Metrika manuscript No.
(will be inserted by the editor)

Integral Transform Methods in Goodness-of-Fit Testing, I: The Gamma Distributions

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Received: 30 October 2018 /Published online: 29 October 2019

Abstract We apply the method of Hankel transforms to develop goodness-of-fit tests for gamma distributions with given shape parameters and unknown rate parameters. We derive the limiting null distribution of the test statistic as an integrated squared Gaussian process, obtain the corresponding covariance operator and oscillation properties of its eigenfunctions, show that the eigenvalues of the operator satisfy an interlacing property, and make applications to two data sets. We prove consistency of the test, provide numerical power comparisons with alternative tests, study the test statistic under several contiguous alternatives, and obtain the asymptotic distribution of the test statistic for gamma alternatives with varying rate or shape parameters and for certain contaminated gamma models. We investigate the approximate Bahadur slope of the test statistic under local alternatives, and we establish the validity of the Wieand condition under which approaches through the approximate Bahadur and the Pitman efficiencies are in accord.

Keywords Bahadur slope · contaminated model · contiguous alternative · Gaussian process · generalized Laguerre polynomial · goodness-of-fit testing · Hankel transform · Hilbert-Schmidt operator · Lipschitz continuity · modified Bessel function · Pitman efficiency.

Mathematics Subject Classification (2010) Primary 33C10 · 62G10; Secondary 62G20 · 62H15

1 Introduction

The topic of goodness-of-fit testing has been intensely studied recently. Consequently, there exists a comprehensive body of results developed, by Henze and other authors, using test statistics based on integral transforms of Fourier, Laplace, Mellin, and related types, and making astute use of related differential equations and distributional characterizations. The resulting test statistics have been shown to be superior in various ways to classical goodness-of-fit statistics, notably in comparisons of power, consistency, and in their behavior with respect to contiguous alternatives.

On reviewing the literature on goodness-of-fit tests we were motivated to develop such tests, for multivariate exponential families, based on integral transforms, and the first step in such a program is to derive such results for the classical gamma distributions. In this paper, we apply Hankel transform methods to develop goodness-of-fit tests for gamma distributions with given shape parameter α and unknown rate parameter. We remark that we were particularly fortuitous

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This paper is dedicated to Professor Norbert Henze, on the occasion of his 67th birthday to have as a constant guide in our investigations the results of Baringhaus and Taherizadeh (2010) and Taherizadeh (2009) for the exponential distributions.

The gamma distributions with known shape parameters arise in queueing theory (Allen, 1990), ion channel activation (Kass, *et al.*, 2014), the analysis of engineering equipment breakdowns (Czaplicki, 2014; Sturgul, 2015), the calculation of insurance premiums for maritime commerce (Postan and Poizner, 2013), and other areas, and goodness-of-fit tests for these distributions date back to Pettitt (1978). For the case of unknown shape parameter, goodness-of-fit tests based on empirical distribution functions were provided by D'Agostino and Stephens (1986) and numerous other authors; in particular, Henze, Meintanis, and Ebner (2012) developed a test based on the empirical Laplace transform and provided an extensive review of the literature.

Let X be a positive random variable with probability density function (p.d.f.) $f(x)$; also, let J_ν be the Bessel function of the first kind of order ν, as defined in (2.1). For $\nu \ge -1/2$, the function

$$\mathcal{H}_{X,\nu}(t) = \Gamma(\nu+1) \int_0^\infty (tx)^{-\nu/2} \, J_\nu\big(2(tx)^{1/2}\big) \, f(x) \, \mathrm{d}x, \tag{1.1}$$

$t \ge 0$, is called the *Hankel transform of order* ν *of* X. For $X \sim Gamma(\alpha, 1)$, a gamma distribution with shape parameter α and scale parameter 1, we have $\mathcal{H}_{X,\alpha-1}(t) = e^{-t/\alpha}$.

Let $X_1, \dots, X_n$ be independent, identically distributed (i.i.d.), positive, continuous random variables with a distribution $\mathcal{P}$. We wish to test the null hypothesis, $H_0 : \mathcal{P} \in \{Gamma(\alpha, \lambda), \lambda > 0\}$ against the alternative, $H_1 : \mathcal{P} \notin \{Gamma(\alpha, \lambda), \lambda > 0\}$, where α is known. Since H_0 does not specify λ then $X_1, \dots, X_n$ cannot be used directly to conduct the test. Let $\overline{X}_n = n^{-1} \sum_{i=1}^n X_j$ be the sample mean and set $Y_j = X_j/\overline{X}_n$, $j = 1, \dots, n$; under H_0, the distribution of $Y_1, \dots, Y_n$ does not depend on λ, so we can base a test on them. Let P_0 denote the distribution function of the $Gamma(\alpha, 1)$ distribution. We define the *empirical Hankel transform of order* ν of $Y_1, \dots, Y_n$ as

$$\mathcal{H}_{n,\nu}(t) = \frac{\Gamma(\nu+1)}{n} \sum_{j=1}^n (tY_j)^{-\nu/2} J_\nu(2\sqrt{tY_j}), \tag{1.2}$$

$t \ge 0$, and then the statistic for testing H_0 against H_1 is

$$T^2_{n,\alpha-1} = n \int_0^\infty \big[\mathcal{H}_{n,\alpha-1}(t) - e^{-t/\alpha}\big]^2 \, \mathrm{d}P_0(t). \tag{1.3}$$

As the Hankel transform is one-to-one we will infer from large values of T_n^2 that $\mathcal{H}_{n,\alpha-1}(t)$ differs significantly from $e^{-t/\alpha}$, hence large values of T_n^2 provide strong evidence against H_0. Therefore, we will obtain the distribution of $T^2_{n,\alpha-1}$ and analyze its properties, e.g., consistency, behavior under contiguous alternatives, efficiency, and compare its power with alternative tests.

We now summarize our results. We give in Section 2 basic results on the Bessel and related special functions, and some properties and examples of Hankel transforms of some probability distributions. In Section 3, we state the limiting null distribution of the statistic T_n^2 as an integral of the square of a centered Gaussian process Z.

We present in Section 4 properties of $\mathcal{S}$, the covariance operator corresponding to Z, oscillation properties of the eigenfunctions of $\mathcal{S}$, and interlacing properties of the eigenvalues of $\mathcal{S}$. In Section 5, we make applications to two data sets, assert the consistency of the test, and provide numerical power comparisons with the Anderson-Darling and Cramér-von Mises statistics. In Section 6, we consider the test statistic under various contiguous alternatives to H_0. In particular, we state the asymptotic distribution of T_n^2 under gamma alternatives with varying rate or shape parameters and for a class of contaminated gamma models.

In Section 7, we present the Bahadur and Pitman efficiency properties of the statistic T_n^2. We investigate the approximate Bahadur slope of T_n^2 under certain local alternatives and establish the validity of the Wieand condition, under which the approaches through the approximate Bahadur efficiency and the Pitman efficiency are in accord. In Section 8, we describe some open problems and directions for future research, while Sections 9-11 are reserved for proofs or definitions.

2 Bessel Functions and Hankel Transforms

Throughout the paper, all needed results on the classical special functions can be found in the books by Erdélyi, *et al.* (1953) and Olver, *et al.* (2010), and we conform to their notation. Thus,

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, \mathrm{d}x,$$

$\mathrm{Re}(\alpha) > 0$, is the gamma function, and for $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}_0$, the set of nonnegative integers, we will make frequent use of the *rising factorial*, $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$.

We write $X \sim Gamma(\alpha, \lambda)$ whenever a random variable X is gamma-distributed with shape parameter $\alpha > 0$, rate parameter $\lambda > 0$, and p.d.f. $f(x) = \lambda^\alpha x^{\alpha-1} e^{-\lambda x}/\Gamma(\alpha)$, $x > 0$.

For $\nu \in \mathbb{R}$, $-\nu \notin \mathbb{N}$, the *Bessel function of the first kind of order* ν is

$$J_\nu(z) = \sum_{j=0}^\infty \frac{(-1)^j}{j!\,\Gamma(\nu+1+j)}\,(z/2)^{2j+\nu}, \tag{2.1}$$

$z \in \mathbb{C}$; see Erdélyi, *et al.* (1953, Chapter 7). In particular, the series (2.1) is continuous, converges absolutely for all z, and converges uniformly on compact subsets of $\mathbb{C}$.

The *modified Bessel function of the first kind of order* ν is defined for $-\nu \notin \mathbb{N}$ and $x \in \mathbb{R}$ as

$$I_\nu(x) = \sum_{j=0}^\infty \frac{1}{j!\,\Gamma(\nu+1+j)}(x/2)^{2j+\nu}, \tag{2.2}$$

Let $a, b \in \mathbb{R}$, where $-b \notin \mathbb{N}_0$. The *confluent hypergeometric function* is defined as

$${}_1F_1(a;b;x) = \sum_{j=0}^\infty \frac{(a)_j}{(b)_j}\,\frac{x^j}{j!}. \tag{2.3}$$

$x \in \mathbb{R}$. We refer to Olver, *et al.* (2010, Chapter 13) for detailed accounts of this function. Especially, we will make repeated use of *Kummer's formula*:

$${}_1F_1(a;b;x) = e^x\,{}_1F_1(b-a;b;-x). \tag{2.4}$$

Let X be a positive random variable with probability density function $f(x)$ and Hankel transform $\mathcal{H}_{X,\nu}$, as defined in (1.1). Then, $\mathcal{H}_{X,\nu}$ satisfies the following properties:

Lemma 1 *For* $\nu \geq -1/2$,
(i) $|\mathcal{H}_{X,\nu}(t)| \leq 1$ *for all* $t \geq 0$.
(ii) $\mathcal{H}_{X,\nu}(0) = 1$.
(iii) $\mathcal{H}_{X,\nu}(t)$ *is a continuous function of* t.

Example 1 Let $X \sim Gamma(\alpha, \lambda)$, where $\alpha, \lambda > 0$. For $t \geq 0$, it follows from the definition (1.1) of the Hankel transform that

$$\mathcal{H}_{X,\nu}(t) = \frac{\Gamma(\nu+1)}{\Gamma(\alpha)}\lambda^\alpha \int_0^\infty (tx)^{-\nu/2}\,J_\nu(2\sqrt{tx})\,x^{\alpha-1}e^{-\lambda x}\,\mathrm{d}x.$$

Writing $(tx)^{-\nu/2}\,J_\nu(2\sqrt{tx})$ as a power series and integrating term-by-term, we obtain

$$\mathcal{H}_{X,\nu}(t) = {}_1F_1(\alpha; \nu+1; -t/\lambda). \tag{2.5}$$

For the case in which $\nu = \alpha - 1$, (2.5) reduces to $\mathcal{H}_{X,\nu}(t) = {}_1F_1(\alpha;\alpha;-t/\lambda) = e^{-t/\lambda}$, $t \geq 0$. In particular, if $\alpha = 1$, so that X has an exponential distribution with rate parameter λ, then $\mathcal{H}_{X,0}(t) = e^{-t/\lambda}$, $t \geq 0$, as shown by Baringhaus and Taherizadeh (2010, Example 2.1).

Example 2 Let $Z \sim Gamma(\alpha, 1)$ independently of a positive random variable X. Then,

$$\mathcal{H}_{XZ,\nu}(t) = E_X\big[{}_1F_1(\alpha; \nu+1; -tX)\big],$$

$t \ge 0$. To prove this result, we again apply (1.1), and the independence of X and Z, obtaining

$$\mathcal{H}_{XZ,\nu}(t) = E_X\, E_Z\big[\Gamma(\nu+1)(tXZ)^{-\nu/2} J_\nu\big(2(tXZ)^{1/2}\big)\big].$$

Applying Example 1 to calculate the expectation with respect to Z, we obtain

$$\mathcal{H}_{XZ,\nu}(t) = E_X\big[{}_1F_1(\alpha; \nu+1; -tX)\big].$$

In particular, if $\nu = \alpha - 1$ then $\mathcal{H}_{XZ,\nu}(t) = E_X\big[e^{-tX}\big]$, the Laplace transform of X, a result shown for $\nu = 0$ in Baringhaus and Taherizadeh (2010, Example 2.2).

The following example, which provides the Hankel transform of a function related to the gamma density, will be needed repeatedly in the sequel.

Example 3 Suppose that $X \sim Gamma(\alpha, 1)$. Then, for $t \ge 0$,

$$E\big[(tX/\alpha)^{1-(\alpha/2)}\, J_\alpha\big(2(tX/\alpha)^{1/2}\big)\big] = \frac{1}{\Gamma(\alpha+1)}\, t\, e^{-t/\alpha}. \tag{2.6}$$

Here again, we write $(tX/\alpha)^{1-(\alpha/2)}\, J_\alpha\big(2(tX/\alpha)^{1/2}\big)$ as a power series in tX/α, integrate term-by-term, and simplify the resulting series to obtain (2.6).

The next result constitutes a characterization of the gamma distributions using Hankel transforms of arbitrary order ν, where $\nu \ge -1/2$. The result allows extension to the gamma case the results of Baringhaus and Taherizadeh (2013) on a supremum norm test statistic.

Theorem 1 *Let X be a positive random variable with Hankel transform $\mathcal{H}_{X,\nu}$. If there exist $\epsilon > 0$ and $\alpha > 0$ such that $\mathcal{H}_{X,\nu}(t) = {}_1F_1(\alpha; \nu+1; -t)$ for all $t \in [0, \epsilon]$, then $X \sim Gamma(\alpha, 1)$.*

We refer to Hadjicosta (2019) for three proofs of this result.

3 The Distribution of the Test Statistic

Let $X_1, \dots, X_n$ be i.i.d., positive, continuous random variables with distribution $\mathcal{P}$. We wish to test the null hypothesis, $H_0 : \mathcal{P} \in \{Gamma(\alpha, \lambda), \lambda > 0\}$ against the alternative hypothesis, $H_1 : \mathcal{P} \notin \{Gamma(\alpha, \lambda), \lambda > 0\}$, where α is known. Using the empirical Hankel transform $\mathcal{H}_{n,\nu}$ given in (1.2), we define the test statistic

$$T^2_{n,\nu} = n \int_0^\infty \big[\mathcal{H}_{n,\nu}(t) - {}_1F_1(\alpha; \nu+1; -t/\alpha)\big]^2\, \mathrm{d}P_0(t). \tag{3.1}$$

Under H_0, $E(X_1) = \alpha/\lambda$ and, for large n, $Y_j = X_j/\overline{X}_n \simeq \lambda X_j/\alpha$, almost surely. By the Continuous Mapping Theorem (Billingsley, 1968, p. 31), for each $t \ge 0$ and for sufficiently large n, the sequence of random variables $(tY_j)^{-\nu/2} J_\nu(2\sqrt{tY_j})$, $j = 1, \dots, n$, approximates the i.i.d. sequence $(\lambda t X_j/\alpha)^{-\nu/2} J_\nu(2(\lambda t X_j/\alpha)^{1/2})$, $j = 1, \dots, n$. Applying to (1.2) the Strong Law of Large Numbers we obtain, for large n, $\mathcal{H}_{n,\nu}(t) \simeq \mathcal{H}_{X_1,\nu}(\lambda t/\alpha)$, almost surely. By Example 1 and the Hankel Uniqueness Theorem 12, $\mathcal{H}_{X_1,\nu}(\lambda t/\alpha) = {}_1F_1(\alpha; \nu+1; -t/\alpha)$, $t \ge 0$, if and only if H_0 is valid. Therefore, large values of $T^2_{n,\nu}$ provide strong evidence against H_0.

We also remark that since the family of gamma distributions is scale-invariant then the test statistic, as a function of $X_1, \dots, X_n$, should satisfy the same property. Since $Y_1, \dots, Y_n$ clearly are scale-invariant in $X_1, \dots, X_n$ then the same holds for $T^2_{n,\nu}$.

Henceforth, we set $\nu = \alpha - 1$; since $\nu \ge -1/2$ then $\alpha \ge 1/2$. We also denote $T^2_{n,\alpha-1}$ and $\mathcal{H}_{n,\alpha-1}$ by T^2_n and $\mathcal{H}_n$, respectively. By Kummer's formula (2.4), the statistic (3.1) reduces to (1.3).

We now evaluate the test statistic T^2_n for a given random sample.

Proposition 1 *The test statistic (1.3) is a V-statistic of order 2. Specifically,*

$$T_n^2 = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} h(Y_i, Y_j)$$

where, for $x, y \geq 0$,

$$\begin{aligned} h(x,y) &= \Gamma(\alpha)\,(xy)^{(1-\alpha)/2}\,\exp(-x-y)\,I_{\alpha-1}\big(2(xy)^{1/2}\big) \\ &\quad - \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha}\Big[\exp\Big(-\frac{\alpha x}{\alpha+1}\Big) + \exp\Big(-\frac{\alpha y}{\alpha+1}\Big)\Big] + \Big(\frac{\alpha}{\alpha+2}\Big)^{\alpha}. \end{aligned} \tag{3.2}$$

Denote by $L^2 = L^2(P_0)$ the space of (equivalence classes of) Borel measurable functions $f : [0,\infty) \to \mathbb{C}$ that are square-integrable with respect to P_0, i.e. $\int_0^\infty |f(t)|^2 \, \mathrm{d}P_0(t) < \infty$. The space L^2 is a separable Hilbert space when equipped with the inner product

$$\langle f, g\rangle_{L^2} = \int_0^\infty f(t)\,\overline{g(t)}\,\mathrm{d}P_0(t),$$

and the corresponding norm, $\|f\|_{L^2} = \langle f, f\rangle_{L^2}^{1/2}$, $f, g \in L^2$. Moreover, it is well-known that the normalized Laguerre polynomials $\{\mathcal{L}_n^{(\alpha-1)} : n \in \mathbb{N}_0\}$, defined in Appendix 9, form an orthonormal basis, i.e. a complete orthonormal system, for L^2; see Szegö (1967, Chapter 5.7).

