A Stable Cut Finite Element Method for Partial Differential Equations on Surfaces: The Helmholtz-Beltrami Operator

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Abstract

We consider solving the surface Helmholtz equation on a smooth two dimensional surface embedded into a three dimensional space meshed with tetrahedra. The mesh does not respect the surface and thus the surface cuts through the elements. We consider a Galerkin method based on using the restrictions of continuous piecewise linears defined on the tetrahedra to the surface as trial and test functions.

Using a stabilized method combining Galerkin least squares stabilization and a penalty on the gradient jumps we obtain stability of the discrete formulation under the condition $hk < C$, where h denotes the mesh size, k the wave number and C a constant depending mainly on the surface curvature κ, but not on the surface/mesh intersection. Optimal error estimates in the H^1 and L^2-norms follow.

1 Introduction

The accurate computation of lateral waves in two dimensional surfaces, embedded in three space dimensions, is an important problem in the mechanics of fluid films [**?**] and bubbles [**?**, **?**]. Similar computational challenges are found in the modelling of waves in cell membranes, see for instance [**?**, **?**]. Despite the importance of the accurate simulation of wave phenomena on two dimensional surfaces for membrane dynamics there appears to be no works in the numerical analysis literature discussing this problem. Our intention in the present contribution is to design and analyse a finite element method for the computational approximation of wave equations in the frequency domain on two-dimensional closed surfaces. We will consider a model from Grinfeld [**?**, Equation (103)], in the frequency domain.

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This leads to an indefinite elliptic problem of Helmholtz-type set on the closed surface that we will consider as a model problem. The extension of the results derived herein to surfaces with boundary, using Dirichlet or impedance conditions, is straightforward using the results from [?], but to limit the length of this manuscript we leave these aspects for further work.

In a previous paper [?] we considered solving the Laplace-Beltrami problem on a smooth two dimensional surface embedded into a three dimensional space partitioned into a mesh consisting of shape regular tetrahedra. The mesh did not respect the surface and thus the surface can cut through the elements in an arbitrary manner. Following Olshanskii, Reusken, and Grande [?] we constructed a Galerkin method by using the restrictions of continuous piecewise linears defined on the tetrahedra to the surface. Observe that the discussion below also holds for the original approach where the surface is meshed [?, ?]. The motivation for the unfitted approach comes from situations where the film changes shape during the simulation as in the fully time-dependent case or in shape optimization.

To alleviate the ill-conditioning of the resulting method we proposed to add a stabilization term penalizing the jump of the gradient of the solution to the formulation. In the case of indefinite elliptic problems a similar stabilization improves the stability of the formulation yielding discrete wellposedness under a weaker condition on the mesh parameter and the wave number than is usually expected. The analysis draws on ideas from [?, ?, ?] for the stabilization of the Helmholtz equation. For background material on standard Galerkin FEM for Helmholtz equations in flat domains we refer to the seminal work by Babuska and Ihlenburg [?] and the later works by Melenk et al. on wave number explicit analysis for hp-methods [?] and nonconforming methods [?].

Typically the finite element analysis of the wave equation in the frequency domain introduces conditions on the size of the meshsize h compared to the wavenumber k. For a standard Galerkin finite element method of indefinite elliptic problems, the condition that hk^2 has to be small for stability and optimal estimates. This follows by the analysis of Schatz [?], using the combination of an H^1 error estimate by Gårdings inequality and a duality argument showing that the L^2-norm error converges at a faster rate than that measured in the H^1-norm. Thanks to the stabilization the mesh-wavenumber condition takes the form hk small instead. This condition appears here only because of the discrete approximation of the surface. In the case of a semi-disrete formulation where the integration takes place on the exact surface the formulation is unconditionally stable. Our estimates are explicit in the mesh size and the wave number, but not in the surface curvature, which we assume is moderate. The conformity error introduced due to the approximation of the surface however leads to a condition hk small. To simplify the presentation we will assume that $k \geqslant 1$ and $h < 1$. Generic constants C may depend on the surface curvature, but not on the wavenumber, the mesh-size or the intersection of the surface with the computational mesh. In cases where we want to highlight a particular dependence, we add a subscript to the constant.

The outline of the remainder of this paper is as follows: In Section 2 we formulate the model problem and the finite element method, in Section 3 we prove a priori error estimates, and finally in Section 4 we present numerical investigations confirming our theoretical results.

2 Model Problem and Finite Element Method

2.1 The Continuous Problem

Let Σ be a smooth two-dimensional closed and orientable surface embedded in $\mathbb{R}^3$ with signed distance function b. We consider the following problem: for a given $k \in \mathbb{R}$, find $u : \Sigma \to \mathbb{C}$ such that

$$-\Delta_\Sigma u - k^2 u = f \quad \text{on } \Sigma. \tag{2.1}$$

Here Δ_Σ is the Laplace-Beltrami operator defined by

$$\Delta_\Sigma = \nabla_\Sigma \cdot \nabla_\Sigma \tag{2.2}$$

where ∇_Σ is the tangent gradient

$$\nabla_\Sigma = \boldsymbol{P}_\Sigma \nabla \tag{2.3}$$

with $\boldsymbol{P}_\Sigma = \boldsymbol{P}_\Sigma(\boldsymbol{x})$ the projection of $\mathbb{R}^3$ onto the tangent plane of Σ at $\boldsymbol{x} \in \Sigma$, defined by

$$\boldsymbol{P}_\Sigma = \boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n} \tag{2.4}$$

where $\boldsymbol{n} = \nabla b$ denotes the exterior unit normal to Σ at $\boldsymbol{x}$, $\boldsymbol{I}$ is the identity matrix, and ∇ the $\mathbb{R}^3$ gradient.

The corresponding weak statement takes the form: find $u \in H^1(\Sigma)$ such that

$$a(u,v) = l(v) \quad \forall v \in H^1(\Sigma) \tag{2.5}$$

where

$$a(u,v) = (\nabla_\Sigma u, \nabla_\Sigma v)_\Sigma - (k^2 u, v)_\Sigma, \qquad l(v) = (f,v)_\Sigma \tag{2.6}$$

and $(v,w)_\Sigma = \int_\Sigma v\overline{w}$ is the L^2 inner product. We will assume that $k \in \mathbb{R}$ is such that the Fredholm alternative yields a unique solution of the problem. Assuming that the following bound holds on the smallest distance to an eigenvalue of Δ_Σ,

$$\min_m |\lambda_m - k^2| \geqslant ck \tag{2.7}$$

we have the following elliptic regularity estimate:

$$k^{-1}|u|_{2,\Sigma} + |u|_{1,\Sigma} + \|ku\|_\Sigma \leqslant C\|f\|_\Sigma. \tag{2.8}$$

Here $\|w\|^2_\Sigma = (w,w)_\Sigma$ denotes the L^2 norm on Σ and

$$|w|^2_{j,\Sigma} = \|(\otimes_{i=1}^j \nabla_\Sigma) w\|^2_\Sigma, \qquad \|w\|^2_{j,\Sigma} = \sum_{i=0}^j |w|^2_{j,\Sigma} \tag{2.9}$$

are the Sobolev semi-norm and norm on Σ for $j = 0, 1, 2$, where the L^2 norm for a matrix is based on the pointwise Frobenius norm. The constant in (**??**) depends on the curvature of the surface. The following L^2-estimate is a consequence of the Fredholm's alternative under the assumption (**??**):

$$\|u\|_\Sigma^2 \leqslant \max_m |\lambda_m - k^2|^{-2} \|f\|_\Sigma^2 \leqslant c^{-2} k^{-2} \|f\|_\Sigma^2. \tag{2.10}$$

Using the equation we also immediately obtain a bound of the H^1-norm of u

$$\|\nabla_\Sigma u\|_\Sigma^2 = (f, u) + k^2 \|u\|_\Sigma^2 \leqslant c^{-1}(k^{-1} + c^{-1}) \|f\|_\Sigma^2. \tag{2.11}$$

The H^2-estimate, finally, is a consequence of the elliptic regularity of the Laplace-Beltrami operator, $|u|_{2,\Sigma} \leqslant C_R \|\Delta_\Sigma u\|_\Sigma$, see [**?**], and the fact that $\Delta_\Sigma u = -f - k^2 u$ implying that

$$\|\Delta_\Sigma u\|_\Sigma^2 = \|f\|_\Sigma^2 + 2(f, k^2 u) + k^4 \|u\|_\Sigma^2 \leqslant (1 + 2c^{-1}k + c^{-2}k^2) \|f\|_\Sigma^2. \tag{2.12}$$

Remark 2.1 *We see that the constant C in* (**??**) *is defined by by the constants in the right hand sides of* (**??**), (**??**) *and, the constant of* (**??**) *multiplied with C_R^2.*

Remark 2.2 *We here consider the equation on complex form to allow for the addition of damping in the form of complex wave number. The complex variables can then be exploited to yield positivity of the stabilization terms. A robust method using real variables, valid only for real wave numbers can be designed similarly using the ideas of [**?**].*

2.2 Detailed Stability Bound on the Sphere

The assumption (**??**) can be checked in special cases such as for the sphere. In that case $\lambda_m = m(m+1)$, $m = 1, 2, \dots$ (see [**?**]) and we can see that a moderately small c, allows for an important range of values of k^2. It is also clear in this case that the scaling proposed in (**??**) is the one that allows for k^2 to take values in a fraction of the real line where (**??**) holds uniformly for increasing wavenumber. To see this denote the desired fraction by $\alpha = 1 - \eta$, where $\eta \in \mathbb{R}^+$ is assumed to be small, and observe that the distance between two eigenvalues λ_m and λ_{m+1} is

$$\delta\lambda_m := \lambda_{m+1} - \lambda_m = 2(m+1).$$

Now assume that

$$\lambda_m(1 + \eta/m) \leqslant k^2 \leqslant \lambda_{m+1}(1 - \eta/(m+2)). \tag{2.13}$$

First we show that this bound results in a constant fraction of the length of the interval $(\lambda_m, \lambda_{m+1})$ admissible for k^2. Writing the length of the admissible interval $(\lambda_m(1 + \eta/m), \lambda_{m+1}(1 - \eta/(m+2))$ we see that

$$\begin{aligned} \lambda_{m+1}(1 - \eta/(m+2)) - \lambda_m(1 + \eta/m) &= \delta\lambda_m - (m+1)\eta - (m+1)\eta \\ &= 2(m+1) - 2\eta(m+1) = 2(m+1)(1-\eta) = \delta\lambda_m \alpha. \end{aligned} \tag{2.14}$$

Therefore, regardless of m, if k^2 takes a value between λ_m and λ_{m+1}, satisfying (**??**), then α is indeed the fraction of the interval $\delta\lambda_m$ that is admissible.

Next we will estimate the value of c in (**??**), assuming (**??**). It follows from this last relation that

$$|\lambda_m - k^2| \geq \lambda_m \eta/m = (m+1)\eta$$

and

$$|\lambda_{m+1} - k^2| \geq \lambda_{m+1}\eta/(m+2) = (m+1)\eta.$$

Using now that

$$(m+1)\eta = \left(\frac{m+1}{m+2}\right)^{\frac{1}{2}} \lambda_{m+1}^{\frac{1}{2}}\eta$$

and by the upper bound of (**??**)

$$\lambda_{m+1}^{\frac{1}{2}} \geq k\left(\frac{m+2}{m+2-\eta}\right)^{\frac{1}{2}}$$

we have that

$$(m+1)\eta \geq \left(\frac{m+1}{m+2}\right)^{\frac{1}{2}} k \left(\frac{m+2}{m+2-\eta}\right)^{\frac{1}{2}} \eta = \left(\frac{m+1}{m+2-\eta}\right)^{\frac{1}{2}} k\eta \geq \sqrt{\frac{2}{3}}k\eta.$$

We conclude that on the sphere the constant c of (**??**) is larger than or equal to $\sqrt{\frac{2}{3}}\eta$.

