta \wedge *\gamma)$$

if γ is a basic k–form. □

By abuse of notation, in the rest of the paper we use the notation $d = d_\nabla|_{\Omega^\bullet(B)}$ and $d^* = d_\nabla^*|_{\Omega^\bullet(B)}$. In particular, we will say that a basic form is *coclosed* if $d_\nabla^*\gamma = 0$. Similarly, we will denote by $\triangle$ the restriction of $d_\nabla d_\nabla^* + d_\nabla^* d_\nabla$ to basic forms and say that a basic form γ is *harmonic* if $\triangle\gamma = 0$.

Remark When M is closed every basic harmonic form is closed and coclosed (in the sense we have just defined), but this is not necessarily the case if M is not compact. We will therefore always keep a distinction between harmonic and closed-and-coclosed (basic) forms.

In order to define the adapted Dirac operator $\not{D}$ we need to assume that M is spin. The spin structure might not be projectable but the associated Spin^c–structure always is (since the frame bundle is projectable). Since every complex representation of $\mathrm{Spin}(n)$

is also a representation of $\mathrm{Spin}^c(n)$, we can always define a twisted Dirac operator acting on spinors with values in a Hermitian vector bundle by

$$\not{D}\psi = \sum_{i=1}^{n+1} \gamma(e_i)\nabla_{e_i}\psi, \tag{2.5}$$

where γ denotes Clifford multiplication. The restriction of $\not{D}$ to basic (complex) spinors plays the role of the Dirac operator of the orbifold B.

The fact that we defined basic elliptic operators as the restriction to basic tensors of elliptic operators on M allows us to extend standard properties of elliptic operators on compact manifolds (elliptic regularity estimates, properties of the spectrum, etc) to transversally elliptic operators.

Proposition 2.6 *Let M be a closed Riemannian principal Seifert circle bundle with orbit space B. Let $P\colon C^\infty(B;E) \to C^\infty(B;F)$ be a self-adjoint basic elliptic operator. Then P has discrete spectrum and there exists an orthonormal basis of $L^2(B;E)$ consisting of eigensections of P. Moreover, every eigensection of P is smooth.*

The second advantage of introducing operators based on the adapted connection is that local computations with basic tensors coincide with local computations in the standard Riemannian case as if B were smooth. In particular, one can define *transverse* (or *basic*) *curvature tensors* and prove Weitzenböck formulas relating squares of Dirac-type operators (such as $\not{D}$ and $d_\nabla + d_\nabla^*$) and the rough Laplacian $\nabla^*\nabla$. Vanishing results and eigenvalue estimates based on positivity properties of curvature terms in these Weitzenböck formulas are then deduced in the usual way; see [86, Chapter 8] for details about this technique. The following proposition is an example of the results obtained using this method.

Remark Strictly speaking, Tondeur [86, Equation (8.1)] replaces the Levi-Civita connection of g with a different choice of connection than the adapted connection (2.1). However, both connections satisfy the key property of [86, Proposition 8.6].

Proposition 2.7 *Let M^{n+1} be a closed Riemannian principal Seifert circle bundle with orbit space B and assume that the transverse Ricci curvature* $\mathrm{Ric}(g_B)$ *satisfies* $\mathrm{Ric}(g_B) \geq (n-1)g_B$.

(i) *The first nonzero eigenvalue of the Laplacian $\triangle$ acting on basic functions is greater than or equal to n.*

(ii) *The first eigenvalue of the Laplacian $\triangle$ acting on coclosed basic 1–forms is greater than or equal to $2(n-1)$ and the eigenspace with eigenvalue $2(n-1)$ consists of basic 1–forms dual to basic Killing vector fields which are also eigenvectors for the transverse Ricci curvature with eigenvalue $2(n-1)$.*

Proof When B is smooth, part (i) is the classical Lichnerowicz–Obata theorem and part (ii) is less well known but also classical; see [21, Theorem 7.6] or [50, Lemma B.2]. In light of the remarks before the proposition, the same proof extends to the case where B is singular. □

Remark 2.8 There is also an analogue of Obata's rigidity result for the eigenvalue estimate in (i): if there is a nontrivial eigenfunction with eigenvalue n then Shioya's orbifold version of Obata's theorem, Theorem 1.1 of [83], implies that B is isometric to a finite quotient of the round n–sphere.

2.1.3 Basic cohomology Absolute de Rham cohomology of a (not necessarily closed) manifold M is the cohomology of the differential complex $(\Omega^\bullet(M), d)$, where $\Omega^\bullet(M)$ is the space of smooth differential forms on M. Basic cohomology is similarly defined using the complex of basic forms. Indeed, note that d preserves basic forms and therefore $(\Omega^\bullet(B), d)$ is a differential chain complex, whose cohomology is called the basic cohomology of M. (Note that $d_\nabla|_{\Omega^\bullet(B)} = d|_{\Omega^\bullet(B)}$ by Lemma 2.4 so there is no ambiguity here on the meaning of d.) We will denote the basic cohomology of M by $H^\bullet(B)$, since when B is smooth it coincides with the de Rham cohomology of the quotient manifold B. We define the compactly supported basic cohomology $H^\bullet_c(B)$ of M in an analogous way.

Remark 2.9 There is a natural Gysin sequence relating the cohomology of M with its basic cohomology [86, Theorem 6.13]. Indeed, there is an exact sequence of complexes

$$0 \to \Omega^\bullet(B) \to \Omega^\bullet_{S^1}(M) \to \Omega^{\bullet-1}(B) \to 0,$$

where $\Omega^\bullet_{S^1}(M)$ is the space of S^1–invariant forms on M and the second map is contraction with ξ. The long exact sequence in cohomology replaces the Gysin sequence of a circle fibration since the cohomology of $(\Omega^\bullet_{S^1}(M), d)$ is isomorphic to the standard de Rham cohomology of M by averaging along the (compact) orbits of ξ.

The following proposition discusses the topological interpretation of basic cohomology.

Proposition 2.10 *Let M be a Riemannian principal Seifert circle bundle with orbit space B. The basic cohomology $H^\bullet(B)$ of M is isomorphic to the equivariant cohomology of M, denoted by $H^\bullet_{\mathrm{orb}}(B;\mathbb{R})$. In particular, if the least common multiple of the orders of the finite stabilisers of points in M is finite, then $H^\bullet(B)$ is isomorphic to the singular cohomology with real coefficients of the topological space B.*

Proof The proposition is a chain of isomorphisms between different cohomology theories for orbifolds and manifolds with a group action. First of all, by H Cartan's generalised Chern–Weil theory the basic cohomology of M is equivalent to the Cartan model for the equivariant cohomology of M; see [44, Chapter 5]. The equivariant version of the de Rham theorem [44, Chapters 1–4] states that the Cartan model is equivalent to Borel's topological construction of the equivariant cohomology of M as the singular cohomology with real coefficients of $ES^1 \times_{S^1} M$, which we denote by $H^\bullet_{\mathrm{orb}}(B;\mathbb{R})$. In order to explain the notation note that the (Haefliger) orbifold cohomology (with arbitrary coefficients) $H^\bullet_{\mathrm{orb}}(B)$ of an orbifold B is defined as the singular cohomology of the classifying space of the (unique up to Morita equivalence) groupoid associated to B, see [1, page 38] and [11, Definition 4.3.6]; the classifying space of a global quotient orbifold M/G is equivalent to $EG \times_G M$ [1, Example 1.53]. The last statement uses the Leray spectral sequence of the fibration $ES^1 \times_{S^1} M \to M/S^1 = B$; see [11, Corollary 4.3.8]. □

We continue this topological parenthesis with two further observations. First of all, the isomorphism classes of principal circle orbibundles (not necessarily Seifert bundles) $\pi\colon M \to B$ are classified by the orbifold first Chern class $c_1^{\mathrm{orb}}(M) \in H^2_{\mathrm{orb}}(B;\mathbb{Z})$; see [11, Theorem 4.3.15]. Secondly, we can define orbifold homotopy groups $\pi_i^{\mathrm{orb}}(B)$ of an orbifold B as the homotopy groups of the classifying space of the associated groupoid. A result of Thurston yields an interpretation of the orbifold fundamental group $\pi_1^{\mathrm{orb}}(B)$ as the group of deck transformations of an orbifold universal cover; see [11, Theorem 4.3.19]. Note also that there is an exact sequence of homotopy groups associated with a Seifert circle bundle $\pi\colon M \to B$ [11, Theorem 4.3.18]

$$\cdots \to \pi_2^{\mathrm{orb}}(B) \to \mathbb{Z} \to \pi_1(M) \to \pi_1^{\mathrm{orb}}(B) \to 1. \tag{2.11}$$

Here the map $\pi_2^{\mathrm{orb}}(B) \to \mathbb{Z} \simeq \pi_1(S^1)$ is determined by the image of $c_1^{\mathrm{orb}}(M)$ in $H^2_{\mathrm{orb}}(B;\mathbb{R})$, ie the orbifold first Chern class modulo torsion.

Remark 2.12 There is an orbifold version of the Bonnet–Myers theorem: if the transverse Ricci curvature of a complete Riemannian Seifert bundle $\pi\colon M \to B$ is strictly positive, then $\pi_1^{\mathrm{orb}}(B)$ is finite. Indeed, by [9, Corollary 21] the diameters of B and its orbifold universal cover must then be bounded.

Returning to basic cohomology, we conclude with a discussion of the basic version of Hodge theory. Exploiting the fact that $d_\nabla + d_\nabla^*$ is an elliptic operator, the same reasoning that led us to Proposition 2.6 allows us to deduce a basic Hodge theorem; see [86, Theorem 7.22].

Proposition 2.13 *Let M be a closed Riemannian principal Seifert circle bundle with orbit space B. Denote by $\mathcal{H}^k(B)$ the space of basic closed and coclosed forms, ie*

$$\mathcal{H}^k(B) = \{\gamma \in \Omega^k(B) \mid d\gamma = 0 = d^*\gamma\}.$$

Then the map that assigns to each closed and coclosed basic form its basic cohomology class is an isomorphism $\mathcal{H}^k(B) \simeq H^k(B)$.

2.2 Seifert bundles that are transversally AC

Let N be a connected, oriented, closed n–manifold and $\pi_\infty\colon N \to \Sigma$ be a Riemannian principal Seifert circle bundle over a closed $(n-1)$–orbifold Σ. Denote by ξ_∞, θ_∞ and g_Σ the choice of a nonsingular vector field, dual 1–form and horizontal metric on N. In particular, N is endowed with the Riemannian metric $g_N = g_\Sigma + \theta_\infty^2$.

The Riemannian cone $\mathrm{C}(N)$ over a Riemannian manifold (N, g_N) is $\mathbb{R}^+ \times N$ endowed with the (incomplete) Riemannian metric $g_{\mathrm{C}} = dr^2 + r^2 g_N$. Instead of this conical metric, exploiting the Seifert bundle structure of N we will consider $\mathrm{BC}(N) = \mathrm{BC}(N, \pi, \theta_\infty, g_\Sigma) = \mathbb{R}^+ \times N$ endowed with the bundle-like transversally conical metric

$$g_{\mathrm{BC}} = dr^2 + r^2 g_\Sigma + \theta_\infty^2. \tag{2.14}$$

Let (M, g) be a complete connected oriented Riemannian manifold with only one end. We assume that $\pi\colon M \to B$ is a Riemannian principal Seifert circle bundle with generating vector field ξ, connection 1–form θ and horizontal metric g_B. Here B denotes the orbit space M/S^1.

Definition 2.15 We say that (M, π, θ, g_B) is *transversally asymptotically conical* (*AC*) asymptotic to $\mathrm{BC}(N)$ with rate $\nu < 0$ if there exists a compact set $K \subset M$, a positive number $R > 0$ and a diffeomorphism

$$f\colon \mathrm{BC}(N) \cap \{r > R\} \to M \setminus K$$

such that for all $j \geq 0$,

$$|\nabla^j_{g_{\mathrm{BC}}}(f^*g - g_{\mathrm{BC}})|_{g_{\mathrm{BC}}} = O(r^{\nu-j}). \tag{2.16}$$

Remark Since $|d\theta_\infty|_{g_{\mathrm{BC}}} = O(r^{-2})$, here we can compute covariant derivatives using either the adapted or the Levi-Civita connection of $\mathrm{BC}(N)$, obtaining equivalent definitions.

By averaging along the circle orbits, the diffeomorphism $f\colon \mathrm{BC}(N)\cap\{r > R\} \to M\setminus K$ can be assumed to intertwine the circle actions, ie $f_*\xi_\infty = \xi$. Indeed, since g and g_{BC} are circle invariant this averaging procedure does not destroy the asymptotic decay of f^*g to g_{BC}. Note that the decay condition (2.16) then is equivalent to

$$|\nabla^j_{g_{\mathrm{BC}}}(\theta_\infty - f^*\theta)|_{g_{\mathrm{BC}}} + |\nabla^j_{g_{\mathrm{BC}}}(dr^2 + r^2 g_\Sigma - f^*g_B)|_{g_{\mathrm{BC}}} = O(r^{\nu-j}). \tag{2.16$'$}$$

Here the decay of $f^*\theta$ to θ_∞ allows one to compare the horizontal metrics since the horizontal spaces are isomorphic for r sufficiently large.

2.2.1 Weighted Banach spaces Let (E_∞, h_∞) be a projectable metric vector bundle on N endowed with a projectable metric connection ∇_∞. Here h_∞ is an S^1–invariant metric on the bundle E_∞ and ∇_∞ preserves it. By abuse of notation we will use the same symbols to denote the pullback of $(E_\infty, h_\infty, \nabla_\infty)$ to $\mathrm{BC}(N)$.

Definition 2.17 Let (M, π, θ, g_B) be a transversally AC principal Seifert circle bundle asymptotic to $\mathrm{BC}(N)$ with rate $\nu < 0$. Let (E, h, ∇) be a projectable metric bundle and connection over M. We say that (E, h, ∇) is *admissible* if, under the identification $f\colon (R,\infty)\times N \to M\setminus K$ of Definition 2.15, there exists an S^1–equivariant bundle isomorphism $f^*E \simeq E_\infty$ such that $f^*h = h_\infty + h'$ and $f^*\nabla = \nabla_\infty + a$, where (h', a) satisfy

$$|\nabla^j_\infty h'|_{g_{\mathrm{BC}}\otimes h_\infty} = O(r^{\nu-j}) \quad \text{and} \quad |\nabla^j_\infty a|_{g_{\mathrm{BC}}\otimes h_\infty} = O(r^{\nu-1-j}).$$

We will mostly be interested in (sub)bundles of $\bigotimes^r \mathcal{H} \otimes \bigotimes^s \mathcal{H}^*$, where $\mathcal{H}$ is the horizontal bundle of M. By (2.16), any such bundle together with the metric induced by g and the connection induced by the adapted connection (2.1) of g is admissible.

Remark In Definition 2.17 we used the same rate of decay ν of the transversally AC Seifert bundle for ease of exposition and because we will mostly be working with tensor bundles and connections induced by the adapted connection. This restriction is of course unnecessary.

Fix once and for all an extension of the radial function f^*r from $M \setminus K$ to the interior of M. By abuse of notation we will denote this extension by r and assume that $r \geq 1$ and that, for each $k \geq 1$, $|\nabla^k r|$ is uniformly bounded on M by a k–dependent constant.

Definition 2.18 Let (E, h, ∇) be an admissible bundle. For all $p \geq 1$, $k \in \mathbb{N}_0$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$ we define the weighted Sobolev space $L^p_{k,\nu}(B)$ and the weighted Hölder space $C^{k,\alpha}_\nu(B)$ of basic sections of E as the closure of $C^\infty_c(B; E)$ with respect to the norms

$$\|u\|_{L^p_{k,\nu}} = \left(\sum_{j=0}^{k} \|r^{-\frac{n}{p}-\nu+j} \nabla^j u\|^p_{L^p} \right)^{\frac{1}{p}},$$

$$\|u\|_{C^{k,\alpha}_\nu} = \sum_{j=0}^{k} \|r^{-\nu+j} \nabla^j u\|_{C^0} + [r^{-\nu+k} \nabla^k u]_\alpha.$$

By dropping the Hölder seminorm $[r^{-\nu+k}\nabla^k u]_\alpha$ in the definition of the $C^{k,\alpha}_\nu$–norm, we obtain the definition of the space of basic sections of E of class C^k_ν. Finally, define

$$C^\infty_\nu(B) = \bigcap_{k \geq 0} C^k_\nu(B).$$

A standard technique to work with weighted Banach spaces on AC manifolds is the scaling and covering argument of [4, Theorem 1.2]. The same technique can be used in the more general context of transversally AC Seifert bundles. Decompose the region $\{r \geq R\}$ in $\mathrm{BC}(N)$ into the union of "annuli" $\mathcal{A}_{2^k R, 2^{k+1} R} = \{2^k R \leq r \leq 2^{k+1} R\}$. Each region $\mathcal{A}_{2^k R, 2^{k+1} R}$ can be covered with the same finite number of open subsets of the form $(U \times S^1)/\Gamma$, for some finite group Γ. The fact that the number of open subsets is independent of the radius of the annulus follows from the fact that $\mathrm{BC}(N)$ (and all the structure it carries) is the radial extension of the compact Seifert bundle $\pi_\infty \colon N \to \Sigma$. Up to a factor of $(2^k R)^{-\nu}$, on each annulus the weighted Sobolev/Hölder norms of *basic* sections are equivalent (with constants independent of R and k) to the standard Sobolev/Hölder norms of basic sections on the fixed annulus $\{1 \leq r \leq 2\}$. Then estimates on the exterior region $\{r \geq R\}$ in $\mathrm{BC}(N)$ (and, via the identification f of Definition 2.15, in M) can be obtained by applying standard estimates for basic sections on these rescaled regions, rescaling back and summing/taking supremums over $k \in \mathbb{Z}_{\geq 0}$. Combined with interior estimates for basic sections on a compact set in M, this method yields estimates for basic sections on the whole manifold.

The following embedding theorem can be proved using this strategy. The necessary local interior estimate is Parker's equivariant Sobolev embedding theorem [79, Section 1], which states that for basic sections the Sobolev inequalities work as if we were in dimension n rather than $n+1$.

Theorem 2.19 *Let M be an $(n+1)$–dimensional transversally AC principal Seifert circle bundle. All Banach spaces below are spaces of basic sections of an admissible vector bundle E.*

(1) *If $k \geq h \geq 0$, $k-\frac{n}{p} \geq h-\frac{n}{q}$, $p \leq q$ and $\nu \leq \nu'$, there is a continuous embedding $L^p_{k,\nu}(B) \subset L^q_{h,\nu'}(B)$. Moreover, if $k > h$, $k-\frac{n}{p} > h-\frac{n}{q}$ and $\nu < \nu'$, then the embedding is compact.*

(2) *If $k-\frac{n}{p} \geq h+\alpha$, then there is a continuous embedding $L^p_{k,\nu}(B) \subset C^{h,\alpha}_{\nu}(B)$.*

Remark 2.20 There are further results about embeddings and products that follow more easily from Definition 2.18. For example the first statement below only uses Hölder's inequality.

(i) If $k \geq h \geq 0$, $k-\frac{n}{p} \geq h-\frac{n}{q}$, $p > q$ and $\nu < \nu'$, there is a continuous embedding $L^p_{k,\nu}(B) \subset L^q_{h,\nu'}(B)$. Moreover, if $k > h$ and $k-\frac{n}{p} > h-\frac{n}{q}$, then the embedding is compact.

(ii) If $\nu \leq \nu'$ and $k+\alpha \geq h+\beta$, then there are continuous embeddings $C^{k+1}_{\nu}(B) \subset C^{k,\alpha}_{\nu}(B) \subset C^{h,\beta}_{\nu'}(B) \subset C^{h}_{\nu'}(B)$. Moreover, if $\nu < \nu'$, the embedding $C^{k,\alpha}_{\nu}(B) \subset C^{h}_{\nu'}(B)$ is compact.

(iii) If $\nu < \nu'$, there is a continuous embedding $C^{h,\alpha}_{\nu}(B) \subset L^q_{h,\nu'}(B)$ for every $q \geq 1$.

(iv) If $\nu_1+\nu_2 \leq \nu$, then the product $C^{k,\alpha}_{\nu_1}(B) \times C^{k,\alpha}_{\nu_2}(B) \to C^{k,\alpha}_{\nu}(B)$ is continuous.

In the next section we will use (iv) to control the nonlinearities in the equations describing circle-invariant Spin(7)–metrics.

2.2.2 Admissible operators Consider a transversally elliptic operator P_∞ of order k on $\mathrm{BC}(N)$. The fact that the conical metric $dr^2+r^2 g_\Sigma$ is conformal to the cylindrical metric dt^2+g_Σ, where $r=e^t$, motivates us to consider the rescaled operator $r^k P_\infty$. Using the conformal equivalence between cones and cylinders, it makes sense to require that $r^k P_\infty$ acting on basic sections is invariant under translations in t.

Definition 2.21 Let $P\colon C^\infty(B;E)\to C^\infty(B;F)$ be a transversally elliptic operator of order k between sections of admissible vector bundles over a transversally AC principal Seifert circle bundle $\pi\colon M\to B$ asymptotic to $\mathrm{BC}(N)$. Let

$$f\colon \mathrm{BC}(N)\cap\{r>R\}\to M\setminus K$$

be the identification of Definition 2.15. Let $P_\infty\colon C^\infty(f^*E)\to C^\infty(f^*F)$ be a transversally elliptic operator on $\mathrm{BC}(N)$ such that $r^k P_\infty$ acting on basic sections is a translation-invariant operator. We say that P is an admissible operator asymptotic to P_∞ if

$$|\nabla^j_\infty(f^*(Pu)-P_\infty f^*u)|_{g_{\mathrm{BC}}\otimes h_\infty}=O(r^{-k+\nu-j})$$

for some $\nu<0$ for every $j\geq 0$ and every smooth basic section u of E on $M\setminus K$.

