ON THE GROWTH OF EIGENFUNCTION AVERAGES: MICROLOCALIZATION AND GEOMETRY

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Abstract. Let (M, g) be a smooth, compact Riemannian manifold and $\{\phi_h\}$ an L^2-normalized sequence of Laplace eigenfunctions, $-h^2\Delta_g\phi_h = \phi_h$. Given a smooth submanifold $H \subset M$ of codimension $k \geq 1$, we find conditions on the pair $(\{\phi_h\}, H)$ for which

$$\Big|\int_H \phi_h d\sigma_H\Big| = o(h^{\frac{1-k}{2}}), \qquad h \to 0^+.$$

One such condition is that the set of conormal directions to H that are recurrent has measure 0. In particular, we show that the upper bound holds for any H if (M, g) is surface with Anosov geodesic flow or a manifold of constant negative curvature. The results are obtained by characterizing the behavior of the defect measures of eigenfunctions with maximal averages.

1. Introduction

On a compact Riemannian manifold (M, g) of dimension n we consider sequences of normalized Laplace eigenfunctions $\{\phi_h\}$ with eigenvalue $\lambda = h^{-2}$, i.e. solving

$$(-h^2\Delta_g - 1)\phi_h = 0, \qquad \|\phi_h\|_{L^2(M)} = 1.$$

We study the average oscillatory behavior of ϕ_h when restricted to a submanifold $H \subset M$. In particular, our goal is to understand conditions on the pair $(\{\phi_h\}, H)$ under which

$$\int_H \phi_h d\sigma_H = o\big(h^{\frac{1-k}{2}}\big), \tag{1.1}$$

as $h \to 0^+$, where σ_H is the volume measure on H induced by the Riemannian metric, and k is the codimension of H.

We note that the bound

$$\Big|\int_H \phi_h d\sigma_H\Big| = O(h^{\frac{1-k}{2}}) \tag{1.2}$$

holds for any pair $(\{\phi_h\}, H)$ [Zel92, Corollary 3.3], and is sharp in general. Therefore, we seek conditions under which the average is sub-maximal. Observe also that if $k = n$, then (1.2) is a pointwise estimate agreeing with the standard L^∞ bounds of [Ava56, Lev52, Hör68]

$$\|\phi_h\|_{L^\infty} = O(h^{\frac{1-n}{2}}).$$

As explained below, by considering the case $k = n$, we include bounds on L^∞ norms in our results. Integrals of the form (1.1), where H is a curve, have a long history. [Goo83, Hej82] study the case in which H is a periodic geodesic in a compact hyperbolic manifold, and prove the bound (1.2) in that case. The work [Zel92] in fact shows that (1.1) holds for a density one subsequence of eigenvalues. Moreover, one can give explicit polynomial improvements on the error term in (1.2) for a density one subsequence of eigenfunctions [JZ16].

These estimates, however, are not generally satisfied for the full sequence of eigenfunctions and the question of when all eigenfunctions satisfy (1.1) has been studied recently for the case of curves in surfaces [CS15, SXZ17, Wym17b, Wym17a] and for submanifolds [Wym17c]. Finally, given a hypersurface, the question of which eigenfunctions satisfy (1.1) was studied in [CGT18]. We address both of these questions, strengthening the results concerning which eigenfunctions can have maximal averages on a given submanifold H, and giving weaker conditions on the submanifold H that guarantee (1.1) for all eigenfunctions.

We improve and extend nearly all existing results regarding averages of eigenfunctions over submanifolds. We recover all conditions in the papers [CS15, SXZ17, Wym17b, Wym17a, Wym17c, GT18, Gal17, CGT18, Bér77, SZ16a, SZ16b] which guarantee that the improved bound (1.1) holds. As far as the authors are aware, these papers contain all previously known conditions ensuring improved averages. Moreover, we give strictly weaker conditions guaranteeing (1.1) when $k < n$; we replace the condition that the set of loop directions has measure zero from [Wym17c] with the condition that the set of recurrent directions has measure zero. This allows us to prove that under conditions on (M, g) including those studied in [Goo83, Hej82, CS15, SXZ17], the improved bound (1.1) holds unconditionally with respect to the submanifold H. These improvements are possible because the main estimate, Theorem 6, gives explicit bounds on averages over submanifolds H which depend only on the microlocalization of a sequence of eigenfunctions in the conormal directions to H. This gives a new proof of (1.2) from [Zel92] with explicit control over the constant C for high energies. In fact, we characterize those defect measures which may support maximal averages. The estimate requires no assumptions on the geometry of H or M and is purely local. It is only with this bound in place that we use dynamical arguments to draw conclusions about the pairs $((M, g), H)$ supporting eigenfunctions with maximal averages. We note, however, that this paper does not obtain logarithmically improved averages as in [Bér77, SXZ17, Wym17a].

Recall that all compact, negatively curved Riemannian surfaces have Anosov geodesic flow [Ano67]. One consequence of the results in this paper is the following.

Theorem 1. *Suppose (M, g) is a compact, Riemannian surface with Anosov geodesic flow and $\gamma : [a, b] \to M$ is a smooth curve segment with $|\gamma'| > 0$. Then*

$$\int_a^b \phi_h(\gamma(s))ds = o(1) \qquad \text{and} \qquad \int_a^b h\partial_\nu \phi_h(\gamma(s))ds = o(1)$$

as $h \to 0^+$ for every sequence $\{\phi_h\}$ of Laplace eigenfunctions. Here ∂_ν denotes the derivative in the normal direction to the curve.

In order to state our more general results we introduce some geometric notation. Let $H \subset M$ be a closed smooth submanifold of codimension k. We denote by N^*H the conormal bundle to H and we write SN^*H for the unit conormal bundle of H, where the metric is induced from that in $N^*H \subset T^*M$. We write σ_{SN^*H} for the measure on SN^*H induced by the Sasaki metric on TM (see e.g. [Ebe73a]). In particular, if

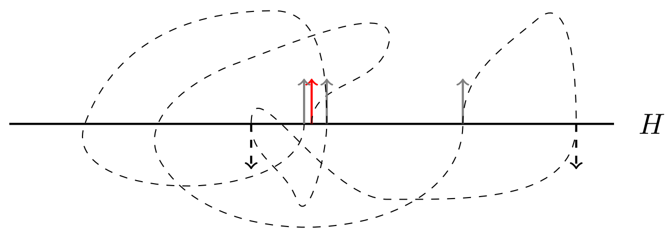

FIGURE 1. The figure shows a recurrent point in SN^*H in red together with its geodesic shown as the dashed line. The intersections of the geodesic with SN^*H are shown in gray and black arrows.