Clearly it is interesting to see how the method performs compared to the standard method in the vicinity of the eigenvalues and therefore the behavior of the method for values of k^2 close to an eigenvalue is explored in Section **??**.

2.3 The Finite Element Method on Σ

Let $\mathcal{K}$ be a quasi uniform partition into shape regular tetrahedra of a domain Ω in $\mathbb{R}^3$ completely containing Σ. Let $\mathcal{K}_h$ be the set of tetrahedra that intersect Σ and denote by Ω_h the domain covered by $\mathcal{K}_h$; that is,

$$\mathcal{K}_h = \{K \in \mathcal{K} : K \cap \Sigma \neq \emptyset\}, \qquad \Omega_h = \cup_{K\in\mathcal{K}_h} K. \tag{2.15}$$

We denote the local mesh size by h_K and define the global mesh size $h = \max_{K\in\mathcal{K}_h}\{h_K\}$. Since $h_K \sim h$ by the quasi uniformity of $\mathcal{K}$, we will simply use h throughout the remaining work. We let $\mathcal{V}_h$ be the space of continuous piecewise linear, complex valued, polynomials defined on $\mathcal{K}_h$. Our finite element method takes the form: find $\tilde{u}_h \in \mathcal{V}_h$ such that

$$A(\tilde{u}_h, v) + \gamma_j j(\tilde{u}_h, v) = l_s(v) \quad \forall v \in \mathcal{V}_h \tag{2.16}$$

where the bilinear form $A(\cdot,\cdot)$ is defined by

$$A(v, w) = a(v, w) + \gamma_s s(v, w) \quad \forall v, w \in \mathcal{V}_h \tag{2.17}$$

with the stabilization terms

$$s(v,w) = \sum_{K \in \mathcal{K}_h} h^2 (\Delta_\Sigma v + k^2 v, \Delta_\Sigma w + k^2 w)_{\Sigma \cap K} \tag{2.18}$$

and

$$j(v,w) = \sum_{F \in \mathcal{F}_I} ([\boldsymbol{n}_F \cdot \nabla v], [\boldsymbol{n}_F \cdot \nabla w])_F. \tag{2.19}$$

Above $\mathcal{F}_I$ denotes the set of internal interfaces in $\mathcal{K}_h$, $\boldsymbol{n}_F$ denotes a fixed unit normal to the face $F \in \mathcal{F}_I$ and $[\boldsymbol{n}_F \cdot \nabla v] = (\boldsymbol{n}_F \cdot \nabla v)^+ - (\boldsymbol{n}_F \cdot \nabla v)^-$ with $w(\boldsymbol{x})^\pm = \lim_{t \to 0^+} w(\boldsymbol{x} \pm t\boldsymbol{n}_F)$, is the jump in the normal gradient across the face F. For consistency the right hand side is modified to read

$$l_s(v) = l(v) - \sum_{K \in \mathcal{K}_h} \gamma_s h^2 (f, \Delta_\Sigma v + k^2 v)_{\Sigma \cap K}. \tag{2.20}$$

The parameters $\gamma_x \in \mathbb{C}$, $x = s, j$ will be assumed to satisfy $\text{Im}(\gamma_x) \geq \gamma_{min} > 0$ for some $\gamma_{min} \in \mathbb{R}$. The choice of sign in the complex part of the stabilization should be made so that coercivity is obtained on the imaginary part of the bilinear form. The real part of the stabilization parameter has been shown to allow for tuning of the numerical wave number so that pollution can be minimized. See [?, ?] for a discussion of these aspects on planar domains. To simplify the presentation and without loss of generality we will assume that $\gamma = \gamma_s = \gamma_j$ and $Re(\gamma) = 0$ below.

2.4 Approximation of the Surface

Next, we recall that for a smooth oriented surface Σ, there is an open δ tubular neighborhood $U_\delta(\Sigma) = \{\boldsymbol{x} \in \mathbb{R}^3 : |b(\boldsymbol{x})| < \delta\}$ of Σ such that for each $\boldsymbol{x} \in U_\delta(\Sigma)$ there is a unique closest point $\boldsymbol{p}(\boldsymbol{x}) \in \Sigma$ minimizing the Euclidean distance to $\boldsymbol{x}$. Note that the closest point mapping $\boldsymbol{x} \mapsto \boldsymbol{p}(\boldsymbol{x})$ satisfies $\boldsymbol{p}(\boldsymbol{x}) = \boldsymbol{x} - b(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{p}(\boldsymbol{x}))$. Using $\boldsymbol{p}$ we extend u outside of Σ by defining

$$u^e(\boldsymbol{x}) = u \circ \boldsymbol{p}(\boldsymbol{x}) \tag{2.21}$$

In the following, a superscript e is also used to denote the extension of other quantities defined on the surface. The extension v^e of $v \in H^s(\Sigma)$ satisfies the stability estimate

$$\|v^e\|_{s,\Omega_h} \leqslant C h^{\frac{1}{2}} \|v\|_{s,\Sigma}, \quad s = 0, 1, 2. \tag{2.22}$$

For h sufficiently small the constant in the inequality (**??**) depends only on the curvature of the surface Σ.

In practice we are typically not able to compute on the exact surface Σ, instead we have to consider an approximate surface Σ_h. Depending on how the surface is described the construction of the approximate surface can be done in different ways. Here we consider, in particular, a simple situation where Σ is described by a level set function b and Σ_h is defined by the zero level set to a piecewise linear approximate level set function $b_h \in \text{Re}(\mathcal{V}_h)$. In this case the approximate surface is a piecewise linear surface since it is the level set to a piecewise linear function. We let the approximate normal $\boldsymbol{n}_h$ be the exact normal to the piecewise linear approximate surface Σ_h. and that the following estimates hold

$$\|b\|_{L^\infty(\Sigma_h)} \leqslant Ch^2, \qquad \|\boldsymbol{n}^e - \boldsymbol{n}_h\|_{L^\infty(\Sigma_h)} \leqslant Ch. \tag{2.23}$$

These properties are, for instance, satisfied if b_h is the Lagrange interpolant of b. Observe that by the properties of the interpolant the discrete interface Σ_h is also contained in $\mathcal{K}_h$. Finally, we define the lift v^l of a function v defined on discrete surface Σ_h to the exact surface Σ by requiring that

$$(v^l)^e = v^l \circ \boldsymbol{p} = v. \tag{2.24}$$

We refer to Figure **??** for an illustration of the relevant geometric concepts.

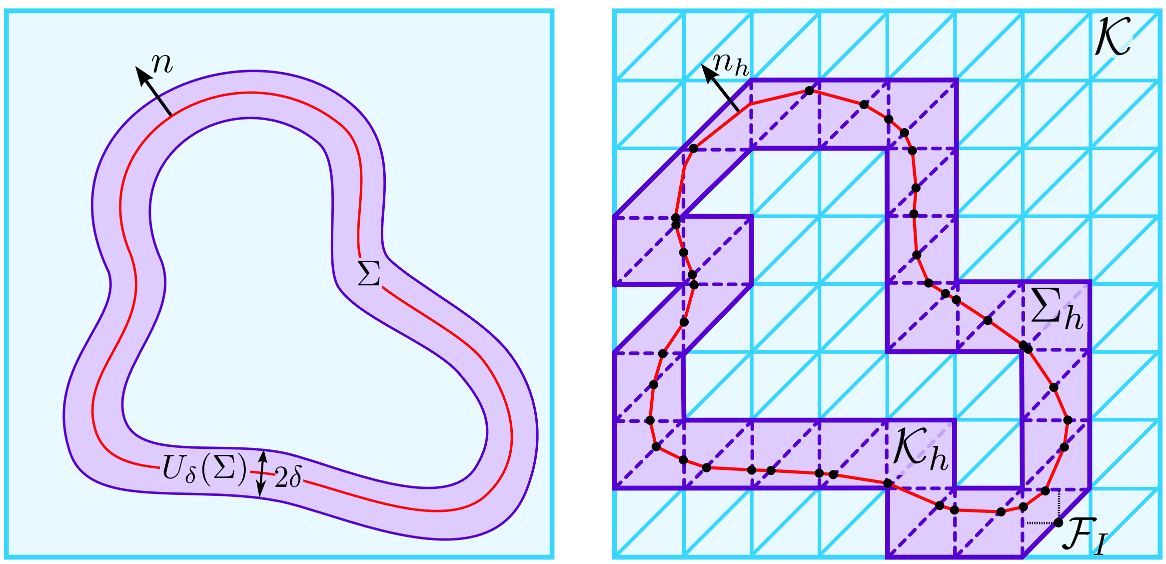

Figure 1: Set-up of the continuous and discrete domains. (Left) Continuous surface Σ enclosed by a δ tubular neighborhood $U_\delta(\Sigma)$. (Right) Discrete manifold Σ_h embedded into a background mesh $\mathcal{K}$ from which the active mesh $\mathcal{K}_h$ is extracted.

2.5 The Finite Element Method on Σ_h

Here let

$$\mathcal{K}_h = \{K \in \mathcal{K} : K \cap \Sigma_h \neq \emptyset\}, \quad \Omega_h = \cup_{K \in \mathcal{K}_h} K \tag{2.25}$$

and $\mathcal{V}_h$ be the continuous piecewise linear, complex valued functions defined on $\mathcal{K}_h$. The finite element method on Σ_h takes the form: find $u_h \in \mathcal{V}_h$ such that

$$A_h(u_h, v) + \gamma_j j(u_h, v) = l_h(v) \quad \forall v \in \mathcal{V}_h. \tag{2.26}$$

The bilinear form $A_h(\cdot,\cdot)$ is defined by

$$A_h(v, w) = a_h(v, w) + \gamma_s s_h(v, w) \quad \forall v, w \in \mathcal{V}_h \tag{2.27}$$

with

$$a_h(v,w) = (\nabla_{\Sigma_h} v, \nabla_{\Sigma_h} w)_{\Sigma_h} - (k^2 v, w)_{\Sigma_h} \tag{2.28}$$

and

$$s_h(v,w) = \sum_{K\in\mathcal{K}_h} h^2(\Delta_{\Sigma_h} v + k^2 v, \Delta_{\Sigma_h} w + k^2 w)_{\Sigma_h\cap K} \tag{2.29}$$

where the tangent gradients are defined using the normal to the discrete surface

$$\nabla_{\Sigma_h} v = \boldsymbol{P}_{\Sigma_h}\nabla v = (\boldsymbol{I} - \boldsymbol{n}_h \otimes \boldsymbol{n}_h)\nabla v. \tag{2.30}$$

The form on the right hand side $l_h(\cdot)$ is given by

$$l_h(v) = (f^e, v)_{\Sigma_h} - \sum_{K\in\mathcal{K}_h} \gamma_s h^2 (f^e, \Delta_{\Sigma_h} w + k^2 w)_{\Sigma_h\cap K}. \tag{2.31}$$

Observe that since the level set function b_h is piecewise linear and defined on $\mathcal{V}_h$, $\Delta_{\Sigma_h} v|_{K\cap\Sigma_h} = 0$. Therefore the stabilization term and the right hand side reduces to

$$s_h(v,w) = \sum_{K\in\mathcal{K}_h} h^2(\Delta_{\Sigma_h} v + k^2 v, \Delta_{\Sigma_h} w + k^2 w)_{\Sigma_h} = (h^2k^2 v, k^2 w)_{\Sigma_h} \tag{2.32}$$

and

$$l_h(v) = (f^e, v)_{\Sigma_h} - \sum_{K\in\mathcal{K}_h} \gamma_s h^2 (f^e, \Delta_{\Sigma_h} v + k^2 v)_{\Sigma_h\cap K} = (f^e, v - \gamma_s h^2 k^2 v)_{\Sigma_h} \tag{2.33}$$

We notice that these simplifications allow us to write the following formulation which is suitable for implementation: find $u_h \in \mathcal{V}_h$ such that

$$(\nabla_{\Sigma_h} u_h, \nabla_{\Sigma_h} v)_{\Sigma_h} - (k^2(1-\gamma_s h^2 k^2)u_h, v)_{\Sigma_h} + \gamma_j j(u_h, v) = (f_e, (1-\gamma_s h^2k^2)v)_{\Sigma_h} \quad \forall v \in \mathcal{V}_h. \tag{2.34}$$

Since this weakly consistent stabilization actually is a norm on u_h, one may prove that the system is invertible for all h as follows. Assume that $f = 0$ in (**??**) and prove that this implies $u_h = 0$ (the system is square), then take $v = u_h$ in (**??**) and take the imaginary part of the equation to obtain

$$\text{Im}(\gamma_s)(hk)^2\|ku_h\|^2_{\Sigma_h} \leqslant 0. \tag{2.35}$$

Therefore $u_h = 0$ and the discrete system is invertible. On planar domains one may prove a similar result using only the gradient penalty term [**?**]. Due to the curved surface it seems difficult to eliminate the lower order term in the stabilization.