By Definition 2.17, if $P\colon C^\infty(E)\to C^\infty(F)$ is an elliptic operator of order k between admissible vector bundles defined as the composition of $\nabla^k\colon C^\infty(E)\to C^\infty\big(\bigotimes^k T^*M\otimes E\big)$ with a constant coefficient bundle map $\bigotimes^k T^*M\otimes E\to F$, then P is admissible. In particular, the operator $d+d^*$ $(=d_\nabla+d^*_\nabla)$ of (2.2), the Laplacian $\triangle$ $(=d_\nabla d^*_\nabla+d^*_\nabla d_\nabla)$ and the Dirac operator (2.5) acting on basic spinors and differential forms on a transversally AC principal Seifert bundle are all admissible operators.

With this definition the theory of admissible transversally elliptic operators on transversally AC principal Seifert bundles acting on basic sections is identical to the standard theory of elliptic operators on AC manifolds [71; 70]. We will briefly state the main analytic results we are going to use in the paper and only comment on the strategy of the proofs since they are identical to the standard case of AC manifolds. We point to the literature without any attempt at being exhaustive.

First of all, the following integration-by-parts formula, which can be proved using a sequence of appropriately chosen "logarithmic" cut-off functions converging to the constant function 1, will be repeatedly used throughout the paper.

Lemma 2.22 *Let $P\colon C^\infty(B;E)\to C^\infty(B;F)$ be an admissible operator of order 1 and let P^* be its formal adjoint. Then for every $u\in L^2_{1,\nu}(B)$ and $v\in L^2_{1,\nu'}(B)$ with $\nu+\nu'\leq -n+1$, we have*

$$\langle Pu,v\rangle_{L^2}=\langle u,P^*v\rangle_{L^2}.$$

The following elliptic regularity estimates can be proved using the scaling and covering technique discussed earlier; cf [4, Proposition 1.6] and also [76, Theorem 3.1] for a closely related early result.

Theorem 2.23 *Let $P\colon C^\infty(B;E)\to C^\infty(B;F)$ be an admissible operator of order k. Then for every $l\geq 0$, $p\geq 1$, $\alpha\in(0,1)$ and $\nu\in\mathbb{R}$, there exists $C>0$ such that*

$$\|u\|_{L^p_{l+k,\nu+k}} \leq C(\|Pu\|_{L^p_{l,\nu}} + \|u\|_{L^p_{0,\nu+k}}),$$
$$\|u\|_{C^{l+k,\alpha}_{\nu+k}} \leq C(\|Pu\|_{C^{l,\alpha}_{\nu}} + \|u\|_{C^{0,\alpha}_{\nu+k}}),$$
$$\|u\|_{C^{l+k,\alpha}_{\nu+k}} \leq C(\|Pu\|_{C^{l,\alpha}_{\nu}} + \|u\|_{L^2_{\nu+k}})$$

for all basic sections $u\in C^\infty_c(B)$.

2.2.3 Fredholm theory for transversally elliptic operators In order to proceed further it is necessary to study in more detail the mapping properties of the model operator P_∞. This can be done explicitly by separation of variables.

Definition 2.24 Let P_∞ be a transversally elliptic operator on $\mathrm{BC}(N)$ acting on basic sections of a projectable bundle $E_\infty\to\mathrm{BC}(N)$, and assume that r^kP_∞ is a translation-invariant operator.

(1) We say that a basic section u of E_∞ is *homogeneous of order* λ if $\mathcal{L}_{r\partial_r}u=\lambda u$. In the particular case of differential forms, however, we adopt the convention that a basic k–form γ on $\mathrm{BC}(N)$ is homogeneous of order λ if $\mathcal{L}_{r\partial_r}\gamma=(\lambda+k)\gamma$.

(2) We say that λ is an *indicial root* of P_∞ if there exists a homogeneous basic section u of rate λ such that $P_\infty u=0$. We denote the set of indicial roots of P_∞ by $\mathcal{D}(P_\infty)$.

(3) For $\lambda\in\mathcal{D}(P_\infty)$ let $d(\lambda)$ denote the dimension of the space of basic sections $u\in\ker P_\infty$ of the form $u=\sum_{j=0}^m u_j(\log r)^j$ with $u_0,\dots,u_m$ homogeneous of order λ.

(4) For $\nu,\nu'\notin\mathcal{D}(P_\infty)$ with $\nu<\nu'$, set $N(\nu,\nu')=\sum_{\lambda\in\mathcal{D}(P_\infty)\cap(\nu,\nu')}d(\lambda)$.

Because of the translation invariance, the indicial roots of P_∞ are completely determined by the spectrum of a transversally elliptic operator $P_\infty|_N$ on the compact Seifert bundle N. In particular, Proposition 2.6 implies that $\mathcal{D}(P_\infty)$ is discrete.

In the rest of the paper we are going to make extensive use of the following results, which are the analogues of the Fredholm theory in the AC setting developed in [71, Theorems 1.1 and 1.2].

Theorem 2.25 *Let $P: C^\infty(B;E) \to C^\infty(B;F)$ be an admissible operator of order k and fix $l \geq 0$, $\alpha \in (0,1)$ and $\nu, \nu' \in \mathbb{R}$ with $\nu < \nu'$.*

(i) *If $\nu \in \mathbb{R} \setminus \mathcal{D}(P_\infty)$, then $P: C^{l+k,\alpha}_\nu(B) \to C^{l,\alpha}_{\nu-k}(B)$ is a Fredholm operator.*

(ii) *Assume that $\nu, \nu' \notin \mathcal{D}(P_\infty)$ and denote by $i(\nu)$ and $i(\nu')$ the indices of $P: C^{k,\alpha}_\nu(B) \to C^{0,\alpha}_{\nu-k}(B)$ and $P: C^{k,\alpha}_{\nu'}(B) \to C^{0,\alpha}_{\nu'-k}(B)$, respectively. Then*

$$i(\nu') - i(\nu) = N(\nu, \nu'),$$

where $N(\nu,\nu') = \sum_{\lambda \in \mathcal{D}(P_\infty) \cap (\nu,\nu')} d(\lambda)$.

We also state explicitly two useful more technical results that are needed to prove Theorem 2.25. First of all, separation of variables on $\mathrm{BC}(N)$ and decomposition of basic sections on N into eigenspaces of $P_\infty|_N$ using Proposition 2.6 allow one to prove the following result about the asymptotic behaviour of solutions to $Pu = v$.

Proposition 2.26 *Let $P: C^\infty(B;E) \to C^\infty(B;F)$ be an admissible operator of order k and fix $l \geq 0$, $\alpha \in (0,1)$ and $\nu, \nu' \in \mathbb{R}$ with $\nu < \nu'$ and $\nu, \nu' \notin \mathcal{D}(P_\infty)$. Set $N = N(\nu,\nu')$. Let $u_1, \dots, u_N$ be a basis of the space of basic sections $u \in \ker P_\infty$ of the form $u = \sum_{j=0}^m \overline{u}_j (\log r)^j$, for basic sections $\overline{u}_0, \dots, \overline{u}_m$ of E_∞ which are homogeneous of rate $\lambda \in (\nu,\nu')$.*

Then there exists a compact set $K \subset M$ such that for every $v \in C^{0,\alpha}_{\nu-k}(B)$ with $v = Du'$ for some $u' \in C^{k,\alpha}_{\nu'}(B)$, there exist $a = (a_1, \dots, a_N) \in \mathbb{R}^N$ and a basic section u on $M \setminus K$ of class $C^{k,\alpha}_\nu$ such that $u'|_{M\setminus K} = u + \sum_{i=1}^N a_i\, u_i$. Moreover, there exists a constant $C > 0$, independent of f, u, u', a, such that

$$\|u\|_{C^{k,\alpha}_\nu} + \|a\| \leq C(\|f\|_{C^{0,\alpha}_{\nu-k}} + \|u'\|_{C^{k,\alpha}_{\nu'}}).$$

Secondly, we describe the obstructions to solving the equation $Pu = v$.

Proposition 2.27 *Let $P: C^\infty(B;E) \to C^\infty(B;F)$ be an admissible operator of order k, and fix $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$. Then for every $v \in C^{0,\alpha}_{\nu-k}(B)$ such that*

$$\langle v, \overline{u} \rangle_{L^2} = 0$$

for all $\overline{u} \in \ker P^* \cap C^\infty_{-n-\nu+k}(B)$, there exists $u \in C^{k,\alpha}_\nu(B)$ such that $Pu = v$ and

$$\|u\|_{C^{k,\alpha}_\nu} \leq C\|Pu\|_{C^{0,\alpha}_{\nu-k}}.$$

The proofs of Theorem 2.25 and Propositions 2.26 and 2.27 are intertwined. The analysis of the asymptotic operator P_∞ by separation of variables as in Proposition 2.26 allows one to deduce the existence of a compact set $K \subset M$ and a constant $C > 0$ such that

$$\|u\|_{C^{l+k,\alpha}_{\nu}} \leq C(\|Pu\|_{C^{l,\alpha}_{\nu-k}} + \|u\|_{L^2(K)})$$

for all smooth basic sections u. From this inequality (and the analogous one for P^*) it follows immediately that P and P^* have finite-dimensional kernel and closed range. The duality result in Proposition 2.27 then follows from abstract functional-analytic arguments, and in turn it allows one to conclude the proof of Theorem 2.25(i). The proof of Theorem 2.25(ii) uses again Proposition 2.26. We refer the reader to [71] and [78, Chapters 9 and 10] for further details.

2.3 Basic weighted L^2–cohomology

As an application of the theory we have introduced, we conclude this section with a calculation of the weighted basic L^2–cohomology of a transversally AC Riemannian principal Seifert circle bundle $\pi\colon M \to B$. The L^2–cohomology of a smooth AC manifold is well known; see [48, Theorem 1.A] and [70, Example 0.15]. For our applications later in the paper it will be important to understand the space of basic closed and coclosed forms that do not necessarily have an L^2–rate of decay; more precisely, we will need to determine all topological contributions to the space of basic closed and coclosed forms. This is not widely known even in the case of AC manifolds and we provide an elementary proof using the tools we have introduced.

Let $\pi\colon M \to B$ be a transversally AC Riemannian principal Seifert circle bundle asymptotic to $\mathrm{BC}(N)$. Recall that the basic de Rham cohomology (with compact support) of M and N is defined as the cohomology of the de Rham complex of smooth basic differential forms (with compact support). As in (2.16) we identify the complement of a compact set in M with an exterior region in $\mathrm{BC}(N)$ using a diffeomorphism f such that $f_*\xi = \xi_\infty$. In particular, basic differential forms on M pull back to basic differential forms on $\mathrm{BC}(N)$.

We introduce the main piece of topological data we will use. Regard M as a manifold with boundary N, and similarly B as a topological space with boundary Σ. Even though we do not require that the singularities of B are contained in a compact set, the circle action on M still has some finiteness properties: outside of a compact set the circle action on M is determined by the circle action on the compact manifold N.

In particular, the finiteness assumption in Proposition 2.10 is certainly satisfied and we can therefore deduce a long exact sequence in basic cohomology from the long exact sequence in singular cohomology (with real coefficients) for the pair (B,Σ):

$$\cdots \to H^{k-1}(\Sigma) \to H^k_c(B) \to H^k(B) \to H^k(\Sigma) \to \cdots . \tag{2.28}$$

We can in fact be completely explicit. The map $H^k_c(B) \to H^k(B)$ is induced by the natural inclusion of compactly supported basic forms in the space of all smooth basic forms. The map $H^k(\Sigma) \to H^{k+1}_c(B)$ is $[\tau] \mapsto [d(\chi\tau)]$, where χ is a basic cut-off function with $\chi \equiv 1$ outside a compact set. In order to define the map $H^k(B) \to H^k(\Sigma)$ we use the following representation for basic cohomology classes on M.

Lemma 2.29 *Every basic cohomology class $[\sigma] \in H^k(B)$ can be represented by a smooth closed form σ that decays as r^{-k}. More precisely, fix $R > 0$ sufficiently large and embed N in M as $f(\{r = R\})$. Let $\beta_0 \in \mathcal{H}^k(\Sigma)$ be the basic harmonic representative on N of the image of $[\sigma]$ in the basic cohomology of N via the restriction map $H^k(B) \to H^k(\Sigma)$. Then we can take σ to be a smooth closed form with $\sigma = \beta_0$ outside a compact set.*

Using the lemma, the map $H^k(B) \to H^k(\Sigma)$ is explicitly defined by $[\sigma] \mapsto [\beta_0]$.

Proof Choose a closed smooth basic representative σ' of $[\sigma]$. Outside a compact set we can think of σ' as a basic form defined on an exterior domain in $\mathrm{BC}(N)$. We can then write

$$\sigma' = dr \wedge \alpha + \beta,$$

where (α,β) is a curve in $\Omega^{k-1}(\Sigma) \oplus \Omega^k(\Sigma)$ parametrised by $r \in [R,\infty)$ for some $R > 0$. A straightforward calculation shows that $d\sigma' = 0$ if and only if

$$\partial_r\beta = d_\Sigma\alpha \quad \text{and} \quad d_\Sigma\beta = 0. \tag{2.30}$$

Here d_Σ denotes the restriction of d_∇ on N to basic forms, where ∇ is the adapted connection of $\pi_\infty\colon N \to \Sigma$. We now use basic Hodge theory, Proposition 2.13, on N to write $\beta = \beta_0 + d_\Sigma\gamma$, where β_0 is basic harmonic on N and γ is a coclosed basic $(k-1)$–form on N depending smoothly on $r \geq R$. Note that β_0 is independent of r because of the first equation in (2.30). This same equation now yields $d_\Sigma(\partial_r\gamma - \alpha) = 0$. Hence $\alpha = \alpha_0 + \partial_r\gamma$ for a smooth curve α_0 in $\mathcal{H}^{k-1}(\Sigma)$ parametrised by $r \geq R$. We then consider the basic $(k-1)$–form Γ on $[R,\infty) \times N$ defined by

$$\Gamma = \int_R^r \alpha_0(s)\,ds + \gamma.$$

Fix a basic cut-off function χ which vanishes for $r \le R$ and is equal to 1 on $r \ge R+1$, and define

$$\sigma = \sigma' - d(\chi\Gamma).$$

Then $d\sigma = 0$ and $[\sigma] = [\sigma'] \in H^k(B)$, $\sigma = \beta_0$ on $r \ge R+1$ since $d\Gamma = \sigma' - \beta_0$ and $[\beta_0]$ is the image of $[\sigma]$ in $H^k(\Sigma)$. □

Fix $\nu \in \mathbb{R}$ and let $\mathcal{H}^k_\nu(B)$ be the space of basic closed and coclosed forms of class L^2_ν. If ν is not an indicial root for the Laplacian on basic k–forms, then by Theorem 2.23 every form in $\mathcal{H}^k_\nu(B)$ is in fact of class C^∞_ν. Note that $-k$ is always an indicial root of the Laplacian acting on basic k–forms. Indeed, every basic closed and coclosed k–form on N pulls back to a basic harmonic form on $\mathrm{BC}(N)$ which is homogeneous of order $-k$. Our main result is the identification of $\mathcal{H}^k_\nu(B)$ as we cross the indicial root $-k$.

Theorem 2.31 *Let $\pi\colon M^{n+1} \to B$ be a transversally AC principal Seifert circle bundle asymptotic to* $\mathrm{BC}(N)$, *where $\pi_\infty\colon N \to \Sigma$ is a closed principal Seifert bundle. In the following statements, $\delta > 0$ is chosen sufficiently small so that the only indicial root in $[-k-\delta, -k+\delta]$ is $-k$.*

(i) *If $k < \frac{n}{2}$, there are natural isomorphisms*

$$\mathcal{H}^k_{-k-\delta}(B) \simeq H^k_c(B),$$
$$\mathcal{H}^k_{-k+\delta}(B)/\mathcal{H}^k_{-k-\delta}(B) \simeq \mathrm{im}(H^k(B) \to H^k(\Sigma)).$$

(ii) *If $k = \frac{n}{2}$* (so n is even), *there are natural isomorphisms*

$$\mathcal{H}^k_{-k-\delta}(B) \simeq \mathrm{im}(H^k_c(B) \to H^k(B)),$$
$$\mathcal{H}^k_{-k+\delta}(B)/\mathcal{H}^k_{-k-\delta}(B) \simeq \mathrm{im}(H^k(B) \to H^k(\Sigma))^{\oplus 2}.$$

(iii) *If $k > \frac{n}{2}$, there are natural isomorphisms*

$$\mathcal{H}^k_{-k-\delta}(B) \simeq \mathrm{im}(H^k_c(B) \to H^k(B)),$$
$$\mathcal{H}^k_{-k+\delta}(B) \simeq H^k(B).$$

The rest of the section contains a proof of this theorem, which involves various steps.

We begin with the following calculation of excluded indicial roots.

Lemma 2.32 *Let $\pi_\infty\colon N^{n+1} \to \Sigma$ be a closed, connected, oriented Riemannian principal Seifert circle bundle and consider homogeneous basic forms on* $\mathrm{BC}(N)$ *in the sense of Definition 2.24.*

(i) *If $k \leq \frac{n}{2} - 1$, there are no basic harmonic k–forms which are homogeneous of order $-n+k+2 < \lambda < -k$. Moreover, every basic harmonic k–form which is homogeneous of order $\lambda = -k$ is closed and coclosed.*

(ii) *If $k = \frac{n}{2}$, every basic harmonic k–form which is homogeneous of order $\lambda = -n+k = -k$ is closed and coclosed.*

(iii) *If $k < \frac{n}{2}$, there are no basic closed and coclosed k–forms which are homogeneous of order $-n+k < \lambda < -k$.*

(iv) *If $k \neq \frac{n}{2}$, every basic closed and coclosed k–form homogeneous of order $\lambda = -k$ is the pullback of a basic harmonic k–form on N. If $k = \frac{n}{2}$, every basic closed and coclosed k–form homogeneous of order $\lambda = -k = -n+k$ is the pullback of a basic harmonic k–form on N or its image under the basic Hodge-star operator $*$.*

(v) *Let γ be a polynomial in $\log r$ with coefficients in the space of basic k–forms which are homogeneous of order λ. If γ is basic harmonic then either $\gamma = \gamma_0$ is constant in $\log r$ or $\lambda = -\frac{n}{2} + 1$ and $\gamma = \gamma_1 \log r + \gamma_0$ with γ_0, γ_1 basic harmonic k–forms homogeneous of order λ.*

Proof With our definitions, d_∇, d_∇^* and $d_\nabla d_\nabla^* + d_\nabla^* d_\nabla$ acting on basic forms on $\mathrm{BC}(N)$ are equivalent to the operators d, d^* and $dd^* + d^*d$ acting on differential forms on a Riemannian cone $\mathrm{C}(\Sigma)$. A characterisation of harmonic and closed and coclosed homogeneous forms on a cone is given in [32, Appendix A]. The lemma follows immediately from these results using the fact that the Laplacian on Σ is a nonnegative operator. □

Since $\triangle * = * \triangle$, one deduces similar statements for $k \geq \frac{n}{2}$ by replacing k with $n-k$.

Remark In view of Lemma 2.32(iii)–(iv), for all $\delta > 0$ sufficiently small we have

$$\mathcal{H}^k_{-k-\delta}(B) = L^2\mathcal{H}^k(B) \quad \text{if } k \leq \tfrac{n}{2},$$
$$\mathcal{H}^k_{-k+\delta}(B) = L^2\mathcal{H}^k(B) \quad \text{if } k > \tfrac{n}{2}.$$

Therefore Theorem 2.31 includes the calculation of the basic L^2–cohomology of a transversally AC principal Seifert circle bundle.

Lemma 2.33 *Let γ be a basic harmonic p–form of class C^{∞}_{λ}. Assume that either*

(1) $\lambda < -\frac{n}{2}+1$, *or*

(2) $p < \frac{n}{2}-1$ *and* $\lambda < -p$.

Then γ is closed and coclosed.

Proof If γ is a basic harmonic p–form of class C^{∞}_{λ} for some $\lambda < -\frac{n}{2}+1$, then the integration by parts formula Lemma 2.22 implies

$$0 = \langle \triangle\gamma, \gamma\rangle_{L^2} = \|d_{\nabla}\gamma\|^2_{L^2} + \|d^*_{\nabla}\gamma\|^2_{L^2}.$$

If $p < \frac{n}{2}-1$, then by Lemma 2.32(i) and elliptic regularity every basic harmonic p–form in C^{∞}_{λ} with $\lambda < -p$ is in fact in $C^{\infty}_{-n+p+2+\epsilon}$ for every $\epsilon > 0$. Choose ϵ sufficiently small so that $2p \le n-2-2\epsilon$. Then $2(-n+p+2+\epsilon)-1 \le -n+1$ and we can apply the first part of the proof. □

We can now prove the first and third parts of Theorem 2.31.

Proof of Theorem 2.31(i) and (iii) Fix $k < \frac{n}{2}$ and $\delta > 0$ as in the statement of part (i) of the theorem and set $\nu = -k-\delta$. Since ν is not an indicial root, every $\sigma \in \mathcal{H}^k_{\nu}(B)$ is in fact of class $C^{\infty}_{\nu}(B)$ by weighted elliptic regularity. As in the proof of Lemma 2.29, outside a compact set we write $\sigma = dr \wedge \alpha + \beta$, where (α,β) is a smooth curve in $\Omega^{k-1}(\Sigma)\oplus\Omega^k(\Sigma)$ parametrised by $r \in [R,\infty)$ for some $R > 0$ and satisfying $r|\alpha| + |\beta| = O(r^{\nu+k})$. In particular, observe that $\gamma = -\int_r^{\infty}\alpha\, ds$ is well-defined and satisfies $d\gamma = \sigma$. Then, for χ a basic cut-off function with $\chi \equiv 1$ for $r \ge R+1$, $\sigma - d(\chi\gamma)$ is closed and compactly supported. We then define $\Phi^k_{-} \colon \mathcal{H}^k_{\nu}(B) \to H^k_c(B)$ by $\Phi^k_{-}(\sigma) = [\sigma - d(\chi\gamma)]$.