(x', x'') are Fermi coordinates in a tubular neighborhood of H, where H is identified with $\{(x', x'') : x'' = 0\}$, we have

$$\sigma_{SN^*H}(x', \xi'') = \sigma_H(x') d\mathrm{Vol}_{S^{k-1}}(\xi''), \tag{1.3}$$

where $x = (x', 0) \in H$, $\xi'' \in SN^*_x H$, and S^{k-1} is the $k-1$ dimensional sphere. We say that H is a closed embedded submanifold of codimension k if H is a manifold of dimension $n-k$, possibly with boundary, that is embedded in M and is closed as a subset of M.

Let $G^t(\rho) : T^*M \to T^*M$ denote the geodesic flow and $T_H : SN^*H \to \mathbb{R} \cup \{\infty\}$ with

$$T_H(\rho) := \inf\{t > 0 : \; G^t(\rho) \in SN^*H\},$$

be the first return time. Define the loop set

$$\mathcal{L}_H := \{\rho \in SN^*H : \; T_H(\rho) < \infty\} \tag{1.4}$$

and first return map $\eta : \mathcal{L}_H \to SN^*H$ by $\eta(\rho) = G^{T_H(\rho)}(\rho)$. Next, consider the infinite loop sets

$$\mathcal{L}_H^{+\infty} := \bigcap_{k \geq 0} \eta^{-k}(\mathcal{L}_H) \qquad \text{and} \qquad \mathcal{L}_H^{-\infty} := \bigcap_{k \geq 0} \eta^{k}(\mathcal{L}_H),$$

and the recurrent set

$$\mathcal{R}_H = \mathcal{R}_H^+ \cap \mathcal{R}_H^-$$

where

$$\mathcal{R}_H^{\pm} := \left\{ \rho \in \mathcal{L}_H^{\pm\infty} : \; \rho \in \bigcap_{N>0} \overline{\bigcup_{k \geq N} \eta^{\pm k}(\rho)} \right\}.$$

In dynamical systems, the sets

$$\bigcap_{N>0} \overline{\bigcup_{k \geq N} \eta^{k}(\rho)} \qquad \text{and} \qquad \bigcap_{N>0} \overline{\bigcup_{k \geq N} \eta^{-k}(\rho)}$$

are known respectively as the ω and α limit sets of the point ρ. The recurrent set consists of the $\rho \in SN^*H$ such that ρ lies in its own α and ω limit sets and should be thought of as the property that the geodesic through ρ is asymptotically closed as its length tends to infinity (see Figure 1).

In what follows we write $\pi_H : SN^*H \to H$ for the canonical projection map onto H, and $\dim_{\text{box}}(B)$ for the Minkowski box dimension of a set B.

Theorem 2. *Let (M, g) be a smooth, compact Riemannian manifold of dimension n. Let $H \subset M$ be a closed embedded submanifold of codimension k, and $A \subset H$ be a subset with boundary ∂A satisfying $\dim_{\text{box}}(\partial A) < n - k - \frac{1}{2}$. Suppose*

$$\sigma_{SN^*H}(\mathcal{R}_H \cap \pi_H^{-1}(A)) = 0.$$

Then

$$\int_A \phi_h d\sigma_H = o(h^{\frac{1-k}{2}})$$

as $h \to 0^+$ for every sequence $\{\phi_h\}$ of Laplace eigenfunctions.

Theorem 2 improves on the work of Wyman [Wym17c], replacing the measure of the loop set $\mathcal{L}_H$, by that of the recurrent set $\mathcal{R}_H$. Taking H to be a single point (i.e. $k = n$) also recovers the results of [STZ11]; see Remark 1.

When H is a hypersurface, i.e. $k = 1$, we can also study the oscillatory behavior of the normal derivative $h\partial_\nu \phi_h$ along H.

Theorem 3. *Suppose (M, g, H, A) satisfy the assumptions of Theorem 2 with $k = 1$. Then for every sequence $\{\phi_h\}$ of Laplace eigenfunctions*

$$\left|\int_A \phi_h d\sigma_H\right| + \left|\int_A h\partial_\nu \phi_h d\sigma_H\right| = o(1)$$

as $h \to 0^+$.

Theorem 2 allows us to derive substantial conclusions about the geometry of submanifolds supporting eigenfunctions with maximal averages. Indeed, if there exists $c > 0$ and a sequence of eigenfunctions $\{\phi_h\}$ for which

$$\left|\int_A \phi_h d\sigma_H\right| > c\, h^{\frac{1-k}{2}},$$

then,

$$\sigma_{SN^*H}(\mathcal{R}_H \cap \pi_H^{-1}(A)) > 0.$$

Next, we present different geometric conditions on (M, g) which imply $\sigma_{SN^*H}(\mathcal{R}_H) = 0$. We recall that strictly negative sectional curvature implies Anosov geodesic flow. Also, both Anosov geodesic flow and non-negative sectional curvature imply that (M, g) has no conjugate points.

Theorem 4. *Let (M, g) be a smooth, compact Riemannian manifold of dimension n. Let $H \subset M$ be a closed embedded submanifold of codimension k. Suppose one of the following assumptions holds:*

- ***A.*** *(M, g) has no conjugate points and H has codimension $k > \frac{n+1}{2}$.*
- ***B.*** *(M, g) has no conjugate points and H is a geodesic sphere.*
- ***C.*** *(M, g) has constant negative curvature.*
- ***D.*** *(M, g) is a surface with Anosov geodesic flow.*

E. *(M, g) has Anosov geodesic flow and non-positive curvature, and H is totally geodesic.*

F. *(M, g) has Anosov geodesic flow and H is a subset M that lifts to a horosphere.*

Then

$$\sigma_{SN^*H}(\mathcal{R}_H) = 0.$$

In addition, condition A implies that $\sigma_{SN^*H}(\mathcal{L}_H) = 0$.

Combining Theorems 2 and 4 gives the following result on the oscillatory behavior of eigenfunctions when restricted to H.