Define the stochastic process

$$Z_n(t) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\Big[\Gamma(\alpha)(tY_j)^{(1-\alpha)/2} J_{\alpha-1}(2\sqrt{tY_j}) - e^{-t/\alpha}\Big], \tag{3.3}$$

$t \geq 0$. We will view $Z_n := \{Z_n(t), t \geq 0\}$ as a random element in L^2 since, as we will observe in Lemma 2 below, its sample paths are in L^2. The proof of the following result follows directly from the definition (1.3) of the statistic T_n^2 and the observation that $n^{1/2}\big[\mathcal{H}_n(t) - e^{-t/\alpha}\big] \equiv Z_n(t)$.

Lemma 2 *The test statistic (1.3) can be written as*

$$T_n^2 = \int_0^\infty \big(Z_n(t)\big)^2 \,\mathrm{d}P_0(t) = \|Z_n\|_{L^2}^2.$$

In particular, $\|Z_n\|_{L^2}^2 < \infty$.

It is well-known that under H_0, $(Y_1, \ldots, Y_n)$ has a Dirichlet distribution which does not depend on λ. Therefore, without loss of generality, we will set $\lambda = 1$ in deriving the null distribution of T_n^2.

Theorem 2 *Let $X_1, X_2, \ldots$ be i.i.d. Gamma$(\alpha, 1)$ random variables, where $\alpha \geq 1/2$, and let $Z_n := \{Z_n(t), t \geq 0\}$ be the stochastic process defined in (3.3). Then there exists a centered Gaussian process $Z := \{Z(t), t \geq 0\}$, with sample paths in L^2 and with covariance function,*

$$K(s,t) = e^{-(s+t)/\alpha}\Big[\Gamma(\alpha)(st/\alpha^2)^{(1-\alpha)/2} I_{\alpha-1}\big(2\sqrt{st}/\alpha\big) - \alpha^{-3}st - 1\Big], \tag{3.4}$$

$s, t \geq 0$, such that $Z_n \xrightarrow{d} Z$ in L^2 as $n \to \infty$. Moreover,

$$T_n^2 \xrightarrow{d} \int_0^\infty [Z(t)]^2 \,\mathrm{d}P_0(t).$$

Remark 1 The proof of Theorem 2 is by an approach similar to that of Baringhaus and Taherizadeh (2010) and is given in Section 10. As $Y_1, \ldots, Y_n$ are not independent, we cannot directly apply a Central Limit Theorem to deduce that $Z_n \to Z$. Instead, we apply a standard method of constructing auxiliary processes, $Z_{n,1}$, $Z_{n,2}$, and $Z_{n,3}$, and then decomposing $Z_n - Z$ into a sum of four parts, viz.,

$$Z_n - Z = (Z_n - Z_{n,1}) + (Z_{n,1} - Z_{n,2}) + (Z_{n,2} - Z_{n,3}) + (Z_{n,3} - Z).$$

Next, we show that $Z_n - Z_{n,1}$, $Z_{n,1} - Z_{n,2}$, and $Z_{n,2} - Z_{n,3}$ each converge to 0 in probability, in L^2; then we apply a Central Limit Theorem to deduce that $Z_{n,3} \xrightarrow{d} Z$ in L^2, and so we obtain $Z_n \xrightarrow{d} Z$ in L^2. Finally, we apply the Continuous Mapping Theorem to conclude that $\|Z_n\|_{L^2}^2 \xrightarrow{d} \|Z\|_{L^2}^2$.

4 Eigenvalues and Eigenfunctions of the Covariance Operator

The covariance operator $\mathcal{S} : L^2 \to L^2$ of the random element Z is defined for $s \ge 0$ and $f \in L^2$ by

$$\mathcal{S}f(s) = \int_0^\infty K(s,t) f(t) \, \mathrm{d}P_0(t),$$

where $K(s,t)$ is the covariance function defined in equation (3.4). Let $\{\delta_k : k \ge 1\}$ be the positive eigenvalues, listed in non-increasing order, of $\mathcal{S}$; also, let $\{\chi^2_{1k} : k \ge 1\}$ be i.i.d. χ^2_1 random variables. It follows from the Karhunen-Loève expansion of the Gaussian process $Z(t)$ that the integrated squared process, $\int_0^\infty Z^2(t) \, \mathrm{d}P_0(t)$, has the same distribution as $\sum_{k=1}^\infty \delta_k \chi^2_{1k}$; see Le Maître and Knio (2010, Chapter 2). Therefore, under H_0, $T_n^2 \to \sum_{k=1}^\infty \delta_k \chi^2_{1k}$.

For $s, t \ge 0$, let

$$K_0(s,t) = e^{-(s+t)/\alpha} \Gamma(\alpha) (st/\alpha^2)^{(1-\alpha)/2} I_{\alpha-1}\big(2\sqrt{st}/\alpha\big), \tag{4.1}$$

the first term in the covariance function defined in equation (3.4); by (10.12),

$$\begin{aligned} K_0(s,t) = \int_0^\infty & \Gamma(\alpha)(sx/\alpha)^{(1-\alpha)/2} J_{\alpha-1}(2(sx/\alpha)^{1/2}) \\ & \times \Gamma(\alpha)(tx/\alpha)^{(1-\alpha)/2} J_{\alpha-1}(2(tx/\alpha)^{1/2}) \, \mathrm{d}P_0(x). \end{aligned}$$

We will find first the eigenvalues and eigenfunctions of the integral operator $\mathcal{S}_0 : L^2 \to L^2$, defined for $s \ge 0$ and f in L^2 by

$$\mathcal{S}_0 f(s) = \int_0^\infty K_0(s,t) f(t) \, \mathrm{d}P_0(t).$$

Before presenting the results on the eigenvalues and eigenfunctions of $\mathcal{S}_0$, we state for the sake of completeness some preliminary definitions pertaining to (linear) operators on L^2. Note that these definitions are provided by Sunder (2015) or Young (1998).

An operator $\mathcal{T} : L^2 \to L^2$ is called *symmetric (self-adjoint)* if, for all $f, g \in L^2$, $\langle \mathcal{T}f, g\rangle_{L^2} = \langle f, \mathcal{T}g\rangle_{L^2}$. A symmetric operator $\mathcal{T}$ is called *positive* if $\langle \mathcal{T}f, f\rangle_{L^2} \ge 0$ for all $f \in L^2$. An operator $\mathcal{T}$ is called *compact* if for every bounded sequence $\{f_k : k \in \mathbb{N}\}$ in L^2, the sequence $\{\mathcal{T}f_k : k \in \mathbb{N}\}$ has a convergent subsequence in L^2. The set of eigenvalues of a compact operator is countable.

An operator $\mathcal{T}$ is *Hilbert-Schmidt* if for every orthonormal basis $\{f_k : k \in \mathbb{N}\}$ in L^2, the series $\sum_{k=1}^\infty \|\mathcal{T}f_k\|^2_{L^2}$ converges. Each Hilbert-Schmidt operator is compact (Young, 1998, p. 93).

An operator $\mathcal{T}$ is *of trace class* if for every orthonormal basis $\{f_k : k \in \mathbb{N}\}$ in L^2, the series $\sum_{k=1}^\infty \|\mathcal{T}f_k\|_{L^2}$ converges. An operator $\mathcal{T}$ is trace-class if and only if it is a product of two Hilbert-Schmidt operators (Sunder, 2015, p. 74). Further, trace-class operators are Hilbert-Schmidt.

Recall that $\alpha \ge 1/2$. Throughout the remainder of the paper, we use the notation

$$\beta = \Big(\frac{\alpha+4}{\alpha}\Big)^{1/2} \quad \text{and} \quad b_\alpha = \big(1 + \tfrac{1}{2}\alpha(1-\beta)\big)^{1/2}. \tag{4.2}$$

We also set

$$\rho_k = \alpha^\alpha b_\alpha^{4k+2\alpha}, \tag{4.3}$$

$k \in \mathbb{N}_0$, and for $s \ge 0$,

$$\mathfrak{L}_k^{(\alpha-1)}(s) = \beta^{\alpha/2} \exp((1-\beta)s/2) \mathcal{L}_k^{(\alpha-1)}(\beta s), \tag{4.4}$$

where $\mathcal{L}_k^{(\alpha-1)}(s)$ is the generalized Laguerre polynomial defined in (9.11).

Theorem 3 *The set $\{(\rho_k, \mathfrak{L}_k^{(\alpha-1)}) : k \in \mathbb{N}_0\}$ is a complete enumeration of the eigenvalues and eigenfunctions, respectively, of $\mathcal{S}_0$, and the eigenfunctions $\{\mathfrak{L}_k^{(\alpha-1)} : k \in \mathbb{N}_0\}$ form an orthonormal basis in L^2. Moreover, $\mathcal{S}_0$ is positive and of trace-class.*

For the proof of this result we refer to Hadjicosta (2019) or Hadjicosta and Richards (2018).

Theorem 4 *Let $\mathcal{S} : L^2 \to L^2$ be the covariance operator of the random element Z defined as*

$$\mathcal{S}f(s) = \int_0^\infty K(s,t)f(t)\,\mathrm{d}P_0(t),$$

for all $s \geq 0$ and for all functions f in L^2, where $K(s,t)$ is the covariance function defined in equation (3.4). Then, $\mathcal{S}$ is positive and of trace-class.

The proof of this result is similar to the proof of Theorem 3, and the complete details are provided by Hadjicosta (2019).

Recall that a non-trivial function $f \in L^2$ is an *eigenfunction* of $\mathcal{S}$ if there exists an *eigenvalue* $\delta \in \mathbb{C}$ such that $\mathcal{S}f = \delta f$. As $\mathcal{S}$ is self-adjoint and positive, its eigenvalues are real and nonnegative. In the next result, whose proof is given in Section 11, we find the positive eigenvalues and corresponding eigenfunctions of $\mathcal{S}$, and we will also show that 0 is not an eigenvalue of $\mathcal{S}$.

Theorem 5 *For $\delta \in \mathbb{R}$, $\delta \neq \rho_k$ for any $k \in \mathbb{N}$, define the functions*

$$A(\delta) = 1 - \beta^\alpha \sum_{k=0}^\infty \frac{(\alpha)_k}{k!(\rho_k - \delta)}\rho_k^2,$$
$$B(\delta) = 1 - \alpha\beta^\alpha \sum_{k=0}^\infty \frac{(\alpha)_k}{k!(\rho_k - \delta)}\rho_k^2(b_\alpha^2 - k\beta)^2,$$

and

$$D(\delta) = \alpha^2\beta^\alpha \sum_{k=0}^\infty \frac{(\alpha)_k}{k!(\rho_k - \delta)}\rho_k^2(b_\alpha^2 - k\beta).$$

Then the positive eigenvalues of $\mathcal{S}$ are the positive roots of the function $G(\delta) := \alpha^3 A(\delta)B(\delta) - D^2(\delta)$. Also, the eigenfunction corresponding to an eigenvalue δ has the Fourier-Laguerre expansion

$$\beta^{\alpha/2} \sum_{k=0}^\infty \frac{\rho_k}{\rho_k - \delta}\Big(\frac{(\alpha)_k}{k!}\Big)^{1/2}\big(c_1 + c_2\alpha^{-1}(b_\alpha^2 - k\beta)\big)\mathfrak{L}_k^{(\alpha-1)},$$

where c_1, c_2 are not both equal to 0, $\alpha^3 c_1 A(\delta) = c_2 D(\delta)$, and $c_2 B(\delta) = c_1 D(\delta)$.

In the previous result, we assumed that $\delta \notin \{\rho_k : k \in \mathbb{N}_0\}$. As stated in the following conjecture, we believe that this assumption is valid for all α.

Conjecture 1 *For δ an eigenvalue of the operator $\mathcal{S}$, there is no $l \in \mathbb{N}_0$ such that $\delta = \rho_l$.*

Conjecture 2 *There is no $l \in \mathbb{N}_0$ such that*

$$\alpha\beta^{\alpha+2} \sum_{\substack{k=0 \\ k\neq l}}^\infty \frac{(\alpha)_k}{k!}\frac{\rho_k^2}{\rho_k - \rho_l}(l-k)^2 = 1 + \alpha(b_\alpha^2 - l\beta)^2. \tag{4.5}$$

We will show in Appendix 11 that Conjecture 2 implies Conjecture 1.

Remark 2 Since $b_\alpha < 1$ then $\rho_k < \rho_0$ for all $k \geq 1$. Therefore, if $l = 0$ then each term in the sum on the left-hand side of (4.5) is negative, hence the sum itself is negative. On the other hand, the right-hand side clearly is positive. Therefore, the conjecture is valid if $l = 0$.

Conjecture 1 was proved by Taherizadeh (2009) for $\alpha = 1$ and by Hadjicosta (2019) for $\alpha = 2$. In both cases, the left-hand side of (4.5) was shown to exceed the right-hand side, so we conjecture that the same holds for all α. We have found that the method of proof for $\alpha = 1, 2$ extends to all integer $\alpha \leq 10$, however the method is inapplicable for integer $\alpha \geq 11$ or for non-integral α.

A difficulty of the eigenvalues δ_k is that they have no closed form expression; hence there is no simple formula for m, the number of terms in the truncated series $\sum_{k=1}^{m} \delta_k \chi_{1k}^2$ that should be used in practice to approximate the asymptotic distribution, $\sum_{k=1}^{\infty} \delta_k \chi_{1k}^2$, of the test statistic T_n^2.

For $\alpha = 1$, Baringhaus and Taherizadeh (2010) calculated several δ_k numerically and found that the truncated sum $\sum_{k=1}^{10} \delta_k$ closely approximates the exact value of $Tr(\mathcal{S})$; hence, the distribution of the truncated sum, $\sum_{k=1}^{10} \delta_k \chi_{1k}^2$ is a good approximation to the asymptotic distribution, $\sum_{k=1}^{\infty} \delta_k \chi_{1k}^2$, of T_n^2. This approach is feasible since, as $\mathcal{S}_0$ is of trace-class then by Brislawn (1991, p. 237, Corollary 3.2), $Tr(\mathcal{S}_0)$ can be calculated by integrating the kernel K_0 or by evaluating the sum of all eigenvalues ρ_k:

$$\int_0^{\infty} K_0(s,s)\,\mathrm{d}P_0(s) = Tr(\mathcal{S}_0) = \sum_{k=0}^{\infty} \rho_k = \alpha^{\alpha} b_{\alpha}^{2\alpha}(1-b_{\alpha}^4)^{-1}. \tag{4.6}$$

Since $\mathcal{S}$ also is of trace-class then, using (3.4), we obtain

$$\sum_{k=1}^{\infty} \delta_k = Tr(\mathcal{S}) = \int_0^{\infty} K(s,s)\,\mathrm{d}P_0(s) = \alpha^{\alpha}\Big[\frac{b_{\alpha}^{2\alpha}}{1-b_{\alpha}^4} - \frac{1}{(\alpha+2)^{\alpha}}\Big(1+\frac{(\alpha+1)}{(\alpha+2)^2}\Big)\Big]. \tag{4.7}$$

To determine for general α the number of terms in the truncated series $\sum_{k=1}^{m} \delta_k \chi_{1k}^2$ that should be used in practice to approximate the asymptotic distribution of T_n^2, we derive bounds for the δ_k in terms of the ρ_k and then obtain a general formula for m as a function of α. In this regard, we are reminded of the concept of a "scree plot" in principal component analysis; see Johnson and Wichern (1998, p. 441), so we refer to the ratio $(\sum_{k=1}^{m} \delta_k)/Tr(\mathcal{S})$ as the *mth scree ratio* for T_n^2.

Since $\mathcal{S}$ is compact and positive then the set of all its eigenvalues is countable and contains only nonnegative values (Young, 1998, p. 98, Theorem 8.12). To prove that the eigenvalues are positive and also are simple, i.e., of multiplicity 1, we will apply the theory of total positivity; see Karlin (1964). In what follows, we denote by $\det(a_{ij})$ the $r \times r$ determinant with (i,j)th entry a_{ij}.

Proposition 2 *The eigenvalues $\{\delta_k : k \geq 1\}$ of $\mathcal{S}$ and the eigenvalues $\{\rho_k : k \geq 0\}$ of $\mathcal{S}_0$ are positive and simple. In particular, $\mathcal{S}$ and $\mathcal{S}_0$ are injective. Further, the corresponding eigenfunctions $\{\phi_k : k \geq 1\}$ of $\mathcal{S}$ satisfy the oscillation property,*

$$(-1)^{r(r-1)/2} \det\big(\phi_i(s_j)\big) \geq 0 \tag{4.8}$$

for all $r \geq 1$ and $0 \leq s_1 < \cdots < s_r < \infty$, and the same property holds for the eigenfunctions $\{\mathfrak{L}_k : k \geq 0\}$ of $\mathcal{S}_0$.

We now state an interlacing property of the eigenvalues δ_k and ρ_k.