(1) Φ^k_{-} is injective. By Lemma 2.32(i), if $\sigma \in \mathcal{H}^k_{\nu}(B)$ then $\sigma \in C^{\infty}_{-n+k+\epsilon}(B)$ for every $\epsilon > 0$, and therefore the form γ defined earlier by radial integration lies in $C^{\infty}_{-n+k+1+\epsilon}$. Hence if $\Phi^k_{-}(\sigma) = 0$, ie $\sigma - d(\chi\gamma)$ is the differential of a basic compactly supported form, then $\sigma = d\gamma'$ with $\gamma' \in C^{\infty}_{-n+k+1+\epsilon}(B)$. Since $2k < n$, as in Lemma 2.33 integration by parts is justified and we obtain

$$\|\sigma\|^2_{L^2} = \langle d\gamma, \sigma\rangle_{L^2} = \langle \gamma, d^*\sigma\rangle_{L^2} = 0.$$

(2) Φ^k_{-} is surjective. If σ is a closed basic smooth compactly supported form, fix $\alpha \in (0,1)$ and consider the equation $\triangle\gamma = d^*\sigma$ for $\gamma \in C^{3,\alpha}_{\nu+1}(B)$. By Proposition 2.27, the obstructions to solving this equation lie in the space of

basic harmonic $(k-1)$–forms in $C^\infty_{-n-\nu+1}(B)$. By Lemma 2.33 every such form is closed, and therefore all obstructions to solving $\triangle\gamma = d^*\sigma$ vanish. Moreover, $d^*\gamma$ is a basic harmonic $(k-2)$–form in $C^{2,\alpha}_\nu(B)$ and therefore a second application of Lemma 2.33 implies that $dd^*\gamma = 0$. We conclude that $\sigma - d\gamma \in \mathcal{H}^k_\nu(B)$ and $\Phi^k_-(\sigma - d\gamma) = \Phi^k_-(\sigma) = [\sigma]$, as desired.

We will now study $\mathcal{H}^k_\nu(B)$ where $\nu = -k+\delta$ and $k < \frac{n}{2}$. By Proposition 2.26 and Lemma 2.32(iv), every $\sigma \in \mathcal{H}^k_\nu(B)$ can be written in the form $\sigma = \tau + \sigma'$ outside a compact set, where $\tau \in \mathcal{H}^k(\Sigma)$ and $\sigma' \in C^\infty_{-k-\epsilon}(B)$ for every $\epsilon > 0$ sufficiently small. It is also clear that τ represents the image of $[\sigma] \in H^k(B)$ in $H^k(\Sigma)$. We then define a map $\Phi^k_+ \colon \mathcal{H}^k_\nu(B) \to H^k(\Sigma)$ by $\Phi^k_+(\sigma) = [\tau]$. By basic Hodge theory Proposition 2.13, on N the kernel of Φ^k_+ is $\mathcal{H}^k_{-k-\epsilon}$. The image of Φ^k_+ is clearly contained in $\operatorname{im}(H^k(B) \to H^k(\Sigma))$; we have to prove it coincides with this subspace.

Fix $\tau \in \mathcal{H}^k(\Sigma)$ with $[\tau] \in \operatorname{im}(H^k(B) \to H^k(\Sigma))$. By assumption and Lemma 2.29, there exists a basic closed k–form σ on M with $\sigma = \tau$ outside a compact set. Moreover, since M is asymptotic to $\mathrm{BC}(N)$ and τ is basic closed and coclosed on $\mathrm{BC}(N)$, we have $d^*\sigma \in C^{0,\alpha}_{-k-1-\epsilon}$ for all sufficiently small $\epsilon > 0$. We now study the equation $\triangle\gamma = d^*\sigma$ for a basic $(k-1)$–form $\gamma \in C^{3,\alpha}_{-k+1-\epsilon}(B)$. As before, if a solution γ exists then $dd^*\gamma = 0$ and the obstructions to solving the equation lie in the space of basic harmonic $(k-1)$–forms in $C^\infty_{-n+k+1+\epsilon}(B)$, which are all closed by Lemma 2.33. Moreover, taking $\epsilon > 0$ smaller if necessary, we can assume that $2k < -n-\epsilon$; then $-n+k+1+\epsilon < -k+1$ and therefore every harmonic $(k-1)$–form $\bar\gamma \in C^\infty_{-n+k+1+\epsilon}(B)$ actually lies in $C^\infty_{-n+k-1+\epsilon'}(B)$ for every $\epsilon' > 0$. Then we can integrate by parts,

$$\langle d^*\sigma, \bar\gamma\rangle_{L^2} = \langle \sigma, d\bar\gamma\rangle_{L^2} = 0,$$

and conclude that all obstructions to solving $\triangle\gamma = d^*\sigma$ vanish. It follows that $\sigma - d\gamma \in \mathcal{H}^k_\nu(B)$ and $\Phi^k_+(\sigma - d\gamma) = [\tau]$, as desired.

The case $k > \frac{n}{2}$ can be understood by duality. Namely, there is a natural pairing

$$\mathcal{H}^k_{-k+\delta}(B) \times \mathcal{H}^{n-k}_{-n+k-\delta}(B) \to \mathbb{R}, \quad (\alpha,\beta) \mapsto \int_M \theta\wedge\alpha\wedge\beta.$$

This pairing is nondegenerate since by Lemma 2.32(iii), for every $\beta \in \mathcal{H}^{n-k}_{-n+k-\delta}(B)$ we have $*\beta \in \mathcal{H}^k_{-k+\delta}(B)$. Thus we have an isomorphism

$$\mathcal{H}^k_{-k+\delta}(B) \simeq \mathcal{H}^{n-k}_{-n+k-\delta}(B)^* \simeq H^{n-k}_c(B)^* \simeq H^k(B),$$

which is simply the map that assigns to a basic closed and coclosed form its basic cohomology class. Using this isomorphism, Proposition 2.26 and Lemma 2.32(iv), it is also clear that

$$\mathcal{H}^k_{-k+\delta}(B)/\mathcal{H}^k_{-k-\delta}(B) \simeq \operatorname{im}(H^k(B) \to H^k(\Sigma))$$

and therefore $\mathcal{H}^k_{-k-\delta}(B) \simeq \operatorname{im}(H^k_c(B) \to H^k(B))$ by exactness of (2.28). □

The case $n = 2k$ is more challenging, but parts of Theorem 2.31 can be proved with very similar arguments to the ones we used.

Lemma 2.34 *Let $2k = n$ and fix $\delta > 0$ sufficiently small. Then there is a natural isomorphism*

$$\mathcal{H}^k_{-k+\delta}(B)/\mathcal{H}^k_{-k-\delta}(B) \to \operatorname{im}(H^k(B) \to H^k(\Sigma))^{\oplus 2},$$

and the natural map $\mathcal{H}^k_{-k-\delta}(B) \to \operatorname{im}(H^k_c(B) \to H^k(\Sigma))$ that assigns to each closed and coclosed form its cohomology class is injective.

Proof Set $k = \frac{n}{2}$. By Proposition 2.26 and Lemma 2.32(iv), outside a compact set every $\sigma \in \mathcal{H}^k_{-k+\delta}(B)$ can be written as $\sigma = \tau_1 + *\tau_2 + \sigma'$ with $\sigma' \in C^\infty_{-k-\delta}(B)$ and $\tau_1, \tau_2 \in \mathcal{H}^k(\Sigma)$. Define $\Phi^k_+ : \mathcal{H}^k_{-k+\delta}(B) \to H^k(\Sigma)^{\oplus 2}$ by $\Phi^k_+(\sigma) = ([\tau_1], [\tau_2])$. Now, clearly $[\tau_1] \in \operatorname{im}(H^k(B) \to H^k(\Sigma))$ and $\Phi^k_+(*\sigma) = (\pm[\tau_2], [\tau_1])$, hence Φ^k_+ induces a map from $\mathcal{H}^k_{-k+\delta}(B)$ to $\operatorname{im}(H^k(B) \to H^k(\Sigma))^{\oplus 2}$ with kernel $\mathcal{H}^k_{-k-\delta}(B)$.

In order to prove that Φ^k_+ is surjective onto $\operatorname{im}(H^k(B) \to H^k(\Sigma))^{\oplus 2}$, one can show that for every $\sigma \in C^\infty_{-k+\delta}(B)$ it is always possible to solve $\triangle\gamma = d^*\sigma$ for $\gamma \in C^\infty_{-k+1+\delta}(B)$ with $dd^*\gamma = 0$. This is done exactly as in the proof of the surjectivity of Φ^k_- in the proof of Theorem 2.31(i); see Proposition 2.35 below.

In order to prove the second part of the lemma, define $\Phi^k_- : \mathcal{H}^k_{-k-\delta}(B) \to H^k(B)$ by $\sigma \mapsto [\sigma]$. The image of Φ^k_- is clearly contained in the kernel of $H^k(B) \to H^k(\Sigma)$, which coincides with $\operatorname{im}(H^k_c(B) \to H^k(B))$ by exactness of (2.28). The proof of the injectivity of Φ^k_- is analogous to the one in the proof of Theorem 2.31(i). □

It remains to prove that $\Phi^k_- : \mathcal{H}^k_{-k-\delta}(B) \to \operatorname{im}(H^k_c(B) \to H^k(B))$ is surjective for $k = \frac{n}{2}$. This requires a refined analysis of the equation $\triangle\gamma = d^*\sigma$ for γ a basic $(k-1)$–form of class $C^\infty_{-k+1-\delta}$.

Proposition 2.35 *Let $n = 2k$ and let $\delta > 0$ be sufficiently small. Let σ be a basic smooth k–form such that $\sigma \in C^{\infty}_{-k+\delta}(B)$ and $d^*\sigma \in C^{\infty}_{-k-1-\delta}(B)$. Then the equation $\triangle\gamma = d^*\sigma$ has a solution $\gamma \in C^{\infty}_{-k+1+\delta}(B)$ with $d\gamma \in C^{\infty}_{-k-\delta}$.*

Proof The obstructions to solving $\triangle\gamma = d^*\sigma$ with $\gamma \in C^{\infty}_{-k+1\mp\delta}(B)$ lie in the space of basic harmonic $(k-1)$–forms of class $C^{\infty}_{-k+1\pm\delta}(B)$ which are not closed. In particular, we can always solve $\triangle\gamma = d^*\sigma$ with $\gamma \in C^{\infty}_{-k+1+\delta}(B)$, since every basic harmonic $(k-1)$–form of class $C^{\infty}_{-k+1-\delta}(B)$ is closed by Lemma 2.33. Note also that by Lemma 2.33 any solution satisfies $dd^*\gamma = 0$ since $d^*\gamma$ is a basic harmonic $(k-2)$–form of class $C^{\infty}_{-k+\delta}(B)$ and we can always assume that δ is small enough so that $-k+\delta = -\frac{n}{2}+\delta < -\frac{n}{2}+1$. We need to show that we can take γ with $d\gamma \in C^{\infty}_{-k-\delta}$. The proof of this fact follows the lines of the proof of [32, Proposition 5.16]. We sketch the key ideas.

The main point is to understand exactly the space of basic harmonic $(k-1)$–forms as we cross the indicial root $-k+1$. Denote by $\triangle^p_\nu$ the Laplacian restricted to basic p–forms of class $C^{l,\alpha}_\nu$ for some $l \geq 2$ and $\alpha \in (0,1)$, and by $i(\triangle^p_\nu)$ its index. By Lemma 2.32(iv)–(v) we have

$$i(\triangle^{k-1}_{-k+1+\delta}) - i(\triangle^{k-1}_{-k+1-\delta}) = 2\dim H^{k-1}(\Sigma).$$

Moreover, $\operatorname{coker} \triangle^{k-1}_{-k+1\pm\delta} \simeq \ker \triangle^{k-1}_{-k+1\mp\delta}$ and therefore

$$\ker \triangle^{k-1}_{-k+1+\delta} \simeq \ker \triangle^{k-1}_{-k+1-\delta} \oplus \mathcal{H}^{k-1}(\Sigma).$$

Now, decompose $\mathcal{H}^{k-1}(\Sigma)$ into a subspace isomorphic to $\operatorname{im}(H^{k-1}(B) \to H^{k-1}(\Sigma))$ and a complementary subspace W, which is isomorphic to $\operatorname{im}(H^k(B) \to H^k(\Sigma))$ via the basic Hodge-star operator on N; see [32, Lemma 5.11]. By Theorem 2.31(i) we have $\ker \triangle^{k-1}_{-k+1+\delta}/\mathcal{H}^{k-1}_{-k+1+\delta}(B) \simeq W$.

Note however that every basic harmonic $(k-1)$–form $\overline{\gamma}$ of class $C^{\infty}_{-k+1+\delta}(B)$ is coclosed. Indeed, outside a compact set we can write

$$\overline{\gamma} = \tau_1 \log r + \tau_2 + \overline{\gamma}', \tag{2.36}$$

with $\tau_1, \tau_2 \in \mathcal{H}^{k-1}(\Sigma)$ and $\overline{\gamma}' \in C^{\infty}_{-k+1-\delta}(B)$. Since $\tau_1 \log r + \tau_2$ is basic coclosed on $\mathrm{BC}(N)$, we conclude that, up to taking δ smaller if necessary, $d^*\overline{\gamma}$ is a basic harmonic $(k-2)$–form of class $C^{\infty}_{-k-\delta}(B)$. By Lemma 2.33, $dd^*\overline{\gamma} = 0$ and then an integration by parts shows that

$$0 = \langle dd^*\overline{\gamma}, \overline{\gamma}\rangle_{L^2} = \|d^*\overline{\gamma}\|^2_{L^2}.$$

Using these facts we conclude that $*d\overline{\gamma}$ is a closed and coclosed k–form in $C^\infty_{-k+\delta}(B)$. More precisely, using (2.36), we have

$$*\,d\overline{\gamma} \in *_\Sigma \tau_1 + C^\infty_{-k-\delta}(B).$$

By Lemma 2.34 we conclude that $[\tau_1] \in W \subset \mathcal{H}^{k-1}(\Sigma)$. Lemma 2.34 also implies that $d\overline{\gamma} = 0$ if $\tau_1 = 0$: indeed, if $\tau_1 = 0$ then $d\overline{\gamma} \in \mathcal{H}^k_{-k-\delta}(B)$ and $[d\overline{\gamma}] = 0$.

Fix an L^2–orthonormal basis $\tau_1, \dots, \tau_m$ of W. The discussion above implies that for each $j = 1, \dots, m$ there exists a basic harmonic form $\overline{\gamma}_j$, unique up to the addition of an appropriately decaying basic closed and coclosed basic form, such that

$$\overline{\gamma}_j = \tau_j \log r + \sum_{i=1}^m \alpha^i_j \tau_i + \overline{\gamma}'_j$$

for some $\alpha^i_j \in \mathbb{R}$ and $\overline{\gamma}'_j \in C^\infty_{-k+1-\delta}(B)$. The collection $\overline{\gamma}_1, \dots, \overline{\gamma}_m$ forms a basis of the space of obstructions to solving $\triangle\gamma = d^*\sigma$ with $\gamma \in C^\infty_{-k+1-\delta}(B)$. Now, fix a basic cut-off function χ with $\chi \equiv 1$ outside of a compact set. An integration by parts shows that

$$\langle \triangle(\chi\tau_i), \overline{\gamma}_j \rangle_{L^2} = \delta_{ij}.$$

Thus we can always solve the equation $\triangle\gamma = d^*\sigma$ with $\gamma \in C^\infty_{-k+1-\delta}(B)$ modulo the span of $\chi\tau_1, \dots, \chi\tau_m$. To conclude, note that $d(\chi\tau_i)$ is of class $C^\infty_{-k-\delta}(B)$ for δ sufficiently small since τ_i is closed on $\mathrm{BC}(\Sigma)$. □

Proof of Theorem 2.31(ii) In view of Lemma 2.34 it remains only to prove that $\Phi^k_- \colon \mathcal{H}^k_{-k-\delta}(B) \to \operatorname{im}(H^k_c(B) \to H^k(\Sigma))$ is surjective for $k = \frac{n}{2}$. Consider then a closed compactly supported basic k–form σ. By Proposition 2.35 we can solve $\triangle\gamma = d^*\sigma$ with $d\gamma \in C^\infty_{-k-\delta}(B)$ and $dd^*\gamma = 0$. Then $\sigma - d\gamma \in \mathcal{H}^k_{-k-\delta}(B)$ and $\Phi^k_-(\sigma - d\gamma) = \Phi^k_-(\sigma) = [\sigma] \in H^k(B)$. □

3 Highly collapsed Spin(7)–metrics on principal Seifert circle bundles

In this section we prove our main existence result, Theorem A: the existence of highly collapsed Spin(7)–metrics on the total space of a suitable principal Seifert circle bundle over an AC G_2–orbifold. In Section 3.1 we establish properties of AC G_2–orbifolds we use in Section 3.2 to prove Theorem A.

3.1 Asymptotically conical G_2–orbifolds

In this section we study 8–dimensional transversally AC principal Seifert circle bundles $\pi\colon M \to B$ carrying a transverse G_2–structure which is parallel with respect to the adapted connection. We introduce the necessary notation and prove results about harmonic and closed and coclosed basic forms on such manifolds.

3.1.1 Basic torsion-free G_2–structures Let $\pi\colon M^8 \to B$ be a principal Seifert circle bundle endowed with a connection 1–form θ. A *basic* G_2*–structure* is a reduction of the structure group of the horizontal subbundle $\mathcal{H} = \ker\theta$ to G_2. Equivalently, a basic G_2–structure is the choice of a basic 3–form $\varphi \in \Omega^3(B)$ that is pointwise equivalent, under an appropriate identification of the fibres of $\mathcal{H}$ with $\mathbb{R}^7$, to the standard flat G_2–structure $(u,v,w) \mapsto \langle uv, w\rangle$ on $\mathbb{R}^7 = \operatorname{Im}\mathbb{O}$. Every basic G_2–structure determines a horizontal Riemannian metric $g_B = g_\varphi$ on M, and (M,θ,g_B) is then a Riemannian principal Seifert circle bundle. We will denote by ψ the basic 4–form $\psi = *\varphi$. The triple (M,π,θ,φ) will be called a G_2 *principal Seifert circle bundle*.

We will now collect important identities for basic forms on G_2 principal Seifert circle bundles. We refer the reader to [16] for their proof. Indeed, while [16] considers the case of 7–manifolds endowed with a G_2–structure, the results we discuss in this subsection only use the representation theory of the compact Lie group G_2, and therefore they immediately generalise to basic G_2–structures on Seifert circle bundles.

Lemma 3.1 *Let (M,π,θ,φ) be a G_2 principal Seifert circle bundle. Then there are pointwise orthogonal decompositions*

$$\Omega^2(B) = \Omega^2_7(B) \oplus \Omega^2_{14}(B), \quad \Omega^3(B) = \Omega^3_1(B) \oplus \Omega^3_7(B) \oplus \Omega^3_{27}(B)$$

of basic forms according to irreducible representations of G_2, where

$$\begin{aligned}
\Omega^2_7(B) &= \{X \lrcorner \varphi = *(X^\flat \wedge \psi) \mid X^\flat \in \Omega^1(B)\},\\
\Omega^3_7(B) &= \{X \lrcorner \psi = -*(X^\flat \wedge \varphi) \mid X^\flat \in \Omega^1(B)\},\\
\Omega^2_{14}(B) &= \{\tau \in \Omega^2(B) \mid \tau\wedge\psi = 0\} = \{\sigma \in \Omega^2(B) \mid *\sigma = -\sigma\wedge\varphi\},\\
\Omega^3_1(B) &= \{f\varphi \mid f \in \Omega^0(B)\},\\
\Omega^3_{27}(B) &= \{\rho \in \Omega^3(B) \mid \rho\wedge\varphi = 0 = \rho\wedge\psi\}.
\end{aligned}$$

By acting with the basic Hodge-star operator $*$, Lemma 3.1 implies similar decompositions of the space of basic 4–forms and 5–forms. The decomposition of 4–forms can be used to describe the linearisation of the map $\varphi \mapsto \psi$; see [16, Proposition 4] and [59, Proposition 10.3.5].

Lemma 3.2 *The linearisation of the map $\varphi \mapsto \psi$ is*

$$\rho \mapsto *\big(\tfrac{4}{3}\pi_1\rho + \pi_7\rho - \pi_{27}\rho\big).$$

Lemma 3.3 *Let (M,π,θ,φ) be a G_2 principal Seifert circle bundle. Then there exists a basic function τ_0, a basic 1–form τ_1, a 2–form $\tau_2 \in \Omega^2_{14}(B)$ and $\tau_3 \in \Omega^3_{27}(B)$ such that*

$$d\varphi = \tau_0\psi + 3\tau_1\wedge\varphi + \tau_3, \quad d\psi = 4\tau_1\wedge\psi + \tau_2\wedge\varphi. \tag{3.4}$$

For a proof, see [16, Proposition 1]. If $d\varphi = 0 = d\psi$, ie $\tau_i = 0$ for all i, we say that φ is *torsion-free*. Linear algebra and representation theory of G_2 then imply that φ is parallel for the adapted connection. Since G_2 is the stabiliser in $\mathrm{GL}(7,\mathbb{R})$ of the flat G_2–structure on $\mathbb{R}^7$, one then concludes that the existence of a torsion-free transverse G_2–structure implies that the holonomy of the adapted connection reduces from SO(7) to G_2. We also note that a crucial consequence of the torsion-free condition is the vanishing of the transverse Ricci curvature; see [16, Section 4.5.3]. From now on we will concentrate on torsion-free basic G_2–structures. In this case we will refer to (M,π,θ,φ) as a G_2*–holonomy* principal Seifert circle bundle.