Corollary 5. *Let (M, g) be a manifold of dimension n and let $H \subset M$ be a closed embedded submanifold of codimension k satisfying one of the assumptions A-F in Theorem 4. Suppose that $A \subset H$ satisfies $\dim_{\text{box}}(\partial A) < n - k - \frac{1}{2}$. Then*

$$\int_A \phi_h d\sigma_H = o(h^{\frac{1-k}{2}})$$

as $h \to 0^+$ for every sequence $\{\phi_h\}$ of Laplace eigenfunctions.

We conjecture that the conclusions of Theorem 4, and hence also Corollary 5, hold in the case that (M, g) is a manifold with Anosov geodesic flow of any dimension.

Conjecture. *Let (M, g) be a manifold of dimension n with Anosov geodesic flow and let $H \subset M$ be a submanifold of codimension k. Then*

$$\sigma_{SN^*H}(\mathcal{R}_H) = 0.$$

1.1. **Relation with L^∞ bounds.** We note again that taking $k = n$ and $H = \{x\}$ for some $x \in M$ the estimate in (1.2) reads,

$$|u_h(x)| \leq Ch^{\frac{1-n}{2}}. \tag{1.5}$$

By Remark 1 the constant C can be chosen independent of x (and indeed, for small h, depending only on the injectivity radius of (M, g) and dimension of M [Gal17]). Estimates of this form are well known, first appearing in [Ava56, Lev52, Hör68] (see also [Zwo12, Chapter 7]), and situations which produce sharp examples for (1.5) are extensively studied. Many works [Bér77, IS95, TZ02, SZ02, STZ11, SZ16a, SZ16b] have studied connections between growth of L^∞ norms of eigenfunctions and the global geometry of the manifold M. More recently [GT18, Gal17] examine the relation between defect measures and L^∞ norms.

We continue in the spirit of [GT18, Gal17, CGT18]; studying the relation between between defect measures and averages over submanifolds. Some of our arguments draw heavily from the ideas in [Gal17] and, in particular, taking $k = n$ in Theorem 6 (together with Remark 1) recovers [Gal17, Theorem 2]. Hence, we also generalize many of the results of [SZ02, STZ11, SZ16a, SZ16b] to manifolds of lower codimension. For example taking $k = n$ in Theorem 2 gives the main results of [STZ11] (see also [Gal17, Corollary 1.2]).

1.2. **Semiclassical operators and a quantitative estimate.** This section contains the key analytic theorem for controlling submanifold averages (Theorem 6) which, in particular, has Theorems 2 and 3 as corollaries. We control the oscillatory behavior of quasimodes of semiclassical pseudodifferential operators using a quantitative estimate relating averages of quasimodes to the behavior of the associated defect measure. As a consequence, we characterize defect measures for which the corresponding quasimodes may have maximal averages.

It is convenient to work with general semiclassical pseudodifferential operators, instead of only with the Laplace operator, for several reasons. First, by generalizing the operators under consideration, we are able to understand the phenomena which underly estimates for averages. Also, we are able to study many types of operators, e.g. Schrödinger operators, simultaneously with the Laplacian. For example, by a simple argument we are able to apply Theorem 6 directly to obtain estimates on normal derivatives of Laplace eigenfunctions to hypersurfaces (see Theorem 3). Finally, since we are able to work in compact subsets of phase space, defect measures appear naturally as a description of the microlocal concentration properties of eigenfunctions.

We say that a sequence of functions $\{\phi_h\}$ is *compactly microlocalized* if there exists $\chi \in C_c^\infty(T^*M)$ so that

$$(1 - Op_h(\chi))\phi_h = O_{C^\infty}(h^\infty \|\phi_h\|_{L^2(M)}). \tag{1.6}$$

Also, we say that $\{\phi_h\}$ is a *quasimode for* $P \in \Psi_h^\infty(M)$ if

$$P\phi_h = o_{L^2}(h), \qquad \|\phi_h\|_{L^2} = 1. \tag{1.7}$$

In addition, for $p \in S^\infty(T^*M;\mathbb{R})$, we say that a submanifold $H \subset M$ of codimension k is *conormally transverse for* p if given $f_1, \dots f_k \in C_c^\infty(M;\mathbb{R})$ such that

$$H = \bigcap_{i=1}^k \{f_i = 0\}, \qquad \{df_i\} \text{ linearly independent on } H,$$

we have

$$N^*H \ \subset \ \{p \neq 0\} \cup \bigcup_{i=1}^k \{H_p f_i \neq 0\}, \tag{1.8}$$

where H_p is the Hamiltonian vector field associated to p.

Finally, we say the p is *Laplace-like* if for all x,

$$T_x^*M \cap \{p = 0\}$$

has positive definite second fundamental form. Let

$$\Sigma_{H,p} = \{p = 0\} \cap N^*H,$$

and consider the Hamiltonian flow

$$\varphi_t := \exp(tH_p).$$

We fix $t_0 > 0$ and define for a Borel measure μ on $\{p = 0\}$, the measure $\mu_{H,p}$ on $\Sigma_{H,p}$ by setting

$$\mu_{H,p}(A) := \frac{1}{2t_0}\mu\Big(\bigcup_{|t|\le t_0} \varphi_t(A)\Big), \qquad \text{for all Borel } A \subset \Sigma_{H,p}.$$

Remark 2 in [CGT18] shows that if μ is a defect measure associated to a quasimode $\{\phi_h\}$ and H is conormally transverse for p, then $\mu_{H,p}(A)$ is independent of the choice of t_0. It is then natural to replace the fixed choice of t_0 with $\lim_{t_0\to 0}$. In particular, for μ a defect measure associated to $\{\phi_h\}$,

$$\mu_{H,p}(A) = \lim_{t_0\to 0} \frac{1}{2t_0}\,\mu\Big(\bigcup_{|t|\le t_0} \varphi_t(A)\Big), \tag{1.9}$$

for all Borel sets $A \subset \Sigma_{H,p}$.

Next, let $r_H : M \to \mathbb{R}$ be the geodesic distance to H. Then, define $|H_p r_H| : \Sigma_{H,p} \to \mathbb{R}$ by

$$|H_p r_H|(\rho) := \lim_{t\to 0} |H_p r_H(\varphi_t(\rho))|.$$

Finally, we write $\mu \perp \lambda$ when μ and λ are mutually singular measures and let $\sigma_{\Sigma_{H,p}}$ be the volume measure induced on $\Sigma_{H,p}$ by the Sasaki metric. Note that in Fermi normal coordinates (x', x''), as in (1.3),

$$\sigma_{\Sigma_{H,p}} = \sigma_H(x')d\mathrm{Vol}_{\Sigma_{H,p}\cap T^*_x M}(\xi''), \tag{1.10}$$

where Vol denotes the volume induced by the Euclidean metric on $N^*_x H$.