For the analysis it will be useful to introduce the weakly consistent formulation also on Σ. We let

$$A^r(u_h, v_h) := a(u_h, v_h) + \gamma s^r(u_h, v_h), \qquad s^r(v,w) = (h^2k^2 v, k^2 w)_\Sigma. \tag{2.36}$$

Remark 2.3 *The penalty on the gradient jumps plays two roles in this work. First, similarly as in [?] it leads to a stable algebraic system. In the present case it is also necessary to obtain error estimates independent of the wavenumber/mesh relation in the semi-discrete case, and under the mild condition $hk \lesssim 1$ in the fully discrete case. Other stabilizations can be applied provided they have similar stabilizing properties as the face penalty. However due to this double role, the face penalty appears to be natural in this context. The application of other stabilizing terms is left for future work, as well as the interesting question if the method can be proven to be stable and accurate using only the stabilizing term j. This is the case for the standard Helmholtz equation in the plane on a conforming mesh.*

For future reference we now recall a key result from [?] that will be useful for the analysis.

Lemma 2.1 *There is a constant $C > 0$ such that for all $v_h \in \mathcal{V}_h$ there holds*

$$h\|\nabla v_h^l\|_\Sigma \leqslant Ch\|\nabla v_h\|_{\Sigma_h} \leqslant C(h\|\nabla_{\Sigma_h} v_h\|_{\Sigma_h} + j(v_h, v_h)^{\frac{1}{2}}) \tag{2.37}$$

$$\leqslant C(h\|\nabla_\Sigma v_h^l\|_\Sigma + j(v_h, v_h)^{\frac{1}{2}}). \tag{2.38}$$

Proof. Identical to the proof of Lemma 3.2 of [?] and using equivalences $\|\nabla v_h^l\|_\Sigma \sim \|\nabla v_h\|_{\Sigma_h}$ and $\|\nabla_\Sigma v_h^l\|_\Sigma \sim \|\nabla v_h\|_{\Sigma_h}$. □

The following Lemma will also be useful

Lemma 2.2 *There is a constant C_κ, depending on the curvature of Σ, such that for all $v \in \mathcal{V}_h$ there holds*

$$\sum_{K \in \mathcal{K}_h} h^2 \|\Delta_\Sigma v_h^l\|^2_{\Sigma \cap K} \leqslant C_\kappa \|n_\Sigma \cdot \nabla v_h^l\|_\Sigma \leqslant C_\kappa (h^2 \|\nabla_\Sigma v_h^l\|^2_\Sigma + |||v_h|||^2_j). \tag{2.39}$$

Proof. First we use the following relation that follows from the arguments in [?]: since v_h is piecewise affine there holds

$$\Delta_\Sigma v_h^l|_{\Sigma\cap K} = -\mathrm{tr}(\kappa) \nabla v_h^l \cdot n_\Sigma|_{\Sigma \cap K}, \forall K \in \mathcal{K}_h. \tag{2.40}$$

To see this we write

$$\Delta_\Sigma v_h^l = \nabla_\Sigma \cdot (I - n_\Sigma \otimes n_\Sigma) \nabla v_h^l = \nabla_\Sigma \cdot (n_\Sigma n_\Sigma \cdot \nabla v_h^l) = -(\nabla \cdot n_\Sigma) n_\Sigma \cdot \nabla v_h^l - \underbrace{n_\Sigma \cdot (\nabla_\Sigma n_\Sigma)(\nabla v_h^l)}_{=0}$$

and the relation follows recalling that $\nabla \cdot n_\Sigma = \mathrm{tr}(\kappa)$. Applying the relation (**??**) followed by Lemma **??** we obtain

$$\sum_{K \in \mathcal{K}_h} h^2 \|\Delta_\Sigma v_h^l\|^2_{\Sigma \cap K} \leqslant C_\kappa h^2 \|\nabla v_h^l\|^2_\Sigma \leqslant C_\kappa (h^2 \|\nabla_\Sigma v_h^l\|^2_\Sigma + |||v_h|||^2_j). \tag{2.41}$$

□

3 A Priori Error Estimates

For the a priori error analysis we will follow the framework for the analysis of stabilized finite element methods for the Helmholtz equation proposed in [**?**]. In order to estimate the error induced by approximating the equations on an approximate surface we need to first recall a number of technical results regarding the mapping from the approximate to the exact surface and the bounds on the error committed when changing the domain of integration. For detailed proofs, we refer to [**?**, **?**, **?**]. We also recall some approximation error estimates.

3.1 Geometric Estimates

First we recall how the tangential gradient of lifted and extended functions can be computed and how the surface measure changes under lifting. Starting with the Hessian of the signed distance function

$$\boldsymbol{\kappa} = \nabla \otimes \nabla b \quad \text{in } U_{\delta_0}(\Sigma) \tag{3.1}$$

the derivative of the closest point projection and of an extended function v^e is given by

$$Dp = \boldsymbol{P}_\Sigma(I - b\boldsymbol{\kappa}) = \boldsymbol{P}_\Sigma - b\boldsymbol{\kappa} \tag{3.2}$$

$$Dv^e = D(v \circ \boldsymbol{p}) = DvDp = Dv\boldsymbol{P}_\Sigma(I - b\boldsymbol{\kappa}). \tag{3.3}$$

The self-adjointness of $\boldsymbol{P}_\Sigma$, $\boldsymbol{P}_{\Sigma_h}$, and $\boldsymbol{\kappa}$, and the fact that $\boldsymbol{P}_\Sigma\boldsymbol{\kappa} = \boldsymbol{\kappa} = \boldsymbol{\kappa}\boldsymbol{P}_\Sigma$ and $\boldsymbol{P}_\Sigma^2 = \boldsymbol{P}_\Sigma$ leads to the identity

$$\nabla_{\Sigma_h} v^e = \boldsymbol{P}_{\Sigma_h}(I - b\boldsymbol{\kappa})\boldsymbol{P}_\Sigma \nabla v = \boldsymbol{B}^T \nabla_\Sigma v \tag{3.4}$$

where $\boldsymbol{B}$ denotes the invertible linear application

$$\boldsymbol{B} = \boldsymbol{P}_\Sigma(I - b\boldsymbol{\kappa})\boldsymbol{P}_{\Sigma_h} : T_x(\Sigma_h) \to T_{\boldsymbol{p}(x)}(\Sigma) \tag{3.5}$$

mapping the tangential space of Σ_h at x to the tangential space of Σ at $\boldsymbol{p}(x)$. Setting $v = w^l$ and using the identity $(w^l)^e = w$, we immediately get that

$$\nabla_\Sigma w^l = \boldsymbol{B}^{-T} \nabla_{\Sigma_h} w \tag{3.6}$$

for any elementwise differentiable function w on Σ_h lifted to Σ. We recall from [**?**, Lemma 14.7] that for $x \in U_{\delta_0}(\Sigma)$, the Hessian $\boldsymbol{\kappa}$ admits a representation

$$\boldsymbol{\kappa}(x) = \sum_{i=1}^{d} \frac{\kappa_i^e}{1 + b(x)\kappa_i^e} a_i^e \otimes a_i^e \tag{3.7}$$

where κ_i are the principal curvatures with corresponding principal curvature vectors a_i. Thus

$$\|\boldsymbol{\kappa}\|_{L^\infty(U_{\delta_0}(\Sigma))} \leqslant C \tag{3.8}$$

for $\delta_0 > 0$ small enough and as a consequence the following bounds for the linear operator $\boldsymbol{B}$ can be derived:

$$\|\boldsymbol{B}\|_{L^\infty(\Sigma_h)} \leqslant C, \quad \|\boldsymbol{B}^{-1}\|_{L^\infty(\Sigma)} \leqslant C, \quad \|\boldsymbol{P}_\Sigma - \boldsymbol{B}\boldsymbol{B}^T\|_{L^\infty(\Sigma)} \leqslant Ch^2. \tag{3.9}$$

Next, we recall that the surface measure $d\sigma$ on Σ is related to the surface measure $d\sigma_h$ on Σ_h by the identity

$$d\sigma = |\boldsymbol{B}| d\sigma_h \tag{3.10}$$

where $|\boldsymbol{B}|$ is the determinant of $\boldsymbol{B}$ which is given by

$$|\boldsymbol{B}| = \Pi_{i=1}^2 (1 - b\kappa_i^e) \boldsymbol{n}^e \cdot \boldsymbol{n}_h. \tag{3.11}$$

Using this the following estimates for the determinant can be proved,

$$\| \, |\boldsymbol{B}| \, \|_{L^\infty(\Sigma_h)} \leqslant C, \quad \| \, |\boldsymbol{B}|^{-1} \, \|_{L^\infty(\Sigma_h)} \leqslant C, \quad \|1 - |\boldsymbol{B}|\|_{L^\infty(\Sigma_h)} \leqslant Ch^2. \tag{3.12}$$

3.2 Interpolation Error Estimates

We let $\pi_h : L^2(\Omega_h) \to \mathcal{V}_h|_{\Sigma_h}$ denote the standard Scott-Zhang interpolation operator and recall the interpolation error estimate

$$\|v - \pi_h v\|_{m,K} \leqslant Ch^{2-m} \|v\|_{2,\mathcal{N}(K)}, \quad m = 0, 1, 2 \tag{3.13}$$

where $\mathcal{N}(K) \subset \Omega_h$ is the union of the neighboring elements of K. We also define an interpolation operator $\pi_h^l : L^2(\Sigma) \to (\mathcal{V}_h|_{\Sigma_h})^l$ as follows

$$\pi_h^l v = ((\pi_h v^e)|_{\Sigma_h})^l. \tag{3.14}$$

We define the energy norm $|||\cdot|||_\Sigma$ associated with the exact surface and the norms $|||\cdot|||_s$ and $|||\cdot|||_j$ associated with the stablizing terms by

$$|||v|||_{\Sigma,k}^2 = \|\nabla_\Sigma v\|_\Sigma^2 + \|kv\|_\Sigma^2, \quad |||v|||_j^2 = j(v,v), \quad |||v|||_s^2 = s(v,v), \quad |||v|||_{s^r}^2 = s^r(v,v). \tag{3.15}$$

From the results of [?] we deduce approximation results needed in the analysis.