We collect some useful identities for basic functions and 1–forms on a G_2–holonomy principal Seifert circle bundle. In the following lemma the curl operator acting on basic 1–forms is defined by

$$\operatorname{curl}\gamma = *(d\gamma\wedge\psi). \tag{3.5}$$

Lemma 3.6 *For $f \in \Omega^0(B)$ and $\gamma \in \Omega^1(B)$, with dual basic vector field $X = \gamma^\sharp$, the following identities hold*:

1. $\pi_1 d(X \lrcorner \varphi) = -\frac{3}{7}(d^*\gamma)\varphi$ *and* $\pi_7 d(X \lrcorner \varphi) = \frac{1}{2}*(\operatorname{curl}\gamma\wedge\varphi)$.
2. $d*(\gamma\wedge\psi) - d^*(\gamma\wedge\varphi) = *(\operatorname{curl}\gamma\wedge\varphi) - (d^*\gamma)\varphi \in \Omega^3_{1\oplus 7}(B)$.
3. *The basic Dirac operator $\not{D}$ can be identified with either of the following two operators*:

$$\not{D}\colon \Omega^0(B)\oplus\Omega^1(B) \to \Omega^0(B)\oplus\Omega^1(B), \quad (f,\gamma) \mapsto (d^*\gamma, df + \operatorname{curl}\gamma),$$
$$\not{D}\colon \Omega^0(B)\oplus\Omega^1(B) \to \Omega^3_{1\oplus 7}(B), \qquad (f,\gamma) \mapsto \pi_{1\oplus 7}(*d(f\varphi) + d*(\gamma\wedge\psi)).$$

Proof The lemma can be deduced from [16, Proposition 3] and the identification of the Dirac operator of a G_2–manifold B via the isomorphism between the spinor bundle with $\mathbb{R} \oplus TB$; see for example [77, Equation (6.2)]. □

3.1.2 Nearly Kähler orbifolds and transversally AC G_2–holonomy Seifert bundles Let N be a closed oriented 7–manifold. Assume that $\pi_\infty \colon N \to \Sigma$ is a principal Seifert circle bundle over a closed 6–orbifold Σ and fix a connection θ_∞ on π_∞. We say that N admits a basic SU(3)–structure if there exist basic forms $\omega \in \Omega^2(\Sigma)$ and $\operatorname{Re}\Omega, \operatorname{Im}\Omega \in \Omega^3(\Sigma)$ such that ω and Ω are, respectively, a nondegenerate 2–form and a complex volume form on the horizontal subspace $\mathcal{H}$, satisfying $\omega \wedge \Omega = 0$ and $2\omega^3 = 3\operatorname{Re}\Omega \wedge \operatorname{Im}\Omega$. Every basic SU(3)–structure defines a horizontal metric g_Σ and, together with θ_∞, the structure of a Riemannian Seifert bundle on N. A basic SU(3)–structure (ω, Ω) is called *nearly Kähler* if

$$d\omega = 3\operatorname{Re}\Omega, \quad d\operatorname{Im}\Omega = -2\omega^2. \tag{3.7}$$

If (ω, Ω) is a basic nearly Kähler structure on N then $\mathrm{BC}(N)$ has a basic torsion-free G_2–structure

$$\varphi_{\mathrm{C}} = r^2 dr \wedge \omega + r^3 \operatorname{Re}\Omega. \tag{3.8}$$

In particular, since the metric on $\mathrm{BC}(N)$ has vanishing transverse Ricci curvature, every basic nearly Kähler structure induces a transversally Einstein metric g_Σ with Einstein constant 5.

Since the horizontal metric induced by (3.8) is of the form $g_{\mathrm{C}} = dr^2 + r^2 g_\Sigma$, it makes sense to talk of a transversally AC G_2–holonomy principal Seifert circle bundle $(M, \pi, \theta, \varphi)$ asymptotic to $(\mathrm{BC}(N), \pi_\infty, \theta_\infty, \varphi_{\mathrm{C}})$. Since φ and φ_{C} determine the horizontal metrics g_B and $g_{\mathrm{C}} = dr^2 + r^2 g_\Sigma$, we can replace (2.16) in Definition 2.15 with a similar polynomial decay for θ and φ (and their covariant derivatives) to θ_∞ and φ_{C}.

In the discussion so far the choice of the connection 1–form θ_∞ on $\pi_\infty \colon N \to \Sigma$ (and, by radial extension, on $\mathrm{BC}(N)$) remained free. In fact, up to gauge transformations there is a canonical choice. Indeed, if (ω, Ω) is a basic nearly Kähler structure on N then it follows from [31, Theorem 3.8] (immediately extended from manifolds to Seifert bundles since the proof only uses pointwise identities based on the representation theory of SU(3) and integration by parts) that every basic closed and coclosed 2–form κ on N satisfies

$$\kappa \wedge \omega^2 = 0 = \kappa \wedge \Omega, \tag{3.9}$$

ie κ is a basic *primitive* $(1,1)$*–form*. Conversely, every basic closed primitive $(1,1)$–form κ is necessarily coclosed since $*\kappa = -\kappa \wedge \omega$. In particular, by adding a basic 1–form to θ_∞ so that $d\theta_\infty$ is closed and coclosed, we can always assume that $d\theta_\infty$ satisfies (3.9) and this requirement uniquely determines θ_∞ up to gauge transformations. In the rest of the section we say that $(N, \pi_\infty, \theta_\infty, \omega, \Omega)$ is a nearly Kähler principal Seifert circle bundle if (ω, Ω) is a basic nearly Kähler structure and θ_∞ is normalised so that $d\theta_\infty$ is a primitive $(1,1)$–form.

In the rest of the paper we will also work under the following assumption.

Assumption The principal Seifert circle bundle $\pi_\infty\colon N \to \Sigma$ is not flat.

Since we assume that $d\theta_\infty$ is the unique basic closed and coclosed representative of its cohomology class, the assumption is equivalent to the requirement that the image of $c_1^{\mathrm{orb}}(N)$ in $H^2_{\mathrm{orb}}(\Sigma;\mathbb{R})$ not vanish. Note that if (M,π,θ,φ) is asymptotic to $\mathrm{BC}(N)$, then necessarily $d\theta \neq 0$ also.

Proposition 3.10 *Let (M,π,θ,φ) be a transversally AC G_2–holonomy principal Seifert circle bundle asymptotic to $\mathrm{BC}(N)$, and assume that $d\theta_\infty \neq 0$. Then*:

(i) *M has finite fundamental group.*

(ii) *M is an irreducible Seifert circle bundle, ie there are no basic 1–forms on M that are parallel with respect to the adapted connection.*

Proof The proof of part (i) is analogous to the one of [32, Proposition 5.9]. Observe that, since the horizontal metric g_Σ is Einstein with uniform positive Einstein constant, the orbifold Σ has finite orbifold fundamental group by Remark 2.12. Since $d\theta_\infty \neq 0$, the homotopy exact sequence (2.11) then presents $\pi_1(N)$ as an extension of $\pi_1^{\mathrm{orb}}(\Sigma)$ by a finite group. We therefore conclude that N has finite fundamental group.

The map $\pi_1(M\setminus K) \to \pi_1(M)$ is surjective, since otherwise, as in [47, Lemma 2.18], a finite cover of M would be a Riemannian submersion over a Ricci-flat orbifold with at least two asymptotically conical ends: this is impossible by the orbifold version of the Cheeger–Gromoll splitting theorem [10] or, considering a sequence of metrics on M that collapse to the orbifold B, the Cheeger–Colding almost-splitting theorem [20, Theorem 6.64].

For part (ii) it is enough to observe that there are no parallel basic 1–forms on $\mathrm{BC}(N)$. In order to show that this is true, observe that there is a correspondence between basic parallel 1–forms on $\mathrm{BC}(N)$ and basic functions f on N satisfying $\nabla df = fg_\Sigma$ (this follows from an explicit calculation of the Levi-Civita connection of a Riemannian cone). In particular, any such f would satisfy $\triangle f = 6f$ and by Remark 2.8, Σ would then be isometric to a finite quotient of the round 6–sphere. In this case however, $\pi_\infty\colon N \to \Sigma$ would be forced to be flat. □

Remark In view of part (i), the irreducibility assumption of part (ii) is equivalent to the requirement that the holonomy of the adapted connection be the whole group G_2 by [15, Lemma 1].

Although we have outlined some of its important consequences, at this stage the assumption $d\theta_\infty \neq 0$ may seem unmotivated. In fact, it is a necessary condition for obtaining nontrivial examples from Theorem A; see Remark 3.34 below.

3.1.3 Basic closed and coclosed forms on AC G_2–orbifolds Consider a nearly Kähler principal Seifert circle bundle $(N, \pi_\infty, \theta_\infty, \omega, \Omega)$, and let $(M, \pi, \theta, \varphi)$ be a transversally AC G_2–holonomy principal Seifert circle bundle asymptotic to $\mathrm{BC}(N)$. In this section we prove a number of results about basic harmonic and closed and coclosed forms on M. The main results we are going to use in the next section are Proposition 3.15 and Corollary 3.16, which describe the obstructions to solving the Poisson equation $\triangle\sigma = \kappa$ for a basic 2–form κ, and provide a normal form for exact basic 5–forms with appropriate decay.

We first analyse basic harmonic functions and 1–forms on M. Using the fact that every nearly Kähler principal Seifert circle bundle is transversally Einstein with Einstein constant 5, Proposition 2.7 yields an improvement of the indicial root computations of Lemma 2.32.

Lemma 3.11 *Let N be a transversally nearly Kähler principal Seifert circle bundle.*

(i) *There are no basic harmonic functions on $\mathrm{BC}(N)$ which are homogeneous of order $\lambda \in [-6, 1]$ except for constant multiples of 1 and r^{-5} (of rates $\lambda = 0$ and $\lambda = -5$, respectively).*

(ii) *There are no basic harmonic 1–forms on $\mathrm{BC}(N)$ which are homogeneous of order $\lambda \in (-5, 0)$.*

The irreducibility in Proposition 3.10(ii) and the fact that g has vanishing transverse Ricci curvature have the following important consequence.

Lemma 3.12 *There are no basic harmonic functions and* 1*–forms on* M *in* $C^{\infty}_{\nu}(B)$ *for* $\nu < 0$.

Proof Let u and γ be a basic harmonic function and 1–form in $C^{\infty}_{\nu}(B)$. If $\nu < -\frac{5}{2}$ then we can integrate by parts:

$$0 = \langle \triangle u, u\rangle_{L^2} = \|\nabla u\|^2_{L^2}, \quad 0 = \langle \triangle \gamma, \gamma\rangle_{L^2} = \|\nabla \gamma\|^2_{L^2},$$

where, since M is transversally Ricci-flat, we used the fact that $\triangle = \nabla^*\nabla$ on basic 1–forms. We conclude that u is constant and therefore vanishes since it decays at infinity, and $\gamma = 0$ since M is irreducible. On the other hand, by Lemma 3.11 there are no indicial roots for the Laplacian acting on functions and 1–forms in the interval $(-5, 0)$. □

Remark 3.13 In particular, if $\sigma \in C^{\infty}_{\nu}(B)$ is a basic harmonic 2–form and $\rho \in C^{\infty}_{\nu}(B)$ is a basic harmonic 3–form on M for $\nu < 0$, then $\sigma \in \Omega^2_{14}(B)$ and $\rho \in \Omega^3_{27}(B)$. Indeed, the basic Laplacian preserves the decomposition of Lemma 3.1, and its restriction to $\Omega^{\bullet}_1(B)$ and $\Omega^{\bullet}_7(B)$ coincides with the basic Laplacian on functions and 1–forms, respectively.

In view of this remark, we can always normalise the choice of θ on M so that $d\theta \in \Omega^2_{14}(B)$. Indeed, by the exact sequence (2.28) and Theorem 2.31, the basic cohomology class of $d\theta$ is represented by a unique basic closed and coclosed form decaying with rate -2. Therefore we can add a decaying basic 1–form to θ so that $d\theta$ is closed and coclosed, thus of type 14 by Remark 3.13, and θ is still asymptotic to θ_{∞}. In the rest of the section we will include this assumption in the definition of a transversally AC G_2–holonomy Seifert bundle $(M, \pi, \theta, \varphi)$.

Proposition 3.14 *The basic Dirac operator*

$$\not{D}\colon C^{1,\alpha}_{\nu}\Omega^0(B) \oplus C^{1,\alpha}_{\nu}\Omega^1(B) \to C^{0,\alpha}_{\nu-1}\Omega^0(B) \oplus C^{0,\alpha}_{\nu-1}\Omega^1(B)$$

is surjective if $\nu > -6$ *and injective if* $\nu < 0$.

Proof Using the vanishing of the transverse Ricci curvature we conclude that every basic (f, γ) in the kernel of $\not{D}$ is actually harmonic. Then Lemma 3.12 implies that $\not{D}$ is injective if $\nu < 0$ and, by duality, surjective if $\nu > -6$. □

Proposition 3.15 *Fix $\alpha \in (0,1)$, $k \geq 1$ and a generic $\nu \in (-4,0)$. Then there exists $C > 0$ with the following significance. Let κ be a basic 2–form of class $C^{k-1,\alpha}_{\nu-1}(B)$. If $\langle \kappa, \overline{\kappa} \rangle_{L^2} = 0$ for all $\overline{\kappa} \in \mathcal{H}^2_{-6-\nu}(B)$ then there exists a unique $\sigma \in C^{k+1,\alpha}_{\nu+1}\Omega^2(B)$ which is $L^2_{\nu+1}$–orthogonal to basic harmonic 2–forms of class $C^\infty_{\nu+1}(B)$ and satisfies $\triangle\sigma = \kappa$ and*

$$\|\sigma\|_{C^{k+1,\alpha}_{\nu+1}(B)} \leq C\|\kappa\|_{C^{k-1,\alpha}_{\nu-1}(B)}.$$

Moreover, if $d^*\kappa = 0$, then $d^*\sigma = 0$ and $\kappa = d^*d\sigma$, and if $d\kappa = 0$, then $d\sigma \in \Omega^3_{27}(B)$.

Proof Since $2 < \frac{7}{2} - 1$ and $\nu > -4$, Lemma 2.33 shows that every basic harmonic 2–form of class $C^\infty_{-6-\nu}(B)$ is closed and coclosed. The first part of the proposition follows immediately. Since $\nu < 0$, the last two statements follow immediately from Lemma 3.12 and Remark 3.13 since $d^*\sigma$ and $d\sigma$ are then a basic (weakly) harmonic 1–form and 3–form, respectively, of class $C^k_\nu(B)$. □

We can now combine the two previous propositions to deduce a normal form for basic exact 5–forms.

Corollary 3.16 *Fix $\alpha \in (0,1)$, $k \geq 1$ and a generic $\nu \in (-4,0)$. Then for every basic closed 5–form τ of class $C^{k-1,\alpha}_{\nu-1}(B)$ which is $L^2(B)$–orthogonal to $\mathcal{H}^5_{-6-\nu}(B)$, there exist unique $\gamma \in \Omega^1(B)$ and $\sigma \in \Omega^2(B)$ of class $C^{k+1,\alpha}_{\nu+1}(B)$ such that $d\sigma \in \Omega^3_{27}(B)$ and*

$$\tau = d(\operatorname{curl}\gamma \wedge \varphi - (d^*\gamma)\psi - *d\sigma).$$

Moreover, there exists a constant $C > 0$ independent of τ, γ, σ such that

$$\|\gamma\|_{C^{k+1,\alpha}_{\nu+1}(B)} + \|\sigma\|_{C^{k+1,\alpha}_{\nu+1}(B)} \leq C\|\tau\|_{C^{k-1,\alpha}_{\nu-1}(B)}.$$

Proof We apply Proposition 3.15 with $\kappa = -*\tau$. The assumptions on τ guarantee that (a) all obstructions to solving $\triangle\sigma' = \kappa$ vanish, and (b) $\kappa = d^*d\sigma'$, ie $\tau = d*d\sigma'$.

Using Proposition 3.14 and Lemma 3.6(3) we now write

$$d\sigma' = *d(f\varphi) + d*(\gamma \wedge \psi) - \rho$$

for $f \in \Omega^0(B)$, $\gamma \in \Omega^1(B)$ and $\rho \in \Omega^3_{27}(B)$ of classes $C^{k+1,\alpha}_{\nu+1}(B)$ and $C^{k,\alpha}_\nu(B)$, respectively. Indeed, since $\nu + 1 > -3 > -6$, the Dirac operator $\not{D}$ is surjective. Since $\nu + 1 < 1$, Lemma 3.11(i) implies that the kernel of $\not{D}$ consists only of constant functions, and therefore we can make our choice unique by normalising f so that it decays at infinity. With this normalisation, we conclude that $f = 0$. Indeed, since $\rho \wedge \varphi_0 = 0$, the vanishing of $d(d\sigma' \wedge \varphi_0)$ implies that f is harmonic. In particular, $\rho = d\sigma$ with $\sigma = *(\gamma \wedge \psi) - \sigma' \in C^{k+1,\alpha}_{\nu+1}(B)$.

In order to conclude the proof we now use Lemma 3.6(2) to rewrite

$$\tau = d*(d*(\gamma \wedge \psi) - d^*(\gamma \wedge \varphi) - d\sigma) = d(\operatorname{curl}\gamma \wedge \varphi - (d^*\gamma)\psi - *d\sigma). \qquad \square$$

Remark 3.17 Integration by parts shows that τ is $L^2(B)$–orthogonal to $\mathcal{H}^5_{-6-\nu}(B)$ whenever $\tau = du$ with $u \in C^{1,\alpha}_\nu(B)$.

3.2 Adiabatic limit of circle-invariant Spin(7)–metrics

In [32] we developed a construction of complete G_2–holonomy metrics on appropriate principal circle bundles over AC Calabi–Yau 3–folds. In [2], Apostolov and Salamon described the dimensional reduction of the nonlinear PDEs for G_2–holonomy in the presence of a Killing field. The resulting equations, called the Apostolov–Salamon equations in [32], form a complicated system of nonlinear PDEs. The strategy of [32] is to construct solutions of the Apostolov–Salamon equations by studying the adiabatic limit of the equations as the orbits of the Killing field shrink to zero size. In this section we describe a similar construction of complete Spin(7)–holonomy metrics on principal Seifert circle bundles over AC G_2–orbifolds.

We consider a Spin(7)–manifold admitting an isometric circle action. We interpret the dimensional reduction of the equations for Spin(7)–holonomy in terms of the intrinsic torsion of the G_2–structure induced on the 7–dimensional orbifold quotient and a coupled abelian G_2–monopole. These equations can be thought of as a Spin(7) analogue of the Gibbons–Hawking construction of 4–dimensional hyperkähler metrics with a triholomorphic circle action. As for the Apostolov–Salamon equations, however, the dimensional reduction of the Spin(7)–holonomy equations to 7 dimensions still consists of *nonlinear* equations and in general it is not clear how to solve them directly. We therefore consider the *adiabatic limit* of these equations when the circle orbits have uniformly small length. This formal picture is then turned into our existence result Theorem A using the analysis of the previous sections. With respect to [32] we are able to give a more streamlined argument by applying the implicit function theorem directly instead of proving convergence of a formal power series solution.

3.2.1 Gibbons–Hawking-type ansatz for Spin(7)–manifolds Let $\pi\colon M^8 \to B$ be a principal Seifert circle bundle over a 7–orbifold B. Denote by ξ the vector field that generates the fibrewise circle action, normalised to have period 2π. A Spin(7)–structure on M is the choice of a 4–form Φ with distinguished algebraic properties at each point (an admissible 4–form in the language of [59, Definition 10.5.1]). Circle-invariant Spin(7)–structures are completely determined by the choice of a connection 1–form θ, a basic positive function h and a basic G_2–structure φ:

$$\Phi = \theta \wedge \varphi + h^{\frac{2}{3}}\psi, \tag{3.18}$$

where $\psi = *\varphi$. The metric g_Φ induced by Φ on M is

$$g_\Phi = h^{\frac{1}{3}}\, g_B + h^{-1}\theta^2, \tag{3.19}$$

where g_B is the horizontal metric induced by the basic G_2–structure φ.

By [59, Definition 10.5.2] a Spin(7)–structure Φ is torsion-free, ie the metric g_Φ has holonomy contained in Spin(7), if and only if $d\Phi = 0$. We now express the torsion-free condition $d\Phi = 0$ for an S^1–invariant Spin(7)–structure Φ (3.18) as a PDE system for the triple (φ, h, θ).

Proposition 3.20 *The S^1–invariant* Spin(7)*–structure Φ on M determined by the triple (φ, h, θ) via* (3.18) *is torsion-free if and only if*

$$d\varphi = 0, \quad d(h^{\frac{2}{3}}\psi) + d\theta \wedge \varphi = 0. \tag{3.21}$$

Proof The proposition follows from direct differentiation of (3.18): $d\varphi = 0$ is equivalent to $\xi \lrcorner\, d\Phi = 0$ and the second equation is equivalent to the vanishing of $d\Phi$ in horizontal directions. □

The aim of this section is to construct solutions to (3.21). The equations are nonlinear, so it is unclear how to find solutions in general. We make however two easy remarks.

First of all, there is a cohomological constraint in order to be able to solve (3.21). Indeed, if a solution exists then $[\varphi]$ is a well-defined basic cohomology class and we have $[d\theta] \cup [\varphi] = 0$ in basic cohomology. Note also that $[d\theta]$ represents the (real) orbifold first Chern class of the orbibundle $\pi\colon M \to B$ in the orbifold cohomology $H^2_{\mathrm{orb}}(B;\mathbb{R})$ of Proposition 2.10, so we can rewrite the constraint as

$$c_1^{\mathrm{orb}}(M) \cup [\varphi] = 0 \in H^5_{\mathrm{orb}}(B). \tag{3.22}$$

Secondly, we can use Lemma 3.3 to reinterpret (3.21) as coupled equations for the torsion of the basic G_2–structure φ and for the pair (h,θ).

Lemma 3.23 *The basic* G_2*–structure* φ *arising from a solution* (φ,h,θ) *of* (3.21) *has torsion*

$$\tau_0=\tau_1=\tau_3=0,\quad \tau_2=-h^{-\frac{2}{3}}\kappa_0,$$

where κ_0 *is the component of the curvature* $d\theta$ *of* θ *in* $\Omega^2_{14}(B)$*. Moreover,* (h,θ) *satisfies*

$$*d\left(\tfrac{3}{2}h^{\frac{2}{3}}\right)+d\theta\wedge\psi=0. \tag{3.24}$$

Proof Decompose $d\theta=U\lrcorner\varphi+\kappa_0$ for a basic vector field U and $\kappa_0\in\Omega^2_{14}(B)$. By [16, Equations (3.2) and (3.5)], we have $d\theta\wedge\varphi=2U^\flat\wedge\psi+\kappa_0\wedge\varphi$. Then it is straightforward to check that (3.21) is equivalent to the stated expressions for the torsion components. Moreover, the vanishing of τ_1 forces $U^\flat=\frac{1}{3}h^{-\frac{1}{3}}dh$. By [16, Equations (3.4) and (3.5)] and the definition of Ω^2_{14} in Lemma 3.1, we have $*(d\theta\wedge\psi)=3U^\flat$ and therefore we arrive at (3.24). □

Equation (3.24) is a known gauge-theoretic equation that arises as the dimensional reduction of the (abelian) Spin(7)–instanton equations to dimension 7; its solutions are called abelian G_2 (or octonionic) monopoles.