Proposition 3 *For $k \geq 1$, $\rho_{k-1} \geq \delta_k \geq \rho_{k+1}$. In particular, $\delta_k = O(\rho_k)$ as $k \to \infty$.*

Remark 3 The preceding result yields the inequalities $\rho_0 \geq \delta_1 \geq \rho_2 \geq \delta_3 \geq \cdots$ and $\rho_1 \geq \delta_2 \geq \rho_3 \geq \delta_4 \geq \cdots$. For the case in which $\alpha = 1$, we have observed from the tables of eigenvalues computed by Taherizadeh (2009, p. 28, 54) that the eigenvalues ρ_k and δ_k satisfy the stronger, strict interlacing property, $\rho_k > \delta_k > \rho_{k+1}$ for all $k \geq 1$, and we therefore conjecture that the strict interlacing property holds for general α. We have not been able to resolve this conjecture using general Hilbert space operator-theoretic methods or using specific properties of the Bessel functions, and it appears that more powerful methods are needed to resolve the problem.

There is also the issue of choosing the value of m so that the mth scree ratio of T_n^2 exceeds $1-\epsilon$, where $0 < \epsilon < 1$. Applying the interlacing inequalities for δ_k, we obtain $\sum_{k=1}^{m} \delta_k \geq \sum_{k=2}^{m+1} \rho_k$. Since $Tr(\mathcal{S}_0) > Tr(\mathcal{S})$, we advise that m be chosen so that

$$\sum_{k=0}^{m+1} \rho_k \geq (1-\epsilon) Tr(\mathcal{S}_0).$$

Table 1 Values of the lower bound on m for the scree ratio of T_n^2.

α	0.5	0.75	1	3	5	10	20	50
m	15	12	10	6	4	3	2	1

This leads to a value for m that is readily applicable. Substituting $\rho_k = \alpha^\alpha b_\alpha^{4k+2\alpha}$, evaluating in closed form the resulting geometric series, and substituting for $Tr(\mathcal{S}_0)$ from (4.6), we obtain

$$\alpha^\alpha b_\alpha^{2\alpha} \frac{1 - b_\alpha^{4(m+2)}}{1 - b_\alpha^4} = \sum_{k=0}^{m+1} \rho_k \geq (1-\epsilon) Tr(\mathcal{S}_0) = (1-\epsilon)\alpha^\alpha b_\alpha^{2\alpha} \frac{1}{1 - b_\alpha^4}.$$

Solving this inequality for m, we obtain

$$m \geq \frac{\log \epsilon}{4 \log b_\alpha} - 2. \tag{4.9}$$

We illustrate this bound by calculating it for various values of α. For $\epsilon = 10^{-10}$, which represents accuracy to ten decimal places, this results in the values displayed in Table 1.

5 Applications, Consistency of the Test, and Numerical Power Calculations

The first data set (Hogg and Tanis, 2009, p. 155) consists of $n = 25$ waiting times (in seconds) for a Geiger counter to observe 100 alpha-particles emitted by barium-133. As noted by Hogg and Tanis (2009, p. 464), a Kolmogorov-Smirnov test that the data were drawn from a $Gamma(\alpha = 100, \lambda = 14.7)$ distribution failed to reject that hypothesis at the 10% level of significance.

We apply the statistic T_n^2 to test H_0, the null hypothesis that the data are drawn from a gamma distribution with $\alpha = 100$ and unspecified λ. The observed value of T_n^2 is 6.301×10^{-10}.

We used the limiting null distribution of T_n^2 to estimate $T^2_{n\,;\,0.05}$. For $\alpha = 100$, it follows from Table 1 that only one eigenvalue is needed to approximate accurately the asymptotic distribution of T_n^2; therefore, $T_n^2 \approx \delta_1 \chi_1^2$. By (4.7), we obtain $\delta_1 \simeq Tr(\mathcal{S}) = 6.722 \times 10^{-6}$. Therefore, $T^2_{n\,;\,0.05} \simeq \delta_1 \chi^2_{1\,;\,0.05}$, where $\chi^2_{1\,;\,0.05}$ is the 95th percentile of the χ_1^2 distribution, so we obtain $T^2_{n\,;\,0.05} = 2.582 \times 10^{-5}$. As this critical value exceeds the observed value of T_n^2, we fail to reject the null hypothesis that the waiting times are drawn from a $Gamma(\alpha = 100, \lambda)$ distribution.

As an alternative approach, we conducted a simulation study to approximate $T^2_{n\,;\,0.05}$, the 95th percentile of the null distribution of T_n^2. We generated $10,000$ random samples of size $n = 25$ from the $Gamma(100, 1)$ distribution, calculated the value of T_n^2 for each sample, and recorded the 95th percentile of all 10,000 simulated values of T_n^2. We repeated this process ten times, finally approximating $T^2_{n\,;\,0.05}$ as the 20%-trimmed mean of all 10 simulated 95th percentiles, viz., $T^2_{n\,;\,0.05} = 2.368 \times 10^{-5}$. Since this critical value exceeds the observed value of T_n^2 then we fail to reject the null hypothesis at the 5% level of significance. Moreover, we derived from our simulation study an approximate P-value of 0.99 for the test.

The second data set, given by Barlow and Campo (1975), provides $n = 107$ failure times (in hours) for the right rear brakes on a sample of tractors. The data were analyzed recently by Cuparić, Milošević and Obradović (2018), where the null hypothesis of exponentiality was rejected.

To test the hypothesis that the data are drawn from a gamma-distributed population, we assume for illustrative purposes that $\alpha = 2.3$. This value was obtained by setting the maximum likelihood estimate of the mode of the $Gamma(\alpha, \lambda)$ density, viz., $(\alpha - 1)\bar{X}_n/\alpha$, equal to a mode of the histogram, and solving the resulting equation for α. Then the observed value of T_n^2 is 0.0053.

For $\alpha = 2.3$, it follows from (4.9) that $T_n^2 \approx \sum_{k=1}^{7} \delta_k \chi_{1k}^2$. We calculated the δ_k numerically as the positive roots of the function $G(\delta)$ in Theorem 5, and then we applied the results of Imhof (1961) or Kotz, Johnson, and Boyd (1967) to derive the distribution of $\sum_{k=1}^{7} \delta_k \chi_{1k}^2$ and carry out the test. A one-term approximation (Kotz, Johnson, and Boyd, 1967, Eqs. (71), (79)),

$$P\Big(\sum_{k=1}^{m} \delta_k \chi_{1k}^2 \geq t\Big) \simeq P\big(\chi_m^2 \geq 2t/(\delta_1 + \delta_m)\big)$$

Table 2 The outcome of testing the tractor brakes data with numerous values of α.

α	1.0	1.8	2.3	3	5	8
Observed T_n^2	0.6965	0.0162	0.0053	0.0534	0.1977	0.3180
$T^2_{n\,;\,0.05}$	0.1406	0.0559	0.0356	0.0228	0.0088	0.0037
$T^2_{\infty\,;\,0.05}$	0.1420	0.0576	0.0376	0.0235	0.0091	0.0037
P-value	0.0000	0.3013	0.4702	0.0026	0.0000	0.0000

is well-known to be accurate for other problems (see Gupta and Richards (1983)) and leads to an explicit formula, $T^2_{n\,;\,0.05} \simeq \frac{1}{2}(\delta_1 + \delta_m)\chi^2_{m\,;\,0.05}$, for an approximate critical value of T_n^2.

As an alternative to calculating $\delta_1, \ldots, \delta_M$, we can apply the interlacing inequalities in Proposition 3 to obtain a stochastic upper bound, $\sum_{k=1}^{M} \delta_k \chi^2_{1k} \le \sum_{k=0}^{M-1} \rho_k \chi^2_{1k}$. By applying results of Kotz, Johnson, and Boyd (1967, loc. cit.) or Imhof (1961) to approximate the critical values of this upper bound, we obtain a conservative test of H_0, i.e., with a level of significance at most 5%.

By performing a simulation study as for the previous data set, we obtained the approximation, $T^2_{n\,;\,0.05} = 0.0356$, which exceeds the observed value of the test statistic. Also, by applying the results of Imhof (1961), the limiting critical value derived from $\sum_{k=1}^{7} \delta_k \chi^2_{1k}$, denoted by $T^2_{\infty\,;\,0.05}$, equals 0.0376. Therefore, we fail to reject the null hypothesis at the 5% level of significance. Moreover, the simulation study provided an approximate P-value of 0.47.

Since it was assumed for illustrative purposes that $\alpha = 2.3$, we repeated the test for several values of α, obtaining the results in Table 2. We note that the null hypothesis is rejected for the case in which $\alpha = 1$ where, under the null hypothesis, the data are exponentially distributed; hence, we deduce in this case the same conclusion as Cuparić, Milošević and Obradović (2018).

With regard to the consistency of the test statistic, we now provide a result that the test is consistent against any fixed alternative distribution.

Theorem 6 *Let $X_1, X_2, \ldots$ be a sequence of positive, i.i.d., random variables with finite mean μ. Let $\gamma \in (0,1)$ denote the level of significance of the test and $c_{n,\gamma}$ be the $(1-\gamma)$-quantile of the test statistic T_n^2 under H_0. If $X_1, X_2, \ldots$ are not Gamma-distributed then*

$$\lim_{n\to\infty} P(T_n^2 > c_{n,\gamma}) = 1.$$

With regard to a proof of this result, if we define

$$\Lambda := \int_0^\infty \Big(E[\Gamma(\alpha)(tX_1/\mu)^{(1-\alpha)/2} J_{\alpha-1}(2(tX_1/\mu)^{1/2})] - e^{-t/\alpha} \Big)^2 \, \mathrm{d}P_0(t),$$

then the essential part of the proof is to establish that $n^{-1}T_n^2 \xrightarrow{p} \Lambda$. The extensive details required to prove this limit are provided by Hadjicosta (2019) or Hadjicosta and Richards (2018).

Remark 4 By applying Theorem 1 of Baringhaus, Ebner, and Henze (2017) we also find that, under fixed alternatives to the null hypothesis, $n^{1/2}(n^{-1}T_n^2 - \Lambda) \xrightarrow{d} N(0, \sigma^2)$ as $n \to \infty$, where σ^2 is a constant that is determined from the alternative distribution.

Turning to numerical calculations of the power of the test, we provide in Table 3 simulated critical values for various n and four levels of significance, denoted by γ, for $\alpha = 2, 5, 10$. The last row of Table 3 is derived using the approximate limiting null distribution $\sum_{k=1}^{7} \delta_k \chi^2_{1k}$ and the method of Imhof (1961) for calculating the distribution of such linear combinations. The eigenvalues $\{\delta_k\}$ are calculated numerically, by applying the results of Theorem 5, using the Newton-Raphson method in the software `R` (R Development Core Team, 2007). All other entries in Table 3 are calculated as the 20%-trimmed mean of 10 simulated $(1-\gamma)$-percentiles, each based on 10000 replications. The values in the table indicate that, as α increases, the critical points converge more rapidly; in particular, for $\alpha = 10$ and $n \ge 20$, the 90th and higher percentiles equal, to three decimal places, the limiting percentiles.

Next, we compare the power of the new test with the Cramér-von Mises (C^2) and Anderson-Darling (A^2) tests. We conducted a Monte Carlo study with 5000 replications at a 5% significance

Table 3 Critical values of T_n^2 for $\alpha = 2, 5, 10$.

n	$\alpha = 2$, $1-\gamma$				$\alpha = 5$, $1-\gamma$				$\alpha = 10$, $1-\gamma$			
	0.90	0.95	0.975	0.99	0.90	0.95	0.975	0.99	0.90	0.95	0.975	0.99
20	0.032	0.044	0.056	0.075	0.006	0.008	0.011	0.015	0.002	0.002	0.003	0.004
50	0.033	0.046	0.059	0.078	0.006	0.009	0.011	0.015	0.002	0.002	0.003	0.004
80	0.033	0.046	0.060	0.079	0.006	0.009	0.011	0.015	0.002	0.002	0.003	0.004
100	0.033	0.046	0.060	0.079	0.006	0.009	0.012	0.015	0.002	0.002	0.003	0.004
∞	0.033	0.048	0.063	0.080	0.006	0.009	0.012	0.016	0.002	0.002	0.003	0.004

level for $n = 20, 50$. The critical values of C^2 and A^2 are calculated in the same way as for T_n^2, *viz.*, as the 20%-trimmed mean of 10 simulated 95%-percentiles, each based on 10000 replications. In Tables 4, 5, and 6, we present for $\alpha = 2, 5, 10$ the percentage points of 5000 Monte Carlo samples found to be significant. An asterisk denotes a power of 100%, and we list in boldface the most powerful test in each case. For $\theta > 0$ and $x > 0$, the alternative distributions considered are the:

Gamma(α): Gamma distribution with shape parameter α and rate parameter 1.
Weibull(θ): Weibull distribution with density function $\theta x^{\theta-1}\exp(-x^\theta)$.
LIFR(θ): Linear Increasing Failure Rate distribution with density function $(1+\theta x)\exp[-x-\frac{1}{2}\theta x^2]$.
LN(θ): Lognormal distribution with density function $(\theta x)^{-1}(2\pi)^{-1/2}\exp[(-\log x)^2/2\theta^2]$.
IG(θ): Inverse Gaussian distribution with density function $(\theta/2\pi x^3)^{1/2}\exp[-\theta(x-1)^2/2x]$.
GO(θ): Gompertz distribution with density function $\theta e^{x+\theta}\exp(-\theta e^x)$.
Rayleigh(θ): Rayleigh distribution with density function $(x/\theta)\exp[-x^2/2\theta]$.

These distributions were chosen from among numerous alternatives for which calculations were done by several authors, e.g., Baringhaus and Taherizadeh (2010), Henze, Meintanis, and Ebner (2012), and Taherizadeh (2009).

In the case of the Gompertz distributions, the test based on A^2 is the most powerful of the three for all tabulated n and α, and the test based on T_n^2 is the next most powerful. For $\alpha = 2$ and $n = 20$, we see from Table 4 that the test based on T_n^2 is at least as powerful as the tests based on C^2 and A^2 for 17 of the 26 alternatives considered. The tables for $\alpha = 2$ and $n = 50$ also indicate that the test based on T_n^2 is comparable in power to the other two tests. Therefore, for small values of α, the new test is a serious competitor to the classical tests, irrespective of the size of n.

For large α and small n, Tables 5 and 6 indicate that the test based on T_n^2 is more powerful than the tests based on C^2 and A^2 for the majority of alternatives considered here. If n is large then T_n^2 becomes less superior to the other two tests; this is a consequence of the consistency of each test, which implies that, as $n \to \infty$, the powers of all three tests converge to 1.

6 Contiguous Alternatives to the Null Hypothesis

For $n \in \mathbb{N}$, let $X_{n1}, \ldots, X_{nn}$ be a triangular array of row-wise independent random variables. As usual, let $P_0 = \mathit{Gamma}(\alpha, 1)$, $\alpha \geq 1/2$, and let Q_{n1} be a probability measure dominated by P_0. We wish to test the null hypothesis

$$H_0: \text{ The marginal distribution of each } X_{ni},\ i = 1, \ldots, n, \text{ is } P_0$$

against the alternative

$$H_1: \text{ The marginal distribution of each } X_{ni},\ i = 1, \ldots, n, \text{ is } Q_{n1}.$$

We write the Radon-Nikodym derivative of Q_{n1} with respect to P_0 in the form $\mathrm{d}Q_{n1}/\mathrm{d}P_0 = 1 + n^{-1/2}h_n$, and then we will need the following

Assumptions 7 We assume that:

(A1) The functions $\{h_n : n \in \mathbb{N}\}$ form a sequence of P_0-integrable functions converging pointwise, P_0-almost everywhere, to a function h, and

Table 4 Percentage points of 5000 Monte Carlo samples found to be significant at 5% level of significance; $\alpha = 2$.

$n = 20$			
Distribution	T_n^2	C^2	A^2
Gamma(1)	62	48	**64**
Gamma(1.5)	**18**	12	**18**
Gamma(2)	5	5	5
Gamma(2.5)	**9**	8	7
Gamma(3)	**19**	18	13
Gamma(3.5)	**33**	29	26
Gamma(4)	**48**	42	40
Weibull(1)	62	47	**67**
Weibull(2)	27	**28**	24
Weibull(2.5)	**73**	70	64
Weibull(3)	**96**	93	91
LIFR(0.02)	61	46	**64**
LIFR(0.05)	56	41	**62**
LIFR(0.1)	53	36	**58**
LN(0.8)	**21**	19	20
LN(0.9)	**39**	34	37
LN(1)	**60**	52	55
LN(1.5)	**98**	96	97
IG(0.5)	**81**	76	78
IG(1)	**32**	31	30
IG(1.5)	10	**13**	10
IG(3)	**31**	**31**	29
IG(3.5)	**47**	44	43
IG(4)	**61**	56	57
GO(2)	24	17	**35**
GO(4)	38	25	**47**
Rayleigh(1)	**27**	**27**	22

$n = 50$			
Distribution	T_n^2	C^2	A^2
Gamma(1)	93	85	**94**
Gamma(1.5)	**33**	22	**33**
Gamma(2)	5	5	5
Gamma(2.5)	**16**	15	13
Gamma(3)	**46**	40	39
Gamma(3.5)	**77**	65	67
Gamma(4)	**92**	85	87
Weibull(1)	92	85	**94**
Weibull(2)	**69**	63	61
Weibull(2.5)	99	99	99
Weibull(3)	*	*	*
LIFR(0.02)	**92**	83	**92**
LIFR(0.05)	89	80	**91**
LIFR(0.1)	86	74	**89**
LN(0.8)	35	**38**	36
LN(0.9)	**71**	66	66
LN(1)	**91**	88	88
LN(1.5)	*	*	*
IG(0.5)	99	99	99
IG(1)	61	**63**	60
IG(1.5)	13	**23**	20
IG(3)	68	69	**78**
IG(3.5)	88	88	**92**
IG(4)	**97**	96	**97**
GO(2)	44	39	**63**
GO(4)	69	57	**79**
Rayleigh(1)	**69**	61	59

Table 5 Percentage points of 5000 Monte Carlo samples found to be significant at 5% level of significance; $\alpha = 5$.