Lemma 3.1 *Let u be the exact solution of (??). Then the following estimates hold*

$$|||u - \pi_h^l u|||_{\Sigma,k}^2 + |||u^e - \pi_h u^e|||_j^2 + \sum_K \|h^{-\frac{1}{2}}(u - \pi_h^l u)\|_{\partial K \cap \Sigma}^2 \leqslant C(hk)^2(1 + h^4k^4)\|f\|_\Sigma^2 \tag{3.16}$$

$$|||u - \pi_h^l u|||_s^2 \leqslant C(hk)^2(1 + h^4k^4)\|f\|_\Sigma^2 \tag{3.17}$$

and,

$$|||\pi_h u^e|||_j^2 + h^2 |||\pi_h^l u|||_{\Sigma,k}^2 + |||\pi_h^l u|||_s^2 \leqslant Ch^2(1 + k^2)(1 + h^2k^2)\|f\|_\Sigma^2. \tag{3.18}$$

Proof. The bound (**??**) follows immediately from the approximation results of [**?**]. For (**??**) we apply the triangle inequality followed by the first inequality of Lemma **??** to obtain

$$\begin{aligned}\sum_{K\in\mathcal{K}} &\|\Delta(u-\pi_h^l u)+k^2(u-\pi_h^l u)\|_{\Sigma\cap K}^2\\ &\leqslant C(\|\Delta_\Sigma u\|_\Sigma^2+\|\mathrm{tr}(\kappa)\nabla(\pi_h^l u_h-u^e)\cdot n_\Sigma\|_\Sigma^2+k^4\|u-\pi_h^l u\|_\Sigma^2)\\ &\leqslant C(1+k^4h^4)\|u\|_{2,\Sigma}^2.\end{aligned}$$

To prove (**??**) we add and subtract u, use a triangle inequality and apply (**??**) and (**??**) and finally observe that, using the regularity (**??**) and the equation (**??**),

$$h^2|||u|||_{\Sigma,k}^2+|||u|||_s^2\leqslant Ch^2\|f\|_\Sigma^2.$$

□

3.3 Error Estimates for the Semi Discretized Formulation

We will first give an analysis for the semi-discretized method (**??**). This is to show how the ideas of [**?**] carries over to the case of approximation of the Helmholtz equation on a surface, without the technicalities introduced by the discretized surface. The analysis is based on the observation that we have coercivity on the stabilization terms that constitute a (very weak) norm on the solution. In this norm we obtain an optimal error estimate. We then proceed using duality to estimate the error in the L^2-norm, independent of the error in energy norm. Then finally we estimate the error in the energy norm. To simplify the notation we assume that hk is bounded by some constant, so that higher powers can be omitted. Observe however that we do not assume that hk is "small enough" here, which will be necessary when also the domain is discretized in the next section. We first prove a preliminary lemma that will be useful in the following analysis.

Lemma 3.2 *(Continuity) For all $v,w\in H^2(\Sigma)$, and $v_h,w_h\in\mathcal{V}_h$, there holds*

$$\begin{aligned}|a(v+v_h,w+w_h)|\leqslant{}& |||v+v_h|||_s\|h^{-1}(w+w_h)\|_\Sigma\\ &+C|||v^e+v_h|||_j\left(\sum_K\|h^{-\frac{1}{2}}(w+w_h)\|_{\partial K\cap\Sigma}^2\right)^{\frac{1}{2}}.\end{aligned}\tag{3.19}$$

Proof. Using an integration by parts we see that

$$\begin{aligned}a(v+v_h,w+w_h)|={}&\sum_K\int_{\partial K\cap\Sigma}[\![\nabla_\Sigma v_h]\!]\cdot n_{\partial K\cap\Sigma}\overline{(w+w_h)}\,\mathrm{d}\sigma\\ &-\sum_K(\Delta_\Sigma(v+v_h)+k^2(v+v_h),w+w_h)_{K\cap\Sigma}.\end{aligned}\tag{3.20}$$

We now multiply and divide and with $h^{\frac{1}{2}}$ in the first term of the right hand side and with h in the second. Then we apply the Cauchy-Schwarz inequality and observe that by using trace inequalities from $\partial K \cap \Sigma$ to $F \in \partial K$,

$$\sum_K (h[\![\nabla_\Sigma v_h]\!], [\![\nabla_\Sigma v_h]\!])_{\partial K \cap \Sigma} \leqslant C|||v_h|||_j^2 = C|||v^e + v_h|||_j^2. \tag{3.21}$$

This completes the proof of (**??**). □

Remark 3.1 *Observe that by the symmetry of the form $a(\cdot,\cdot)$ the claim holds also when v, v_h and w, w_h are interchanged.*

Lemma 3.3 *Let u be the solution of* (**??**) *and $\tilde{u}_h$ the solution of* (**??**)*. Assume that the regularity estimate* (**??**) *holds, then*

$$|||u - \tilde{u}_h|||_s + |||u^e - \tilde{u}_h|||_j \leqslant C(\gamma_i^{-1} + 1)hk\|f\|_\Sigma, \tag{3.22}$$

where $\gamma_i := Im(\gamma)$.

Proof. Note that by definition

$$A(v,v) = \underbrace{\|\nabla_\Sigma v\|_\Sigma^2 - k^2\|v\|_\Sigma}_{\in \mathbb{R}} + \gamma \underbrace{|||v|||_s^2)}_{\in \mathbb{R}}.$$

Therefore,

$$\text{Im}(A(v,v)) = \gamma_i |||v|||_s^2$$

By the condition $\gamma_i > 0$, and the regularity of u we note that there holds

$$\begin{aligned}\gamma_i(&|||u - \tilde{u}_h|||_s^2 + |||u^e - \tilde{u}_h|||_j^2) \\ &= \text{Im}(A(u - \tilde{u}_h, u - \tilde{u}_h) + \gamma j(u^e - \tilde{u}_h, u^e - \tilde{u}_h)).\end{aligned} \tag{3.23}$$

Using now the consistency of the formulation we have by Galerkin orthogonality

$$\gamma_{min}(|||u - \tilde{u}_h|||_s^2 + |||u^e - \tilde{u}_h|||_j^2)$$
$$= \text{Im}(A(u - \tilde{u}_h, u - \pi_h u^e) + \gamma j(u^e - \tilde{u}_h, u^e - \pi_h u^e)) \tag{3.24}$$
$$\leqslant |a(u - \tilde{u}_h, u - \pi_h u^e) + \gamma s(u - \tilde{u}_h, u - \pi_h u^e) + \gamma j(u^e - \tilde{u}_h, u^e - \pi_h u^e)|. \tag{3.25}$$

By Lemma **??** there holds

$$\begin{aligned}|a(u - \tilde{u}_h, u - \pi_h u^e)| &\leqslant |||u - \tilde{u}_h|||_s \|h^{-1}(u - \pi_h u^e)\|_\Sigma \\ &+ C|||u^e - \tilde{u}_h|||_j \left(\sum_K \|h^{-\frac{1}{2}}(u - \pi_h u^e)\|_{\partial K \cap \Sigma}^2\right)^{\frac{1}{2}}.\end{aligned} \tag{3.26}$$

For the stabilization terms we use the Cauchy-Schwarz inequality to obtain

$$|\gamma s(u - \tilde{u}_h, u - \pi_h u^e) + \gamma j(u^e - \tilde{u}_h, u^e - \pi_h u^e)| \\ \leqslant |\gamma|(|||u - \tilde{u}_h|||_s + |||u^e - \tilde{u}_h|||_j)(|||u - \pi_h u^e|||_s + |||u^e - \pi_h u^e|||_j). \quad (3.27)$$

The claim now follows by applying Lemma **??** and the regularity estimate (**??**). □

Theorem 3.1 *Let u be the solution of* (**??**) *and $\tilde{u}_h$ the solution of* (**??**)*. Assume that the regularity estimate* (**??**) *holds, then*

$$|||u - \tilde{u}_h|||_{\Sigma,k} \leqslant C_\gamma Im(\gamma)^{-1}(|\gamma| + 1)(hk + h^2k^3)\|f\|_\Sigma. \quad (3.28)$$

Proof. First let z be the solution of (**??**) with the right hand side $f = u - \tilde{u}_h$. Then by the finite element formulation (**??**) there holds

$$\|u - \tilde{u}_h\|_\Sigma^2 = a(u - \tilde{u}_h, z - \pi_h z^e) - \gamma s(u - \tilde{u}_h, \pi_h z^e) - \gamma j(u^e - \tilde{u}_h, \pi z^e). \quad (3.29)$$

Using Lemma **??** in the first term of the right hand side and the Cuachy-Schwarz inequality in the second and third we obtathe bound

$$\|u - \tilde{u}_h\|_\Sigma^2 \leqslant |||u - \tilde{u}_h|||_s \|h^{-1}(z - \pi_h z^e)\|_\Sigma \quad (3.30)$$

$$+ C|||u^e - \tilde{u}_h|||_j \left(\sum_K \|h^{-\frac{1}{2}}(z - \pi_h z^e)\|_{\partial K \cap \Sigma}^2\right)^{\frac{1}{2}} \quad (3.31)$$

$$+ \gamma_i(|||u - \tilde{u}_h|||_s + |||u^e - \tilde{u}_h|||_j)(|||\pi_h z^e|||_s + |||\pi_h z^e|||_j). \quad (3.32)$$

By interpolation, the definition of z and the regularity of z we obtain

$$\|h^{-1}(z - \pi_h z^e)\|_\Sigma \leqslant Chk\|u - \tilde{u}_h\|_\Sigma \quad (3.33)$$

$$|||\pi_h z^e|||_s \leqslant |||\pi_h z^e - z|||_s + Ch\|u - \tilde{u}_h\|_\Sigma \leqslant Ch(1 + k)\|u - \tilde{u}_h\|_\Sigma \quad (3.34)$$

and

$$|||\pi_h z^e|||_j = |||\pi_h z^e - z^e|||_j \leqslant Chk\|u - \tilde{u}_h\|_\Sigma. \quad (3.35)$$

Collecting the above bounds and using Lemma **??** we obtain

$$\|k(u - \tilde{u}_h)\|_\Sigma \leqslant C(1 + |\gamma|)hk^2(|||u - \tilde{u}_h|||_s + |||u^e - \tilde{u}_h|||_j) \leqslant C_\gamma h^2k^3\|f\|_\Sigma. \quad (3.36)$$

We may now proceed to bound $|||u - \tilde{u}_h|||_{\Sigma,k}^2$ using the real part of the bilinear form, Galerkin orthogonality, and the control of the L^2-norm of the error.

$$|||u - \tilde{u}_h|||_{\Sigma,k}^2 = \mathrm{Re}(A(u - \tilde{u}_h, u - \pi_h u^e) - \gamma j(\tilde{u}_h, \tilde{u}_h - \pi_h u^e)) + 2\|k(u - \tilde{u}_h)\|_\Sigma^2. \quad (3.37)$$

In the first term of the right hand side we now proceed as for (**??**) using the inequality (**??**) and Lemma **??** to conclude that

$$|A(u - \tilde{u}_h, u - \pi_h u^e) - \gamma j(\tilde{u}_h, \tilde{u}_h - \pi_h u^e)| \leqslant C_\gamma (hk)^2 \|f\|_\Sigma^2. \quad (3.38)$$

We conclude by combining this bound with (**??**). □

Lemma 3.4 *Under the same assumptions as for Lemma* **??** *and Theorem* **??** *there holds*

$$\|u - \tilde{u}_h\|_\Sigma \leqslant C_\gamma (hk)^2 \|f\|_\Sigma \tag{3.39}$$

and

$$|||\tilde{u}_h|||_{\Sigma,k} \leqslant C_\gamma (1+k)\|f\|_\Sigma, \quad |||\tilde{u}_h|||_s \leqslant C_\gamma (1+k) h \|f\|_\Sigma. \tag{3.40}$$

Proof. The first claim follows directly from equation (**??**). The remaining inequalities are immediate by adding and subtracting the exact solution u in the norms of the left hand side, followed by a triangle inequality and then applying the results of Lemma **??** and Theorem **??**. □

3.4 Error Estimates for the Fully Discrete Formulation

To obtain an error estimate for the fully discrete scheme we need an equivalent to Lemma **??** for the formulation on the discrete surface and we also need upper bounds of the conformity error that we commit by approximating the surface. We start by proving these technical lemmas.