3.2.2 Formal analysis of the adiabatic-limit equations Lacking a better understanding of (3.21), our strategy for producing solutions to (3.21) is to degenerate the equations by introducing a small parameter $\epsilon>0$ and studying solutions in the limit $\epsilon\to 0$: we assume the existence of a solution to the formal limit of the equations when $\epsilon=0$ and prove that it can be perturbed to a solution of the system for small $\epsilon>0$. The particular degeneration we introduce is geometrically very natural: we consider a 1–parameter family $\{\Phi_\epsilon\}_{\epsilon>0}$ of S^1–invariant torsion-free Spin(7)–structures on M such that the circle orbits shrink to zero length as $\epsilon\to 0$. By rescaling along the circle orbits we write

$$\Phi_\epsilon=\epsilon\,\theta\wedge\varphi+h^{\frac{2}{3}}\psi,\quad g_{\Phi_\epsilon}=h^{\frac{1}{3}}\,g_B+\epsilon^2h^{-1}\theta^2,$$

where g_B is the horizontal metric induced by the basic G_2–structure φ. The PDE system (3.21) for Φ_ϵ then becomes

$$d\varphi=0,\quad d(h^{\frac{2}{3}}\psi)+\epsilon\,d\theta\wedge\varphi=0. \tag{3.25}$$

For $\epsilon > 0$ the system (3.25) is equivalent to (3.21). In the limit $\epsilon \to 0$, however, the equations simplify: Lemma 3.23 with $d\theta = 0$ implies that solutions to (3.25) with $\epsilon = 0$ satisfy $dh_0 = 0$ and $d\varphi_0 = 0 = d\psi_0$. Assume then the existence of a basic torsion-free G_2–structure φ_0 on M and set $h_0 = 1$. We want to perturb this solution of the limiting equations (3.25) with $\epsilon = 0$ to a solution of the system with $\epsilon > 0$.

To this end, we reinterpret (3.25) as the vanishing of a nonlinear map Ψ defined by

$$\xi = (\varphi, h, \kappa) \mapsto d(h^{\frac{2}{3}}\psi) + \kappa \wedge \varphi. \tag{3.26}$$

Here φ is a closed basic G_2–structure, ψ is its dual 4–form, h is a basic function and κ is a basic closed 2–form (satisfying additional conditions that we will impose below). In particular, note that $\Psi(\xi)$ is a closed 5–form.

The triple $\xi_0 = (\varphi_0, 1, 0)$ is a solution to (3.26). In order to understand nearby solutions we are going to linearise (3.25) at ξ_0. Consider a perturbation $\xi = \xi_0 + \zeta$ with $\zeta = (\rho, f, \kappa)$. We assume that ρ is sufficiently small in $C^0(B)$–norm so that $\varphi = \varphi_0 + \rho$ still defines a basic G_2–structure. We write $\psi = \psi_0 + \widehat{\rho} + Q_{\varphi_0}(\rho)$ for the dual 4–form, where $\widehat{\rho}$ is the image of ρ under the linear map of Lemma 3.2 and Q_{φ_0} is a smooth map satisfying

$$|Q_{\varphi_0}(\rho)| \leq C|\rho|^2 \quad \text{and} \quad |\nabla Q_{\varphi_0}(\rho)| \leq C|\rho|\,|\nabla\rho| \tag{3.27}$$

for a uniform constant C; see [59, Proposition 10.3.5]. In fact, as in the proof of [62, Lemma 5.16], exploiting the transverse AC structure we can also control all higher derivatives of $Q_{\varphi_0}(\rho)$: for each $k \geq 2$ there is a constant C_k independent of ρ such that

$$|\nabla^k Q_{\varphi_0}(\rho)| \leq C_k \sum_{\substack{m,n\geq 0\\ m+n\leq k}} r^{-m}|\rho|^{\max(0,2-n)} \Bigg(\sum_{\substack{k_1,\dots,k_n\geq 1\\ k_1+\dots+k_n=k-m}} \prod_{j=1}^{n} |\nabla^{k_j}\rho| \Bigg). \tag{3.28}$$

Here the adapted connection and the norms are defined using the metric induced by φ_0.

We then write $\Psi(\xi_0 + \zeta) = \mathcal{L}_0(\zeta) + \mathcal{N}_0(\zeta)$, where

$$\mathcal{L}_0(\rho, f, \kappa) = d\big(\widehat{\rho} + \tfrac{2}{3} f\,\psi_0\big) + \kappa \wedge \varphi_0 \tag{3.29}$$

is linear and

$$\mathcal{N}_0(\rho, f, \kappa) = d\big((1+f)^{\frac{2}{3}}(\psi_0 + \widehat{\rho} + Q_{\varphi_0}(\rho)) - \psi_0 - \widehat{\rho} - \tfrac{2}{3} f\,\psi_0\big) + \kappa \wedge \rho \tag{3.30}$$

contains the nonlinearities.

Suppose we are given a *bounded* solution $\zeta_0 = (\rho_0, f_0, \kappa_0)$ to the linearised equation $\mathcal{L}_0(\zeta_0) = 0$. Then for $\epsilon > 0$ small, $\xi_0 + \epsilon\zeta_0 = (\varphi_0 + \epsilon\rho_0, 1 + \epsilon f_0, \epsilon\kappa_0)$ is an approximate solution to (3.26). Here we assume that ϵ is sufficiently small that $\varphi_\epsilon = \varphi_0 + \epsilon\rho_0$ is still a basic G_2–structure. In order for ξ_ϵ to be a geometrically meaningful approximate solution we must require that the basic closed form κ_0 represents the orbifold first Chern class of the Seifert circle bundle $\pi\colon M \to B$, ie $\kappa_0 = d\theta_0$ is the curvature of a connection 1–form θ_0 on the Seifert bundle. Note also that, linearising (3.24), (f_0, θ_0) satisfies $*df_0 + d\theta_0 \wedge \psi_0 = 0$, ie (f_0, θ_0) is an abelian G_2–monopole. In particular, f_0 is harmonic and, in the context of this paper, the assumption that ζ_0 is bounded forces f_0 to be constant. In this case it makes sense to further normalise ζ_0 by requiring that $f_0 \equiv 0$. We work under this assumption in the rest of the section.

Remark It is also interesting to consider the case where ζ_0 is *not* bounded, which corresponds to families of Spin(7)–metrics collapsing in codimension 1 with unbounded curvature. A natural assumption is then to assume that (f_0, θ_0) is an abelian G_2–monopole with Dirac-type singularities along a coassociative submanifold. This more challenging case is beyond the scope of this paper.

We now aim to exponentiate the infinitesimal deformation ζ_0 to an exact solution to (3.26). To this end, we look for a further perturbation $\xi_0 + \epsilon\zeta_0 + \zeta$ such that

$$\Psi(\xi_0 + \epsilon\zeta_0) + \mathcal{L}_\epsilon(\zeta) + \mathcal{N}_\epsilon(\zeta) = 0, \tag{3.31}$$

where (using $\mathcal{L}_0(\rho_0) = 0$ and $f_0 \equiv 0$)

$$\Psi(\xi_0 + \epsilon\zeta_0) = dQ_{\varphi_0}(\epsilon\rho_0) + \epsilon^2 d\theta_0 \wedge \rho_0 \tag{3.32}$$

and $\mathcal{L}_\epsilon$ and $\mathcal{N}_\epsilon$ are defined by the same formulas (3.29) and (3.30) with φ_ϵ in place of φ_0. Since $\varphi_\epsilon = \varphi_0 + \epsilon\rho_0$ is arbitrarily C^0–close to φ_0 as $\epsilon \to 0$, the solvability of (3.31) reduces to showing that, under additional conditions on ζ, the linearised equation $\mathcal{L}_0(\zeta) = \Psi(\xi_0 + \epsilon\zeta_0)$ has a solution and $\mathcal{L}_0$ can be inverted on the image of $\mathcal{N}_\epsilon$.

3.2.3 Implementation of the adiabatic-limit strategy In the rest of the section we exploit the analysis on AC G_2–orbifolds developed earlier in the paper to implement this strategy. Let $\pi\colon M^8 \to B$ be a principal Seifert circle bundle with connection 1–form θ_0. We assume that M carries a basic torsion-free G_2–structure φ_0 inducing a transversally AC metric on M.

There are three main points we need to address in order to apply the implicit function theorem:

(1) Construct a bounded solution of the linearised problem $\mathcal{L}_0(\zeta_0) = 0$.

(2) Understand the mapping properties of $\mathcal{L}_0$.

(3) Prove that $\mathcal{L}_0$ can be inverted on $\Psi(\xi_0 + \epsilon\zeta_0)$ and on the image of $\mathcal{N}_\epsilon$.

The infinitesimal deformation As explained after Remark 3.13, we can assume without loss of generality that $d\theta_0 \in \Omega^2_{14}(B)$. By Lemma 3.1 this means that $*d\theta_0 = -d\theta_0 \wedge \varphi_0$ and therefore $d\theta_0$ is closed and coclosed. Moreover, $|\nabla^k(d\theta_0)| = O(r^{-2-k})$ for all $k \geq 0$.

We want to find a basic closed 3–form ρ_0 such that $\mathcal{L}_0(\rho_0, 0, d\theta_0) = 0$, where $\mathcal{L}_0$ is the linear operator (3.29). Since $d\theta_0 \in \Omega^2_{14}(B)$, the equation $\mathcal{L}_0(\rho_0, 0, d\theta_0) = 0$ is equivalent to

$$d\rho_0 = 0, \quad d\widehat{\rho}_0 - *d\theta_0 = 0, \tag{3.33}$$

which is not obviously elliptic as $\widehat{\rho}_0$ involves the basic Hodge-star operator and the type decomposition of basic forms.

Fix $\mu = -1 + \delta$ for an arbitrarily small $\delta > 0$ and consider instead the elliptic equation $\triangle\sigma_0 = d\theta_0$ for a basic 2–form σ_0 of class $C^\infty_{\mu+1}(B)$. By Proposition 3.15, if a solution σ_0 existed then $d\theta_0 = d^* d\sigma_0$ and $d\sigma_0 \in \Omega^3_{27}(B)$. We would therefore conclude that $\rho_0 = d\sigma_0$ solves (3.33).

By Proposition 3.15 the equation $\triangle\sigma_0 = d\theta_0$ has a solution if and only if $d\theta_0$ is $L^2(B)$–orthogonal to basic closed and coclosed 2–forms on M in $\mathcal{H}^2_{-6-\mu}(B)$. Now, since $-6-\mu = -5+\delta < -2$, we have $\mathcal{H}^2_{-6-\mu}(B) \simeq H^2_c(B)$ by Theorem 2.31(i) and Lemma 2.32(iii). Therefore by duality a solution σ_0 exists if and only if $[*d\theta_0] = -[d\theta_0 \wedge \varphi_0] = 0 \in H^5(B)$. Note that this condition coincides with the necessary topological constraint (3.22).

Remark 3.34 Our standing assumption $d\theta_\infty \neq 0$ is in fact necessary for (3.22) to be satisfied in a nontrivial way. For otherwise $d\theta_0$ would represent an L^2–integrable basic cohomology class by Theorem 2.31(i) and therefore $*d\theta_0 = -d\theta_0 \wedge \varphi_0$ could never be exact unless $d\theta_0 = 0$. However, if $d\theta_0 = 0$ then $\Phi_\epsilon = \epsilon\theta_0 \wedge \varphi_0 + \psi_0$ is already torsion-free for all $\epsilon > 0$, albeit locally reducible and therefore not interesting from a Spin(7)–geometry point of view. In fact, in this case one can argue that $M = (B \times S^1)/\,\Gamma$ for a freely acting finite group Γ and a *smooth* AC G_2–manifold B.

Mapping properties of the linearisation We now consider the linear operator $\mathcal{L}_0$ acting on the space of triples (ρ, f, κ) of a basic closed 3–form ρ, a basic function f and a basic closed 2–form κ. In fact we must further restrict κ to be exact, ie we vary $d\theta_0$ in the fixed basic cohomology class $c_1^{\mathrm{orb}}(M)$.

We now fix $\nu = -2+\delta$ for an arbitrarily small $\delta > 0$, $k \geq 1$ and $\alpha \in (0,1)$. Given a basic 5–form τ of class $C^{k-1,\alpha}_{\nu-1}(B)$, we look for a basic closed 3–form ρ, basic 1–form η and basic function f of class $C^{k,\alpha}_{\nu}(B)$ such that

$$\mathcal{L}_0(\rho, f, d\eta) = d\big(\hat{\rho} + \tfrac{2}{3} f\psi_0 + \eta\wedge\varphi_0\big) = \tau.$$

Clearly τ must be closed (in fact, exact) if a solution exists.

Corollary 3.16 describes sufficient conditions for solving this equation. If τ is closed and L^2–orthogonal to $\mathcal{H}^5_{-6-\nu}(B)$, then there exist a basic 2–form σ and a basic 1–form γ of class $C^{k+1,\alpha}_{\nu+1}(B)$ such that $\tau = \mathcal{L}_0\big(d\sigma, -\tfrac{3}{2}d^*\gamma, d\,\mathrm{curl}\,\gamma\big)$.

The nonlinear equation This discussion suggests writing $\zeta = \big(d\sigma, -\tfrac{3}{2}d^*\gamma, d\,\mathrm{curl}\,\gamma\big)$ and considering (3.31) as an equation for a basic 2–form σ and a basic 1–form γ of class $C^{k+1,\alpha}_{\nu+1}(B)$, where $\nu = -2+\delta$ with $\delta > 0$ small.

In order to control the nonlinearities we use Remark 2.20(iv) and the estimates (3.27) and (3.28): if σ and γ are of class $C^{k+1,\alpha}_{\nu+1}(B)$ with $\nu = -2+\delta$, then

$$\begin{aligned}\mathcal{N}_\epsilon\big(d\sigma, -\tfrac{3}{2}d^*\gamma, d\ \mathrm{curl}\,\gamma\big)\\ = d\big((1+f)^{\frac{2}{3}}(\psi_\epsilon + \hat{\rho} + Q_{\varphi_\epsilon}(\rho)) - \psi_\epsilon - \hat{\rho} - \tfrac{2}{3} f\,\psi_\epsilon + \mathrm{curl}\,\gamma\wedge d\sigma\big)\end{aligned}$$

lies in $d\,C^{k,\alpha}_{\nu}(B)$; cf [62, Lemma 5.16]. By Remark 3.17, $\mathcal{L}_0$ can therefore be inverted on the image of $\mathcal{N}_\epsilon$.

Similarly, using the fact that $\rho_0 = d\sigma_0$ is exact, we observe that

$$\Psi(\xi_0 + \epsilon\zeta_0) = d(Q_{\varphi_0}(\epsilon\rho_0) + \epsilon^2 d\theta_0\wedge\sigma_0).$$

Moreover, for all $j \geq 0$, $|\nabla^j(Q_{\varphi_0}(\epsilon\rho_0) + \epsilon^2 d\theta_0\wedge\sigma_0)| \leq C_j r^{-2-j+\delta}$ for $\delta > 0$ sufficiently small by (3.27) and (3.28) and Remark 2.20(iv). Therefore the error (3.32) is also in the image of $\mathcal{L}_0$ by Remark 3.17.

The existence result We now have all the ingredients to apply the implicit function theorem and guarantee the existence of solutions to (3.31) for $\epsilon > 0$ sufficiently small. We now summarise the resulting existence result for solutions to (3.21).

Let $(N, \pi_\infty, \theta_\infty, \omega, \Omega)$ be a nearly Kähler principal Seifert circle bundle, and let $(M, \pi, \theta_0, \varphi_0)$ be a transversally AC G_2–holonomy principal Seifert circle bundle asymptotic to $\mathrm{BC}(N)$. Given $\epsilon > 0$ we modify the metric on $\mathrm{BC}(N)$ by rescaling g_{BC} in the direction of the circle orbits:

$$g_{\mathrm{BC},\epsilon} = dr^2 + r^2 g_\Sigma + \epsilon^2 \theta_\infty^2.$$

A Riemannian metric g on M is called *ALC* (*asymptotically locally conical*) if there exists a compact set $K \subset M$, $\epsilon > 0$, $R > 0$ and a diffeomorphism $f : (R, \infty) \times N \to M \setminus K$ such that

$$|\nabla^j (f^* g - g_{\mathrm{BC},\epsilon})| = O(r^{-j+\mu}) \tag{3.35}$$

for all $j \geq 0$ and some $\mu < 0$. Here covariant derivatives and norms are computed using the Riemannian metric $g_{\mathrm{BC},\epsilon}$.

Theorem 3.36 *Let $\pi : M^8 \to B$ be a principal Seifert circle bundle endowed with a transversally AC basic torsion-free G_2–structure φ_0 and let θ_0 be the (unique up to diffeomorphism) connection 1–form on π such that $d\theta_0 \in \Omega^2_{14}(B)$.*

If $\pi : M \to B$ is nontrivial and in basic cohomology

$$[d\theta_0 \wedge \varphi_0] = 0 \in H^5(B),$$

then there exists $\epsilon_0 = \epsilon_0(M, \varphi_0)$ with the following significance. For all $\epsilon \in (0, \epsilon_0)$, there exists an S^1–invariant torsion-free $\mathrm{Spin}(7)$–structure Φ_ϵ on M such that:

(i) *The induced metric g_{Φ_ϵ} has holonomy $\mathrm{Spin}(7)$ and ALC asymptotics.*
(ii) *As $\epsilon \to 0$, (M, g_{Φ_ϵ}) is arbitrarily close to $g_{\varphi_0} + \epsilon^2 \theta_0^2$ in $C^{k,\alpha}_{\mathrm{loc}}$ for every $k \geq 0$. In particular, (M, g_{Φ_ϵ}) collapses with bounded curvature to the orbifold (B, g_{φ_0}) as $\epsilon \to 0$.*

Proof The existence of torsion-free S^1–invariant $\mathrm{Spin}(7)$–structures Φ_ϵ on M for all $\epsilon > 0$ follows from the previous discussion. We start with the solution $(\varphi_0, 1, 0)$ of the limiting equation (3.31) with $\epsilon = 0$. We have explained how to construct a solution $(d\sigma_0, 0, d\theta_0)$ to the linearised problem $\mathcal{L}_0(d\sigma_0, 0, d\theta_0)$. We then consider (3.31) as an equation for a basic 2–form σ and a basic 1–form γ of class $C^{k+1,\alpha}_{\nu+1}(B)$, where $\nu = -2 + \delta$ with $\delta > 0$ small, and $k \geq 1$ and $\alpha \in (0, 1)$ are arbitrary. An application of the implicit function theorem — in the quantitative version stated for example in [7, Lemma 1.3] — yields the existence of a torsion-free $\mathrm{Spin}(7)$–structure Φ_ϵ on M for every $\epsilon > 0$ sufficiently small.

Now, the induced Spin(7)–metric g_{Φ_ϵ} is ALC with μ the maximum between the rate of decay of (π, θ, φ) to $(\pi_\infty, \theta_\infty, \varphi_{\mathrm{C}})$ and $-1+\delta$. Indeed, the implicit function theorem already implies that (3.35) holds with this choice of μ for all $0 \leq j \leq k$. For fixed $\epsilon > 0$ it is then possible to deduce the decay of higher-order derivatives by a bootstrap argument exploiting the fact that Φ_ϵ is a closed and coclosed form with respect to a metric which differs from the model metric $g_{\mathrm{BC},\epsilon}$ by $C^{k,\alpha}_\mu$–terms, and that weighted elliptic estimates analogous to those in Theorem 2.23 hold for $g_{\mathrm{BC},\epsilon}$. The convergence statement in (ii) follows immediately from the application of the implicit function theorem, since on any given compact set in M, weighted Hölder norms are equivalent to standard Hölder norms.

Finally, in order to prove that g_{Φ_ϵ} has holonomy $\mathrm{Spin}(7)$ we use [15, Lemma 2], which states that the holonomy of a metric induced by a torsion-free Spin(7)–structure on a simply connected 8–manifold M is equal to $\mathrm{Spin}(7)$ if and only if there are no parallel 1–forms and 2–forms on M. By Proposition 3.10(i), up to a finite cover we can assume that M is simply connected. We first consider parallel forms of degree 1 and 2 on $\mathrm{BC}(N)$, since these determine the asymptotic behaviour of parallel forms on M. In fact, since $|d\theta_\infty|_{g_{\mathrm{BC}}} = O(r^{-2})$ we can consider forms on $\mathrm{BC}(N)$ which are parallel with respect to the adapted connection. Since the holonomy of the adapted connection of $\mathrm{BC}(N)$ is G_2, there are no parallel 2–forms and the only parallel 1–form is θ_∞. Since we assume that $d\theta_\infty \neq 0$, however, θ_∞ cannot extend to a parallel 1–form on M. Hence every parallel form on M of degree 1 and 2 must decay and therefore vanish. □

4 Examples

In this final section we use our existence result, Theorem 3.36, to construct concrete examples of complete Spin(7)–metrics. Given the currently limited knowledge of AC G_2–manifolds, using orbifolds is essential. We are able to use Theorem 3.36 to produce infinitely many different topological types of complete Spin(7)–metrics and examples of 8–manifolds carrying infinitely many distinct families of ALC Spin(7)–metrics (Theorems B and C in the introduction). Previously only a handful of complete noncompact Spin(7)–metrics was known.

In Section 4.3 we use the analysis on AC orbifolds developed in this paper to extend the construction in [32] of ALC G_2–manifolds from AC Calabi–Yau 3–folds to the case of AC Calabi–Yau orbifolds. As an illustrative example, we use this extension to produce infinitely many distinct families of ALC G_2–metrics on $S^3 \times \mathbb{R}^4$.

4.1 Self-dual Einstein 4–orbifolds and Bryant–Salamon AC G_2–metrics

The starting point for the construction of Theorem 3.36 is an AC G_2–manifold or orbifold satisfying appropriate topological conditions. AC G_2–manifolds are hard to construct. All currently known examples of AC G_2–manifolds admit a symmetry group that acts with cohomogeneity one, ie with generic orbits of codimension 1. The large symmetry group affords a reduction of the PDE for the holonomy reduction to G_2 to a system of nonlinear ODEs. Studying solutions to these ODE systems is still nontrivial: in 1989 Bryant and Salamon [17] constructed three explicit AC G_2–metrics, but only very recently have further examples of AC G_2–manifolds been found [33, Theorem C]; these examples are not explicit and their existence is based on the qualitative analysis of the relevant ODE system.