Theorem 6. *Let (M, g) be a smooth, compact Riemannian manifold of dimension n and $P \in \Psi^\infty(M)$ have real valued principal symbol $p(x,\xi)$. Suppose that $H \subset M$ is a closed embedded submanifold of codimension k conormally transverse for p, and that $\{\phi_h\}$ is a compactly microlocalized quasimode for P with defect measure μ. Let $f \in L^1(H, \sigma_{\Sigma_{H,p}})$ and $\lambda_H \perp \sigma_{\Sigma_{H,p}}$ be such that*

$$\mu_{H,p} = f d\sigma_{\Sigma_{H,p}} + \lambda_H. \tag{1.11}$$

Let $w \in C^\infty_c(H^o)$. Then there exists $C(n,k) = C_{n,k} > 0$, depending only on n and k, so that

$$\limsup_{h\to 0^+} h^{\frac{k-1}{2}} \left|\int_H w\phi_h d\sigma_H\right| \le C_{n,k}\int_H |w|\sqrt{f|H_p r_H|^{-1}}d\sigma_{\Sigma_{H,p}}. \tag{1.12}$$

If in addition p is Laplace-like, then for $w \in C^\infty(H)$ and $A \subset H$ with $\dim_{\mathrm{box}}(\partial A) < n - k - \frac{1}{2}$,

$$\limsup_{h\to 0^+} h^{\frac{k-1}{2}} \left|\int_A w\phi_h d\sigma_H\right| \le C_{n,k}\int_{\pi_H^{-1}(A)} |w|\sqrt{f|H_p r_H|^{-1}}d\sigma_{\Sigma_{H,p}}. \tag{1.13}$$

In addition to relating the L^2 microlocalization of quasimodes to averages on submanifolds, Theorem 6 gives a quantitative version of the bound (1.2) proved in [Zel92, Corollary 3.3] and generalizes the work of the second author [Gal17, Theorem 2] to manifolds of any codimension. Note also that the estimate (1.13) is saturated for every $0 < k \le n$ on the round sphere S^n.

Remark 1. Let $\varepsilon(h) > 0$ satisfy $e(h) = o(1)$. We actually prove the stronger statement that (1.13) can be replaced with

$$\limsup_{h\to 0^+} h^{\frac{k-1}{2}} \sup_{(A_1,H_1)\in\mathcal{A}(A,H,\varepsilon(h))} \left|\int_{A_1} \phi_h d\sigma_{H_1}\right| \le C_{n,k}\int_{\pi_H^{-1}(A)} \sqrt{f|H_p r_H|^{-1}}d\sigma_{\Sigma_{H,p}}$$

where

$$\mathcal{A}(A,H,\varepsilon(h)) =$$
$$\{(A_1,H_1)\mid A_1\subset H_1,\ \dim_{\rm box}(\partial A_1)<n-k-\tfrac12,\ d(A,A_1)=\varepsilon(h),\ d_s(\Sigma_{H,p},\Sigma_{H_1})=\varepsilon(h)\}$$

and d_s is the distance induced by the Sasaki metric. That is, our estimate is locally uniform in $o_{C^1}(1)$ neighborhoods of H (see Remark 4 for an explanation). This also implies that all of our other estimates are uniform in $o_{C^1}(1)$ neighborhoods.

A direct consequence of Theorem 6 is the following.

Theorem 7. *Let (M,g) be a smooth, compact Riemannian manifold of dimension n. Let $H\subset M$ be a closed embedded submanifold of codimension k, and let $A\subset H$ be a subset with boundary ∂A satisfying $\dim_{\rm box}(\partial A)<n-k-\frac12$. If $\{\phi_h\}$ is a sequence of Laplace eigenfunctions with defect measure μ so that $\mu_H\perp 1_A\,\sigma_{SN^*H}$, then*

$$\int_A \phi_h d\sigma_H = o(h^{\frac{1-k}{2}}).$$

Theorem 7 strengthens the results of [CGT18]. In particular, in [CGT18], the measure μ is said to be conormally diffuse if $\mu_H(SN^*H)=0$, which implies $\mu_H\perp\sigma_{SN^*H}$.

We note that Theorem 7 is an immediate consequence of Theorem 6. To see this, first observe that if we take $P=-h^2\Delta_g-1$, set $p(x,\xi)=|\xi|^2_{g(x)}-1=\sigma(P)$, and let $\{\phi_h\}$ satisfy $P\phi_h=0$, then

$$(1-Op_h(\chi))\phi_h = O_{C^\infty}(h^\infty\|\phi_h\|_{L^2}),$$

for any $\chi\in C_c^\infty(T^*M)$ with $\chi\equiv1$ on $|\xi|_g\le 2$ (see e.g. [DZ16, Appendix E] for the elliptic parametrix construction). Next, note that in this setting we have $\sigma_{\Sigma_{H,p}}=\sigma_{SN^*H}$. Hence, if

$$\int_{\pi_H^{-1}(A)}\sqrt{f}d\sigma_{\Sigma_{H,p}}=0,$$

then by Theorem 6,

$$\int_A \phi_h d\sigma_H = o(h^{\frac{1-k}{2}}).$$

To see that any $H\subset M$ is conormally transverse (recall the definition (1.8)), observe that if $H=\cap_{i=1}^k f_i$, then $N^*H=\operatorname{span}\{df_i:\ i=1,\dots,k\}$. In particular, given $(x,\xi)\in N^*H\cap\{p=0\}$ there exists $i\in\{1,\dots,k\}$ for which $H_pf_i(x,\xi)=2\langle df_i(x),\xi\rangle\neq0$.

1.3. **Manifolds with no focal points or Anosov geodesic flow.** In order to prove parts C, D, E and F of Theorem 4, we assume either that (M,g) has no focal points or that the geodesic flow on (M,g) is Anosov. We show that these structures allow us to restrict to working on the set of points $\mathcal{A}_H$ in SN^*H at which the tangent space to SN^*H splits into a sum of bounded and unbounded directions. To make this sentence precise we introduce some notation.