$n = 20$			
Distribution	T_n^2	C^2	A^2
Gamma(4)	**15**	9	13
Gamma(4.5)	**9**	7	7
Gamma(5)	5	5	5
Gamma(6)	**7**	**7**	6
Gamma(7)	**13**	12	10
Gamma(8)	**23**	19	17
Gamma(10)	**47**	37	35
Weibull(3)	15	**17**	15
Weibull(4)	**65**	62	59
Weibull(5)	**94**	93	92
LIFR(2)	**94**	72	93
LIFR(4)	**88**	60	87
LN(0.5)	**18**	13	16
LN(0.7)	**83**	63	76
LN(0.9)	**98**	92	97
LN(1)	**99**	96	**99**
IG(2)	**71**	53	65
IG(2.5)	**49**	36	43
IG(3)	**31**	22	27
GO(2)	79	54	**84**
GO(4)	54	35	**67**
Rayleigh(1)	32	15	**33**

$n = 50$			
Distribution	T_n^2	C^2	A^2
Gamma(4)	**23**	13	21
Gamma(4.5)	**9**	6	**9**
Gamma(5)	5	5	5
Gamma(6)	**11**	**11**	10
Gamma(7)	**30**	25	25
Gamma(8)	**55**	44	48
Gamma(10)	**91**	79	82
Weibull(3)	31	41	**38**
Weibull(4)	98	98	98
Weibull(5)	*	*	*
LIFR(2)	*	98	*
LIFR(4)	*	95	*
LN(0.5)	**27**	25	26
LN(0.7)	**99**	95	98
LN(0.9)	*	*	*
LN(1)	*	*	*
IG(2)	**95**	89	94
IG(2.5)	**81**	72	76
IG(3)	**55**	49	52
GO(2)	98	92	**99**
GO(4)	82	75	**95**
Rayleigh(1)	56	33	**59**

(A2) $\sup_{n\in\mathbb{N}} E_{P_0}|h_n|^4 < \infty$.

Table 6 Percentage points of 5000 Monte Carlo samples found to be significant at 5% level of significance; $\alpha = 10$.

$n = 20$			
Distribution	T_n^2	C^2	A^2
Gamma(5)	**68**	37	58
Gamma(8)	**16**	8	13
Gamma(10)	5	5	5
Gamma(12)	**7**	**7**	5
Gamma(15)	**19**	14	12
Gamma(20)	**48**	36	32
Weibull(5)	**38**	37	34
Weibull(6)	**72**	70	67
Weibull(7)	**91**	90	90
LIFR(50)	**98**	81	96
LIFR(100)	**97**	78	96
LN(0.2)	**73**	59	57
LN(0.4)	**42**	23	33
LN(0.6)	**97**	86	95
IG(4)	**79**	55	70
IG(5)	**59**	37	48
IG(6)	**39**	24	32
IG(7)	**23**	14	19
IG(8)	**14**	10	11
GO(10)	82	57	**83**
GO(20)	59	42	**67**
Rayleigh(1)	**94**	68	91

$n = 50$			
Distribution	T_n^2	C^2	A^2
Gamma(5)	**96**	72	89
Gamma(8)	**28**	13	20
Gamma(10)	5	5	5
Gamma(12)	**16**	10	9
Gamma(15)	**52**	31	34
Gamma(20)	**93**	76	82
Weibull(5)	**80**	81	**80**
Weibull(6)	99	99	99
Weibull(7)	*	*	*
LIFR(50)	*	99	*
LIFR(100)	*	99	*
LN(0.2)	*	97	98
LN(0.4)	**73**	50	61
LN(0.6)	*	99	*
IG(4)	**99**	91	96
IG(5)	**90**	72	82
IG(6)	**69**	47	59
IG(7)	**43**	29	33
IG(8)	**23**	15	18
GO(10)	98	93	**99**
GO(20)	84	81	**93**
Rayleigh(1)	*	97	*

Since $\int (\mathrm{d}Q_{n1}/\mathrm{d}P_0)\,\mathrm{d}P_0 = 1$ then we also have $\int h_n\,\mathrm{d}P_0 = 0$, for all $n \in \mathbb{N}$. Denote the indicator function of an event A by $I(A)$. By applying (A2), we deduce the uniform integrability of $|h_n|^2$:

$$\begin{aligned}\lim_{k\to\infty}\sup_{n\in\mathbb{N}} E_{P_0}\big(|h_n|^2 I(|h_n|^2 > k)\big) &= \lim_{k\to\infty}\sup_{n\in\mathbb{N}} \int |h_n|^2\; I(|h_n|^2 > k)\,\mathrm{d}P_0 \\ &\le \lim_{k\to\infty} k^{-1} \sup_{n\in\mathbb{N}} E_{P_0}|h_n|^4 = 0.\end{aligned}$$

By Bauer (1981, p. 95, Theorem 2.11.4), the P_0-almost everywhere convergence of h_n to h implies the P_0-stochastic convergence of h_n to h. Again by Bauer (1981, p. 104, Theorem 2.12.4), the uniform integrability of h_n^2 together with the P_0-stochastic convergence of h_n to h imply the convergence of h_n in mean square, viz.

$$\lim_{n\to\infty}\int |h_n - h|^2\,\mathrm{d}P_0 = 0.$$

By the triangle and the Cauchy-Schwarz inequalities,

$$0 \le \lim_{n\to\infty}\Big|\int (h_n - h)\,\mathrm{d}P_0\Big| \le \lim_{n\to\infty}\int |h_n - h|\,\mathrm{d}P_0 \le \lim_{n\to\infty}\Big(\int |h_n - h|^2\mathrm{d}P_0\Big)^{1/2} = 0,$$

therefore

$$\lim_{n\to\infty}\int h_n\,\mathrm{d}P_0 = \int h\,\mathrm{d}P_0 = 0.$$

Hadjicosta (2019) has shown that Assumptions 7 hold for several contiguous alternatives, e.g.,

(1) Gamma alternatives with shape parameter $\alpha \ge 1/2$ and rate parameter $\lambda_n = 1 + n^{-1/2}$.
(2) Gamma alternatives with shape parameter $\alpha_n = \alpha + n^{-1/2}$ and rate parameter 1.
(3) Contaminated models, $Q_{n1} = (1 - n^{-1/2})P_0 + n^{-1/2}P_1$, where the probability measure P_1 is dominated by P_0 and $\int (\mathrm{d}P_1/\mathrm{d}P_0)^4\,\mathrm{d}P_0 < \infty$.

Let $\boldsymbol{P}_n = P_0 \otimes \cdots \otimes P_0$ and $\boldsymbol{Q}_n = Q_{n1} \otimes \cdots \otimes Q_{n1}$, where $P_0 = Gamma(\alpha, 1)$, $\alpha \ge 1/2$, be the n-fold product probability measures of P_0 and Q_{n1} respectively.

Theorem 8 *Let $X_{n1}, \ldots, X_{nn}$, $n \in \mathbb{N}$, be a triangular array of positive row-wise i.i.d. random variables, where $X_{nj} \equiv X_j$, $j = 1, \ldots, n$. We assume that the distribution of X_{nj} is Q_{n1} for every $j = 1, \ldots, n$. Further, let $Z_n := \{Z_n(t), t \geq 0\}$ be a stochastic process with*

$$Z_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \Big[\Gamma(\alpha)(tX_{nj}/\bar{X}_n)^{(1-\alpha)/2} J_{\alpha-1}\big(2(tX_{nj}/\bar{X}_n)^{1/2}\big) - e^{-t/\alpha} \Big],$$

$t \geq 0$. Under Assumptions 7, there exists a centered Gaussian process $Z := \{Z(t), t \geq 0\}$ with sample paths in L^2 and the covariance function $K(s,t)$ in (3.4), and a function

$$c(t) = \int_0^\infty \big[\Gamma(\alpha)(tx/\alpha)^{(1-\alpha)/2} J_{\alpha-1}(2(tx/\alpha)^{1/2}) + \alpha^{-2} t e^{-t/\alpha}\, x\big] h(x)\, \mathrm{d}P_0(x),$$

$t \geq 0$, such that $Z_n \xrightarrow{d} Z + c$ in L^2. Moreover, as $n \to \infty$,

$$T_n^2 \xrightarrow{d} \int_0^\infty \big(Z(t) + c(t)\big)^2\, \mathrm{d}P_0(t).$$

A detailed proof of this theorem is provided by Hadjicosta (2019) who followed the approach of Taherizadeh (2009, pp. 79–91).

7 The Efficiency of the Test

Let $X_1, X_2, \ldots$ be i.i.d., positive random variables with a distribution $\mathcal{P}$ that is indexed by a parameter $\theta \in \Theta := (-\eta, \eta)$ or $\Theta := [0, \eta)$, for some $\eta > 0$. We represent H_0 by $\Theta_0 = \{0\}$ and H_1 by $\Theta_1 = \Theta \setminus \{0\}$. In Section 3, we showed that T_n^2 is scale-invariant, i.e. it does not depend on the unknown rate parameter λ. Under the null hypothesis, we assume that $X_1, X_2, \ldots$ are i.i.d., positive P_0-distributed random variables; further, under the local alternative, represented by $\theta \in \Theta_1$, we assume that $X_1, X_2, \ldots$ are i.i.d., positive P_θ-distributed random variables.

The Radon-Nikodym derivative of P_θ with respect to P_0 is $\mathrm{d}P_\theta/\mathrm{d}P_0 = 1 + \theta h_\theta$. We assume that as $\theta \to 0$, the function h_θ converges to some function h in L^2. Since $\int (\mathrm{d}P_\theta/\mathrm{d}P_0)\, \mathrm{d}P_0 = 1$, we obtain $\int_0^\infty h_\theta(x)\, \mathrm{d}P_0(x) = 0$, for $\theta \in \Theta_1$. Further, we shall assume that for $\theta \in \Theta_1$,

$$\int_0^\infty x h_\theta(x)\, \mathrm{d}P_0(x) = 0. \tag{7.1}$$

Let Θ_0 and Θ_1 be null and alternative parameter spaces, respectively, and $\{U_n : n \in \mathbb{N}\}$ be a sequence of test statistics. For $\theta \in \Theta_0$, $F_n(t) = P_\theta(U_n < t)$, $t \in \mathbb{R}$, is the null distribution of U_n, and *the level attained by* U_n is $L_n := 1 - F_n(U_n)$. For $\theta \in \Theta_1$, the *exact Bahadur slope* of the sequence $\{U_n : n \in \mathbb{N}\}$ is

$$c(\theta) = -2 \lim_{n \to \infty} n^{-1} \log L_n,$$

whenever the limit exists (almost surely). For $\theta \in \Theta_0$, this limit exists with $c(\theta) = 0$.

For a sequence $\{U_{j,n} : n \in \mathbb{N}\}$ of test statistics with exact Bahadur slope $c_j(\theta)$, $j = 1, 2$, the *exact Bahadur asymptotic relative efficiency* of $\{U_{1,n} : n \in \mathbb{N}\}$ with respect to $\{U_{2,n} : n \in \mathbb{N}\}$ is $e^B_{1,2}(\theta) = c_1(\theta)/c_2(\theta)$, $\theta \in \Theta_1$. If $e^B_{1,2}(\theta) > 1$, then we prefer the sequence $\{U_{1,n} : n \in \mathbb{N}\}$.

In general, it is difficult to calculate the exact Bahadur slope (Bahadur, 1971, Theorem 7.2), so we investigate the approximate Bahadur slope. We note that Bahadur (1967) showed that for $\Theta_0 = \{\theta_0\}$, the approximate Bahadur slope is close to the exact Bahadur slope for θ in a neighborhood of θ_0, i.e., under local alternatives.

To obtain the approximate Bahadur slope of our test statistic T_n^2 under local alternatives, we need to show that the sequence $\{T_n : n \in \mathbb{N}\}$ is a *standard sequence* (Bahadur, 1960, Section 2).

Theorem 9 *The sequence of test statistics $\{T_n : n \in \mathbb{N}\}$ is a standard sequence. The approximate Bahadur slope of the test is $c^{(a)}(\theta) := \delta_1^{-1} b^2(\theta)$, where δ_1 is the largest eigenvalue of $\mathcal{S}$ and*

$$b^2(\theta) = \theta^2 \int_0^\infty \Big[\int_0^\infty \Gamma(\alpha)(tx/\alpha)^{(1-\alpha)/2} J_{\alpha-1}(2(tx/\alpha)^{1/2}) h_\theta(x)\, \mathrm{d}P_0(x) \Big]^2 \mathrm{d}P_0(t). \tag{7.2}$$

Table 7 Limiting approximate Bahadur slopes for the contaminated gamma models.

k	2	3	4	5	6
c_T	0.149	0.284	0.373	0.433	0.477

Moreover,

$$\lim_{\theta\to 0}\frac{c^{(a)}(\theta)}{\theta^2} = \delta_1^{-1}\int_0^\infty \Big[\int_0^\infty \Gamma(\alpha)(tx/\alpha)^{(1-\alpha)/2} J_{\alpha-1}(2(tx/\alpha)^{1/2})h(x)\,\mathrm{d}P_0(x)\Big]^2\,\mathrm{d}P_0(t). \tag{7.3}$$

A complete proof of this result is given by Hadjicosta (2019) following the approach of Taherizadeh (2009, p. 98, Theorem 5.4).

If we write the squared term in (7.3) as a product of two integrals, one in x and one in y, interchange the order of integration, and apply Weber's integral (10.1), then (7.3) reduces to

$$\lim_{\theta\to 0}\frac{c^{(a)}(\theta)}{\theta^2} = \delta_1^{-1}\int_0^\infty\int_0^\infty \Gamma(\alpha)(xy/\alpha^2)^{(1-\alpha)/2}e^{-(x+y)/\alpha}I_{\alpha-1}(2(xy/\alpha^2)^{1/2})h(x)h(y)\,\mathrm{d}P_0(x)\,\mathrm{d}P_0(y),$$

and a similar result holds for (7.2). These expressions provide an alternative way to calculate the approximate Bahadur slope of the test.

We now obtain the limiting approximate Bahadur slope for several alternatives. In the following calculations, we will take $\alpha = 2$ as the general case can be treated similarly. Consider the contaminated models $P_\theta = (1-\theta)P_0 + \theta P_1$, where P_1 is a probability measure dominated by P_0; for $\alpha = 1$, these alternatives were considered earlier by Baringhaus and Taherizadeh (2010). It is straightforward to verify that assumption (7.1) is satisfied if $\int x\,\mathrm{d}P_1(x) = \int x\,\mathrm{d}P_0(x) = 2$. Also, $h_\theta = (\mathrm{d}P_1/\mathrm{d}P_0) - 1$. By (7.3), the limiting Bahadur slope of the sequence $\{T_n : n \in \mathbb{N}\}$ is

$$c_T := \lim_{\theta\to 0}\frac{c^{(a)}(\theta)}{\theta^2} = \delta_1^{-1}\int_0^\infty \big(\mathcal{H}_{P_1}(t/2) - e^{-t/2}\big)^2\,\mathrm{d}P_0(t),$$

where $\mathcal{H}_{P_1}$ denotes the Hankel transform of P_1. Further, by applying the results of Theorem 5 for calculating the eigenvalues of $\mathcal{S}$, we obtain $\delta_1^{-1} = 83.242$.

Consider the contaminated gamma models $P_\theta = (1-\theta)P_0 + \theta\cdot Gamma(2k,k)$, $k \in \mathbb{N}$, $k \ge 2$. By equation (2.5) and Kummer's formula (2.4), we obtain

$$\mathcal{H}_{Gamma(2k,k)}(t/2) = {}_1F_1(2k;2;-t/2k) = e^{-t/2k}{}_1F_1(2-2k;2;t/2k).$$

In Table 7, we provide the limiting approximate Bahadur slopes for $k = 2,3,4,5,6$.

Next, consider the contaminated model $P_\theta = (1-\theta)P_0 + \theta\cdot U(0,4)$, where $U(0,4)$ denotes the uniform distribution on the interval $(0,4)$. By Olver, *et al.* (2010, (10.22.9)), $\mathcal{H}_{U(0,4)}(t/2) = (1 - J_0((8t)^{1/2})/(2t)$ and the limiting approximate slope equals 0.018.