Lemma 3.5 *(Continuity) For all $v, w \in H^2(\Sigma)$, $v_h, w_h \in \mathcal{V}_h$ there holds*

$$\begin{aligned} |a(v + v_h^l, w + w_h^l)| \leqslant{}& |||v + v_h^l|||_s \|h^{-1}(w + w_h^l)\|_\Sigma \\ &+ C(|||v^e + v_h|||_j + h|||v_h^l|||_{\Sigma,k}) \left(\sum_K \|h^{-\frac{1}{2}}(w + w_h^l)\|^2_{\partial K \cap \Sigma} \right)^{\frac{1}{2}}. \end{aligned} \tag{3.41}$$

Proof. The proof of (**??**) is similar to that of (**??**), but this time we instead need to prove the inequality

$$\sum_K (h[\![\nabla_\Sigma v_h^l]\!], [\![\nabla_\Sigma v_h^l]\!])_{\partial K \cap \Sigma} \leqslant C|||v_h|||_j^2 = C|||v^e + v_h|||_j^2 \tag{3.42}$$

to conclude. This leads to a slightly different argument since $\nabla_\Sigma v_h^l = \boldsymbol{B}^{-T}\boldsymbol{P}_{\Sigma_h}\nabla v_h$. It follows that

$$\sum_{K \in \mathcal{T}_h} \int_{\Sigma \cap \partial K} h |[\![\nabla_\Sigma v_h^l]\!]|^2 \, \mathrm{d}\sigma \leqslant \sum_{K \in \mathcal{T}_h} \|h^{\frac{1}{2}} |[\![\boldsymbol{B}^{-T}\boldsymbol{P}_{\Sigma_h}\nabla v_h]\!]|\|^2_{\Sigma_h \cap \partial K}. \tag{3.43}$$

The right hand side may be bounded as follows

$$\begin{aligned} &\sum_{K \in \mathcal{T}_h} \|h^{\frac{1}{2}} |[\![\boldsymbol{B}^{-T}\boldsymbol{P}_{\Sigma_h}\nabla v_h]\!]|\|^2_{\Sigma_h \cap \partial K} \\ &\qquad \leqslant C \sum_{K \in \mathcal{T}_h} \left(\|h^{\frac{1}{2}} |[\![\boldsymbol{B}^{-T}\boldsymbol{P}_{\Sigma_h}]\!] \nabla v_h|\|^2_{\Sigma_h \cap \partial K} + \|h^{\frac{1}{2}} |[\![\nabla v_h]\!]|\|^2_{\Sigma_h \cap \partial K} \right). \end{aligned} \tag{3.44}$$

For the second term in the right hand side we have by a trace inequality from $\Sigma_h \cap \partial K$ to $F \in \partial K$,

$$\sum_{K \in \mathcal{T}_h} \|h^{\frac{1}{2}} |[\![\nabla v_h]\!]|\|^2_{\Sigma_h \cap \partial K} \leqslant |||v_h|||_j. \tag{3.45}$$

For the first term observe that also by repeated trace inequalities, first from $\Sigma_h \cap \partial K$ to ∂K and then from ∂K to K,

$$\|h^{\frac{1}{2}} |[\![\boldsymbol{B}^{-T}\boldsymbol{P}_{\Sigma_h}]\!]\nabla v_h|\|_{\Sigma_h \cap \partial K} \leqslant C\|[\![\boldsymbol{B}^{-T}\boldsymbol{P}_{\Sigma_h}]\!]\|_{L^\infty(\Sigma_h \cap \partial K)} h^{-\frac{1}{2}} \|\nabla v_h\|_{\Omega_h}. \tag{3.46}$$

Now using the regularity of Σ we may write $[\![\boldsymbol{B}^{-T}\boldsymbol{P}_{\Sigma_h}]\!] = [\![\boldsymbol{B}^{-T}\boldsymbol{P}_{\Sigma_h} - \tilde{\boldsymbol{B}}^{-T}\boldsymbol{P}_{\Sigma}]\!]$ where we have introduced $\tilde{\boldsymbol{B}} := \boldsymbol{P}_\Sigma(I - b\boldsymbol{\kappa})\boldsymbol{P}_\Sigma$. Expanding this relation we get

$$\boldsymbol{B}^{-T}\boldsymbol{P}_{\Sigma_h} - \tilde{\boldsymbol{B}}^{-T}\boldsymbol{P}_\Sigma = (\boldsymbol{B}^{-T} - \tilde{\boldsymbol{B}}^{-T})\boldsymbol{P}_{\Sigma_h} + \tilde{\boldsymbol{B}}^{-T}(\boldsymbol{P}_{\Sigma_h} - \boldsymbol{P}_\Sigma) = I + II. \tag{3.47}$$

The term II can be bounded observing that as a consequence of (**??**)

$$\|\tilde{\boldsymbol{B}}^{-T}(\boldsymbol{P}_{\Sigma_h} - \boldsymbol{P}_\Sigma)\|_{L^\infty(\Sigma_h \cap \partial K)} \leqslant C\|\boldsymbol{P}_{\Sigma_h} - \boldsymbol{P}_\Sigma)\|_{L^\infty(\Sigma_h \cap \partial K)} \leqslant Ch \tag{3.48}$$

For the first term is follows that

$$\begin{aligned} I = (\boldsymbol{B}^{-T} - \tilde{\boldsymbol{B}}^{-T})\boldsymbol{P}_{\Sigma_h} &= \tilde{\boldsymbol{B}}^{-T}(\tilde{\boldsymbol{B}}^{T}\boldsymbol{B}^{-T} - I) \\ &= \tilde{\boldsymbol{B}}^{-T}(\tilde{\boldsymbol{B}}^{T} - \boldsymbol{B}^T)\boldsymbol{B}^{-T} = \tilde{\boldsymbol{B}}^{-T}(\boldsymbol{P}_\Sigma(I - b\boldsymbol{\kappa})(\boldsymbol{P}_\Sigma - \boldsymbol{P}_{\Sigma_h}))^T\boldsymbol{B}^{-T}. \end{aligned}$$

Therefore

$$\|(\boldsymbol{B}^{-T} - \tilde{\boldsymbol{B}}^{-T})\boldsymbol{P}_{\Sigma_h}\|_{L^\infty(\Sigma_h \cap \partial K)} \leqslant C\|\boldsymbol{P}_\Sigma - \boldsymbol{P}_{\Sigma_h}\|_{L^\infty(\Sigma_h \cap \partial K)} \leqslant Ch. \tag{3.49}$$

The bounds (**??**) and (**??**) show that and $\|[\![\boldsymbol{B}^{-T}\boldsymbol{P}_{\Sigma_h}]\!]\|_{L^\infty(\partial K)} \leqslant Ch$. Using this bound together with (**??**) and (**??**) we may write

$$\sum_{K \in \mathcal{T}_h} \|h^{\frac{1}{2}} |[\![\boldsymbol{B}^{-T}\boldsymbol{P}_{\Sigma_h}\nabla v_h]\!]|\|^2_{\Sigma_h \cap \partial K} \leqslant C(h\|\nabla v_h\|^2_{\Omega_h} + |||v_h|||^2_j). \tag{3.50}$$

The bound (**??**) then follows using the arguments of Lemma 4.2 of [**?**] (see also Lemma 5.3 of [**?**]) leading to

$$h\|\nabla v_h\|^2_{\Omega_h} \leqslant C(h^2\|\nabla_{\Sigma_h} v_h\|^2_{\Sigma_h} + |||v_h|||^2_j) \tag{3.51}$$

and the norm equivalence $\|\nabla_\Sigma v_h^l\|_\Sigma \sim \|\nabla_{\Sigma_h} v_h\|_{\Sigma_h}$. □

We will first prove some conformity error bounds that we collect in a lemma.

Lemma 3.6 *Let u_h be the solution of (**??**) and assume that $hk < 1$. Then*

$$|a_h(u_h, v_h) - a(u_h^l, v_h^l)| \leqslant Ch^2 |||u_h^l\|_{\Sigma,k} |||v_h^l|||_{\Sigma,k} \tag{3.52}$$

$$|l_s(v_h^l) - l_h(v_h)| \leqslant C_f(h^2 |||v_h^l|||_{\Sigma,k} + h|||v_h|||_{s^r}) \tag{3.53}$$

and

$$|s_h(u_h, v_h) - s^r(u_h^l, v_h^l)| \leqslant Ch^2(hk)^2 |||u_h^l|||_{\Sigma,k} |||kv_h^l|||_{\Sigma,k} \tag{3.54}$$

Proof. For the first term we observe that

$$|a_h(u_h, v_h) - a(u_h^l, v_h^l)| \leqslant |(\nabla_{\Sigma_h} u_h, \nabla_{\Sigma_h} v_h)_{\Sigma_h} - (\nabla_\Sigma u_h^l, \nabla_\Sigma v_h^l)_\Sigma| \qquad (3.55)$$
$$+ |(k^2 u_h, v_h)_{\Sigma_h} - (k^2 u_h, v_h)_\Sigma|$$
$$\leqslant Ch^2 \|\nabla_\Sigma u_h^l\|_\Sigma \|\nabla_\Sigma v_h^l\|_\Sigma + \int_{\Sigma_h} k^2 u_h \bar{v}_h (1 - |\boldsymbol{B}|) \mathrm{d}\sigma_h \qquad (3.56)$$

where we used the result on the Laplace-Beltrami part from [?]. For the zero order term we observe that by (??)

$$\left| \int_{\Sigma_h} k^2 u_h \bar{v}_h (1 - |\boldsymbol{B}|) \mathrm{d}\sigma_h \right| \leqslant Ch^2 \|k u_h^l\|_\Sigma \|k v_h^l\|_\Sigma. \qquad (3.57)$$

For the control of the conformity error of the right hand side we observe that

$$l(v_h^l) - l_h(v_h) = \int_{\Sigma_h} f_h \bar{v}_h (|\boldsymbol{B}| - 1) \; \mathrm{d}\sigma_h - \int_{\Sigma_h} f_h \gamma_s h^2 k^2 \bar{v}_h \; \mathrm{d}\sigma_h \qquad (3.58)$$

The first term on the right hand side was bounded in [?],

$$\int_{\Sigma_h} f_e \bar{v}_h (|\boldsymbol{B}| - 1) \; \mathrm{d}\sigma_h \leqslant C_f h^2 \|v_h\|_\Sigma. \qquad (3.59)$$

The second term may be bounded using the Cauchy-Schwarz inequality

$$\int_{\Sigma_h} f_h \gamma_s h^2 k^2 \bar{v}_h \; \mathrm{d}\sigma_h \leqslant C_f h \|k^2 h v_h\|_{\Sigma_h} \leqslant C_f h \|k^2 h v_h^l\|_\Sigma \qquad (3.60)$$

For the Galerkin least squares term we may write

$$s_h(u_h, v_h) - s^r(u_h^l, v_h^l) = (h^2 k^2 u_h, k^2 v_h)_{\Sigma_h} - (h^2 k^2 u_h^l, k^2 v_h^l)_\Sigma. \qquad (3.61)$$

Using the bounds (??) we have

$$(h^2 k^2 u_h, k^2 v_h)_{\Sigma_h} - (h^2 k^2 u_h^l, k^2 v_h^l)_\Sigma \leqslant Ch^2 (hk)^2 \|k u_h^l\|_\Sigma \|k v_h^l\|_\Sigma$$

□

An immediate consequence of the previous result is the following bounds on the conformity error of the form $A_h(\cdot, \cdot)$.