If (M,g) has no conjugate points, then for any $\rho\in S^*M$, there exist stable and unstable subspaces $E_\pm(\rho)\subset T_\rho S^*M$ so that

$$dG^t: E_\pm(\rho)\to E_\pm(G^t(\rho))$$

and

$$|dG^t(v)| \leq C|v| \text{ for } v \in E_\pm \text{ and } t \to \pm\infty.$$

We recall that a manifold has no focal points if for every geodesic γ, and every Jacobi field $Y(t)$ along γ with $Y(0) = 0$ and $Y'(0) \neq 0$, $Y(t)$ satisfies $\frac{d}{dt}\|Y(t)\|^2 > 0$ for $t > 0$, where $\|\cdot\|$ denotes the norm with respect to the Riemannian metric. In particular, if (M, g) has non-positive curvature, then it has no focal points (see e.g. [Ebe73a, page 440]). It is also known that if (M, g) has no focal points then (M, g) has no conjugate points and that $E_\pm(\rho)$ vary continuously with ρ. (See for example [Ebe73a, Proposition 2.13 and remarks thereafter].) See e.g. [Rug07, Ebe73b, Pes77] for further discussion of manifolds without focal points.

In what follows we write

$$N_\pm(\rho) := T_\rho(SN^*H) \cap E_\pm(\rho). \tag{1.14}$$

We define the *mixed* and *split* subsets of SN^*H respectively by

$$\begin{aligned} \mathcal{M}_H &:= \Big\{\rho \in SN^*H : \ N_-(\rho) \neq \{0\} \text{ and } N_+(\rho) \neq \{0\}\Big\}, \\ \mathcal{S}_H &:= \Big\{\rho \in SN^*H : \ T_\rho(SN^*H) = N_-(\rho) + N_+(\rho)\Big\}. \end{aligned} \tag{1.15}$$

Then we write

$$\mathcal{A}_H := \mathcal{M}_H \cap \mathcal{S}_H, \qquad \mathcal{N}_H := \mathcal{M}_H \cup \mathcal{S}_H, \tag{1.16}$$

where we will use $\mathcal{A}_H$ when considering manifolds with Anosov geodesic flow and $\mathcal{N}_H$ when considering those with no focal points.

Next, we recall that any manifold with no focal points in which every geodesic encounters a point of negative curvature has Anosov geodesic flow [Ebe73a, Corollary 3.4]. In particular, the class of manifolds with Anosov geodesic flows includes those with negative curvature. We also recall that a manifold with Anosov geodesic flow does not have conjugate points [Kli74] and for all $\rho \in S^*M$

$$T_\rho(S^*M) = E_+(\rho) \oplus E_-(\rho) \oplus \mathbb{R}H_p.$$

where E_+, E_- are the stable and unstable directions as before. (For other characterizations of manifolds with Anosov geodesic flow, see [Ebe73a, Theorem 3.2], [Ebe73b].) An equivalent definition of Anosov flow is that there exists $C > 0$ so that for all $\rho \in S^*M$,

$$|dG^t(v)| \leq Ce^{\mp\frac{t}{C}}|v|, \qquad v \in E_\pm(\rho), \quad t \to \pm\infty, \tag{1.17}$$

and the spaces $E_\pm(\rho)$ are Hölder continuous in ρ [Ano67].

Theorem 8. *Let $H \subset M$ be a closed embedded submanifold. If (M, g) has no focal points, then*

$$\sigma_{SN^*H}(\mathcal{R}_H \cap \mathcal{N}_H) = \sigma_{SN^*H}(\mathcal{R}_H).$$

If (M, g) has Anosov geodesic flow, then

$$\sigma_{SN^*H}(\mathcal{R}_H \cap \mathcal{A}_H) = \sigma_{SN^*H}(\mathcal{R}_H).$$

Theorem 8 combined with Theorem 2 give the following result.

Corollary 9. *Let $H \subset M$ be a closed embedded submanifold of codimension k, and let $A \subset H$ satisfy $\dim_{\mathrm{box}}(\partial A) < n - k - \frac{1}{2}$. Then if (M, g) has no focal points and*

$$\sigma_{SN^*H}(\mathcal{N}_H \cap \pi_H^{-1}(A)) = 0$$

we have

$$\int_A \phi_h d\sigma_H = o(h^{\frac{1-k}{2}}) \tag{1.18}$$

as $h \to 0^+$ for every sequence $\{\phi_h\}$ of Laplace eigenfunctions. If instead (M, g) has Ansov geodesic flow then (1.18) *holds when*

$$\sigma_{SN^*H}(\mathcal{A}_H \cap \pi_H^{-1}(A)) = 0.$$

Note that if $\dim M = 2$, then $\mathcal{N}_H = \mathcal{A}_H$ since $\dim T_\rho(SN^*H) = 1$. Indeed, it is not possible to have both $N_+(\rho) \neq \{0\}$ and $N_-(\rho) \neq \{0\}$ unless $N_+(\rho) = N_-(\rho) = T_\rho(SN^*H)$ and hence $\mathcal{M}_H \subset \mathcal{S}_H$. In [Wym17b, Wym17a] the author works with (M, g) non-positively curved (and hence having no focal points), $\dim M = 2$ and $H = \gamma$ a curve. He then imposes the condition that for all time t the curvature of γ, $\kappa_\gamma(t)$, avoids two special values determined by the tangent vector to γ, $\mathbf{k}_\pm(\gamma'(t))$. He shows that under this condition

$$\int_\gamma \phi_h d\sigma_\gamma = o(1).$$

If $\kappa_\gamma(t) = \mathbf{k}_\pm(\gamma'(t))$, then the lift of γ to the universal cover of M is tangent to a stable or unstable horosphere at $\gamma(t)$ and $\kappa_\gamma(t)$ is equal to the curvature of that horosphere. Since this implies that $T_{(\gamma(t),\gamma'(t))}SN^*\gamma$ is stable or unstable, the condition there is that $\mathcal{N}_\gamma = \emptyset$. Thus, the condition $\sigma_{SN^*H}(\mathcal{N}_H \cap \pi_H^{-1}(A)) = 0$ is the generalization to higher codimensions of that in [Wym17b, Wym17a]. We note that [Wym17a] obtains the improved upper bound $O(|\log h|^{-\frac{1}{2}})$.

1.4. **Organization of the paper.** We divide the paper into two major parts. The first part of the paper contains all of the analysis of solutions to $Pu = o(h)$. The sections in this part, Section 2 and Section 3, contain the proofs of Theorem 6 and Theorem 3 respectively. The second part of our paper, consists of an analysis of the geodesic flow and in particular a study of the recurrent set of SN^*H. Theorem 2 is proved in Section 4, and Theorems 4 and 8 are proved in Section 5.