From a different point of view, the Bryant–Salamon examples of AC G_2–manifolds in [17] fall into the class of constructions, pioneered by Calabi [18] in the Calabi–Yau and hyperkähler case, of complete Ricci-flat metrics on total spaces of vector bundles over compact manifolds satisfying appropriate curvature conditions. In particular, the Bryant–Salamon construction yields an AC G_2–metric (unique up to scale) on the total space of the bundle of anti-self-dual 2–forms over a self-dual Einstein 4–manifold Q with positive scalar curvature. By a theorem of Hitchin [51], the only such manifolds are S^4 and $\mathbb{CP}^2$ with their standard metrics. On the other hand, there are infinitely many self-dual Einstein 4–orbifolds with positive scalar curvature and the Bryant–Salamon construction extends immediately (as observed in the introduction of [17]) to the case where Q is an orbifold and B is the total space of the orbibundle of anti-self-dual 2–forms on Q.

The most powerful known method of construction of self-dual Einstein 4–orbifolds is the quaternionic Kähler quotient construction of Galicki and Lawson [37]. We briefly recall this here. The construction is based on the tight relationship between quaternionic Kähler and hyperkähler geometry [85]. Let C be a hyperkähler cone acted upon by a group K of triholomorphic isometries. Using the conical structure one can always find a hyperkähler moment map $\mu\colon \mathrm{C} \to \mathfrak{k}^* \otimes \operatorname{Im}\mathbb{H}$; see [11, Proposition 13.6.1]. Besides the triholomorphic action of K, there is an action of $\mathbb{H}^*$ on C generated by the Euler vector field $r\partial_r$. Letting $\mathbb{H}^*$ act on $\mathfrak{k}^* \otimes \operatorname{Im}\mathbb{H}$ via conjugation on $\operatorname{Im}\mathbb{H}$, the moment map μ is equivariant with respect to the action of $K\cdot\mathbb{H}^*$, where $K\cdot\mathbb{H}^* = (K\times\mathbb{H}^*)/\mathbb{Z}_2$ if $-1\in K$ and $K\cdot\mathbb{H}^* = K\times\mathbb{H}^*$ otherwise. The hyperkähler quotient construction [55] yields the existence of a hyperkähler structure on (the smooth part of) $\mu^{-1}(0)/K$.

Using the $\mathbb{H}^*$–equivariance property of the moment map one can show that $\mu^{-1}(0)/K$ is a new hyperkähler cone C'.

Now, to each hyperkähler cone C there is a naturally associated positive quaternionic Kähler "space" Q, obtained as the quotient of C by the $\mathbb{H}^*$ action generated by $r\partial_r$. Here we say that Q is a positive quaternionic Kähler space if its smooth part carries a Riemannian metric with holonomy contained in $\mathrm{Sp}(1)\cdot\mathrm{Sp}(n)$ (in particular the dimension of Q must be a multiple of 4); any such metric is Einstein and the qualification "positive" refers to the sign of the Einstein constant. In dimension 4 this definition must be modified: we say that Q^4 is quaternionic Kähler if it is self-dual and Einstein. Because of the $\mathbb{H}^*$–equivariance of the moment map $\mu\colon \mathrm{C}\to\mathfrak{k}^*\otimes\mathrm{Im}\,\mathbb{H}$, the construction of the hyperkähler cone C' as a hyperkähler quotient of the cone C yields a quaternionic Kähler structure on $Q'=\mathrm{C}'/\mathbb{H}^*$ as a quaternionic Kähler quotient of $Q=\mathrm{C}/\mathbb{H}^*$; see [37].

Many interesting self-dual Einstein 4–orbifolds with positive scalar curvature can be obtained in this way even from the simplest hyperkähler cone, $\mathrm{C}=\mathbb{H}^{n+1}$. (Here we regard $\mathbb{H}^{n+1}$ as the hyperkähler cone over the round sphere $\mathbb{S}^{4n+3}$; the associated quaternionic Kähler manifold is $\mathbb{HP}^n=\mathbb{S}^{4n+3}/\mathrm{Sp}(1)$.) For example, all *toric* self-dual Einstein 4–orbifolds with positive scalar curvature, ie those self-dual Einstein 4–orbifolds with a T^2–symmetry, arise in this way [19]; see [11, Sections 12.4–12.5 and 13.7] for further details.

Remark There are also self-dual Einstein 4–orbifolds with positive scalar curvature that are *not* obtained via quaternionic Kähler reduction, for example the families of SO(3)–invariant self-dual Einstein orbifold metrics on S^4 and $\mathbb{CP}^2$ constructed by Hitchin in [52]. We will not use these metrics in this paper.

Now, in view of the assumptions of Theorem 3.36, amongst all self-dual Einstein 4–orbifolds with positive scalar curvature we are interested in those that satisfy the following additional property.

Definition 4.1 Let Q be a self-dual Einstein 4–orbifold with positive scalar curvature. We say that Q is Spin(7)*–admissible* if there exists a principal Seifert circle bundle $S\to Q$.

The relevance of this assumption is explained by the following lemma.

Lemma 4.2 *Let Q be a self-dual Einstein 4–orbifold with positive scalar curvature and denote by B the total space of the orbibundle of anti-self-dual 2–forms on Q endowed with Bryant–Salamon AC torsion-free G_2–structure φ_0. Then Q is* Spin(7)*–admissible if and only if there exists a principal Seifert circle bundle $\pi\colon M \to B$ such that $c_1^{\mathrm{orb}}(M) \cup [\varphi_0] = 0 \in H^5_{\mathrm{orb}}(B)$.*

Proof If $\pi\colon S \to Q$ is a principal Seifert circle bundle and $p\colon B \to Q$ is the orbibundle of anti-self-dual 2–forms then $M = p^*S$ is a principal circle orbibundle over B. Over a small enough uniformising chart $U/\Gamma \subset Q$ we can trivialise both π and p. Then there exist representations of Γ in $U(1)$ and $\mathrm{SO}(3)$ such that $\pi^{-1}(U/\Gamma) = (U \times S^1)/\Gamma$ and locally M can be described as $(U \times \mathbb{R}^3 \times S^1)/\Gamma$. If S is smooth then Γ acts freely on S^1 and therefore M is smooth as well. Moreover, the topological constraint $c_1^{\mathrm{orb}}(M) \cup [\varphi_0] = 0$ is automatically satisfied since $H^5_{\mathrm{orb}}(B) \simeq H^5_{\mathrm{orb}}(Q) = 0$.

Conversely, if $\pi\colon M \to B$ is a principal Seifert circle bundle over the total space of the orbibundle of anti-self-dual 2–forms on Q, then Q is Spin(7)–admissible since the restriction S of the orbibundle $M \to B$ to the zero-section Q in B yields a principal Seifert circle bundle $S \to Q$. □

4.2 Concrete examples of ALC Spin(7)–metrics

In order to apply our existence result, Theorem 3.36, we need to understand to what extent known constructions of self-dual Einstein 4–orbifolds with positive scalar curvature yield examples that are Spin(7)–admissible. In the rest of the section we will limit ourselves to discussing families of examples that give a sense of the rich variety of ALC Spin(7)–metrics that can be obtained using Theorem 3.36, and defer a more systematic study to elsewhere. We discuss three sets of examples. Our first theorem provides a proof of the existence of a 1–parameter family of ALC Spin(7)–metrics conjectured by Gukov, Sparks and Tong [46]; currently this is the only example that can be obtained from Theorem 3.36 starting from a smooth AC G_2–manifold. Using AC G_2–orbifolds, we then prove that the same smooth 8–manifold in fact carries infinitely many distinct families of ALC Spin(7)–metrics. Finally, we construct infinitely many smooth 8–manifolds carrying complete ALC Spin(7)–metrics.

4.2.1 An example from a smooth AC G_2–manifold Amongst the known examples of smooth AC G_2–manifolds, only $\Lambda^- T^*\mathbb{CP}^2$ endowed with the Bryant–Salamon AC G_2–metric can be used in Theorem 3.36. Indeed, the other two Bryant–Salamon examples of AC G_2–manifolds [17] have vanishing second cohomology, while all the infinitely many examples constructed in [33] only have compactly supported second cohomology. Thus the topological constraint (3.22) can be satisfied in a nontrivial way only in the case of $\Lambda^- T^*\mathbb{CP}^2$.

Theorem 4.3 *The total space of the nontrivial rank-3 real vector bundle over S^5 carries a 1–parameter family of ALC $\mathrm{Spin}(7)$–metrics. The Bryant–Salamon AC G_2–metric on $\Lambda^- T^*\mathbb{CP}^2$ arises as a collapsed limit of this family.*

Proof The Bryant–Salamon AC G_2–manifold $B = \Lambda^- T^*\mathbb{CP}^2$ has torsion-free one-dimensional second cohomology and contains no 5–cycle. Hence up to a change of orientation and finite quotients there is a unique nontrivial circle bundle M over B and the topological constraint (3.22) is automatically satisfied. The Bryant–Salamon AC G_2–metric is $\mathrm{SU}(3)$–invariant and this $\mathrm{SU}(3)$–action lifts to an isometric action of the ALC $\mathrm{Spin}(7)$–metrics produced by Theorem 3.36. We give a description of all the manifolds involved in terms of this $\mathrm{SU}(3)$–action: $\mathbb{CP}^2 = \mathrm{SU}(3)/U(2)$ and therefore $B = \mathrm{SU}(3) \times_{U(2)} \mathfrak{su}_2$ (here the action of $U(2)$ on $\mathfrak{su}_2$ is induced by the adjoint representation); the unique simply connected circle bundle over $\mathbb{CP}^2$ is $S^5 = \mathrm{SU}(3)/\mathrm{SU}(2)$ and the 8–manifold carrying ALC $\mathrm{Spin}(7)$–metrics by Theorem 3.36 is $M = \mathrm{SU}(3) \times_{\mathrm{SU}(2)} \mathfrak{su}_2$. □

Remark The existence of this 1–parameter family of ALC $\mathrm{Spin}(7)$–metrics was conjectured by Gukov, Sparks and Tong [46]. The family is expected to be part of a geometric transition in $\mathrm{Spin}(7)$–geometry which physically corresponds to a duality between Type IIA string theory on $\Lambda^- T^*\mathbb{CP}^2$ with D6 branes/Ramond–Ramond fluxes. The family of ALC metrics of Theorem 4.3 provides the M theory lift of Type IIA theory on $\Lambda^- T^*\mathbb{CP}^2$ with fluxes, while the lift of Type IIA theory on $\Lambda^- T^*\mathbb{CP}^2$ with a D6–brane wrapping the coassociative $\mathbb{CP}^2$ corresponds to an explicit ALC $\mathrm{Spin}(7)$–metric found by Gukov and Sparks [45] and its conjectural 1–parameter family of deformations up to scale. A complete geometric explanation of the physical duality would involve constructing AC $\mathrm{Spin}(7)$–metrics arising as limits of the two 1–parameter families of ALC $\mathrm{Spin}(7)$–metrics as well as an ALC $\mathrm{Spin}(7)$–metric on $\mathbb{R}_+ \times \mathrm{SU}(3)/U(1)$ with an isolated conical singularity. This is completely analogous to the analysis of [33] in the G_2 setting. In fact, since the metrics in Theorem 4.3 admit a cohomogeneity one action of $\mathrm{SU}(3)$, Lehmann [69] has recently used ODE methods to address these conjectures.

4.2.2 Infinitely many families of ALC Spin(7)–metrics on the same smooth 8–manifold The first examples of self-dual Einstein 4–orbifolds obtained by Galicki and Lawson [37] via the quaternionic Kähler construction were weighted projective planes arising from quotients of $\mathbb{HP}^2$ by a circle. In this section we use these examples to produce infinitely many distinct families of ALC Spin(7)–metrics on the smooth 8–manifold of Theorem 4.3.

In order to describe the main features of Galicki–Lawson's self-dual Einstein metrics we follow [12, Sections 7 and 8], which generalises the construction to circle quotients of quaternionic projective spaces of arbitrary dimension; see also [11, Proposition 12.5.3]. Fix $p_1, p_2, p_3 \in \mathbb{Z}_{>0}$ with $\gcd(p_1, p_2, p_3) = 1$ and consider the circle action

$$e^{i\theta} \cdot [u_1 : u_2 : u_3] = [e^{ip_1\theta}u_1 : e^{ip_2\theta}u_2 : e^{ip_3\theta}u_3]$$

on $\mathbb{HP}^2$. The quaternionic Kähler quotient of $\mathbb{HP}^2$ by the circle action is

$$Q = \{[u_1 : u_2 : u_3] \in \mathbb{HP}^2 \mid p_1\bar{u}_1 i u_1 + p_2\bar{u}_2 i u_2 + p_3\bar{u}_3 i u_3 = 0\}/S^1.$$

By [12, Proposition 7.5], the equation $p_1\bar{u}_1 i u_1 + p_2\bar{u}_2 i u_2 + p_3\bar{u}_3 i u_3 = 0$ cuts out in $\mathbb{S}^{11} \subset \mathbb{H}^3$ a smooth 8–manifold diffeomorphic to the complex Stiefel manifold $V_2(\mathbb{C}^3) \simeq \mathrm{SU}(3)$. There is an action of $S^1 \cdot \mathrm{SU}(2)$ on $V_2(\mathbb{C}^3)$ arising from the circle action on $\mathbb{HP}^2$ and the standard $\mathrm{Sp}(1)$–action on the 3–Sasakian structure on the sphere $\mathbb{S}^{11}$. The quotient $Q = V_2(\mathbb{C}^3)/S^1 \cdot \mathrm{SU}(2)$ then has a quaternionic Kähler metric. Note that in general only the maximal torus of $\mathrm{SU}(3)$ commutes with the S^1–action on $V_2(\mathbb{C}^3)$ and therefore, contrary to the example of Theorem 4.3, Q is only toric and not $\mathrm{SU}(3)$–invariant. We also obtain two natural orbibundles over Q: the Konishi bundle $V_2(\mathbb{C}^3)/S^1$ and the principal circle orbibundle $S^5 = V_2(\mathbb{C}^3)/\mathrm{SU}(2)$. Studying the fibrewise S^1–action on S^5 one can show that Q is isomorphic to the weighted projective plane $\mathbb{WCP}^2[q_1, q_2, q_3]$, with $q_i = p_j + p_k$ if $p_1 + p_2 + p_3$ is odd and $2q_i = p_j + p_k$ otherwise. Here (ijk) runs through cyclic permutations of (123). It is now clear that Q is Spin(7)–admissible and that the 8–dimensional principal Seifert circle bundle over $\Lambda^- T^*Q$ is $M = V_2(\mathbb{C}^3) \times_{\mathrm{SU}(2)} \mathfrak{su}_2$.

Theorem 4.4 *The total space of the nontrivial rank-*3 *real vector bundle over* S^5 *carries infinitely many distinct families of ALC* $\mathrm{Spin}(7)$*–metrics.*

Proof The existence of highly collapsed ALC Spin(7)–metrics follows from applying Theorem 3.36 to the Spin(7)–admissible self-dual Einstein 4–orbifolds Q obtained as quotients of $\mathbb{HP}^2$ by circles labelled by (p_1, p_2, p_3). The fact that, up to obvious symmetries, different choices of (p_1, p_2, p_3) give rise to nonisometric families of Spin(7)–metrics follows from the fact that their tangent cones at infinity, the Bryant–Salamon AC orbifold G_2–metrics on $\Lambda^- T^* Q$, are distinct. □

It is likely that many more examples of toric self-dual Einstein 4–orbifolds are Spin(7)–admissible. For example, an infinite family of orbifolds Q with 2–dimensional $H^2_{\mathrm{orb}}(Q)$ (therefore not weighted projective planes) can be obtained by quaternionic Kähler quotient of $\mathrm{Gr}_2(\mathbb{C}^4)$ by a circle; these examples can also be described as particular reductions of $\mathbb{HP}^3$ by a 2–torus, since $\mathrm{Gr}_2(\mathbb{C}^4)$ is itself a circle quotient of $\mathbb{HP}^3$. The generic Q constructed in this way is Spin(7)–admissible: indeed, the zero-level set of the 3–Sasakian moment map in the Konishi bundle of $\mathrm{Gr}_2(\mathbb{C}^4)$, a circle orbibundle over the 4–orbifold, is smooth for a generic choice of embedding of S^1 into the symmetry group SU(4) of $\mathrm{Gr}_2(\mathbb{C}^4)$. In general, however, we do not know how to recognise which toric self-dual Einstein 4–orbifolds are Spin(7)–admissible. In [14] very clear combinatorial conditions for such an orbifold to admit a smooth 3–Sasaki Konishi bundle are given and it is likely that similar combinatorial conditions characterise Spin(7)–admissibility. Instead of pursuing such a systematic combinatorial approach, in the next section we construct by hand an explicit family of examples with unbounded second orbifold Betti number.

Remark Let Q be a toric self-dual Einstein 4–orbifold, which by [19] must be obtained as a quotient of $\mathbb{HP}^n$ by an $(n-1)$–torus T^{n-1}. Let $\mu\colon \mathbb{H}^{n+1} \to \mathbb{R}^{n-1} \otimes \mathrm{Im}\,\mathbb{H}$ denote the associated hyperkähler moment map and recall that, thinking of μ as a section of the bundle $\mathbb{S}^{4n+3} \times_{\mathrm{SU}(2)} \mathfrak{su}_2 \to \mathbb{HP}^n$, we have $Q = \mu^{-1}(0)/T^{n-1}$. The orbifold Q is Spin(7)–admissible if there exists a subtorus $T^{n-2} \subset T^{n-1}$ such that $S = \mu^{-1}(0)/T^{n-2}$ is a smooth 5–manifold. If this were the case, then we could consider the principal G–bundle $P = \{\boldsymbol{u} \in \mathbb{S}^{4n+3} \mid \mu(\boldsymbol{u}) = 0\}/T^{n-2} \to S$ with $G = \mathrm{SU}(2)$ or SO(3) depending on whether $-1 \in T^{n-2}$ or not, and $M = P \times_{\mathrm{SU}(2)} \mathfrak{su}_2$ would carry ALC Spin(7)–metrics by Theorem 3.36. In general P is an orbifold. Motivated by the fact that P carries a natural hypercomplex structure, Boyer, Galicki and Mann [13] determine conditions on $T^{n-2} \subset T^{n-1}$ under which P is a smooth 8–manifold, but they do not study which additional conditions guarantee that the action of G on P is free.

4.2.3 Examples with arbitrarily large second Betti number from A_n ALE spaces

In this section we prove the existence of infinitely many diffeomorphism types of simply connected 8–manifolds carrying complete Spin(7) metrics. The examples we will consider arise from an extension of Kronheimer's construction of ALE spaces [66; 67] to the quaternionic Kähler setting due to Galicki and Nitta [38].

Let Γ be a finite subgroup of SU(2) acting freely on $\mathbb{C}^2 \setminus \{0\}$. Kronheimer [66] constructed ALE hyperkähler metrics on the minimal resolution of $\mathbb{C}^2/\Gamma$ using the hyperkähler quotient construction. Let $R_0, \dots, R_r$ be the irreducible representations of Γ, with R_0 the trivial representation. Set $n_i = \dim R_i$. The regular representation R of Γ decomposes as

$$R = \bigoplus_{i=0}^{r} \mathbb{C}^{n_i} \otimes R_i.$$

Kronheimer considered $\mathrm{Hom}_\Gamma(R, R \otimes \mathbb{H}) \simeq \mathbb{H}^n$, where Γ acts on $\mathbb{H} \simeq \mathbb{C}^2$ via its embedding in $\mathrm{SU}(2)$. The McKay correspondence implies that $n = |\Gamma|$. Define $\widehat{K}$ as the group of unitary transformations of R that commute with the action of Γ; by the Schur lemma $\widehat{K} = \prod_{i=0}^r U(n_i)$. Then $K = \widehat{K}/\triangle U(1) \simeq \prod_{i=1}^r U(n_i)$ acts effectively and triholomorphically on $\mathbb{H}^n$. Let $\mu\colon \mathbb{H}^n \to \mathfrak{k}^* \otimes \mathrm{Im}\,\mathbb{H}$ denote the hyperkähler moment map for the action of K on $\mathbb{H}^n$ and let $\mathfrak{z}$ denote the Lie algebra of the centre of K. By the hyperkähler quotient construction, for each $\boldsymbol{\zeta} \in \mathfrak{z}^* \otimes \mathrm{Im}\,\mathbb{H}$ the smooth part of the quotient $X_{\boldsymbol{\zeta}} = \mu^{-1}(\boldsymbol{\zeta})/K$ carries a natural hyperkähler structure. Kronheimer showed that for generic $\boldsymbol{\zeta} \in \mathfrak{z}^* \otimes \mathrm{Im}\,\mathbb{H}$, $X_{\boldsymbol{\zeta}}$ is a smooth manifold diffeomorphic to the minimal resolution of $\mathbb{C}^2/\Gamma$ and that its natural hyperkähler structure is asymptotic at infinity to the flat hyperkähler structure on $\mathbb{C}^2/\Gamma$ (with rate -4). Conversely, Kronheimer showed in [67] that any asymptotically locally Euclidean (ALE) hyperkähler 4–manifold is obtained from this quotient construction.

For the extension of this construction to the quaternionic Kähler setting in [38], Galicki and Nitta thought of $\boldsymbol{\zeta} \in \mathfrak{k}^* \otimes \mathrm{Im}\,\mathbb{H}$ as a map $\mathfrak{k} \to \mathrm{Im}\,\mathbb{H} \simeq \mathfrak{su}_2$ and assumed that there exists a group homomorphism $\rho\colon K \to \mathrm{SU}(2)$ with $\rho_* = -\boldsymbol{\zeta}$. We use ρ to define an action of K on $\mathbb{HP}^n$ by

$$g \cdot [u_0 : \boldsymbol{u}] = [\rho(g)u_0 : g\boldsymbol{u}].$$

The quaternionic Kähler quotient of $\mathbb{HP}^n$ by K is $Q = \widehat{\mu}^{-1}(0)/K$, where

$$\widehat{\mu}([u_0 : \boldsymbol{u}]) = -\overline{u}_0 \boldsymbol{\zeta} u_0 + \mu(\boldsymbol{u}).$$

Galicki and Nitta [38, Theorem 3.2] showed that if $\boldsymbol{\zeta}$ is generic in the sense of Kronheimer (ie $\mu^{-1}(\boldsymbol{\zeta})/K$ is smooth) then $\widehat{\mu}^{-1}(0)/K$ is a quaternionic Kähler 4–orbifold.