Wieand (1976) showed that if two standard sequences $\{U_{1,n} : n \in \mathbb{N}\}$ and $\{U_{2,n} : n \in \mathbb{N}\}$ satisfy an additional criterion, now called *Wieand's condition*, then the limiting approximate Bahadur efficiency is in accord with the limiting Pitman efficiency, as the level of significance decreases to 0. In the next theorem, we state that Wieand's condition is valid for our sequence of test statistics $\{T_n : n \in \mathbb{N}\}$. The proof of this theorem is omitted; we refer to Hadjicosta (2019) or Hadjicosta and Richards (2018) for full details.

Theorem 10 *The sequence $\{T_n : n \in \mathbb{N}\}$ satisfies Wieand's condition: There exists a constant $\theta^* > 0$ such that for any $\epsilon > 0$ and $\gamma \in (0,1)$, there exists a constant $C > 0$ such that, for any $\theta \in \Theta_1 \cap (-\theta^*, \theta^*)$ and $n > C/b^2(\theta)$, $P(|n^{-1/2}T_n - b(\theta)| < \epsilon b(\theta)) > 1 - \gamma$.*

8 Concluding Remarks

In constructing the test statistic T_n^2 in (1.3), we set $\nu = \alpha - 1$. In this case, the test statistic has a relatively simple expression as a V-statistic, so the test can be carried out in a straightforward way.

The resulting test statistic also is consistent, has good power performance, and extensive results can be obtained for the eigenvalues and eigenfunctions of the corresponding covariance operator.

For general ν, the calculations in the proof of Proposition 1 can be extended. However, the final expression for the resulting V-statistic will be more complicated, for it will involve the generalized hypergeometric series. Under H_0, we will again obtain $T_n^2 \xrightarrow{d} \sum_{k=1}^{\infty} \delta_k \chi_{1k}^2$, but the eigenfunctions of the corresponding covariance operator will be more complex and may be unavailable.

We remark that there are many choices, other than P_0, for the weight measure. Our choice of P_0 is motivated by classical tests, such as the Anderson-Darling and Cramér-von Mises statistics, for which the weight measure is determined by H_0. In the gamma case, the orthogonal polynomials for P_0 are well-known; however, this may not hold for more general weight measures. We note that Henze, Meintanis, and Ebner (2012) and Taherizadeh (2009, p. 65) provided results for weight measures w of the form $\mathrm{d}w(t) = e^{-\beta t}\,\mathrm{d}P_0(t)$, where β is a "tuning parameter;" similar results for the testing problem considered in this paper can be obtained using our methods.

The entries in Table 2 reflect the dependency of T_n^2 on the value of α. This raises the problem of extending our results to the case in which α is unknown. This problem appears to be formidable; if we replace each α in (3.2) with a suitable estimator $\hat{\alpha}$, then parametric bootstrap procedures can be used to estimate the resulting critical values and the power of the test. However, it may be more difficult to derive the analytical properties of the test statistic.

If $\hat{\alpha}$ is *scale-invariant*, the results of Henze, Meintanis, and Ebner (2012) lead us to believe that, under certain regularity conditions, the asymptotic distribution of the resulting statistic can be obtained. However, the comments of Henze, Meintanis, and Ebner (2012, Remark 2.3) also apply to our problem, viz., a finite-sample implementation of the test will require knowledge of the value of α which, however, is unknown. We also note that there is a substantial literature on the problem of inserting a parameter estimator into a V-statistic; cf., de Wet and Randles (1987), Leucht and Neumann (2013), and Matsui and Takemura (2008); it is an open problem to apply those approaches to our test statistic.

Acknowledgments: We are grateful to the reviewers and the editors for helpful and constructive comments on the initial version of the manuscript.

Conflict of interest statement: On behalf of both authors, the corresponding author states that there is no conflict of interest.

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9 Appendix: Bessel Functions and Hankel Transforms

For the special case in which $\nu = -\frac{1}{2}$, it follows from (2.1) that, for $x \in \mathbb{R}$,

$$x^{1/2} J_{-1/2}(x) = \Big(\frac{2}{\pi}\Big)^{1/2} \cos x, \tag{9.1}$$

For $\nu > -1/2$, the Bessel function is also given by the *Poisson integral*,

$$J_\nu(x) = \frac{(x/2)^\nu}{\pi^{1/2}\,\Gamma\big(\nu+\frac{1}{2}\big)} \int_0^\pi \cos(x\cos\theta)(\sin\theta)^{2\nu}\,\mathrm{d}\theta, \tag{9.2}$$

$x \in \mathbb{R}$; see Erdélyi, *et al.* (1953, 7.12(9)), Olver, *et al.* (2010, (10.9.4)). This result can be proved by expanding $\cos(x\cos\theta)$ as a power series in $x\cos(\theta)$ and integrating term-by-term.

The Bessel function J_ν also satisfies the inequality,

$$|J_\nu(z)| \le \frac{1}{\Gamma(\nu+1)}\,|z/2|^\nu \,\exp(\mathrm{Im(z)}), \tag{9.3}$$

$\nu \ge -1/2$, $z \in \mathbb{C}$; see Erdélyi, *et al.* (1953, 7.3.2(4)) or Olver, *et al.* (2010, (10.14.4)).

Henceforth, we assume that $\nu \ge -1/2$. For $t, x \ge 0$, we set $z = 2(tx)^{1/2}$ in (9.3) to obtain

$$\big|(tx)^{-\nu/2} J_\nu\big(2(tx)^{1/2}\big)\big| \le \frac{1}{\Gamma(\nu+1)}. \tag{9.4}$$

Although the next two results may be known, we were unable to find them in the literature.

Lemma 3 *For $\nu \ge -1/2$ and $t \ge 0$,*

$$\big|t^{-\nu} J_{\nu+1}(t)\big| \le \frac{1}{2^\nu \pi^{1/2}\Gamma\big(\nu+\frac{3}{2}\big)}. \tag{9.5}$$

Proof. By Olver, *et al.* (2010, (10.6.6)),

$$t^{-\nu} J_{\nu+1}(t) = -\big(t^{-\nu} J_\nu(t)\big)', \tag{9.6}$$

$t \ge 0$. For $\nu > -1/2$, it follows by differentiating the Poisson integral (9.2) that

$$\begin{aligned} 2^\nu \pi^{1/2}\Gamma\big(\nu+\tfrac{1}{2}\big)\,|t^{-\nu}J_{\nu+1}(t)| &= \Big|\int_0^\pi \cos\theta\ \sin(t\cos\theta)\ (\sin\theta)^{2\nu}\,\mathrm{d}\theta\Big| \\ &\le \int_0^\pi |\cos\theta|\ |(\sin\theta)^{2\nu}|\,\mathrm{d}\theta. \end{aligned}$$

By a substitution, $s = \sin^2\theta$, the latter integral reduces to a *beta integral*,

$$\int_0^1 s^{a-1}(1-s)^{b-1}\,\mathrm{d}s = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)},$$

$a, b > 0$. This produces (9.5).

For $\nu = -1/2$, it follows from (9.6) and (9.1) that

$$t^{1/2} J_{1/2}(t) = (2/\pi)^{1/2}\,\sin t; \tag{9.7}$$

cf. Olver, *et al.* (2010, (10.16.1)). Then, $|t^{1/2}J_{1/2}(t)| \le (2/\pi)^{1/2}$, as stated in (9.5). □

Remark 5 Substituting $\nu = 0$ in Lemma 3, we obtain $|J_1(t)| \le 2/\pi$, $t \ge 0$. This bound is sharper than a bound given in Olver, *et al.* (2010, (10.14.1)), *viz.*, $|J_1(t)| \le 2^{-1/2}$, $t \ge 0$.

Lemma 4 *For $\nu \ge -1/2$, the function $t^{-\nu} J_{\nu+1}(t)$, $t \ge 0$, is Lipschitz continuous, satisfying for $u, v \in \mathbb{R}$, the inequality*

$$\big|u^{-\nu} J_{\nu+1}(u) - v^{-\nu} J_{\nu+1}(v)\big| \le \frac{1}{2^{\nu+1}\Gamma(\nu+2)}\,|u-v|. \tag{9.8}$$

Proof. For $\nu > -1/2$ we apply (9.6), (9.2), and the triangle inequality to obtain

$$\begin{aligned} &2^{\nu}\pi^{1/2}\Gamma\big(\nu+\tfrac{1}{2}\big)\,\big|u^{-\nu}J_{\nu+1}(u) - v^{-\nu}J_{\nu+1}(v)\big| \\ &\qquad\qquad\le \int_0^{\pi} |\sin(u\cos\theta) - \sin(v\cos\theta)|\;|\cos\theta|\;(\sin\theta)^{2\nu}\,\mathrm{d}\theta. \end{aligned}$$

By a well-known trigonometric identity, and the inequality $|\sin t| \le |t|$, $t \in \mathbb{R}$,

$$\begin{aligned} |\sin(u\cos\theta) - \sin(v\cos\theta)| &= 2\big|\sin\big(\tfrac{1}{2}(u-v)\cos\theta\big)\;\cos\big(\tfrac{1}{2}(u+v)\cos\theta\big)\big| \\ &\le |u-v|\;|\cos\theta|\;\big|\cos\big(\tfrac{1}{2}(u+v)\cos\theta\big)\big| \\ &\le |u-v|\;|\cos\theta|. \end{aligned} \tag{9.9}$$

Therefore,

$$\big|u^{-\nu}J_{\nu+1}(u) - v^{-\nu}J_{\nu+1}(v)\big| \le \frac{2}{2^{\nu}\pi^{1/2}\Gamma\big(\nu+\frac{1}{2}\big)}|u-v|\int_0^{\pi/2}(\cos\theta)^2\;(\sin\theta)^{2\nu}\,\mathrm{d}\theta.$$

Substituting $t = \sin^2\theta$ reduces the latter integral to a beta integral, and then we obtain (9.8).

For $\nu = -1/2$, we apply (9.7) to obtain

$$\big|u^{1/2}J_{1/2}(u) - v^{1/2}J_{1/2}(v)\big| = (2/\pi)^{1/2}\;|\sin u - \sin v| \le (2/\pi)^{1/2}\;|u-v|,$$

the latter inequality following from (9.9) with $\theta = 0$. Then, we obtain (9.8) for $\nu = -1/2$. □

As regards the modified Bessel function I_ν, defined in (2.2), with $\mathrm{i} = \sqrt{-1}$ we find from (2.1) that $I_\nu(x) = \mathrm{i}^{-\nu}\,J_\nu(\mathrm{i}x)$, $x \in \mathbb{R}$; hence, by (9.3),

$$|\Gamma(\nu+1)\,(x/2)^{-\nu}\,I_\nu(x)| \le 1. \tag{9.10}$$

For $n \in \mathbb{N}_0$ and $\alpha > 0$, the (*generalized*) *Laguerre polynomial* of order $\alpha - 1$ and degree n is

$$L_n^{(\alpha-1)}(x) = \frac{(\alpha)_n}{n!}\,{}_1F_1(-n;\alpha;x) = \sum_{k=0}^{n}\frac{(\alpha+k)_{n-k}}{(n-k)!}\,\frac{(-x)^k}{k!},$$

$x \in \mathbb{R}$; see Olver, *et al.* (2010, Chapter 18) or Szegö (1967, Chapter 5). The *normalized (generalized) Laguerre polynomial* of order $\alpha - 1$ and degree n is defined by

$$\mathcal{L}_n^{(\alpha-1)}(x) := \left(\frac{n!}{(\alpha)_n}\right)^{1/2} L_n^{(\alpha-1)}(x), \tag{9.11}$$

$x \in \mathbb{R}$. It is well-known (see Olver, *et al.* (2010, Chapter 18.3) or Szegö (1967, Chapter 5.1)) that the polynomials $\mathcal{L}_n^{(\alpha-1)}$ are orthonormal with respect to the *Gamma*$(\alpha, 1)$ distribution:

$$\int_0^\infty \mathcal{L}_n^{(\alpha-1)}(x)\mathcal{L}_m^{(\alpha-1)}(x)\frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)}\,\mathrm{d}x = \begin{cases}1, & \text{if } n = m\\ 0, & \text{if } n \ne m\end{cases}$$

Lemma 5 *For $v > 0$ and $\alpha > 0$,*

$$\int_0^\infty x^\alpha e^{-vx} L_n^{(\alpha-1)}(x)\,\mathrm{d}x = \frac{\Gamma(\alpha+n)}{n!}(v-1)^{n-1}v^{-(\alpha+n+1)}\big(\alpha(v-1) - n\big).$$

Proof. Starting with the known integral (Olver, *et al.*, 2010, (18.17.34)),

$$\int_0^\infty x^{\alpha-1} e^{-vx} L_n^{(\alpha-1)}(x)\,\mathrm{d}x = \frac{\Gamma(\alpha+n)}{n!}\,(v-1)^n\,v^{-(\alpha+n)},$$

we differentiate each side with respect to v and simplify the outcome to obtain the result. □

Proof of Lemma 1. (i) By (9.4) for $J_\nu(x)$, $\Gamma(\nu+1)\big|(tx)^{-\nu/2}J_\nu(2\sqrt{tx})\big| \le 1$ for all $x, t > 0$. Therefore, by the triangle inequality, $|\mathcal{H}_{X,\nu}(t)| \le 1$.

(ii) It follows from the series expansion (2.1) that

$$\Gamma(\nu+1)(tx)^{-\nu/2}J_\nu\big(2(tx)^{1/2}\big)\Big|_{t=0} = 1,$$

for all x, so we obtain $\mathcal{H}_{X,\nu}(0) = 1$.

(iii) As the function $(tx)^{-\nu/2}J_\nu(2\sqrt{tx})$ is a power series in tx, it is continuous in $t \ge 0$ for every fixed $x \ge 0$. As it is also bounded, then $\Gamma(\nu+1)(tx)^{-\nu/2}J_\nu(2\sqrt{tx})f(x)$ is bounded by the Lebesgue integrable function $f(x)$ for all $x, t \ge 0$. Therefore, the conclusion follows from the Dominated Convergence Theorem. □

The following Hankel transform inversion theorem is a classical result that can be obtained from many sources, e.g., Sneddon (1972, p. 309, Theorem 1).

Theorem 11 (Hankel Inversion) *Let X be a positive, continuous random variable with probability density function $f(x)$ and Hankel transform $\mathcal{H}_{X,\nu}$. For $x > 0$,*

$$f(x) = \frac{1}{\Gamma(\nu+1)}\int_0^\infty (tx)^{\nu/2}J_\nu(2\sqrt{tx})\ \mathcal{H}_{X,\nu}(t)\,\mathrm{d}t,$$

As a consequence of the inversion formula, we obtain the uniqueness of the Hankel transform.

Theorem 12 (Hankel Uniqueness) *Let X and Y be positive random variables with corresponding Hankel transforms $\mathcal{H}_{X,\nu}$ and $\mathcal{H}_{Y,\nu}$. Then $\mathcal{H}_{X,\nu} = \mathcal{H}_{Y,\nu}$ if and only if $X \stackrel{d}{=} Y$.*

The next result, on the continuity of the Hankel transform, is analogous to Theorem 2.3 of Baringhaus and Taherizadeh (2010). Therefore, we will omit the proof.

Theorem 13 (Hankel Continuity) *Let $\{X_n, n \in \mathbb{N}\}$ be a sequence of positive random variables with corresponding Hankel transforms $\{\mathcal{H}_n, n \in \mathbb{N}\}$. If there exists a positive random variable X, with Hankel transform $\mathcal{H}$, such that $X_n \xrightarrow{d} X$, then for all $t \ge 0$,*

$$\lim_{n\to\infty} \mathcal{H}_n(t) = \mathcal{H}(t) \tag{9.12}$$

Conversely, suppose there exists $\mathcal{H} : [0,\infty) \to \mathbb{R}$ such that $\mathcal{H}(0) = 1$, $\mathcal{H}$ is continuous at 0, and (9.12) holds. Then $\mathcal{H}$ is the Hankel transform of a positive random variable X, and $X_n \xrightarrow{d} X$.