Note that as already explained, Corollary 5 is an immediate consequence of combining Theorems 2 and 4. Also, Theorem 7 is a direct consequence of Theorem 6 and Corollary 9 is a consequence of Theorem 2 and Theorem 8. Finally, Theorem 1 is exactly part D of Theorem 4.

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2. Quantitative estimate: Proof of Theorem 6

In Section 2.1 we present the ground work needed for the proof of Theorem 6. In particular, we state the main technical result, Proposition 10, on which the proof of Theorem 6 hinges. We then divide the proof of Theorem 6 in two parts. Assuming the main technical proposition, we first prove the theorem for the case $A = H$ and $w \in C_c^\infty(H^o)$ in Section 2.2, and then generalize it to any subset $A \subset H$ in Section 2.3. Finally, Section 2.4 is dedicated to the proof of Proposition 10.

Throughout this section we assume that P has principal symbol p and H is conormally transverse for p as defined in (1.8). We also assume throughout this section that $\{\phi_h\}$ is a compactly microlocalized quasimode for P (see (1.6) and (1.7)).

2.1. **Preliminaries.** Let $H \subset M$ be a smooth closed submanifold and let U_H be an open neighborhood of H described in local coordinates as $U_H = \{(x'', x') : x \in V \subset \mathbb{R}^d\}$, where these coordinates are chosen so that $H \cap U_H = \{(0, x') : (0, x') \in V\}$. The coordinates $(x'', x') \in U_H$ induce coordinates (x'', x', ξ'', ξ') on $\Sigma^*_{U_H} M = \{(x, \xi) \in \{p = 0\} : x \in U_H\}$ with $(\xi'', \xi') \in \{p = 0\} \cap T^*_{(x'',x')}M$. In these coordinates, ξ' is cotangent to H while ξ'' is conormal to H. Since H is conormally transverse see (1.8) for p, we may assume, without loss of generality, that $x'' = (x_1, \bar{x})$ with dual coordinates $\xi'' = (\xi_1, \bar{\xi})$, where

$$\partial_{\xi_1} p(x, \xi) \neq 0 \text{ on } \{p = 0\} \cap N^*H.$$

Consider the cut-off function $\chi_\alpha \in C_c^\infty(\mathbb{R}, [0, 1])$ with

$$\chi_\alpha(t) = \begin{cases} 0 & |t| \geq \alpha \\ 1 & |t| \leq \frac{\alpha}{2}, \end{cases} \tag{2.1}$$

with $|\chi'_\alpha(t)| \leq 3/\alpha$ for all $t \in \mathbb{R}$. For $\varepsilon > 0$ consider the symbol

$$\beta_\varepsilon(x', \xi') = \chi_\varepsilon(|\xi'|_{g_H(x')}) \in C_c^\infty(T^*H), \tag{2.2}$$

where g_H is the Riemannian metric on H induced by g. Let $w \in C_c^\infty(H^o)$, where H^o denotes the interior of H. We start splitting the period integral as

$$\int_H w\phi_h \, d\sigma_H = \int_H Op_h(\beta_\varepsilon)[w\phi_h] \, d\sigma_H + \int_H Op_h(1 - \beta_\varepsilon)[w\phi_h] \, d\sigma_H.$$

The same proof as [CGT18, Lemma 8] yields that for all $u \in L^2_{\text{comp}}(H^o)$

$$\left| \int_H Op_h(1 - \beta_\varepsilon) u \, d\sigma_H \right| = O_\varepsilon(h^\infty) \, \|u\|_{L^2(H)}.$$

(see also Lemma 12).

Choosing $u = w\phi_h$, and using the restriction bound $\|\phi_h\|_{L^2(H)} = O(h^{-\frac{k}{2}})$ obtained from the standard L^∞ bounds for compactly microlocalized functions [Zwo12, Lemma 7.10], we have

$$\int_H w\phi_h \, d\sigma_H = \int_H Op_h(\beta_\varepsilon)[w\phi_h] \, d\sigma_H + O_\varepsilon(h^\infty). \tag{2.3}$$

We control the integral of $Op_h(\beta_\varepsilon)w\phi_h$ using the following lemma. To shorten notation, we write

$$\Lambda_{H,T} := \bigcup_{|t|\le T} \varphi_t(\Sigma_{H,p})$$

where we recall that $\Sigma_{H,p} := N^*H \cap \{p=0\}$ and continue to write $\varphi_t := \exp(tH_p)$.

Proposition 10. *Let $\chi \in C_c^\infty(T^*M)$ so that $H_p\chi \equiv 0$ on $\Lambda_{H,T}$ for some $T>0$. Let $w \in C_c^\infty(H^o)$. There exists $C_{n,k} = C(n,k) > 0$ depending only on n and k so that*

$$\lim_{\varepsilon\to 0}\limsup_{h\to 0^+} h^{k-1}\left|\int_H Op_h(\beta_\varepsilon w)\big[Op_h(\chi)\phi_h\big]d\sigma_H\right|^2 \le C_{n,k}\,\sigma_{\Sigma_{H,p}}\big(\operatorname{supp}(\chi 1_{\Sigma_{H,p}})\big)\int_{\Sigma_{H,p}} w^2\chi^2|H_pr_H|^{-1}d\mu_H.$$

The proof of Proposition 10 is given in Section 2.4. The purpose of this proposition is to allow us to use χ to localize quasimodes to the support of λ_H and its complement. Recall that λ_H is defined by (1.11). Since λ_H and $\sigma_{\Sigma_{H,p}}$ are mutually singular, it is not difficult to see that Proposition 10 gives the bound

$$\limsup_{h\to 0^+} h^{\frac{k-1}{2}}\left|\int_H w\phi_h d\sigma_H\right| \le C\left(\int_{\Sigma_{H,p}} w^2 f d\sigma_{\Sigma_{H,p}}\right)^{1/2}.$$

By further restricting χ to shrinking balls inside $\Sigma_{H,p}$ an application of the Lebesgue differentiation theorem allows us to obtain a bound of the form $C\int_{\Sigma_{H,p}}|w|\sqrt{f}d\sigma_{\Sigma_{H,p}}$ as claimed. This improvement will be needed when passing to subsets $A \subset H$. The factor $|H_pr_H|^{-1}$ measures the cost of restricting to a hypersurface containing H which is microlocally transversal to H_p. In particular, we choose coordinates so that $H \subset \{x_1 = 0\}$ and $|H_pr_H| = \partial_{\xi_1}p \neq 0$ at a point $\rho \in \Sigma_{H,p}$. This is possible since H is conormally transverse see (1.8) for p.