Now, denote by K_ρ the kernel of ρ and assume that $K/K_\rho \simeq U(1)$, ie $\rho\colon K \to U(1) \subset \mathrm{SU}(2)$. In this case $S = \hat{\mu}^{-1}(0)/K_\rho \to \hat{\mu}^{-1}(0)/K$ is a principal circle orbibundle over Q. Up to rotations and using a K–invariant metric to identify $\mathfrak{k}$ with its dual, we must have $\boldsymbol{\zeta} = 2\pi i \zeta$, where $\zeta \in \mathfrak{z}$. Since $K = \prod_{i=1}^r U(n_i)$ we can identify $\mathfrak{z}$ with $\mathbb{R}^r$ and since $\boldsymbol{\zeta}$ integrates to a group homomorphism $K \to U(1)$, we must have $\zeta \in \mathbb{Z}^r \subset \mathbb{R}^r$. We then define a 1–dimensional representation R_ζ of Γ by $R_\zeta = \bigotimes_{i=1}^r \det(R_i)^{\zeta_i}$.

Proposition 4.5 *Assume that $\boldsymbol{\zeta} = 2\pi i \zeta$ is generic in the sense of Kronheimer and that the homomorphism $\Gamma \to U(1)$ corresponding to the representation R_ζ is injective. Then $S = \hat{\mu}^{-1}(0)/K_\rho$ is smooth, ie $Q = \hat{\mu}^{-1}(0)/K$ is a* Spin(7)*–admissible quaternionic Kähler 4–orbifold.*

Proof Write $\mathbb{HP}^n$ as $\mathbb{H}^n \cup \mathbb{HP}^{n-1}$, where $\mathbb{H}^n$ is identified with the open set where $u_0 \neq 0$ and $\mathbb{HP}^{n-1} = \{[0:\boldsymbol{u}] \in \mathbb{HP}^n\}$.

First work on the open set $\mathbb{H}^n$. We introduce affine quaternionic coordinates $\boldsymbol{v} = \boldsymbol{u}\bar{u}_0$. Note that the open set $\hat{\mu}^{-1}(0) \cap \mathbb{H}^n/K$ of Q is identified with $\{\boldsymbol{v} \in \mathbb{H}^n \mid \mu(\boldsymbol{v}) = \boldsymbol{\zeta}\}/K$, where K acts by $g \cdot \boldsymbol{v} = g\boldsymbol{v}\overline{\rho(g)}$. Thus a dense open set of Q and the ALE manifold $X_{\boldsymbol{\zeta}}$ are obtained as quotients of $\mu^{-1}(\boldsymbol{\zeta})$ by K, but the K–action is different in the two cases. However, the K–actions agree when restricted to K_ρ, so $\hat{\mu}^{-1}(0) \cap \mathbb{H}^n/K_\rho$ coincides with the principal circle bundle $\mu^{-1}(\boldsymbol{\zeta})/K_\rho \to X_{\boldsymbol{\zeta}}$. In particular, the dense open set $\hat{\mu}^{-1}(0) \cap \mathbb{H}^n/K_\rho$ of S is smooth.

Consider now a point $[0:\boldsymbol{u}] \in \mathbb{HP}^{n-1}$ such that $\mu(\boldsymbol{u}) = 0$. We must show that the stabiliser of $[0:\boldsymbol{u}]$ in K_ρ is trivial. Now, in [66, Lemma 3.1] Kronheimer showed that there exists a copy of $\mathbb{C}^2$ in $\mu^{-1}(0) \subset \mathbb{H}^n$ such that every orbit of the K–action on $\mu^{-1}(0)$ meets a single orbit of the Γ–action on $\mathbb{C}^2$. Even if not explicitly mentioned in [66], the identification of $\mu^{-1}(0)/K$ with $\mathbb{C}^2/\Gamma$ can be made equivariant with respect to the action of Sp(1) given by the (diagonal) right quaternionic multiplication on $\mathbb{C}^2 = \mathbb{H}$ and $\mu^{-1}(0) \subset \mathbb{H}^n$. Thus Q is obtained by adding a single point ∞ with isotropy Γ to the open set $\hat{\mu}^{-1}(0) \cap \mathbb{H}^n/K$. To show that S is smooth, we need to show that the induced action of Γ on the fibre S^1 over ∞ of the orbibundle $S \to Q$ is free. This follows from the assumption that R_ζ is an effective Γ–representation, since the action of Γ on the fibre of S over ∞ is precisely given by R_ζ, by [68, Proposition 2.2(ii)].

In other words, to obtain S, compactify the principal circle bundle $\mu^{-1}(\boldsymbol{\zeta})/K_\rho \to X_{\boldsymbol{\zeta}}$ by adding the circle fibre over the orbifold point in the natural orbifold compactification of X_{ζ}. The circle bundle $\mu^{-1}(\zeta)/K_\rho \to X_{\zeta}$ carries a natural anti-self-dual connection, which is asymptotic to the flat connection on $\mathbb{C}^2/\Gamma$ with monodromy R. Hence if R is an effective representation of Γ, S is smooth. $\square$

The condition that $\Gamma \to U(1)$ is injective forces us to restrict to the abelian case $\Gamma = \mathbb{Z}_n$ for some $n \geq 2$. We can then be completely explicit. The irreducible representations of Γ are all 1–dimensional and labelled by an integer $0 \leq i \leq n-1$. The group $K = T^{n-1}$ acts on $\mathbb{H}^n$ by

$$(e^{i\theta_1}, \dots, e^{i\theta_{n-1}}) \cdot (u_1, \dots, u_n) \\ = (e^{i\theta_1} u_1, e^{i(\theta_2-\theta_1)} u_2, \dots, e^{i(\theta_{n-1}-\theta_{n-2})} u_{n-1}, e^{-i\theta_{n-1}} u_n).$$

The moment map $\mu\colon \mathbb{H}^n \to \mathbb{R}^{n-1} \otimes \operatorname{Im}\mathbb{H}$ is therefore

$$\mu(u_1, \dots, u_n) = (\overline{u}_1 i u_1 - \overline{u}_2 i u_2, \dots, \overline{u}_{n-1} i u_{n-1} - \overline{u}_n i u_n).$$

Fix $\zeta = (\zeta_1, \dots, \zeta_{n-1}) \in \mathbb{Z}^{n-1}$. Kronheimer's genericity conditions are — see for example [14, Example 2.22] —

(4.6) $$\zeta_i + \zeta_{i+1} + \dots + \zeta_{i+j} \neq 0 \quad \text{for all } 1 \leq i \leq n-1,\ 0 \leq j \leq n-1-i.$$

Moreover, the representation R_ζ induces an injective homomorphism $\Gamma \to U(1)$ if and only if

(4.7) $$\gcd(|\zeta|, n) = 1, \quad \text{where } |\zeta| = \zeta_1 + 2\,\zeta_2 + \dots + (n-1)\,\zeta_{n-1}.$$

We can also explicitly see that (4.7) guarantees that K_ρ acts freely on points in $\mathbb{HP}^n$ of the form $[0 : \boldsymbol{u}]$ with $\mu(\boldsymbol{u}) = 0$. Consider the action of $T^n \times \mathbb{H}$ on $\mathbb{H}^n$ defined by $(e^{i\psi_1}, \dots, e^{i\psi_n}, u) \cdot \boldsymbol{u} = (e^{i\psi_1} u_1 \overline{u}, \dots, e^{i\psi_n} u_n \overline{u})$. It is immediate to check that $\mu^{-1}(0)$ is the orbit of $(1, \dots, 1)$ and therefore $\{u_0 = 0\} \cap \hat{\mu}^{-1}(0)$ reduces to a single point $[0:1:\dots:1]$. We now choose $(\tilde{\zeta}_1, \dots, \tilde{\zeta}_n) \in \mathbb{Z}^n$ such that $\tilde{\zeta}_i - \tilde{\zeta}_{i+1} = \zeta_i$ for all $i = 1, \dots, n-1$. Note that $\tilde{\zeta}_1 + \dots + \tilde{\zeta}_n \equiv |\zeta|$ modulo n. Then K_ρ can be described as the subgroup of T^n cut out by the constraints

$$e^{i(\psi_1 + \dots + \psi_n)} = 1 = e^{i(\tilde{\zeta}_1 \psi_1 + \dots + \tilde{\zeta}_n \psi_n)}.$$

We conclude that the stabiliser of $[0:1:\dots:1]$ in $K_\rho \times \mathbb{H}$ consists of elements of the form $(\lambda, \dots, \lambda)$ with $\lambda \in S^1$ and such that

$$\lambda^n = 1 = \lambda^{\tilde{\zeta}_1 + \dots + \tilde{\zeta}_n} = \lambda^{|\zeta|}.$$

If n and $|\zeta|$ are coprime, then necessarily $\lambda = 1$.

Note that there exists a suitable choice for ζ for all $n \geq 2$. Indeed, if n is odd, consider $\zeta = (2, 1, \dots, 1)$, while if $n \geq 4$ is even, consider $\zeta = (2, 1, \dots, 1, 2, 1, \dots, 1)$, where the second 2 is the $\frac{1}{2}n^{\text{th}}$ coordinate of ζ. Since $\zeta_i > 0$ for all i, the genericity conditions (4.6) are certainly satisfied. Condition (4.7) is also satisfied since $|\zeta| = \frac{1}{2}n(n-1)+1$ if n is odd and $|\zeta| = \frac{1}{2}n^2+1$ if n is even. When $n=2$ we choose $\zeta = 1$. Note that in each case our choice for ζ satisfies the additional constraint $\gcd(\zeta_1, \dots, \zeta_{n-1}) = 1$.

We now consider the 8–manifold $M = \pi^*\Lambda^- T^* Q$, where $\pi\colon S \to Q$ is the orbibundle map. By abuse of notation, think of $\hat{\mu}$ as a map $\hat{\mu}\colon \mathbb{S}^{4n+3} \to \mathfrak{k}^* \otimes \operatorname{Im}\mathbb{H}$. Then $\hat{\mu}^{-1}(0)/K$ is a 3–Sasaki orbifold. Even though $\hat{\mu}^{-1}(0)/K$ may be singular, $P = \hat{\mu}^{-1}(0)/K_\rho$ is a smooth 8–manifold. Indeed, note that the action of K_ρ on $\mathbb{H}^{n+1}$ coincides with the restriction of the action of K on $\mathbb{H} \oplus \mathbb{H}^n$ which is trivial on the first factor. Since K acts freely on $\mu^{-1}(\bar{u}_0 \boldsymbol{\zeta} u_0)$ when $u_0 \neq 0$ by the genericity assumption on $\boldsymbol{\zeta}$ and on $\mu^{-1}(0) \setminus \{0\}$ since $\mu^{-1}(0)/K = \mathbb{C}^2/\Gamma$, we conclude that K_ρ acts freely on P. Note that P is a principal SU(2)–bundle over S; the fact that the structure group is certainly SU(2) rather than SO(3) is because K_ρ does not contain $-1 \in \mathrm{Sp}(n+1)$ (since $\gcd(|\zeta|, n) = 1$). Then the 8–manifold M is the total space of the associated adjoint bundle $P \times_{\mathrm{SU}(2)} \mathfrak{su}_2 \to S$.

Proposition 4.8 *Let* $\Gamma = \mathbb{Z}_n$ *and assume that* $\boldsymbol{\zeta} = 2\pi i \zeta \in 2\pi i \mathbb{Z}^{n-1}$ *satisfies* (4.6) *and* (4.7) *together with the additional constraint* $\gcd(\zeta_1, \dots, \zeta_r) = 1$. *Then S is simply connected, spin and* $b_2(S) = n-2$. *It follows that S is diffeomorphic to* $\sharp_{n-2}(S^2 \times S^3)$.

Proof As in the proof of Proposition 4.5, we write $S = S_0 \cup S_\infty$, where S_0 is a principal circle bundle over the 4–manifold $X_{\boldsymbol{\zeta}}$, and $S_\infty = (\mathbb{C}^2 \times S^1)/\mathbb{Z}_n$, with intersection $S_0 \cap S_\infty = (S^3 \times S^1)/\mathbb{Z}_n$.

Consider first the fundamental group of S. We have $\pi_1(S_\infty) = \pi_1(S_0 \cap S_\infty) \simeq \mathbb{Z}$. Moreover, if $\gcd(\zeta_1, \dots, \zeta_r) = 1$ then the first Chern class of $S_0 \to X_{\boldsymbol{\zeta}}$ is primitive in $H^2(X_{\boldsymbol{\zeta}}, \mathbb{Z})$ and therefore S_0 is simply connected. Van Kampen's theorem then implies that S is simply connected.

Taking into account the isomorphism $H^1(S_0) \oplus H^1(S_\infty) \to H^1(S_0 \cap S_\infty)$ and the fact that $H^2(S_\infty) = H^2(S_0 \cap S_\infty) = 0$, the Mayer–Vietoris sequence for $S = S_0 \cup S_\infty$ yields an isomorphism $H^2(S) \simeq H^2(S_0)$. Since $c_1(S_0) \neq 0$, the Gysin sequence for the circle fibration $S_0 \to X_{\boldsymbol{\zeta}}$ immediately yields $H^2(S_0) \simeq \mathbb{R}^{n-2}$.

In order to prove that S is spin, note that $w_2(S) \in H^2(S, \mathbb{Z}_2)$ must be the image of $w_2^{\mathrm{orb}}(Q) \in H^2_{\mathrm{orb}}(Q, \mathbb{Z}_2)$ since $\pi\colon S \to Q$ is a principal circle orbibundle and therefore $TS \simeq \pi^*TQ \oplus \mathbb{R}$. On the other hand, in dimension 4 we have $w_2^{\mathrm{orb}}(Q) = \varepsilon(Q) \in H^2_{\mathrm{orb}}(Q, \mathbb{Z}_2)$, where $\varepsilon(Q)$ is the Marchiafava–Romani class of Q, the obstruction to lifting the structure group of the standard $\mathbb{R}^3$–orbibundle $\hat{\mu}^{-1}(0)/K \times_{\mathrm{SU}(2)} \mathbb{R}^3$ over Q from SO(3) to SU(2); here we think of $\hat{\mu}$ as defined on $\mathbb{S}^{4n+3}$. As already noticed, since K_ρ does not contain $-1 \in \mathrm{Sp}(n+1)$ we have $\varepsilon(Q) = 0$.

The diffeomorphism-type of S now follows from the Smale classification of simply connected spin 5–manifolds [84]. □

Since the 8–manifold M retracts onto S, applying Theorem 3.36 to the Spin(7)–admissible self-dual Einstein 4–orbifolds we have constructed immediately yields:

Theorem 4.9 *There exist infinitely many smooth, noncompact, simply connected 8–manifolds carrying complete* Spin(7)*–metrics.*

Remark 4.10 As an aside, the self-dual Einstein 4–orbifolds we considered in this section and their relationship with ALE gravitational instantons are instances of the so-called hyperkähler/quaternionic Kähler correspondence [49; 53]. The correspondence relates a hyperkähler manifold X endowed with a circle action that fixes one complex structure and rotates the other two (a *rotating* circle action) with a quaternionic Kähler manifold Q (in general incomplete) endowed with circle symmetry. Consider for example the triholomorphic circle action $e^{i\theta} \cdot (u_1, u_2) = (e^{i\theta}u_1, e^{i\theta}u_2)$ on $\mathbb{H}^2$, the corresponding moment map $\mu(u) = \bar{u}_1 i u_1 + \bar{u}_2 i u_2$ and the Eguchi–Hanson space $X = \mu^{-1}(-i)/S^1$. The particular choice of level set of the moment map implies that X admits a rotating circle action, denoted by S^1_R. We fix a choice of lift of S^1_R to $\mathbb{H}^2$ by $e^{i\theta} \cdot (u_1, u_2) = (e^{im\theta}u_1 e^{-i\theta}, e^{im\theta}u_2 e^{-i\theta})$ for some $m \in \mathbb{Z}$. Haydys [49] and Hitchin [53; 54] considered the principal circle bundle $\mu^{-1}(-i)$ over X and constructed a quaternionic Kähler metric on $\mu^{-1}(-i)/S^1_R$. When $m \neq 0$ we can realise this quaternionic Kähler metric as a quaternionic Kähler quotient of $\mathbb{HP}^2$. Indeed, define an action of S^1 on $\mathbb{HP}^2$ by $e^{i\theta} \cdot [u_0 : u_1 : u_2] = [e^{i\theta}u_0 : e^{im\theta}u_1 : e^{im\theta}u_2]$. The quaternionic Kähler quotient of $\mathbb{HP}^1$ by this action is $Q = \{[u_0 : u_1 : u_2] \in \mathbb{HP}^2 \mid m\,\mu(u) + \bar{u}_0 i u_0 = 0\}/S^1$ and therefore the open set $Q_0 \subset Q$ where $u_0 \neq 0$ is identified with $\mu^{-1}(-i)/S^1_R$ via the map $[u_0 : u_1 : u_2] \mapsto (\sqrt{m}/|u_0|^2)(u_1\bar{u}_0, u_2\bar{u}_0)$. In the limiting case $m = 0$ we can identify $\mu^{-1}(-i)/S^1_R$ with (the complement of $\mu^{-1}(0)$ in) $\mathbb{HP}^1 \simeq S^4$: if $(v_1, v_2) \in \mathbb{H}^2$ satisfies $\mu(v) \neq 0$ then there exists $v_0 \in \mathbb{H}^*$, unique up to a circle factor, such that $\mu(v_1\bar{v}_0, v_2\bar{v}_0) = -i$. According to [49, Theorem 3], the general hyperkähler/quaternionic Kähler correspondence consists in replacing $\mathbb{H}^2$ with an arbitrary hyperkähler cone C admitting a triholomorphic circle action and $\mathbb{HP}^2$ with the (singular) quaternionic Kähler space $(\mathbb{H}\times \mathrm{C})/\mathbb{H}^*$ associated with the split hyperkähler cone $\mathbb{H}\times \mathrm{C}$. Here $\mathbb{H}^*$ acts diagonally on $\mathbb{H}$ and C. The hyperkähler space X is the hyperkähler quotient of C by S^1 at level set $-i$, say, of the hyperkähler moment map. The corresponding quaternionic Kähler space Q can be realised as the quaternionic Kähler quotient of $(\mathbb{H}\times \mathrm{C})/\mathbb{H}^*$ by the circle action

$$e^{i\theta}[u_0:u] = [e^{i\theta}u_0:e^{im\theta}\cdot u]. \tag{4.11}$$

Here $[u_0:u]$, where $u_0\in\mathbb{H}$ and $u\in \mathrm{C}$, denotes a point in $(\mathbb{H}\times \mathrm{C})/\mathbb{H}^*$, while $e^{i\psi}\cdot u$ denotes the triholomorphic circle action on C, and $m\in\mathbb{Z}$. In particular, if X is a hyperkähler quotient $\mu^{-1}(\zeta)/K$ of $\mathbb{H}^n$ by a subgroup K of $\mathrm{Sp}(n)$, as in the cases we have considered, for X to admit a rotating circle action we must require that ζ exponentiate to a (nontrivial) group homomorphism $\rho\colon K\to U(1)$. The cone C is then the hyperkähler quotient of $\mathbb{H}^n$ by $\ker\rho$ at the zero level-set (with induced triholomorphic action of $K/\ker\rho\simeq S^1$), and similarly $(\mathbb{H}\times \mathrm{C})/\mathbb{H}^*$ is the quaternionic Kähler quotient of $\mathbb{HP}^n$ by the action of $\ker\rho$ induced by

$$K \xrightarrow{\rho\times\mathrm{id}} U(1)\times K \subset \mathrm{Sp}(1)\times\mathrm{Sp}(n)\subset \mathrm{Sp}(n+1).$$

Choosing $m=1$ in (4.11), the quaternionic Kähler space Q corresponding to X is the quaternionic Kähler quotient of $\mathbb{HP}^n$ by K.

4.3 Complete G_2–manifolds from Calabi–Yau orbifolds

The framework introduced in this paper to do weighted analysis on AC orbifolds allows us to extend the results of [32] to the construction of highly collapsed ALC G_2–holonomy metrics on principal Seifert circle bundles over AC Calabi–Yau orbifolds of complex dimension 3. Given the language introduced in Section 2 and our calculation of the basic weighted L^2–cohomology of a Seifert circle bundle of arbitrary dimension in Theorem 2.31, the main result of [32] and its proof can be extended to the orbifold setting without any further complication.

In contrast to the Spin(7)–case, the construction of [32], which uses only circle bundles over smooth AC Calabi–Yau manifolds, already yielded infinitely many complete G_2–manifolds. The freedom to consider AC Calabi–Yau orbifolds is simply an addition of further examples to this already rich landscape. However, using Calabi–Yau orbifolds we will now construct infinitely many families of ALC G_2–metrics on a manifold as simple as $S^3\times\mathbb{R}^4$: given the wealth of examples arising in [32] it seemed likely that many different families of ALC G_2–metrics would end up being defined on the same underlying smooth 7–manifold, but no concrete example was given.