10 Appendix: The Test Statistic

Proof of Proposition 1. By squaring the integrand in (1.3), there are three terms to be calculated. First,

$$\begin{aligned} n\int_0^\infty \mathcal{H}_n^2(t)\,\mathrm{d}P_0(t) &= \frac{1}{n}\int_0^\infty \Big(\sum_{i=1}^n \Gamma(\alpha)(Y_i t)^{(1-\alpha)/2}\,J_{\alpha-1}(2\sqrt{tY_i})\Big)^2\,\mathrm{d}P_0(t) \\ &= \frac{\Gamma(\alpha)}{n}\sum_{i=1}^n\sum_{j=1}^n (Y_iY_j)^{(1-\alpha)/2}\int_0^\infty J_{\alpha-1}(2\sqrt{tY_i})J_{\alpha-1}(2\sqrt{tY_j})e^{-t}\,\mathrm{d}t. \end{aligned}$$

These integrals are of the form of Weber's exponential integral (Olver, *et al.*, 2010, (10.22.67)):

$$\int_0^\infty \exp(-pt)\,J_\nu(2\sqrt{at})\,J_\nu(2\sqrt{bt})\,\mathrm{d}t = p^{-1}\exp\big(-(a+b)/p\big)\,I_\nu(2\sqrt{ab}/p), \tag{10.1}$$

valid for $\nu > -1$ and $a, b, p > 0$. Simplifying the resulting expressions, we obtain

$$n \int_0^\infty \mathcal{H}_n^2(t)\, \mathrm{d}P_0(t) = \frac{\Gamma(\alpha)}{n} \sum_{i=1}^n \sum_{j=1}^n (Y_i Y_j)^{(1-\alpha)/2} \exp(-Y_i - Y_j)\, I_{\alpha-1}\big(2(Y_i Y_j)^{1/2}\big).$$

Second, by proceeding as in Example 1, it is straightforward to deduce

$$\begin{aligned} 2n \int_0^\infty \mathcal{H}_n(t)\, e^{-t/\alpha}\, \mathrm{d}P_0(t) &= 2 \sum_{i=1}^n (1+\alpha^{-1})^{-\alpha}\, e^{-\alpha Y_i/(\alpha+1)} \\ &\equiv \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \left(\frac{\alpha}{\alpha+1}\right)^\alpha \left[e^{-\alpha Y_i/(\alpha+1)} + e^{-\alpha Y_j/(\alpha+1)}\right]. \end{aligned}$$

Third, we have a gamma integral:

$$n \int_0^\infty e^{-2t/\alpha}\, \mathrm{d}P_0(t) = n \left(\frac{\alpha}{\alpha+2}\right)^\alpha = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \left(\frac{\alpha}{\alpha+2}\right)^\alpha.$$

Collecting together all three terms, we obtain the desired result. □

Proof of Theorem 2. By (9.6), $(s^{1-\alpha} J_{\alpha-1}(s))' = -s^{1-\alpha} J_\alpha(s)$. Therefore, the Taylor expansion of order 1 of the function $s^{1-\alpha} J_{\alpha-1}(s)$, at a point s_0, is

$$s^{1-\alpha} J_{\alpha-1}(s) = s_0^{1-\alpha} J_{\alpha-1}(s_0) + (s_0 - s) u^{1-\alpha} J_\alpha(u),$$

where u lies between s and s_0. Setting $s = 2(tY_j)^{1/2}$ and $s_0 = 2(tX_j/\alpha)^{1/2}$, we obtain

$$\begin{aligned} 2^{1-\alpha} (tY_j)^{(1-\alpha)/2}\, J_{\alpha-1}\big(2(tY_j)^{1/2}\big) &= 2^{1-\alpha} (tX_j/\alpha)^{(1-\alpha)/2} J_{\alpha-1}\big(2(tX_j/\alpha)^{1/2}\big) \\ &\quad + 2\big[(tX_j/\alpha)^{1/2} - (tY_j)^{1/2}\big]\, u_j^{1-\alpha}\, J_\alpha(u_j), \end{aligned} \tag{10.2}$$

where u_j lies between $2(tY_j)^{1/2}$and $2(tX_j/\alpha)^{1/2}$. Define

$$W_n = \alpha^{-1/2} - \overline{X}_n^{-1/2} = \frac{\overline{X}_n - \alpha}{(\alpha \overline{X}_n)^{1/2} (\alpha^{1/2} + \overline{X}_n^{1/2})}; \tag{10.3}$$

then

$$(tX_j/\alpha)^{1/2} - (tY_j)^{1/2} = (tX_j/\alpha)^{1/2} - (tX_j/\overline{X}_n)^{1/2} = W_n\, (tX_j)^{1/2},$$

and (10.2) reduces to

$$\begin{aligned} &2^{1-\alpha} (tY_j)^{(1-\alpha)/2}\, J_{\alpha-1}\big(2(tY_j)^{1/2}\big) \\ &\qquad = 2^{1-\alpha} (tX_j/\alpha)^{(1-\alpha)/2}\, J_{\alpha-1}\big(2(tX_j/\alpha)^{1/2}\big) + 2W_n\, (tX_j)^{1/2}\, u_j^{1-\alpha}\, J_\alpha(u_j). \end{aligned} \tag{10.4}$$

Multiplying both sides of (10.4) by $2^{\alpha-1}$, adding and subtracting the term

$$2(tX_j)^{1/2} W_n\, (tX_j/\alpha)^{(1-\alpha)/2}\, J_\alpha\big(2(tX_j/\alpha)^{1/2}\big)$$

on the right-hand side, and then simplifying the result, we obtain

$$\begin{aligned} &(tY_j)^{(1-\alpha)/2} J_{\alpha-1}\big(2(tY_j)^{1/2}\big) \\ &\quad = (tX_j/\alpha)^{(1-\alpha)/2}\, J_{\alpha-1}\big(2(tX_j/\alpha)^{1/2}\big) \\ &\qquad + 2\alpha^{1/2}\, W_n\, (tX_j/\alpha)^{1-(\alpha/2)}\, J_\alpha\big(2(tX_j/\alpha)^{1/2}\big) \\ &\qquad + 2^\alpha\, W_n\, (tX_j)^{1/2} \left(u_j^{1-\alpha}\, J_\alpha(u_j) - \big(2(tX_j/\alpha)^{1/2}\big)^{1-\alpha} J_\alpha\big(2(tX_j/\alpha)^{1/2}\big)\right). \end{aligned} \tag{10.5}$$

Define the processes $Z_{n,1}(t)$, $Z_{n,2}(t)$, and $Z_{n,3}(t)$, $t \geq 0$, by

$$\begin{aligned} Z_{n,1}(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \big[\Gamma(\alpha)\, (tX_j/\alpha)^{(1-\alpha)/2}\, J_{\alpha-1}\big(2(tX_j/\alpha)^{1/2}\big) \\ + 2\Gamma(\alpha)\alpha^{1/2}\, W_n\, (tX_j/\alpha)^{1-(\alpha/2)}\, J_{\alpha}\big(2(tX_j/\alpha)^{1/2}\big) - e^{-t/\alpha}\big], \end{aligned}$$

$$Z_{n,2}(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \big[\Gamma(\alpha)\, (tX_j/\alpha)^{(1-\alpha)/2}\, J_{\alpha-1}\big(2(tX_j/\alpha)^{1/2}\big) + 2\alpha^{-1/2}\, W_n\, te^{-t/\alpha} - e^{-t/\alpha}\big],$$

$$Z_{n,3}(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \big[\Gamma(\alpha)\, (tX_j/\alpha)^{(1-\alpha)/2}\, J_{\alpha-1}\big(2(tX_j/\alpha)^{1/2}\big) + \alpha^{-2}(X_j - \alpha)\, te^{-t/\alpha} - e^{-t/\alpha}\big].$$

We will show that

$$Z_{n,3} \xrightarrow{d} Z \ \text{ in } L^2, \tag{10.6}$$

$$\|Z_n - Z_{n,1}\|_{L^2} \xrightarrow{p} 0, \tag{10.7}$$

$$\|Z_{n,1} - Z_{n,2}\|_{L^2} \xrightarrow{p} 0, \tag{10.8}$$

$$\|Z_{n,2} - Z_{n,3}\|_{L^2} \xrightarrow{p} 0. \tag{10.9}$$

To establish (10.6), let

$$Z_{n,3,j}(t) := \Gamma(\alpha)\, (tX_j/\alpha)^{(1-\alpha)/2}\, J_{\alpha-1}\big(2(tX_j/\alpha)^{1/2}\big) + \alpha^{-2}(X_j - \alpha)te^{-t/\alpha} - e^{-t/\alpha}, \tag{10.10}$$

$t \geq 0$, $j = 1, \ldots, n$. Since $X_j \sim Gamma(\alpha, 1)$ then $E(X_j - \alpha) = 0$; also, by Example 1,

$$E\big[\Gamma(\alpha)(tX_j/\alpha)^{(1-\alpha)/2} J_{\alpha-1}\big(2(tX_j/\alpha)^{1/2}\big)\big] = e^{-t/\alpha}.$$

Therefore $E(Z_{n,3,j}(t)) = 0$, $t \geq 0$ and $j = 1, \ldots, n$, and $Z_{n,3,1}, \ldots, Z_{n,3,n}$ clearly are i.i.d. random elements in L^2. Applying the Cauchy-Schwarz inequality and (9.4), we obtain $E(\|Z_{n,3,1}\|_{L^2}^2) < \infty$. Thus, by the Central Limit Theorem in L^2 (Ledoux and Talagrand, 1991, p. 281),

$$Z_{n,3} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} Z_{n,3,j} \xrightarrow{d} Z,$$

where $Z := (Z(t), t \geq 0)$ is a centered Gaussian random element in L^2. This proves (10.6) and shows that Z has the same covariance operator as $Z_{n,3,1}$.

It is well-known that the covariance operator of the random element $Z_{n,3,1}$ is uniquely determined by the covariance function of the stochastic process $Z_{n,3,1}(t)$ (Gīkhman and Skorokhod, 1980, pp. 218-219). We now show that the function $K(s, t)$ in (3.4) is the covariance function of $Z_{n,3,1}$. Noting that $E[Z_{n,3,1}(t)] = 0$ for all t, we obtain

$$\begin{aligned} K(s,t) &= \operatorname{Cov}\big[Z_{n,3,1}(s), Z_{n,3,1}(t)\big] \\ &= \operatorname{Cov}\big[Z_{n,3,1}(s) + e^{-s/\alpha}, Z_{n,3,1}(t) + e^{-t/\alpha}\big] \\ &= E\big[\big(Z_{n,3,1}(s) + e^{-s/\alpha}\big)\big(Z_{n,3,1}(t) + e^{-t/\alpha}\big)\big] - e^{-(s+t)/\alpha}. \end{aligned}$$

By (10.10),

$$\begin{aligned} & E\big(Z_{n,3,1}(s) + e^{-s/\alpha}\big)\big(Z_{n,3,1}(t) + e^{-t/\alpha}\big) \\ &\quad = E\big[\Gamma(\alpha)\, (sX_1/\alpha)^{(1-\alpha)/2}\, J_{\alpha-1}\big(2(sX_1/\alpha)^{1/2}\big) + \alpha^{-2}(X_1 - \alpha)se^{-s/\alpha}\big] \\ &\qquad \times \big[\Gamma(\alpha)\, (tX_1/\alpha)^{(1-\alpha)/2}\, J_{\alpha-1}\big(2(tX_1/\alpha)^{1/2}\big) + \alpha^{-2}(X_1 - \alpha)te^{-t/\alpha}\big], \end{aligned} \tag{10.11}$$

so the calculation of $K(s, t)$ reduces to evaluating the four terms obtained by expanding the product on the right-hand side of (10.11).

The first term in the product in (10.11) is evaluated using Weber's integral (10.1):

$$\begin{aligned} & E\,[\Gamma(\alpha)]^2 (sX_1/\alpha)^{(1-\alpha)/2} (tX_1/\alpha)^{(1-\alpha)/2} J_{\alpha-1}(2(sX_1/\alpha)^{1/2}) J_{\alpha-1}(2(tX_1/\alpha)^{1/2}) \\ &\qquad = \Gamma(\alpha)(st/\alpha^2)^{(1-\alpha)/2} e^{-(s+t)/\alpha} I_{\alpha-1}(2\sqrt{st}/\alpha). \end{aligned} \tag{10.12}$$

The second term in the product in (10.11) is a Hankel transform of the type in Example 1,

$$E\big[\Gamma(\alpha)(sX_1/\alpha)^{(1-\alpha)/2}J_{\alpha-1}(2(sX_1/\alpha)^{1/2})\,\alpha^{-2}(X_1-\alpha)te^{-t/\alpha}\big] = -\alpha^{-3}st\exp\big(-(s+t)/\alpha\big),$$

and the third term in the product is the same as the second term but with s and t interchanged.

The fourth term in the product in (10.11) is

$$E\big[\alpha^{-4}(X_1-\alpha)^2 ste^{-(s+t)/\alpha}\big] = \alpha^{-4}ste^{-(s+t)/\alpha}\operatorname{Var}(X_1) = \alpha^{-3}ste^{-(s+t)/\alpha}.$$

Combining all four terms, we obtain (3.4).

To establish (10.7), we begin by showing that

$$(\sqrt{n}W_n)^2 = \bigg(\frac{\sqrt{n}(\overline{X}_n-\alpha)}{(\alpha\overline{X}_n)^{1/2}(\alpha^{1/2}+\overline{X}_n^{1/2})}\bigg)^2 \xrightarrow{d} \chi_1^2/4\alpha^2,$$

where χ_1^2 denotes a chi-square random variable with one degree of freedom. By the Central Limit Theorem, $\sqrt{n}(\overline{X}_n-\alpha) \xrightarrow{d} \mathcal{N}(0,\alpha)$, and by the Law of Large Numbers and the Continuous Mapping Theorem, $(\alpha\overline{X}_n)^{1/2}(\alpha^{1/2}+\overline{X}_n^{1/2}) \xrightarrow{p} 2\alpha^{3/2}$. By Slutsky's theorem (Chow and Teicher, 1988, p. 249), $\sqrt{n}W_n \xrightarrow{d} \mathcal{N}(0,\frac{1}{4}\alpha^{-2})$, hence $(\sqrt{n}W_n)^2 \xrightarrow{d} \chi_1^2/4\alpha^2$.

By the Taylor expansion in (10.5),

$$\begin{aligned} Z_n - Z_{n,1} &= \frac{\Gamma(\alpha)}{\sqrt{n}}\sum_{j=1}^n \Big[(tY_j)^{(1-\alpha)/2}J_{\alpha-1}(2(tY_j)^{1/2}) - (tX_j/\alpha)^{(1-\alpha)/2}J_{\alpha-1}(2(tX_j/\alpha)^{1/2}) \\ &\qquad - 2\alpha^{1/2}\,W_n\,(tX_j/\alpha)^{1-(\alpha/2)}\,J_\alpha\big(2(tX_j/\alpha)^{1/2}\big)\Big] \\ &= \frac{2^\alpha\Gamma(\alpha)}{n}(\sqrt{n}W_n)\,\sum_{j=1}^n (tX_j)^{1/2}\Big[u_j^{1-\alpha}\,J_\alpha(u_j) - \big(2(tX_j/\alpha)^{1/2}\big)^{1-\alpha}\,J_\alpha\big(2(tX_j/\alpha)^{1/2}\big)\Big]. \end{aligned}$$

Define

$$V_n := \frac{1}{n^2}\int_0^\infty \Big[\sum_{j=1}^n (tX_j)^{1/2}\Big(u_j^{1-\alpha}\,J_\alpha(u_j) - \big(2(tX_j/\alpha)^{1/2}\big)^{1-\alpha}\,J_\alpha\big(2(tX_j/\alpha)^{1/2}\big)\Big)\Big]^2\,\mathrm{d}P_0(t).$$

Then, $\|Z_n - Z_{n,1}\|_{L^2}^2 = 4^\alpha[\Gamma(\alpha)]^2(\sqrt{n}W_n)^2\,V_n$. By the Cauchy-Schwarz inequality,

$$V_n \le \frac{1}{n}\int_0^\infty t\sum_{j=1}^n X_j\,\big|u_j^{1-\alpha}\,J_\alpha(u_j) - \big(2(tX_j/\alpha)^{1/2}\big)^{1-\alpha}\,J_\alpha\big(2(tX_j/\alpha)^{1/2}\big)\big|^2\,\mathrm{d}P_0(t).$$

Recall that u_j lies between $2(tY_j)^{1/2}$ and $2(tX_j/\alpha)^{1/2}$, so we can write

$$u_j = 2(1-\theta_{n,j,t})(tX_j/\alpha)^{1/2} + 2\theta_{n,j,t}(tY_j)^{1/2} = 2(tX_j)^{1/2}\big(\alpha^{-1/2} + \theta_{n,j,t}(\overline{X}_n^{-1/2} - \alpha^{-1/2})\big),$$

where $\theta_{n,j,t} \in [0,1]$. By Lemma 4, the Lipschitz property of the Bessel functions,

$$\begin{aligned} 4^\alpha[\Gamma(\alpha+1)]^2\,\big|u_j^{1-\alpha}\,J_\alpha(u_j) - \big(2(tX_j/\alpha)^{1/2}\big)^{1-\alpha}\,J_\alpha\big(2(tX_j/\alpha)^{1/2}\big)\big|^2 & \\ &\le \big|u_j - 2(tX_j/\alpha)^{1/2}\big|^2 \\ &= \big|2(tX_j)^{1/2}\theta_{n,j,t}(\overline{X}_n^{-1/2} - \alpha^{-1/2})\big|^2 \\ &\le 4tX_j\,(\overline{X}_n^{-1/2} - \alpha^{-1/2})^2, \end{aligned}$$

since $\theta_{n,j,t} \in [0,1]$. Therefore,

$$V_n \le \frac{1}{4^{\alpha-1}[\Gamma(\alpha+1)]^2}\Big(\frac{1}{n}\sum_{j=1}^n X_j^2\Big)(\overline{X}_n^{-1/2} - \alpha^{-1/2})^2\int_0^\infty t^2\,\mathrm{d}P_0(t).$$

By the Law of Large Numbers, $(\overline{X}_n^{-1/2} - \alpha^{-1/2})^2 \xrightarrow{p} 0$ and $n^{-1}\sum_{j=1}^n X_j^2 \xrightarrow{p} E(X_1^2) = \alpha(\alpha+1)$, so it follows that $V_n \xrightarrow{p} 0$. By Slutsky's theorem, $\|Z_n - Z_{n,1}\|_{L^2}^2 = 4^\alpha[\Gamma(\alpha)]^2(\sqrt{n}W_n)^2\cdot V_n \xrightarrow{d} 0$, therefore $\|Z_n - Z_{n,1}\|_{L^2} \xrightarrow{p} 0$, as asserted in (10.7).