To apply Proposition 10 it is key to work with cut-off functions $\chi \in C_c^\infty(T^*M)$ so that $H_p\chi \equiv 0$ on $\Lambda_{H,T}$ for some $T>0$. Therefore, the following lemma is dedicated to extending cut-off functions on $\Sigma_{H,p}$ to cut-off functions on T^*M that are invariant under the Hamiltonian flow inside $\Lambda_{H,T}$. Let $T_{\Sigma_{H,p}} > 0$ be such that

$$\varphi:\ [-2T,2T]\times\Sigma_{H,p}\to\Lambda_{H,2T}$$

is a diffeomorphism for all $0\le T\le T_{\Sigma_{H,p}}$. Such a $T_{\Sigma_{H,p}}$ exists since H is compact and conormally transverse for p. Moreover, for $T < T_{\Sigma_{H,p}}$, $\Lambda_{H,2T}$ is a closed embedded submanifold in T^*M.

Lemma 11. *For all $\tilde\chi \in C_c^\infty(\Sigma_{H,p};[0,1])$ and $0\le T\le T_{\Sigma_{H,p}}$ there exists $\chi\in C_c^\infty(T^*M;[0,1])$ so that*

$$\chi(\varphi_t(x,\xi)) = \tilde\chi(x,\xi)$$

for all $|t|\le T$ and $(x,\xi)\in\Sigma_{H,p}$. In particular, $H_p\chi\equiv 0$ on $\Lambda_{H,T}$.

Proof. Let $\psi \in C_c^\infty(\mathbb{R};[0,1])$ be a fixed function supported on $(-2T,2T)$ with $\psi \equiv 1$ on $[-T,T]$. Then, using that $\varphi:\ [-2T,2T]\times\Sigma_{H,p} \to \Lambda_{H,2T}$ is a diffeomorphism, define the smooth cut-off $\chi:\Lambda_{H,2T}\to[0,1]$ by the relation

$$\chi(\varphi_t(x,\xi)) = \psi(t)\tilde{\chi}(x,\xi).$$

Finally, extend χ to all of T^*M so that $\chi \in C_c^\infty(T^*M;[0,1])$. We can make such an extension since $\Lambda_{H,T}$ is a closed embedded submanifold in T^*M. □

2.2. **Proof of Theorem 6 for $A = H$ and $w \in C_c^\infty(H^o)$.** Fix $\delta > 0$. Recall that λ_H is defined by (1.11). Since $\sigma_{\Sigma_{H,p}}$ and λ_H are two Radon measures on $\Sigma_{H,p}$ that are mutually singular, there exist $K_\delta \subset \Sigma_{H,p}$ compact and $U_\delta \subset \Sigma_{H,p}$ with $K_\delta \subset U_\delta$ and so that

$$\sigma_{\Sigma_{H,p}}(U_\delta) \le \delta \qquad \text{and} \qquad \lambda_H(\Sigma_{H,p}\backslash K_\delta) \le \delta.$$

Indeed, by definition of mutual singularity, there exist $V, W \subset \Sigma_{H,p}$ so that $\lambda_H(W) = \sigma_{\Sigma_{H,p}}(V) = 0$ and $V \cup W = \Sigma_{H,p}$. Hence, by outer regularity of $\sigma_{\Sigma_{H,p}}$, there exists $U_\delta \supset V$ open with $\sigma_{\Sigma_{H,p}}(U_\delta) \le \delta$. Next, by inner regularity, of λ_H, there exists $K_\delta \subset U_\delta$ compact with $\lambda_H(\sigma_{\Sigma_{H,p}} \setminus K_\delta) = \lambda_H(U_\delta \setminus K_\delta) \le \delta$. Let $\tilde{\kappa}_\delta \in C_c^\infty(\Sigma_{H,p};[0,1])$ be a cut-off function with

$$\tilde{\kappa}_\delta \equiv 1 \text{ on } K_\delta \qquad \text{and} \qquad \operatorname{supp}\tilde{\kappa}_\delta \subset U_\delta.$$

Let $\kappa_\delta \in C_c^\infty(T^*M;[0,1])$ be the cut-off extension of $\tilde{\kappa}_\delta$ given in Lemma 11 with

$$H_p\kappa_\delta \equiv 0 \text{ on } \Lambda_{H,T},$$

where we have fixed $T > 0$ so that $2T \le T_{\Sigma_{H,p}}$. We use (2.3) and split the period integral as

$$\int_H w\phi_h\, d\sigma_H = \int_H Op_h(\beta_\varepsilon w)[Op_h(\kappa_\delta)\phi_h]\, d\sigma_H + \int_H Op_h(\beta_\varepsilon w)[Op_h(1-\kappa_\delta)\phi_h]\, d\sigma_H + O_\varepsilon(h^\infty).$$

Applying Proposition 10 with $\chi = \kappa_\delta$, we have that

$$\lim_{\varepsilon\to 0}\limsup_{h\to 0} h^{k-1}\left|\int_H Op_h(\beta_\varepsilon w)[Op_h(\kappa_\delta)\phi_h]\, d\sigma_H\right|^2 \le C\,\sigma_{\Sigma_{H,p}}\big(\operatorname{supp}\kappa_\delta 1_{\Sigma_{H,p}}\big)\int_{\Sigma_{H,p}} \kappa_\delta^2 w^2 d\mu_{H,p} \le C\,\delta. \tag{2.4}$$

Here we have used that $\sigma_{\Sigma_{H,p}}(U_\delta) \le \delta$ and that by construction $\operatorname{supp}\kappa_\delta 1_{\Sigma_{H,p}} = \operatorname{supp}\tilde{\kappa}_\delta \subset U_\delta$.

We dedicate the rest of the proof to showing that

$$\lim_{\varepsilon\to 0}\limsup_{h\to 0} h^{\frac{k-1}{2}}\left|\int_H Op_h(\beta_\varepsilon w)[Op_h(1-\kappa_\delta)\phi_h] d\sigma_H\right| \le C_{n,k}\int_{\Sigma_{H,p}} |w|\sqrt{f_1}\, d\sigma_{\Sigma_{H,p}} + C\delta^{\frac{1}{2}}. \tag{2.5}$$

where $f_1 := f|H_p r_H|^{-1}$. Putting (2.4) together with (2.5) would conclude the proof of the theorem.