Theorem 4.12 *$S^3\times\mathbb{R}^4$ carries infinitely many distinct families of ALC G_2–metrics.*

Proof We will describe $S^3\times\mathbb{R}^4$ as the total space of a Seifert circle bundle over an AC Calabi–Yau orbifold B in infinitely many different ways. Forgetting for the moment the AC Calabi–Yau metric, we construct B as a Kähler manifold with trivial canonical bundle as the Kähler quotient of $\mathbb{C}^4$ by a circle in $\mathrm{SU}(4)$. Up to conjugation we assume that the circle is embedded in $\mathrm{SU}(4)$ via $e^{i\theta}\mapsto \mathrm{diag}(e^{ip_1\theta},e^{ip_2\theta},e^{-iq_1\theta},e^{-iq_2\theta})$ for nonnegative integers p_1,p_2,q_1,q_2 such that $p_1+p_2=q_1+q_2$. Then

$$B=B_\zeta=\{(z_1,z_2,z_3,z_4)\in\mathbb{C}^4 : p_1|z_1|^2+p_2|z_2|^2-q_1|z_3|^2-q_2|z_4|^2=\zeta\}/S^1,$$

where $\zeta\in\mathbb{R}$ is a parameter. If $\zeta\neq 0$ then B is an orbifold; indeed, the level set

$$M=\{(z_1,z_2,z_3,z_4)\in\mathbb{C}^4 : p_1|z_1|^2+p_2|z_2|^2-q_1|z_3|^2-q_2|z_4|^2=\zeta\}$$

is a smooth manifold and S^1 acts on M with nontrivial finite stabilisers: we have that $M\to M/S^1=B$ is a principal Seifert circle bundle over the orbifold B.

If we further assume that $\gcd(p_i,q_j)=1$ for all $i,j=1,2$ (in particular, $p_i,q_j>0$), then M is diffeomorphic to $S^3\times\mathbb{R}^4$ and B is smooth outside a compact set. In order to see this, assume without loss of generality that $\zeta>0$. Then up to an anisotropic rescaling $(z_1,z_2,z_3,z_4)\mapsto(\sqrt{p_1}z_1,\sqrt{p_2}z_2,\sqrt{q_1}z_3,\sqrt{q_2}z_4)$, M is cut out by the equation $|z_1|^2+|z_2|^2=|z_3|^2+|z_4|^2+\zeta$ and therefore, identifying $\mathbb{C}^2$ and $\mathbb{R}^4$ with $\mathbb{H}$ and S^3 with the unit sphere in $\mathbb{H}$, can be parametrised by $(x,y)\mapsto(\sqrt{|y|^2+\zeta}\,x,y)$ for $(x,y)\in S^3\times\mathbb{R}^4$. Moreover, the assumptions $\gcd(p_i,q_j)=1$ for all $i,j=1,2$ also imply that the only points in M with nontrivial stabiliser are those with $z_1=0=z_2$ or $z_3=0=z_4$, and since $\zeta>0$, the former case is impossible. The circle action is therefore free on the complement of $S^3\times\{0\}\subset S^3\times\mathbb{R}^4\simeq M$.

Still assuming that $\zeta>0$, to fix ideas, B is a $\mathbb{C}^2$–orbibundle over the weighted projective line $\mathbb{WCP}^1[p_1,p_2]$. In particular, $H^4_{\mathrm{orb}}(B)=0$ so that $c_1^{\mathrm{orb}}(M)\cup[\omega_0]=0$. Here $[\omega_0]$ is the Kähler class of the orbifold Kähler metric on B induced by the Kähler quotient construction. In the main existence result of [32] the topological constraint $c_1^{\mathrm{orb}}(M)\cup[\omega_0]=0\in H^4_{\mathrm{orb}}(B)$ plays the role of (3.22) in Theorem 3.36.

It remains to show that B carries an AC orbifold Calabi–Yau metric in the same Kähler class. First of all, note that under our assumptions the Kähler reduction B_0 of $\mathbb{C}^4$ at the zero-level set of the moment map has an isolated singularity at the origin and is in fact a Gorenstein toric Kähler cone. The existence of a Calabi–Yau cone metric on B_0 follows from a general result of Futaki, Ono and Wang [36]. The Calabi–Yau cone metric on B_0 is in fact explicit: the case where $p_1 = p_2 = p$, $q_1 = p - q$ and $q_2 = p + q$ where $p > q > 0$ and $\gcd(p,q) = 1$ coincides with the $Y^{p,q}$ Sasaki–Einstein manifolds of [40]; the general case was considered in [29; 73].

The existence of an AC Calabi–Yau metric on B asymptotic to the Calabi–Yau cone metric on B_0 now follows from the general existence theory for AC Calabi–Yau metrics on crepant resolutions of Calabi–Yau cones, in particular [43] (since the orbifold Kähler class $[\omega_0]$ is not compactly supported). Indeed, we can regard B as an orbifold partial small (therefore necessarily crepant) resolution of the cone B_0. Strictly speaking the existence result of [43] applies to a smooth manifold B; however, since the orbifold singularities of B are contained in a compact set, the extension to the orbifold setting should pose no additional difficulty. In fact, in the special case where $p_1 = p_2 = p$, $q_1 = p - q$ and $q_2 = p + q$, Martelli and Sparks [74, Theorems 1.1 and 1.3] constructed an explicit AC Calabi–Yau metric (unique up to scale) on B asymptotic to B_0.

In summary, $M = S^3 \times \mathbb{R}^4$ is a principal Seifert circle bundle over the AC Calabi–Yau orbifold B, and the topological constraint $c_1^{\mathrm{orb}}(M) \cup [\omega_0] = 0 \in H^4_{\mathrm{orb}}(B)$ is automatically satisfied. The main existence result of [32] guarantees the existence of a 1–parameter family up to scale of highly collapsed ALC G_2–metrics on M. Up to obvious symmetries, families corresponding to different choices of p_1, p_2, q_1, q_2 cannot be isometric to each other since their (unique) tangent cones at infinity are distinct. □

Remarks (1) All the complete G_2–metrics constructed in the proof of the theorem are *toric* in the sense of Madsen and Swann [72], ie they admit a multi-Hamiltonian isometric action of a 3–torus preserving the G_2–structure. Indeed, the AC Calabi–Yau orbifold metric on B is itself toric (in the usual sense of Kähler and symplectic geometry), but only a 2–dimensional subtorus also preserves the holomorphic volume form. The 2–torus symmetry lifts to a symmetry of the ALC G_2–metrics because of uniqueness results in the construction of [32]. Finally, the G_2–metrics are also invariant under the circle action on the fibres of the Seifert bundle.

(2) Since in Theorem 4.12 B has no orbifold singularities outside a compact set, the application of [32] does not really require the machinery of Section 2. However, there are likely very many other examples with noncompact singular set.

References

[1] **A Adem**, **J Leida**, **Y Ruan**, *Orbifolds and stringy topology*, Cambridge Tracts Math. 171, Cambridge Univ. Press (2007) MR Zbl

[2] **V Apostolov**, **S Salamon**, *Kähler reduction of metrics with holonomy G_2*, Comm. Math. Phys. 246 (2004) 43–61 MR Zbl

[3] **C Bär**, *Real Killing spinors and holonomy*, Comm. Math. Phys. 154 (1993) 509–521 MR Zbl

[4] **R Bartnik**, *The mass of an asymptotically flat manifold*, Comm. Pure Appl. Math. 39 (1986) 661–693 MR Zbl

[5] **Y V Bazaikin**, *On new examples of complete noncompact metrics with* $\mathrm{Spin}(7)$ *holonomy group*, Sibirsk. Mat. Zh. 48 (2007) 11–32 MR Zbl In Russian; translated in Siberian Math. J. 48 (2007) 8–25

[6] **Y V Bazaikin**, *Noncompact Riemann spaces with the holonomy group* $\mathrm{Spin}(7)$ *and* 3*–Sasakian manifolds*, Tr. Mat. Inst. Steklova 263 (2008) 6–17 MR Zbl In Russian; translated in Proc. Steklov Inst. Math. 263 (2008), 2–12

[7] **O Biquard**, **V Minerbe**, *A Kummer construction for gravitational instantons*, Comm. Math. Phys. 308 (2011) 773–794 MR Zbl

[8] **J-M Bismut**, *The Atiyah–Singer index theorem for families of Dirac operators: two heat equation proofs*, Invent. Math. 83 (1986) 91–151 MR Zbl

[9] **J E Borzellino**, *Orbifolds of maximal diameter*, Indiana Univ. Math. J. 42 (1993) 37–53 MR Zbl

[10] **J E Borzellino**, **S-H Zhu**, *The splitting theorem for orbifolds*, Illinois J. Math. 38 (1994) 679–691 MR Zbl

[11] **C P Boyer**, **K Galicki**, *Sasakian geometry*, Oxford Univ. Press (2008) MR Zbl

[12] **C P Boyer**, **K Galicki**, **B M Mann**, *The geometry and topology of* 3*–Sasakian manifolds*, J. Reine Angew. Math. 455 (1994) 183–220 MR Zbl

[13] **C P Boyer**, **K Galicki**, **B M Mann**, *Hypercomplex structures from* 3*–Sasakian structures*, J. Reine Angew. Math. 501 (1998) 115–141 MR Zbl

[14] **C P Boyer**, **K Galicki**, **B M Mann**, **E G Rees**, *Compact* 3*–Sasakian* 7*–manifolds with arbitrary second Betti number*, Invent. Math. 131 (1998) 321–344 MR Zbl

[15] **R L Bryant**, *Metrics with exceptional holonomy*, Ann. of Math. 126 (1987) 525–576 MR Zbl

[16] **R L Bryant**, *Some remarks on G_2–structures*, from "Proceedings of Gökova Geometry-Topology Conference 2005" (S Akbulut, T Önder, R J Stern, editors), GGT, Gökova, Turkey (2006) 75–109 MR Zbl

[17] **R L Bryant**, **S M Salamon**, *On the construction of some complete metrics with exceptional holonomy*, Duke Math. J. 58 (1989) 829–850 MR Zbl

[18] **E Calabi**, *Métriques kählériennes et fibrés holomorphes*, Ann. Sci. École Norm. Sup. 12 (1979) 269–294 MR Zbl

[19] **D M J Calderbank**, **M A Singer**, *Toric self-dual Einstein metrics on compact orbifolds*, Duke Math. J. 133 (2006) 237–258 MR Zbl

[20] **J Cheeger**, **T H Colding**, *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Ann. of Math. 144 (1996) 189–237 MR Zbl

[21] **J Cheeger**, **G Tian**, *On the cone structure at infinity of Ricci flat manifolds with Euclidean volume growth and quadratic curvature decay*, Invent. Math. 118 (1994) 493–571 MR Zbl

[22] **C van Coevering**, *Ricci-flat Kähler metrics on crepant resolutions of Kähler cones*, Math. Ann. 347 (2010) 581–611 MR Zbl

[23] **T C Collins**, **G Székelyhidi**, *Sasaki–Einstein metrics and K–stability*, Geom. Topol. 23 (2019) 1339–1413 MR Zbl

[24] **R J Conlon**, **H-J Hein**, *Asymptotically conical Calabi–Yau manifolds, I*, Duke Math. J. 162 (2013) 2855–2902 MR Zbl

[25] **M Cvetič**, **G W Gibbons**, **H Lü**, **C N Pope**, *Almost special holonomy in type IIA and M–theory*, Nuclear Phys. B 638 (2002) 186–206 MR Zbl

[26] **M Cvetič**, **G W Gibbons**, **H Lü**, **C N Pope**, *Cohomogeneity one manifolds of* $\mathrm{Spin}(7)$ *and G_2 holonomy*, Phys. Rev. D 65 (2002) art. id. 106004 MR

[27] **M Cvetič**, **G W Gibbons**, **H Lü**, **C N Pope**, *New complete noncompact* $\mathrm{Spin}(7)$ *manifolds*, Nuclear Phys. B 620 (2002) 29–54 MR Zbl

[28] **M Cvetič**, **G W Gibbons**, **H Lü**, **C N Pope**, *New cohomogeneity one metrics with* $\mathrm{Spin}(7)$ *holonomy*, J. Geom. Phys. 49 (2004) 350–365 MR Zbl

[29] **M Cvetič**, **H Lü**, **D N Page**, **C N Pope**, *New Einstein–Sasaki spaces in five and higher dimensions*, Phys. Rev. Lett. 95 (2005) art. id. 071101 MR

[30] **J Eells**, **S Salamon**, *Twistorial construction of harmonic maps of surfaces into four-manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 12 (1985) 589–640 MR Zbl

[31] **L Foscolo**, *Deformation theory of nearly Kähler manifolds*, J. Lond. Math. Soc. 95 (2017) 586–612 MR Zbl

[32] **L Foscolo**, **M Haskins**, **J Nordström**, *Complete non-compact G_2–manifolds from asymptotically conical Calabi–Yau 3–folds*, preprint (2017) arXiv

[33] **L Foscolo**, **M Haskins**, **J Nordström**, *Infinitely many new families of complete cohomogeneity one G_2–manifolds: G_2 analogues of the Taub–NUT and Eguchi–Hanson spaces*, preprint (2018) arXiv To appear in J. Eur. Math. Soc.

[34] **T Friedrich**, **I Kath**, **A Moroianu**, **U Semmelmann**, *On nearly parallel G_2–structures*, J. Geom. Phys. 23 (1997) 259–286 MR Zbl

[35] **K Fukaya**, *Hausdorff convergence of Riemannian manifolds and its applications*, from "Recent topics in differential and analytic geometry" (T Ochiai, editor), Adv. Stud. Pure Math. 18, Academic, Boston (1990) 143–238 MR Zbl

[36] **A Futaki**, **H Ono**, **G Wang**, *Transverse Kähler geometry of Sasaki manifolds and toric Sasaki–Einstein manifolds*, J. Differential Geom. 83 (2009) 585–635 MR Zbl

[37] **K Galicki**, **H B Lawson, Jr**, *Quaternionic reduction and quaternionic orbifolds*, Math. Ann. 282 (1988) 1–21 MR Zbl

[38] **K Galicki**, **T Nitta**, *Nonzero scalar curvature generalizations of the ALE hyper-Kähler metrics*, J. Math. Phys. 33 (1992) 1765–1771 MR Zbl

[39] **K Galicki**, **S Salamon**, *Betti numbers of* 3*–Sasakian manifolds*, Geom. Dedicata 63 (1996) 45–68 MR Zbl

[40] **J P Gauntlett**, **D Martelli**, **J Sparks**, **D Waldram**, *Sasaki–Einstein metrics on* $S^2 \times S^3$, Adv. Theor. Math. Phys. 8 (2004) 711–734 MR Zbl

[41] **G Gibbons**, **S Hawking**, *Gravitational multi-instantons*, Phys. Lett. B 78 (1978) 430–432

[42] **S Goette**, *Adiabatic limits of Seifert fibrations, Dedekind sums, and the diffeomorphism type of certain* 7*–manifolds*, J. Eur. Math. Soc. 16 (2014) 2499–2555 MR Zbl

[43] **R Goto**, *Calabi–Yau structures and Einstein–Sasakian structures on crepant resolutions of isolated singularities*, J. Math. Soc. Japan 64 (2012) 1005–1052 MR Zbl

[44] **V W Guillemin**, **S Sternberg**, *Supersymmetry and equivariant de Rham theory*, Springer (1999) MR Zbl

[45] **S Gukov**, **J Sparks**, M*–theory on* $\mathrm{Spin}(7)$ *manifolds*, Nuclear Phys. B 625 (2002) 3–69 MR Zbl

[46] **S Gukov**, **J Sparks**, **D Tong**, *Conifold transitions and five-brane condensation in* M*–theory on* $\mathrm{Spin}(7)$ *manifolds*, Classical Quantum Gravity 20 (2003) 665–705 MR Zbl

[47] **M Haskins**, **H-J Hein**, **J Nordström**, *Asymptotically cylindrical Calabi–Yau manifolds*, J. Differential Geom. 101 (2015) 213–265 MR Zbl

[48] **T Hausel**, **E Hunsicker**, **R Mazzeo**, *Hodge cohomology of gravitational instantons*, Duke Math. J. 122 (2004) 485–548 MR Zbl

[49] **A Haydys**, *HyperKähler and quaternionic Kähler manifolds with* S^1*–symmetries*, J. Geom. Phys. 58 (2008) 293–306 MR Zbl

[50] **H-J Hein**, **S Sun**, *Calabi–Yau manifolds with isolated conical singularities*, Publ. Math. Inst. Hautes Études Sci. 126 (2017) 73–130 MR Zbl

[51] **N J Hitchin**, *Kählerian twistor spaces*, Proc. Lond. Math. Soc. 43 (1981) 133–150 MR Zbl

[52] **N J Hitchin**, *A new family of Einstein metrics*, from "Manifolds and geometry" (P de Bartolomeis, F Tricerri, E Vesentini, editors), Sympos. Math. 36, Cambridge Univ. Press (1996) 190–222 MR Zbl

[53] **N Hitchin**, *On the hyperKähler/quaternion Kähler correspondence*, Comm. Math. Phys. 324 (2013) 77–106 MR Zbl

[54] **N Hitchin**, *The hyperholomorphic line bundle*, from "Algebraic and complex geometry" (A Frühbis-Krüger, R N Kloosterman, M Schütt, editors), Springer Proc. Math. Stat. 71, Springer (2014) 209–223 MR Zbl

[55] **N J Hitchin**, **A Karlhede**, **U Lindström**, **M Roček**, *Hyper-Kähler metrics and supersymmetry*, Comm. Math. Phys. 108 (1987) 535–589 MR Zbl

[56] **G R Jensen**, *Einstein metrics on principal fibre bundles*, J. Differential Geom. 8 (1973) 599–614 MR Zbl

[57] **D D Joyce**, *Compact 8–manifolds with holonomy* $\mathrm{Spin}(7)$, Invent. Math. 123 (1996) 507–552 MR Zbl

[58] **D Joyce**, *A new construction of compact 8–manifolds with holonomy* $\mathrm{Spin}(7)$, J. Differential Geom. 53 (1999) 89–130 MR Zbl

[59] **D D Joyce**, *Compact manifolds with special holonomy*, Oxford Univ. Press (2000) MR Zbl

[60] **H Kanno**, **Y Yasui**, *On* $\mathrm{Spin}(7)$ *holonomy metric based on* $\mathrm{SU}(3)/\mathrm{U}(1)$*, I*, J. Geom. Phys. 43 (2002) 293–309 MR Zbl

[61] **H Kanno**, **Y Yasui**, *On* $\mathrm{Spin}(7)$ *holonomy metric based on* $\mathrm{SU}(3)/\mathrm{U}(1)$*, II*, J. Geom. Phys. 43 (2002) 310–326 MR Zbl

[62] **S Karigiannis**, **J D Lotay**, *Deformation theory of* G_2 *conifolds*, Comm. Anal. Geom. 28 (2020) 1057–1210 MR Zbl

[63] **P Kaste**, **R Minasian**, **M Petrini**, **A Tomasiello**, *Kaluza–Klein bundles and manifolds of exceptional holonomy*, J. High Energy Phys. 2002 (2002) art. id. 033 MR

[64] **M Konishi**, *On manifolds with Sasakian 3–structure over quaternion Kähler manifolds*, Kōdai Math. Sem. Rep. 26 (1975) 194–200 MR Zbl

[65] **A Kovalev**, *Asymptotically cylindrical manifolds with holonomy* $\mathrm{Spin}(7)$*, I*, preprint (2013) arXiv

[66] **P B Kronheimer**, *The construction of ALE spaces as hyper-Kähler quotients*, J. Differential Geom. 29 (1989) 665–683 MR Zbl

[67] **P B Kronheimer**, *A Torelli-type theorem for gravitational instantons*, J. Differential Geom. 29 (1989) 685–697 MR Zbl

[68] **P B Kronheimer**, **H Nakajima**, *Yang–Mills instantons on ALE gravitational instantons*, Math. Ann. 288 (1990) 263–307 MR Zbl

[69] **F Lehmann**, *Geometric transitions with Spin(7) holonomy via a dynamical system*, preprint (2020) arXiv

[70] **R Lockhart**, *Fredholm, Hodge and Liouville theorems on noncompact manifolds*, Trans. Amer. Math. Soc. 301 (1987) 1–35 MR Zbl

[71] **R B Lockhart**, **R C McOwen**, *Elliptic differential operators on noncompact manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 12 (1985) 409–447 MR Zbl

[72] **T B Madsen**, **A Swann**, *Toric geometry of G_2–manifolds*, Geom. Topol. 23 (2019) 3459–3500 MR Zbl

[73] **D Martelli**, **J Sparks**, *Toric Sasaki–Einstein metrics on $S^2 \times S^3$*, Phys. Lett. B 621 (2005) 208–212 MR Zbl

[74] **D Martelli**, **J Sparks**, *Resolutions of non-regular Ricci-flat Kähler cones*, J. Geom. Phys. 59 (2009) 1175–1195 MR Zbl

[75] **P Molino**, *Riemannian foliations*, Progr. Math. 73, Birkhäuser, Boston (1988) MR Zbl

[76] **L Nirenberg**, **H F Walker**, *The null spaces of elliptic partial differential operators in $\mathbb{R}^n$*, J. Math. Anal. Appl. 42 (1973) 271–301 MR Zbl

[77] **J Nordström**, *Deformations of asymptotically cylindrical G_2–manifolds*, Math. Proc. Cambridge Philos. Soc. 145 (2008) 311–348 MR Zbl

[78] **F Pacard**, *Connected sum constructions in geometry and nonlinear analysis*, lecture notes (2008) Available at `https://tinyurl.com/pacardlect`

[79] **T H Parker**, *A Morse theory for equivariant Yang–Mills*, Duke Math. J. 66 (1992) 337–356 MR Zbl

[80] **F Reidegeld**, *Exceptional holonomy and Einstein metrics constructed from Aloff–Wallach spaces*, Proc. Lond. Math. Soc. 102 (2011) 1127–1160 MR Zbl

[81] **B L Reinhart**, *Differential geometry of foliations*, Ergeb. Math. Grenzgeb. 99, Springer (1983) MR Zbl

[82] **S Salamon**, *Quaternionic Kähler manifolds*, Invent. Math. 67 (1982) 143–171 MR Zbl

[83] **T Shioya**, *Eigenvalues and suspension structure of compact Riemannian orbifolds with positive Ricci curvature*, Manuscripta Math. 99 (1999) 509–516 MR Zbl

[84] **S Smale**, *On the structure of 5–manifolds*, Ann. of Math. 75 (1962) 38–46 MR Zbl

[85] **A Swann**, *Hyper-Kähler and quaternionic Kähler geometry*, Math. Ann. 289 (1991) 421–450 MR Zbl

[86] **P Tondeur**, *Geometry of foliations*, Monogr. Math. 90, Birkhäuser, Basel (1997) MR Zbl

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