To establish (10.8), define

$$\varDelta_j(t) := \varGamma(\alpha)(tX_j/\alpha)^{1-(\alpha/2)} J_\alpha\big(2(tX_j/\alpha)^{1/2}\big) - \alpha^{-1} t e^{-t/\alpha},$$

$t \ge 0$, $j = 1, \ldots, n$. Then it is straightforward to verify that

$$Z_{n,1} - Z_{n,2} = \frac{2\alpha^{1/2}}{\sqrt{n}} W_n \sum_{j=1}^{n} \varDelta_j(t)$$

and therefore

$$\|Z_{n,1} - Z_{n,2}\|_{L^2}^2 = (2\alpha^{1/2} W_n)^2 \int_0^\infty \Big[\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \varDelta_j(t)\Big]^2 \, \mathrm{d}P_0(t). \tag{10.13}$$

By the Law of Large Numbers, $W_n \xrightarrow{p} 0$. Also, as shown in Example 3,

$$E\big[\varGamma(\alpha)(tX_j/\alpha)^{1-(\alpha/2)} J_\alpha\big(2(tX_j/\alpha)^{1/2}\big)\big] = \alpha^{-1} t e^{-t/\alpha};$$

hence $E(\varDelta_j(t)) = 0$, $t \ge 0$, $j = 1, \ldots, n$. Also, $\varDelta_1(t), \ldots, \varDelta_n(t)$ are i.i.d. random elements in L^2. We now show that $E(\|\varDelta_1\|_{L^2}^2) < \infty$. We have

$$\begin{aligned} E(\|\varDelta_1\|_{L^2}^2) &= E \int_0^\infty \varDelta_1^2(t) \, \mathrm{d}P_0(t) \\ &= E \int_0^\infty \big[\varGamma(\alpha)(tX_1/\alpha)^{1-(\alpha/2)} J_\alpha\big(2(tX_1/\alpha)^{1/2}\big) - \alpha^{-1} t e^{-t/\alpha}\big]^2 \, \mathrm{d}P_0(t). \end{aligned}$$

To show that $E(\|\varDelta_1\|_{L^2}^2) < \infty$ it suffices, by the Cauchy-Schwarz inequality, to prove that

$$E \int_0^\infty \big[\varGamma(\alpha)(tX_1/\alpha)^{1-(\alpha/2)} \, J_\alpha\big(2(tX_1/\alpha)^{1/2}\big)\big]^2 \, \mathrm{d}P_0(t) \ < \ \infty \tag{10.14}$$

and

$$E \int_0^\infty (\alpha^{-1} t e^{-t/\alpha})^2 \, \mathrm{d}P_0(t) \ < \ \infty. \tag{10.15}$$

To establish (10.14), we apply the inequality (9.5) to obtain

$$|J_\alpha\big(2(tX_1/\alpha)^{1/2}\big)| \le (tX_1/\alpha)^{-(1-\alpha)/2} / \pi^{1/2} \varGamma(\alpha + \tfrac{1}{2}),$$

for $t \ge 0$. Therefore,

$$\begin{aligned} E \int_0^\infty \big[\varGamma(\alpha)(tX_1/\alpha)^{1-(\alpha/2)} \, J_\alpha\big(2(tX_1/\alpha)^{1/2}\big)\big]^2 \, \mathrm{d}P_0(t) & \\ \le \Big(\frac{\varGamma(\alpha)}{\pi^{1/2}\varGamma(\alpha + \frac{1}{2})}\Big)^2 \, E(X_1/\alpha) \, \int_0^\infty t \, \mathrm{d}P_0(t) < \infty. & \end{aligned}$$

As for (10.15), that expectation is a convergent gamma integral. Hence, $E(\|\varDelta_1\|_{L^2}^2) < \infty$.

By the Central Limit Theorem in L^2, $n^{-1/2} \sum_{j=1}^{n} \varDelta_j(t)$ converges to a centered Gaussian random element in L^2. Thus, by the Continuous Mapping Theorem,

$$\Big\|\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \varDelta_j(t)\Big\|_{L^2}^2 := \int_0^\infty \Big[\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \varDelta_j(t)\Big]^2 \, \mathrm{d}P_0(t)$$

converges in distribution to a random variable which has finite variance. Since $W_n \xrightarrow{p} 0$ then by (10.13) and Slutsky's Theorem, we obtain $\|Z_{n,1} - Z_{n,2}\|_{L^2}^2 \xrightarrow{d} 0$; therefore, $\|Z_{n,1} - Z_{n,2}\|_{L^2} \xrightarrow{p} 0$.

To prove (10.9), we observe that

$$\begin{aligned} Z_{n,2} - Z_{n,3} &= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \Big(2\alpha^{-1/2} W_n t e^{-t/\alpha} - \alpha^{-2}(X_j - \alpha) t e^{-t/\alpha}\Big) \\ &= t e^{-t/\alpha} \sqrt{n}(\overline{X}_n - \alpha) R_n, \end{aligned}$$

where

$$R_n = \frac{2}{\alpha \overline{X}_n^{1/2}(\alpha^{1/2} + \overline{X}_n^{1/2})} - \frac{1}{\alpha^2}.$$

Therefore,

$$\|Z_{n,2} - Z_{n,3}\|_{L^2}^2 = \left[\sqrt{n}(\overline{X}_n - \alpha) R_n\right]^2 \int_0^\infty (te^{-t/\alpha})^2 \, \mathrm{d}P_0(t).$$

As noted earlier, $\int_0^\infty (te^{-t/\alpha})^2 \, \mathrm{d}P_0(t) < \infty$. Also, by the Central Limit Theorem, $\sqrt{n}(\overline{X}_n - \alpha) \xrightarrow{d} \mathcal{N}(0, \alpha)$; and by the Law of Large Numbers, $R_n \xrightarrow{p} 0$. By Slutsky's theorem, $\left[\sqrt{n}(\overline{X}_n - \alpha) R_n\right]^2 \xrightarrow{d} 0$; hence $\left[\sqrt{n}(\overline{X}_n - \alpha) R_n\right]^2 \xrightarrow{p} 0$, and therefore $\|Z_{n,2} - Z_{n,3}\|_{L^2} \xrightarrow{p} 0$.

Finally, by the Continuous Mapping Theorem in L^2, $\|Z_n\|_{L^2}^2 \xrightarrow{d} \|Z\|_{L^2}^2$, i.e.

$$T_n^2 = \int_0^\infty Z_n^2(t) \, \mathrm{d}P_0(t) \xrightarrow{d} \int_0^\infty Z^2(t) \, \mathrm{d}P_0(t).$$

The proof now is complete. □

11 Appendix: Eigenvalues and Eigenfunctions of the Covariance Operator

Proof of Theorem 5. Since the set $\{\mathfrak{L}_k^{(\alpha-1)} : k \in \mathbb{N}_0\}$ is an orthonormal basis for L^2, the eigenfunction $\phi \in L^2$ corresponding to an eigenvalue δ can be written as

$$\phi = \sum_{k=0}^\infty \langle \phi, \mathfrak{L}_k^{(\alpha-1)} \rangle_{L^2} \mathfrak{L}_k^{(\alpha-1)}.$$

We restrict ourselves temporarily to eigenfunctions for which this series is pointwise convergent. Substituting this series into the equation $\mathcal{S}\phi = \delta\phi$, we obtain

$$\int_0^\infty K(s,t) \sum_{k=0}^\infty \langle \phi, \mathfrak{L}_k^{(\alpha-1)} \rangle_{L^2} \mathfrak{L}_k^{(\alpha-1)}(t) \, \mathrm{d}P_0(t) = \delta \sum_{k=0}^\infty \langle \phi, \mathfrak{L}_k^{(\alpha-1)} \rangle_{L^2} \mathfrak{L}_k^{(\alpha-1)}(s). \tag{11.1}$$

Substituting the covariance function $K(s,t)$ in the left-hand side of (11.1), writing K in terms of K_0, and assuming that we can interchange the order of integration and summation, we obtain

$$\begin{aligned} &\delta \sum_{k=0}^\infty \langle \phi, \mathfrak{L}_k^{(\alpha-1)} \rangle_{L^2} \mathfrak{L}_k^{(\alpha-1)}(s) \\ &\qquad = \sum_{k=0}^\infty \langle \phi, \mathfrak{L}_k^{(\alpha-1)} \rangle_{L^2} \int_0^\infty \Big[K_0(s,t) - e^{-(s+t)/\alpha}(\alpha^{-3}st + 1)\Big] \mathfrak{L}_k^{(\alpha-1)}(t) \, \mathrm{d}P_0(t). \end{aligned} \tag{11.2}$$

By Theorem 3,

$$\int_0^\infty K_0(s,t) \mathfrak{L}_k^{(\alpha-1)}(t) \, \mathrm{d}P_0(t) = \rho_k \mathfrak{L}_k^{(\alpha-1)}(s).$$

On writing $\mathfrak{L}_k^{(\alpha-1)}$ in terms of $L_k^{(\alpha-1)}$, the generalized Laguerre polynomial, applying the well-known formula (Olver, *et al.*, 2010, (18.17.34)) for the Laplace transform of $L_k^{(\alpha-1)}$, and making use of (4.2) and (4.3), we obtain

$$\langle e^{-t/\alpha}, \mathfrak{L}_k^{(\alpha-1)} \rangle_{L^2} := \int_0^\infty e^{-t/\alpha} \mathfrak{L}_k^{(\alpha-1)}(t) \, \mathrm{d}P_0(t) = \Big(\frac{(\alpha)_k}{k!}\Big)^{1/2} \beta^{\alpha/2} \rho_k. \tag{11.3}$$

Again writing $\mathfrak{L}_k^{(\alpha-1)}$ in terms of $L_k^{(\alpha-1)}$, applying Lemma 5, and (4.2) and (4.3), we obtain

$$\langle te^{-t/\alpha}, \mathfrak{L}_k^{(\alpha-1)} \rangle_{L^2} := \int_0^\infty te^{-t/\alpha} \mathfrak{L}_k^{(\alpha-1)}(t) \, \mathrm{d}P_0(t) = \Big(\frac{(\alpha)_k}{k!}\Big)^{1/2} \alpha^2 \beta^{\alpha/2} \rho_k (b_\alpha^2 - k\beta). \tag{11.4}$$

In summary, (11.2) reduces to

$$\delta \sum_{k=0}^{\infty} \langle \phi, \mathfrak{L}_k^{(\alpha-1)} \rangle_{L^2} \mathfrak{L}_k^{(\alpha-1)}(s) = \sum_{k=0}^{\infty} \rho_k \langle \phi, \mathfrak{L}_k^{(\alpha-1)} \rangle_{L^2} \Big[\mathfrak{L}_k^{(\alpha-1)}(s) - e^{-s/\alpha} \Big(\frac{(\alpha)_k}{k!} \Big)^{1/2} \beta^{\alpha/2} \big(\alpha^{-1} s (b_\alpha^2 - k\beta) + 1 \big) \Big]. \tag{11.5}$$

By applying (11.3), we also obtain the Fourier-Laguerre expansion of $e^{-s/\alpha}$ with respect to the orthonormal basis $\{\mathfrak{L}_k^{(\alpha-1)} : k \in \mathbb{N}_0\}$; indeed,

$$e^{-s/\alpha} = \sum_{k=0}^{\infty} \langle e^{-s/\alpha}, \mathfrak{L}_k^{(\alpha-1)} \rangle_{L^2} \mathfrak{L}_k^{(\alpha-1)}(s) = \beta^{\alpha/2} \sum_{k=0}^{\infty} \Big(\frac{(\alpha)_k}{k!} \Big)^{1/2} \rho_k \mathfrak{L}_k^{(\alpha-1)}(s).$$

Similarly, by applying (11.4), we have

$$s e^{-s/\alpha} = \sum_{k=0}^{\infty} \langle s e^{-s/\alpha}, \mathfrak{L}_k^{(\alpha-1)} \rangle_{L^2} \mathfrak{L}_k^{(\alpha-1)}(s) = \alpha^2 \beta^{\alpha/2} \sum_{k=0}^{\infty} \Big(\frac{(\alpha)_k}{k!} \Big)^{1/2} \rho_k (b_\alpha^2 - k\beta) \mathfrak{L}_k^{(\alpha-1)}(s),$$

Let

$$c_1 = \int_0^\infty e^{-t/\alpha} \phi(t) \, \mathrm{d}P_0(t) = \beta^{\alpha/2} \sum_{k=0}^{\infty} \langle \phi, \mathfrak{L}_k^{(\alpha-1)} \rangle_{L^2} \Big(\frac{(\alpha)_k}{k!} \Big)^{1/2} \rho_k, \tag{11.6}$$

and

$$c_2 = \int_0^\infty t e^{-t/\alpha} \phi(t) \, \mathrm{d}P_0(t) = \alpha^2 \beta^{\alpha/2} \sum_{k=0}^{\infty} \langle \phi, \mathfrak{L}_k^{(\alpha-1)} \rangle_{L^2} \Big(\frac{(\alpha)_k}{k!} \Big)^{1/2} \rho_k (b_\alpha^2 - k\beta). \tag{11.7}$$

Combining (11.5)-(11.7), we find that (11.1) reduces to

$$\delta \sum_{k=0}^{\infty} \langle \phi, \mathfrak{L}_k^{(\alpha-1)} \rangle_{L^2} \mathfrak{L}_k^{(\alpha-1)}(s) = \sum_{k=0}^{\infty} \rho_k \Big[\langle \phi, \mathfrak{L}_k^{(\alpha-1)} \rangle_{L^2} - \beta^{\alpha/2} \Big(\frac{(\alpha)_k}{k!} \Big)^{1/2} \big(c_1 + c_2 \alpha^{-1} (b_\alpha^2 - k\beta) \big) \Big] \mathfrak{L}_k^{(\alpha-1)}(s), \tag{11.8}$$

and now comparing the coefficients of $\mathfrak{L}_k^{(\alpha-1)}(s)$, we obtain

$$\delta \, \langle \phi, \mathfrak{L}_k^{(\alpha-1)} \rangle_{L^2} = \rho_k \Big[\langle \phi, \mathfrak{L}_k^{(\alpha-1)} \rangle_{L^2} - \beta^{\alpha/2} \Big(\frac{(\alpha)_k}{k!} \Big)^{1/2} \big(c_1 + c_2 \alpha^{-1} (b_\alpha^2 - k\beta) \big) \Big], \tag{11.9}$$

for all $k \in \mathbb{N}_0$. Since we have assumed that $\delta \neq \rho_k$ for any k then we can solve this equation for $\langle \phi, \mathfrak{L}_k^{(\alpha-1)} \rangle_{L^2}$ to obtain

$$\langle \phi, \mathfrak{L}_k^{(\alpha-1)} \rangle_{L^2} = \beta^{\alpha/2} \frac{\rho_k}{\rho_k - \delta} \Big(\frac{(\alpha)_k}{k!} \Big)^{1/2} \big(c_1 + c_2 \alpha^{-1} (b_\alpha^2 - k\beta) \big). \tag{11.10}$$

Substituting (11.10) into (11.6), we get

$$\begin{aligned} c_1 &= c_1 \beta^\alpha \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!(\rho_k - \delta)} \rho_k^2 + c_2 \alpha^{-1} \beta^\alpha \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!(\rho_k - \delta)} \rho_k^2 (b_\alpha^2 - k\beta) \\ &= c_1 \big(1 - A(\delta) \big) + c_2 \alpha^{-3} D(\delta); \end{aligned}$$

therefore,

$$\alpha^3 c_1 A(\delta) = c_2 D(\delta). \tag{11.11}$$

Similarly, by substituting (11.10) into (11.7), we obtain

$$\begin{aligned} c_2 &= c_1 \alpha^2 \beta^\alpha \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!(\rho_k - \delta)} \rho_k^2 (b_\alpha^2 - k\beta) + c_2 \alpha \beta^\alpha \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!(\rho_k - \delta)} \rho_k^2 (b_\alpha^2 - k\beta)^2 \\ &= c_1 D(\delta) + c_2 \big(1 - B(\delta) \big); \end{aligned}$$

hence,

$$c_2 B(\delta) = c_1 D(\delta). \tag{11.12}$$

Suppose $c_1 = c_2 = 0$; then it follows from (11.10) that $\langle \phi, \mathfrak{L}_k^{(\alpha-1)} \rangle_{L^2} = 0$ for all k and so $\phi = 0$, which is a contradiction since ϕ is a non-trivial eigenfunction. Hence, c_1 and c_2 cannot be both equal to 0. Combining (11.11) and (11.12), and using the fact that c_1, c_2 are not both 0, it is straightforward to deduce that $\alpha^3 A(\delta)B(\delta) = D^2(\delta)$. Therefore, if δ is a positive eigenvalue of $\mathcal{S}$ then it is a positive root of the function $G(\delta) = \alpha^3 A(\delta)B(\delta) - D^2(\delta)$.

Conversely, suppose that δ is a positive root of $G(\delta)$ with $\delta \neq \rho_k$ for any $k \in \mathbb{N}_0$. Define

$$\gamma_k := \beta^{\alpha/2} \Big(\frac{(\alpha)_k}{k!}\Big)^{1/2} \frac{\rho_k}{\rho_k - \delta} \big(c_1 + c_2 \alpha^{-1}(b_\alpha^2 - k\beta)\big), \tag{11.13}$$

$k \in \mathbb{N}_0$, where c_1 and c_2 are real constants that are not both equal to 0 and which satisfy (11.11) and (11.12). That such constants exist can be shown by following a case-by-case argument similar to Taherizadeh (2009, p. 48); for example, if $D(\delta) \neq 0$, $A(\delta) \neq 0$, and $B(\delta) \neq 0$, then we can choose c_2 to be any non-zero number and then set $c_1 = c_2 B(\delta)/D(\delta)$.