Corollary 3.1 *Let u_h be the solution of (??) and assume that $hk < 1$. Then for all $\epsilon > 0$,*

$$|A_h(v_h, w_h) - A^r(v_h^l, w_h^l)| \leqslant C_\gamma h^2 |||v_h^l|||_{\Sigma,k} |||w_h^l|||_{\Sigma,k} \qquad (3.62)$$

Proof. Follows directly from the previous lemma. □

The proof of convergence of the fully discrete scheme now follows the same model as that of the semi-discrete scheme, estimating this time also the error induced by integrating the equations on the discrete representation of the surface.

Lemma 3.7 *Let $u \in H^2(\Sigma)$ be the solution of (??) and $u_h \in \mathcal{V}_h$ be the solution of (??). Then*

$$|||\pi_h^l u - u_h^l|||_{s^r}^2 + |||\pi_h u^e - u_h|||_j \leqslant C_{f,\gamma}(hk) + C_\gamma h|||\pi_h^l u - u_h^l|||_{\Sigma,k}. \tag{3.63}$$

Proof. Using the short-hand notation $\pi_h^l u := ((\pi_h u^e)|_{\Sigma_h})^l$, we define the discrete error on Σ_h and its corresponding lift to Σ by $\xi_h := \pi_h u^e - u_h$ and $\xi_h^l := \pi_h^l u - u_h^l$, respectively.

Recalling the definition of the scheme on the exact and the discrete surfaces we may write

$$\begin{aligned}\gamma_i(|||\xi_h^l|||_{s^r}^2 + |||\xi_h|||_j^2) &= \mathrm{Im}[A^r(\xi_h^l, \xi_h^l) + \gamma j(\xi_h, \xi_h)] \\ &= \mathrm{Im}[a(\pi_h^l u - u, \xi_h^l) + \gamma(s^r(\pi_h^l u - u, \xi_h^l) + j(\pi_h u^e, \xi_h) + s^r(u, \xi_h^l)) \\ &\quad + l(\xi_h^l) - l_h(\xi_h) + A_h(u_h, \xi_h) - A^r(u_h^l, \xi_h^l)].\end{aligned}$$

Note that by applying the triangle inequality followed by Lemma ?? the following bound holds

$$|||\xi_h^l|||_s \leqslant C(h|||\xi_h^l|||_{\Sigma,k} + |||\xi_h|||_j + |||\xi_h^l|||_{s^r}) \tag{3.64}$$

which together with Lemma ?? and an arithmetic–geometric inequality with suitable weights leads to,

$$\begin{aligned}a(\pi_h^l u - u, \xi_h^l) &\leqslant C|||\xi_h^l|||_s\|h^{-1}(\pi_h^l u - u)\|_\Sigma \\ &\quad + C(|||\xi_h|||_j + h|||\xi_h^l|||_{\Sigma,k})\left(\sum_K \|h^{-\frac{1}{2}}(\pi_h^l u - u)\|_{\partial K\cap\Sigma}^2\right)^{\frac{1}{2}} \\ &\leqslant C_f^2(hk)^2 + Ch^2|||\xi_h^l|||_{\Sigma,k}^2 + \frac{1}{8}\gamma_i(|||\xi_h^l|||_{s^r}^2 + |||\xi_h|||_j^2).\end{aligned}$$

Using this bound and the Cauchy-Schwarz inequality, for the first three terms in the right hand side we then have

$$\begin{aligned}&|a(\pi_h^l u - u, \xi_h^l) + \gamma s^r(\pi_h^l u - u, \xi_h^l) + \gamma j(\pi_h u^e, \xi_h)| \\ &\qquad \leqslant C_{f,\gamma}^2(hk)^2 + Ch^2|||\xi_h^l|||_{\Sigma,k}^2 + \frac{1}{4}\gamma_i(|||\xi_h^l|||_{s^r}^2 + |||\xi_h|||_j^2).\end{aligned} \tag{3.65}$$

Once again using the Cauchy-Schwarz inequality, the bound (??) and the arithmetic–geometric inequality leads to

$$s^r(u, \xi_h^l) \leqslant h\|k^2 u\|_\Sigma|||\xi_h^l|||_{s^r} \leqslant C_f^2(hk)^2 + \frac{1}{4}\gamma_i|||\xi_h^l|||_{s^r}^2$$

For the remaining terms we use the result of Lemma **??** to deduce

$$l(\xi_h^l) - l_h(\xi_h) \leqslant C_f(h^2 |||\xi_h^l|||_{\Sigma,k} + h|||\xi_h|||_{s^r})$$
$$\leqslant C_{f,\gamma}^2 h^2 + h^2 |||\xi_h^l|||_{\Sigma,k}^2 + \gamma_i \frac{1}{4} |||\xi_h|||_{s^r}^2. \tag{3.66}$$

To bound the conformity error of $A_h(\cdot,\cdot)$ it is convenient to start from (**??**) and write

$$A_h(u_h, \xi_h) - A^r(u_h^l, \xi_h^l) \leqslant Ch^2 |||u_h^l|||_{\Sigma,k} |||\xi_h^l|||_{\Sigma,k} \leqslant Ch^2 (|||\xi_h^l|||_{\Sigma,k}^2 + |||\pi_h^l u|||_{\Sigma,k}^2). \tag{3.67}$$

By collecting the above bounds and applying (**??**) we conclude that

$$C_\gamma(|||\xi_h^l|||_s^2 + |||\xi_h|||_j^2) \leqslant C_{f,\gamma}^2 (hk)^2 + C_\gamma h^2 |||\xi_h^l|||_{\Sigma,k}^2. \tag{3.68}$$

□

Lemma 3.8 *For the error in the L^2-norm there holds*

$$\|u - u_h^l\|_\Sigma \leqslant C_{f,\gamma}(hk)^2 + Ch^2 k |||\pi_h^l u - u_h^l|||_{\Sigma,k}. \tag{3.69}$$

Proof. We let z be the solution of (**??**) with right hand side $f = u - u_h^l$. It follows that

$$\|u - u_h^l\|_\Sigma^2 = a(u - u_h^l, z - \pi_h^l z) + a(u - u_h^l, \pi_h^l z) = I + II. \tag{3.70}$$

By the continuity of $a(\cdot,\cdot)$ (Lemma **??**), arguments similar to that of (**??**), the approximation properties of $\pi_h^l z$ and the regularity estimate (**??**) we have for the first term

$$I \leqslant Chk(h|u|_{2,\Sigma} + h\|\nabla_\Sigma(\pi_h^l u - u_h)\|_\Sigma + |||u - u_h^l|||_{s^r} + |||\pi_h u^e - u_h|||_j)\|u - u_h^l\|_\Sigma. \tag{3.71}$$

Using the definition of the finite element method and (**??**) and the definition of the reduced operator on the exact surface (**??**), we have for the second term

$$II = l(\pi_h^l z) - l_h(\pi_h z^e) + A_h(u_h, \pi_h z^e) - A^r(u_h^l, \pi_h^l z) \tag{3.72}$$
$$+ \gamma_s s^r(u_h^l, \pi_h^l z) + \gamma_j j(u_h, \pi_h z^e). \tag{3.73}$$

Using Lemma **??** in the two first terms and the Cauchy-Schwarz inequality in the two last we have

$$II \leqslant C_f(h^2 |||\pi_h^l z|||_{\Sigma,k} + h|||\pi_h^l z|||_{s^r}) + C_\gamma h^2 |||u_h^l|||_{\Sigma,k} |||\pi_h^l z|||_{\Sigma,k} \tag{3.74}$$
$$+ |||u_h^l|||_{s^r} |||\pi_h^l z|||_{s^r} + |||u_h|||_j |||\pi_h z^e|||_j \tag{3.75}$$

Recalling the equations (**??**) and (**??**) and (**??**) we have that

$$|||\pi_h^l z|||_{s^r} + |||\pi_h z^e|||_j \leqslant C(hk)\|u - u_h^l\|_\Sigma, \quad |||\pi_h^l z|||_{\Sigma,k} \leqslant C(1 + hk)\|u - u_h^l\|_\Sigma \tag{3.76}$$

and

$$|||\pi_h^l u|||_{s^r} + |||\pi_h u^e|||_j \leqslant C_f(hk). \tag{3.77}$$

Adding and subtracting $\pi_h^l u$ in all the norms on u_h^l in (**??**)–(**??**) and using a triangle inequality and the above bounds on norms of $\pi_h^l z$ and $\pi_h^l u$ we have the bound

$$II \leqslant C_{f,\gamma} hk((hk) + |||\pi_h^l u - u_h^l|||_{s^r} + |||\pi_h u^e - u_h|||_j + h|||\pi_h^l u - u_h^l|||_{\Sigma,k})\|u - u_h\|_{L^2(\Sigma)}. \tag{3.78}$$

By summing up the bounds (**??**) and (**??**) we arrive at the inequality

$$\|u - u_h^l\|_\Sigma \leqslant C\Big((hk)^2 + hk(|||\pi_h^l u - u_h^l|||_s + |||\pi_h u^e - u_h|||_j) + h^2 k|||\pi_h^l u - u_h^l|||_{\Sigma,k}\Big).$$

Using the result of Lemma **??** the conclusion follows. □

We now use the above lemmas for the fully discrete formulation to prove our main result, an a priori error estimate in the $||| \cdot |||_{\Sigma,k}$-norm. This result may then be used to prove stability of the discrete solution under the condition hk small, similarly as in Lemma **??**. We leave the details to the reader.

Theorem 3.2 *Let u be the solution of* (**??**) *satisfying the estimate* (**??**) *and let u_h be the solution of* (**??**). *Then for hk sufficiently small*

$$|||u - u_h^l|||_{\Sigma,k} \leqslant C_{f,\gamma} hk(1 + hk^2). \tag{3.79}$$

Proof. First we observe that by the triangle inequality there holds

$$|||u - u_h^l|||_{\Sigma,k} \leqslant |||\pi_h^l u - u_h^l|||_{\Sigma,k} + |||u - \pi_h^l u|||_{\Sigma,k}. \tag{3.80}$$

Since the bound was proven for the second term in the right hand side in Lemma **??** we only need to consider the first term. Once again we use the notation $\xi_h^l := \pi_h^l u - u_h^l$ and $\xi_h := \pi_h u^e - u_h$.

It follows by the definition of $A(\cdot,\cdot)$ and the assumption that $Re[\gamma] = 0$ that

$$|||\xi_h^l|||_{\Sigma,k}^2 = \mathrm{Re}(A^r(\xi_h^l, \xi_h^l)) + 2\|k(\xi_h^l)\|_\Sigma^2 \leqslant |A^r(\xi_h^l, \xi_h^l)| + 2\|k\xi_h^l\|_\Sigma^2. \tag{3.81}$$

The first term on the right hand side may be decomposed as before

$$A^r(\xi_h^l, \xi_h^l) = A^r(\pi_h^l u - u, \xi_h^l) + s^r(u, \xi_h^l) + l(\xi_h^l) - l_h(\xi_h) + A_h(u_h, \xi_h) - A^r(u_h^l, \xi_h^l) + \gamma_j j(u_h, \xi_h). \tag{3.82}$$

Using the result of (**??**) and the inequality (**??**), we have

$$A^r(\pi_h^l u - u, \xi_h^l) \leqslant C_{f,\gamma}^2 (hk)^2 + h^2 |||\xi_h^l|||_{\Sigma,k}^2 + |||\xi_h^l|||_{s^r}^2 + |||\xi_h|||_j^2. \tag{3.83}$$

Recalling the bound (**??**) we also have

$$l(\xi_h^l) - l_h(\xi_h) \leqslant C_{f,\gamma}^2 h^2 + h^2 |||\xi_h^l|||_{\Sigma,k}^2 + |||\xi_h|||_{s^r}^2. \tag{3.84}$$

Finally, using the bound (**??**) and after adding and subtracting $\pi_h^l u$ and applying the triangular inequality and the arithmetic–geometric inequality we have

$$A_h(u_h,\xi_h) - A(u_h^l,\xi_h^l) \leqslant Ch^2 |||u_h^l|||_{\Sigma,k} |||\xi_h^l|||_{\Sigma,k} \leqslant Ch^2(|||\xi_h^l|||_{\Sigma,k}^2 + |||\pi_h^l u_h|||_{\Sigma,k}^2). \quad (3.85)$$

Applying the results of (**??**)–(**??**), (**??**), the convergence of the stabilizing terms (Lemma **??**) and the L^2-error estimate (Lemma **??**) in (**??**) we obtain

$$|||\xi_h^l|||_{\Sigma,k}^2 \leqslant C_f(1+h^2k^4)(hk)^2 + Ch^2(1+k^2)|||\xi_h^l|||_{\Sigma,k}^2. \quad (3.86)$$

Since hk is assumed to be small, so that $Ch^2(1+k^2) < 1$, the last term in the right hand side can be absorbed in the left hand side and the proof is complete. □

Remark 3.2 *Similar estimates as those obtained in Theorems* **??** *and* **??** *can be obtained using the standard approach due to Schatz [**?**], under the condition hk^2 sufficiently small where the surface curvature also comes into play in the bound. So the above analysis does not show any reduction of pollution through stabilization, contrary to the one dimensional situation studied in [**?**]. Nevertheless we show in the numerical section that the method has superior performance compared to the unstabilized version for problems close to resonance. This will be illustrated numerically below in Section* **??**.

4 Numerical Examples

In the numerical examples below, the L^2 errors on the exact surface are approximated by the corresponding expression on the discrete surface,

$$\|u - u_h^l\|_\Sigma \approx \|u^e - u_h\|_{\Sigma_h}. \quad (4.1)$$

4.1 Varying Wave Number

We consider a sphere with radius $r = 1/2$ and the following stabilization parameters: $\gamma_s = i$, $\gamma_j = 10^{-3} i$, with i the imaginary unit. We use a fabricated solution

$$u = (x-1/2)(y-1/2)(z-1/2) \quad (4.2)$$

and construct the right-hand side accordingly. In Fig. **??** we show a typical discretization and corresponding approximate solution. In Fig. **??** we show the convergence patterns for different wave numbers and note that the rate is unaffected though the error constant increases with increasing wave number.

4.2 Varying Geometry

In this example, we consider the spheroid with one main axis having length $\mathrm{R}_{\max} = 1/2$ constant and the other with length $\mathrm{R}_{\min}$ varying. The data are the same as in the previous example but with constant wave number $k^2 = 1$. In Fig. **??** we show two different spheroids and in Fig. **??** we show the convergence which is optimal independent of geometry. Finally, in Fig. **??**, we consider a more demanding geometry, defined as the zero isoline of

$$\phi = (x^2 + y^2 - 4)^2 + (z^2 - 2)^2 + (y^2 + z^2 - 4)^2 + (x^2 - 1)^2 + (z^2 + x^2 - 4)^2 + (y^2 - 1)^2 - 15$$

and in Fig. **??** the corresponding observed convergence using the same parameters as for the spheroids. Similarly as in the previous example we here observe that the rate is unaffected by the geometry.

4.3 Stability Close to Eigenvalues

To illustrate the enhanced stability of the stabilized method, we consider the unit sphere (of radius 1). On this sphere, the non–zero eigenvalues of the Laplace–Beltrami operator can be analytically computed as $\lambda = m(m+1)$, $m = 1, 2, \ldots$ [**?**]. We consider again the exact solution (**??**) and compute the L^2 error on a fixed mesh under varying k^2 close to the lowest eigenvalue. In Fig. **??** we show how the error behaves using the same stabilization parameters as above. In Fig. **??** we give a close-up of the error closer to the eigenvalue, and in Fig. **??** we give the corresponding errors without stabilization. Note that further closeups would result in further increases of the error for the unstabilized approximation. With stabilization, the L^2 error increases but remains bounded as we pass the eigenvalue, unlike the case where no stabilization is added. Note that resonance occurs, in the unstabilized method, for a k^2–value slightly higher than $k^2 = 2$, which is to be expected in a conforming Galerkin finite element method (cf., e.g, [**?**]).

4.4 Varying Stability Parameters

Here we illustrate the effect of varying the stability parameters. We again consider the problem in Section **??** with $k = 8.3$ (close to a discrete eigenvalue). On a fixed mesh (meshsize twice that of Fig. **??**) we let keep one of the γ_j and γ_s zero and let the other vary. The results are shown in Fig. **??**. We note that γ_s has a distinct optimum value close to 1 and leads to rapidly increasing error when chosen too large, whereas γ_j reaches a plateau after passing 1 and no detrimental effect can be seen by increasing it further.

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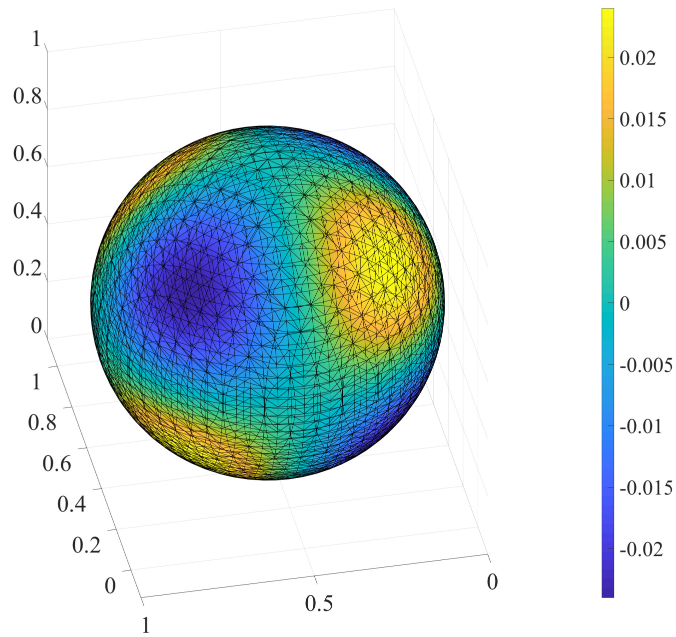

Figure 2: A discretization of the sphere with corresponding discrete solution.

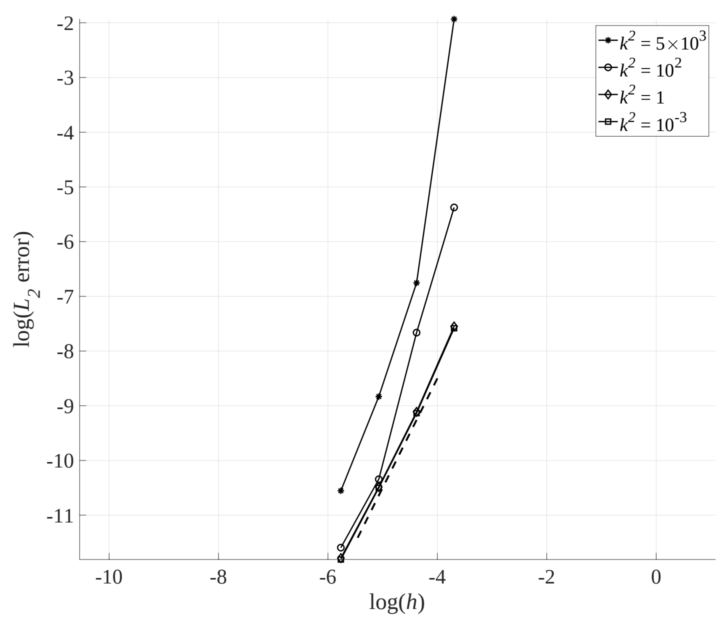

Figure 3: Convergence for different wave numbers. Dotted line has inclination 2:1.

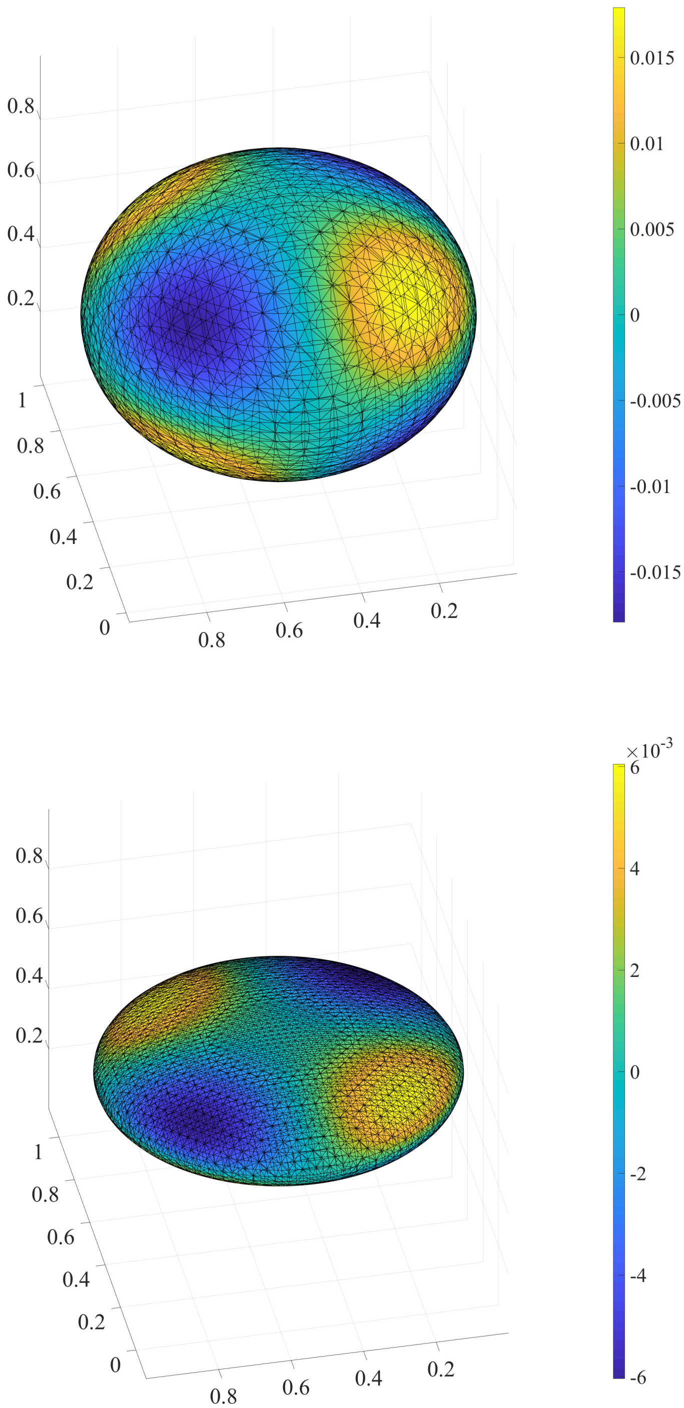

Figure 4: Discretization of spheroids with corresponding discrete solutions.