Define

$$\widetilde{\phi}(s) := \sum_{k=0}^{\infty} \gamma_k \mathfrak{L}_k^{(\alpha-1)}(s), \tag{11.14}$$

$s \geq 0$. By applying the ratio test, we find that $\sum_{k=0}^{\infty} \gamma_k^2 < \infty$; therefore, $\widetilde{\phi} \in L^2$.

To show also that (11.14) converges pointwise, we apply (9.11), (4.4), and a Laguerre polynomial inequality (Erdélyi, *et al.*, 1953, p. 207) to obtain

$$\begin{aligned} \big|\mathfrak{L}_k^{(\alpha-1)}(s)\big| &= \beta^{\alpha/2} \exp((1-\beta)s/2) \Big(\frac{k!}{(\alpha)_k}\Big)^{1/2} \, |L_k^{(\alpha-1)}(\beta s)| \\ &\leq \begin{cases} \beta^{\alpha/2} \Big[2\Big(\frac{k!}{(\alpha)_k}\Big)^{1/2} - \Big(\frac{(\alpha)_k}{k!}\Big)^{1/2}\Big] e^{s/2}, & 1/2 \leq \alpha < 1 \\ \beta^{\alpha/2} \Big(\frac{(\alpha)_k}{k!}\Big)^{1/2} e^{s/2}, & \alpha \geq 1 \end{cases} \end{aligned} \tag{11.15}$$

for $s \geq 0$. Thus, to establish that (11.14) pointwise converges pointwise, we need to show that

$$\sum_{k=0}^{\infty} \Big(\frac{(\alpha)_k}{k!}\Big)^{1/2} |\gamma_k| < \infty \quad \text{and} \quad \sum_{k=0}^{\infty} \Big(\frac{k!}{(\alpha)_k}\Big)^{1/2} |\gamma_k| < \infty. \tag{11.16}$$

However, the convergence of each of these series follows from the ratio test.

Next, we justify the interchange of summation and integration in our calculations. By a corollary to Theorem 16.7 in Billingsley (1979, p. 224), we need to verify that

$$\sum_{k=0}^{\infty} |\gamma_k| \int_0^{\infty} K(s,t) \, |\mathfrak{L}_k^{(\alpha-1)}(t)| \, \mathrm{d}P_0(t) < \infty. \tag{11.17}$$

By (9.10) and (4.1),

$$0 \leq K_0(s,t) \leq \exp(-(s+t)/\alpha) \exp(2\sqrt{st}/\alpha) = \exp(-(\sqrt{s} - \sqrt{t})^2/\alpha) \leq 1. \tag{11.18}$$

By the triangle inequality and by (11.18), we have

$$0 \leq K(s,t) \leq K_0(s,t) + (\alpha^{-3}st + 1)\exp(-(s+t)/\alpha) \leq 2 + \alpha^{-3}st,$$

$s, t \geq 0$. Thus, to prove (11.17), we need to establish that

$$\sum_{k=0}^{\infty} |\gamma_k| \int_0^{\infty} (2 + \alpha^{-3}st) \, |\mathfrak{L}_k^{(\alpha-1)}(t)| \, \mathrm{d}P_0(t) < \infty.$$

By applying the bound (11.15), we see that it suffices to prove that

$$\sum_{k=0}^{\infty}\Big(\frac{(\alpha)_k}{k!}\Big)^{1/2}|\gamma_k|\,\int_0^{\infty} t^j\,\mathrm{d}P_0(t)<\infty$$

and

$$\sum_{k=0}^{\infty}\Big(\frac{k!}{(\alpha)_k}\Big)^{1/2}|\gamma_k|\,\int_0^{\infty} t^j\,\mathrm{d}P_0(t)<\infty,$$

$j=0,1$. As these integrals are finite, the convergence of both series follows from (11.16).

To calculate $\mathcal{S}\widetilde{\phi}(s)$ from (11.14), we follow the same steps as before to obtain

$$\begin{aligned}\mathcal{S}\widetilde{\phi}(s)&=\int_0^{\infty}K(s,t)\sum_{k=0}^{\infty}\gamma_k\mathfrak{L}_k^{(\alpha-1)}(t)\,\mathrm{d}P_0(t)\\&=\sum_{k=0}^{\infty}\rho_k\gamma_k\mathfrak{L}_k^{(\alpha-1)}(s)-c_1\beta^{\alpha/2}\sum_{k=0}^{\infty}\Big(\frac{(\alpha)_k}{k!}\Big)^{1/2}\rho_k\mathfrak{L}_k^{(\alpha-1)}(s)\\&\qquad\qquad-c_2\alpha^{-1}\beta^{\alpha/2}\sum_{k=0}^{\infty}\Big(\frac{(\alpha)_k}{k!}\Big)^{1/2}\rho_k(b_\alpha^2-k\beta)\mathfrak{L}_k^{(\alpha-1)}(s).\end{aligned}$$

By the definition (11.13) of γ_k, and noting that

$$\frac{\rho_k}{\rho_k-\delta}-1=\frac{\delta}{\rho_k-\delta},$$

we have

$$\begin{aligned}\mathcal{S}\widetilde{\phi}(s)&=\beta^{\alpha/2}\sum_{k=0}^{\infty}\Big[\frac{\rho_k}{\rho_k-\delta}-1\Big]\Big(\frac{(\alpha)_k}{k!}\Big)^{1/2}\rho_k(c_1+c_2\alpha^{-1}(b_\alpha^2-k\beta))\mathfrak{L}_k^{(\alpha-1)}(s)\\&=\beta^{\alpha/2}\delta\sum_{k=0}^{\infty}\frac{\rho_k}{\rho_k-\delta}\Big(\frac{(\alpha)_k}{k!}\Big)^{1/2}(c_1+c_2\alpha^{-1}(b_\alpha^2-k\beta))\mathfrak{L}_k^{(\alpha-1)}(s)\\&=\delta\sum_{k=0}^{\infty}\gamma_k\mathfrak{L}_k^{(\alpha-1)}(s)=\delta\widetilde{\phi}(s).\end{aligned}$$

Therefore, δ is an eigenvalue of $\mathcal{S}$ with corresponding eigenfunction $\widetilde{\phi}$. □

A proof that Conjecture 2 implies Conjecture 1. Suppose there exists $l\in\mathbb{N}_0$ such that $\delta=\rho_l$. Substituting $k=l$ in (11.9) and simplifying the outcome, we obtain

$$c_1=c_2\alpha^{-1}(l\beta-b_\alpha^2). \tag{11.19}$$

Substituting $\delta=\rho_l$ in (11.8), applying (11.19), and cancelling common terms in (11.8), we obtain

$$\langle\phi,\mathfrak{L}_k^{(\alpha-1)}\rangle_{L^2}=c_2\,\alpha^{-1}\beta^{(2+\alpha)/2}\Big(\frac{(\alpha)_k}{k!}\Big)^{1/2}\frac{l-k}{\rho_k-\rho_l}\rho_k, \tag{11.20}$$

for $k\neq l$. Substituting this result for the inner product into (11.6), we obtain

$$\begin{aligned}c_1&=\beta^{\alpha/2}\Big[\sum_{\substack{k=0\\k\neq l}}^{\infty}\langle\phi,\mathfrak{L}_k^{(\alpha-1)}\rangle_{L^2}\Big(\frac{(\alpha)_k}{k!}\Big)^{1/2}\rho_k+\langle\phi,\mathfrak{L}_l^{(\alpha-1)}\rangle_{L^2}\Big(\frac{(\alpha)_l}{l!}\Big)^{1/2}\rho_l\Big]\\&=\beta^{\alpha/2}\Big[\sum_{\substack{k=0\\k\neq l}}^{\infty}c_2\,\alpha^{-1}\beta^{(2+\alpha)/2}\,\frac{(\alpha)_k}{k!}\,\frac{l-k}{\rho_k-\rho_l}\rho_k^2+\langle\phi,\mathfrak{L}_l^{(\alpha-1)}\rangle_{L^2}\Big(\frac{(\alpha)_l}{l!}\Big)^{1/2}\rho_l\Big].\end{aligned}$$

Similarly, substituting (11.20) into (11.7), we obtain

$$c_2 = \alpha^2\beta^{\alpha/2}\Big[\sum_{\substack{k=0\\k\neq l}}^{\infty}\langle\phi,\mathfrak{L}_k^{(\alpha-1)}\rangle_{L^2}\Big(\frac{(\alpha)_k}{k!}\Big)^{1/2}\rho_k(b_\alpha^2-k\beta)+\langle\phi,\mathfrak{L}_l^{(\alpha-1)}\rangle_{L^2}\Big(\frac{(\alpha)_l}{l!}\Big)^{1/2}\rho_l(b_\alpha^2-l\beta)\Big]$$

$$= \alpha^2\beta^{\alpha/2}\Big[c_2\,\alpha^{-1}\beta^{(2+\alpha)/2}\sum_{\substack{k=0\\k\neq l}}^{\infty}\frac{(\alpha)_k}{k!}\frac{l-k}{\rho_k-\rho_l}\rho_k^2(b_\alpha^2-k\beta)+\langle\phi,\mathfrak{L}_l^{(\alpha-1)}\rangle_{L^2}\Big(\frac{(\alpha)_l}{l!}\Big)^{1/2}\rho_l(b_\alpha^2-l\beta)\Big].$$

On simplifying the above expressions and substituting for c_1 from (11.19), we obtain

$$\beta^{\alpha/2}\Big(\frac{(\alpha)_l}{l!}\Big)^{1/2}\rho_l\langle\phi,\mathfrak{L}_l^{(\alpha-1)}\rangle_{L^2} = c_2\Big[\alpha^{-1}(l\beta-b_\alpha^2)-\alpha^{-1}\beta^{\alpha+1}\sum_{\substack{k=0\\k\neq l}}^{\infty}\frac{(\alpha)_k}{k!}\frac{l-k}{\rho_k-\rho_l}\rho_k^2\Big], \tag{11.21}$$

and

$$\alpha^2\beta^{\alpha/2}\Big(\frac{(\alpha)_l}{l!}\Big)^{1/2}\rho_l(b_\alpha^2-l\beta)\langle\phi,\mathfrak{L}_l^{(\alpha-1)}\rangle_{L^2} = c_2\Big[1-\alpha\beta^{\alpha+1}\sum_{\substack{k=0\\k\neq l}}^{\infty}\frac{(\alpha)_k}{k!}\frac{l-k}{\rho_k-\rho_l}\rho_k^2(b_\alpha^2-k\beta)\Big]. \tag{11.22}$$

Suppose that $c_2 = 0$ then it follows from (11.19) that $c_1 = 0$, which contradicts the earlier observation that c_1 and c_2 are not both zero; therefore, $c_2 \neq 0$. Also, by (4.2), $b_\alpha^2 < 1 < \beta$, so $b_\alpha^2 - k\beta \neq 0$ for all $k \in \mathbb{N}_0$. Solving (11.21) and (11.22) for the inner product $\langle\phi,\mathfrak{L}_l^{(\alpha-1)}\rangle_{L^2}$ and equating the two expressions, we obtain

$$1-\alpha\beta^{\alpha+1}\sum_{\substack{k=0\\k\neq l}}^{\infty}\frac{(\alpha)_k}{k!}\frac{l-k}{\rho_k-\rho_l}\rho_k^2(b_\alpha^2-k\beta) = \alpha(b_\alpha^2-l\beta)\Big[(l\beta-b_\alpha^2)-\beta^{\alpha+1}\sum_{\substack{k=0\\k\neq l}}^{\infty}\frac{(\alpha)_k}{k!}\frac{l-k}{\rho_k-\rho_l}\rho_k^2\Big].$$

Simplifying the above equation, we obtain (4.5). □

A C^∞ kernel $K : \mathbb{R}^2 \to \mathbb{R}$ is *extended totally positive* (ETP) if for all $r \geq 1$, all $s_1 \geq \cdots \geq s_r$, all $t_1 \geq \cdots \geq t_r$, there holds

$$\frac{\det\big(K(s_i,t_j)\big)}{\prod_{1\leq i<j\leq r}(s_i-s_j)(t_i-t_j)} > 0, \tag{11.23}$$

where instances of equality for the variables s_i and t_j are to be understood as limiting cases, and then L'Hospital's rule is to be used to evaluate this ratio.

Proof of Proposition 2. By (3.4), the kernel $K(s,t)$ is of the form

$$K(s,t) = e^{-(s+t)/\alpha}s^2t^2\sum_{k=0}^{\infty}c_k s^k t^k,$$

where the coefficients c_k are positive for all $k = 0, 1, 2, \ldots$. Therefore,

$$\det\big(K(s_i,t_j)\big) = \det\Big(e^{-(s_i+t_j)/\alpha}s_i^2t_j^2\sum_{k=0}^{\infty}c_k s_i^k t_j^k\Big)$$

$$= \Big(\prod_{i=1}^{r}e^{-(s_i+t_i)/\alpha}s_i^2t_i^2\Big)\cdot\det\Big(\sum_{k=0}^{\infty}c_k s_i^k t_j^k\Big).$$

By Karlin (1964, p. 101) the series $\sum_{k=0}^{\infty}c_k s^k t^k$ is ETP so, by (11.23), $K(s,t)$ is ETP.

In the case of K_0, we have

$$K_0(s,t) = e^{-(s+t)/\alpha}\sum_{k=0}^{\infty}c_k s^k t^k,$$

where $c_k > 0$ for all k. Then it follows by a similar argument that $K_0(s,t)$ is ETP.

By a result of Karlin (1964), the eigenvalues of an integral operator are simple and positive if the kernel of the operator is ETP. Therefore, the eigenvalues of $\mathcal{S}$ and $\mathcal{S}_0$ are simple and positive.

In particular, 0 is not an eigenvalue of $\mathcal{S}$ or $\mathcal{S}_0$, so both operators are injective. Also, the oscillation property (4.8) follows from Karlin (1964, Theorem 3). □

Proof of Proposition 3. Define the kernels $k_0(s,t) = -e^{-(s+t)/\alpha}$ and $k_1(s,t) = -e^{-(s+t)/\alpha}\alpha^{-3}st$, $s,t \ge 0$. Also, define on L^2 the corresponding integral operators,

$$\mathcal{U}_j f(s) = \int_0^\infty k_j(s,t) f(t) \, \mathrm{d}P_0(t),$$

$j = 0, 1$, $s \ge 0$. Then it follows from (3.4) that $\mathcal{S} = \mathcal{S}_0 + \mathcal{U}_0 + \mathcal{U}_1$.

Each $\mathcal{U}_j$ clearly is self-adjoint and of rank one, i.e., the range of $\mathcal{U}_j$ is a one-dimensional subspace of L^2. Also, $\mathcal{S}_0 + \mathcal{U}_0$ is compact and self-adjoint and its kernel, $K_0 + k_0$, is of the form

$$K_0(s,t) + k_0(s,t) = e^{-(s+t)/\alpha} st \sum_{j=0}^{\infty} c_j s^j t^j,$$

where $c_j > 0$ for all j. Arguing as in the proof of Proposition 2, we find that the eigenvalues of $\mathcal{S}_0 + \mathcal{U}_0$ are simple and positive; hence, $\mathcal{S}_0 + \mathcal{U}_0$ is injective.

Let ω_k, $k \ge 1$, be the eigenvalues of $\mathcal{S}_0 + \mathcal{U}_0$, where $\omega_1 > \omega_2 > \cdots$. Since $\mathcal{S}_0$ is compact, self-adjoint, and injective, and since $\mathcal{U}_0$ is self-adjoint and of rank one then, by Hochstadt (1973) or Dancis and Davis (1987), the eigenvalues of $\mathcal{S}_0$ and $\mathcal{S}_0 + \mathcal{U}_0$ are interlaced: $\rho_{k-1} \ge \omega_k \ge \rho_k$ for all $k \ge 1$. Also, there is exactly one eigenvalue of $\mathcal{S}_0 + \mathcal{U}_0$ in one of the intervals $[\rho_k, \rho_{k-1})$, (ρ_k, ρ_{k-1}), or $(\rho_k, \rho_{k-1}]$.

Since $\mathcal{U}_1$ is self-adjoint and of rank one then by Hochstadt's theorem, the eigenvalues of $\mathcal{S}_0 + \mathcal{U}_0$ and $\mathcal{S}_0 + \mathcal{U}_0 + \mathcal{U}_1 \equiv \mathcal{S}$ are interlaced: $\omega_k \ge \delta_k \ge \omega_{k+1}$ for all $k \ge 1$. Also, there is exactly one eigenvalue of $\mathcal{S}$ in one of the intervals $[\omega_{k+1}, \omega_k)$, (ω_{k+1}, ω_k), or $(\omega_{k+1}, \omega_k]$.

Combining these interlacing results, we obtain $\rho_{k-1} \ge \delta_k \ge \rho_{k+1}$, $k \ge 1$. Also, since $\rho_k = \alpha^\alpha b_\alpha^{4k+2\alpha}$ then, by the interlacing inequalities, $\delta_k = O(b_\alpha^{4k})$, hence $\delta_k = O(\rho_k)$. □