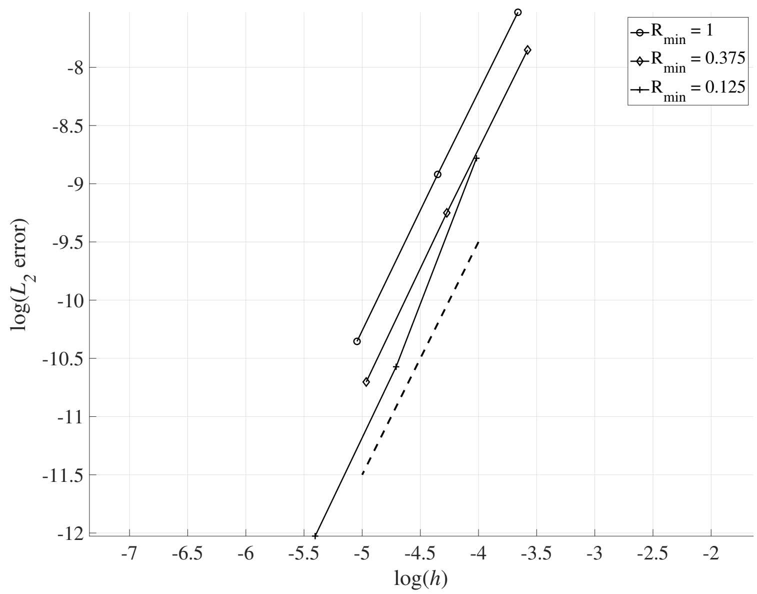

Figure 5: Convergence for different spheroid geometries. Dotted line has inclination 2:1.

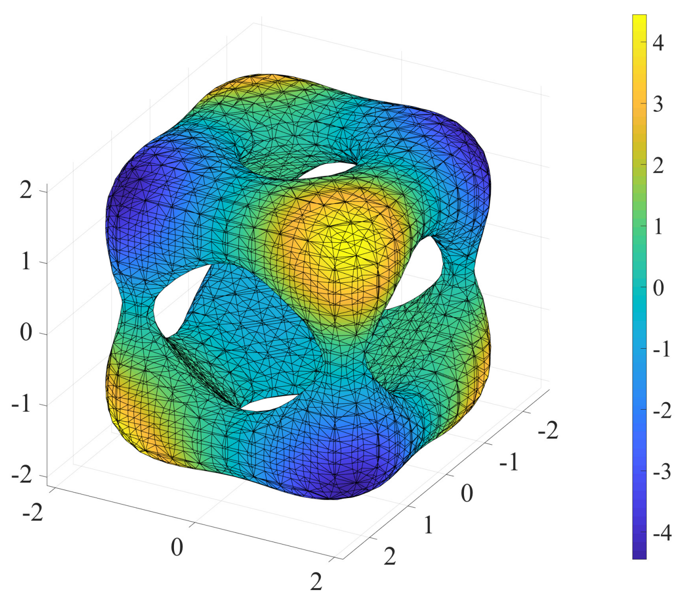

Figure 6: Discretization of a more complex geometry with corresponding discrete solution.

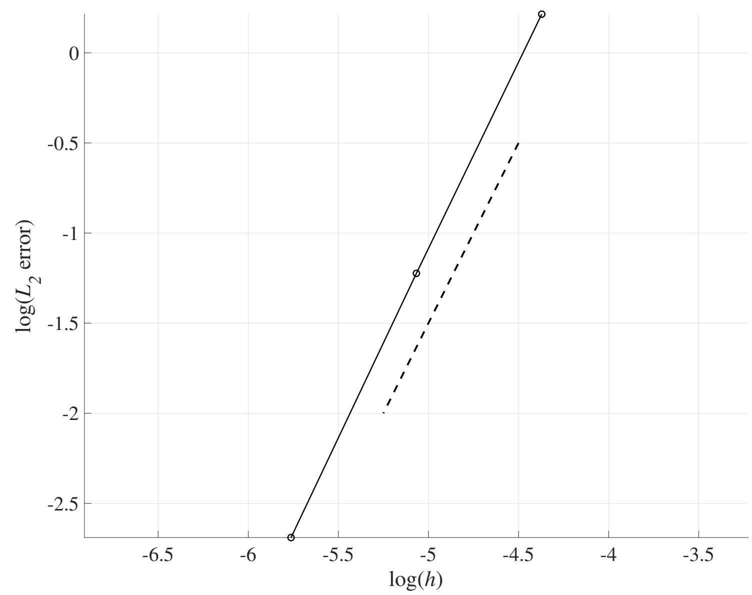

Figure 7: Convergence for the geometry of Fig. ??. Dotted line has inclination 2:1.

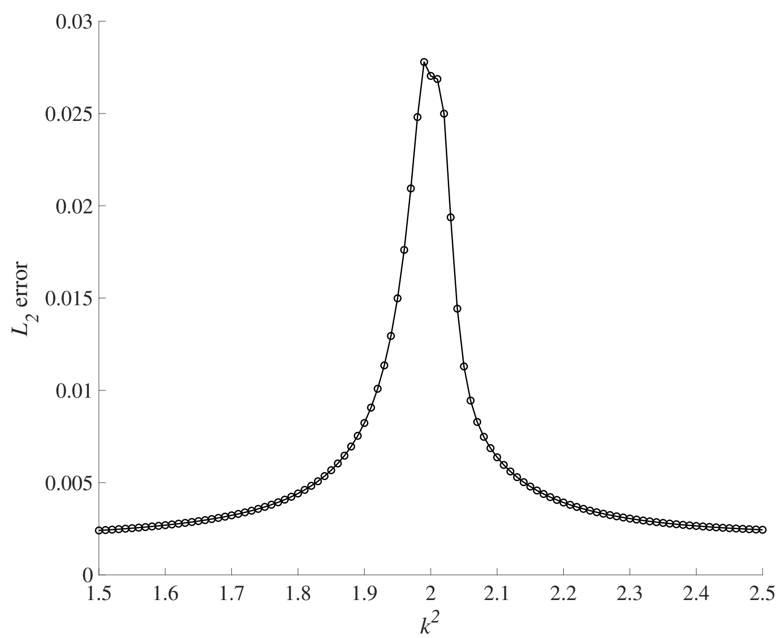

Figure 8: Error close to the lowest eigenvalue of the Laplace–Beltrami operator.

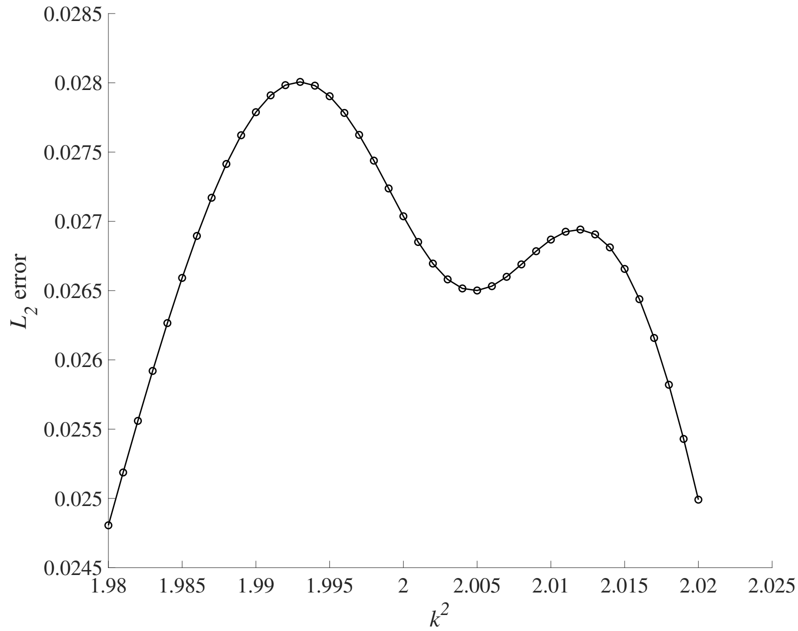

Figure 9: Close-up of the error, with stabilization.

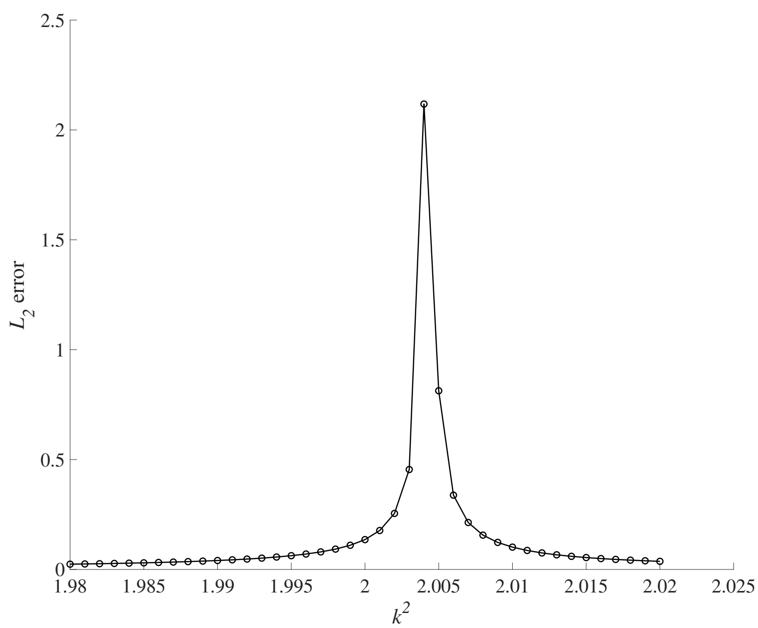

Figure 10: Close-up of the error, without stabilization.

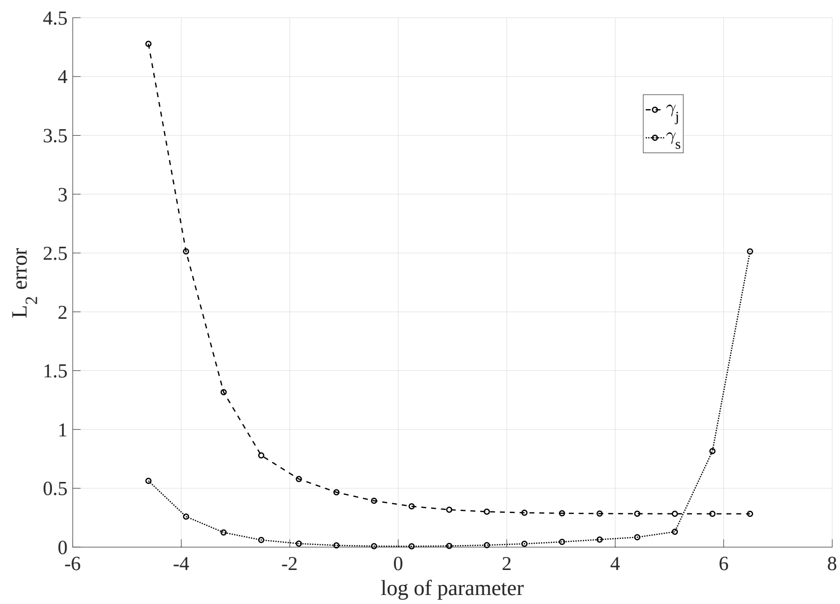

Figure 11: Effect on L^2 errors when varying the stability parameters.