We start by splitting the left hand side in (2.5) into an integral over small balls. Note that since $\dim M = n$, $\dim \Sigma_{H,p} = n-1$ regardless of the codimension k. By the Besicovitch–Federer Covering Lemma [Hei01, Theorem 1.14, Example (c)], there exists a constant $c_n > 0$ depending only on n and $r_0 = r_0(H)$ so that for all $0 < r < r_0$, there exist open balls $\{B_1, \dots, B_{N(r)}\} \subset \Sigma_{H,p}$ of radius r with

$$N(r) \le c_n r^{1-n} \qquad \text{and} \qquad \sigma_{\Sigma_{H,p}}(B_j) \le c_n r^{n-1}, \tag{2.6}$$

so that

$$\Sigma_{H,p} \subset \bigcup_{j=1}^{N(r)} B_j$$

and each point in $\Sigma_{H,p}$ lies in at most c_n balls. Let $\{\tilde{\psi}_j\}$ with $\tilde{\psi}_j \in C_c^\infty(\Sigma_{H,p}; [0,1])$ be a partition of unity associated to $\{B_j\}$, and write ψ_j for the extensions $\psi_j \in C_c^\infty(T^*M; [0,1])$ given in Lemma 11 so that $\psi_j(\varphi_t(x,\xi)) = \tilde{\psi}_j(x,\xi)$ for all $|t| \le 2T$ and $(x,\xi) \in \Sigma_{H,p}$. With this construction, $H_p \psi_j \equiv 0$ on $\Lambda_{H,2T}$,

$$\sum_{j=1}^{N(r)} \psi_j \equiv 1 \text{ on } \Lambda_{H,2T}, \qquad \text{and} \qquad \operatorname{supp}(\psi_j 1_{\Sigma_{H,p}}) \subset B_j. \tag{2.7}$$

Let $\Psi := \sum_{j=1}^{N(r)} \psi_j$. Setting $\chi = (1-\Psi)(1-\kappa_\delta)$ we have $H_p \chi = 0$ on $\Lambda_{H,T}$ and $\operatorname{supp}(\chi 1_{\Sigma_{H,p}}) = \emptyset$ (since $1 - \Psi \equiv 0$ on $\Lambda_{H,2T}$). We then apply Lemma 10 to χ, to obtain

$$\lim_{\varepsilon \to 0} \limsup_{h \to 0} h^{\frac{k-1}{2}} \left| \int_H Op_h(\beta_\varepsilon w)[Op_h((1-\Psi)(1-\kappa_\delta))\phi_h] d\sigma_H \right| = 0.$$

On the other hand, by the triangle inequality we have

$$\left| \int_H Op_h(\beta_\varepsilon w)[Op_h(\Psi(1-\kappa_\delta))\phi_h] d\sigma_H \right| \le \sum_{j=1}^{N(r)} \left| \int_H Op_h(\beta_\varepsilon w)[Op_h(\psi_j(1-\kappa_\delta))\phi_h]\, d\sigma_H \right|.$$

By construction we have that $H_p[\psi_j(1-\kappa_\delta)] \equiv 0$ on $\Lambda_{H,T}$. We may therefore apply Proposition 10 with $\chi = \psi_j(1-\kappa_\delta)$ to find that there exist $\varepsilon_0, C_{n,k} > 0$ so that

$$\lim_{\varepsilon \to 0} \limsup_{h \to 0} h^{k-1} \left| \int_H Op_h(\beta_\varepsilon w)[Op_h(\psi_j(1-\kappa_\delta))\phi_h]\, d\sigma_H \right|^2 \le C_{n,k}\, r^{n-1} \int_{\Sigma_{H,p}} \psi_j^2 w^2 (1-\kappa_\delta)^2 |H_p r_H|^{-1} d\mu_{H,p}.$$

Here we have used that $\operatorname{supp}(\psi_j 1_{\Sigma_{H,p}}) \subset B_j$ and for $r_j > 0$ small enough $\sigma_{\Sigma_{H,p}}(B_j) \le c_n r^{n-1}$ for all $j = 1, \dots, N(r)$, and some $c_n > 0$ depending only on n. It follows that there is $C_{n,k} > 0$ for which

$$\lim_{\varepsilon\to 0}\limsup_{h\to 0} h^{\frac{k-1}{2}}\left|\int_H Op_h(\beta_\varepsilon w)[Op_h(\Psi(1-\kappa_\delta))\phi_h]d\sigma_H\right| \leq$$
$$\leq C_{n,k}\, r^{\frac{n-1}{2}}\sum_{j=1}^{N(r)}\left(\int_{\Sigma_{H,p}}\psi_j^2 w^2(1-\kappa_\delta)^2\,|H_p r_H|^{-1}d\mu_{H,p}\right)^{\frac{1}{2}}.$$

Decomposing $\mu_{H,p} = f\sigma_{\Sigma_{H,p}} + \lambda_H$, and using that

$$\mathrm{supp}((1-\kappa_\delta)1_{\Sigma_{H,p}}) \subset \Sigma_{H,p}\backslash K_\delta$$

while $\lambda_H(\Sigma_{H,p}\backslash K_\delta) \leq \delta$, we conclude that there exists $C > 0$ so that

$$\lim_{\varepsilon\to 0}\limsup_{h\to 0} h^{\frac{k-1}{2}}\left|\int_H Op_h(\beta_\varepsilon w)[Op_h(1-\kappa_\delta)\phi_h]d\sigma_H\right| \leq C_{n,k}F_r f_1 + C\delta^{1/2} \qquad (2.8)$$

where

$$F_r f_1 := r^{\frac{n-1}{2}}\sum_{j=1}^{N(r)}\left(\int_{\Sigma_{H,p}}\psi_j^2 w^2 f_1\, d\sigma_{\Sigma_{H,p}}\right)^{\frac{1}{2}}.$$

Indeed, applying the triangle inequality,

$$C_{n,k} r^{\frac{n-1}{2}}\sum_{j=1}^{N(r)}\left(\int_{\Sigma_{H,p}}\psi_j^2 w^2(1-\kappa_\delta)^2|H_p r_H|^{-1}\, d\mu_{H,p}\right)^{\frac{1}{2}} \leq$$
$$C_{n,k}F(r) + Cr^{\frac{n-1}{2}}\sum_{j=1}^{N(r)}\left(\int_{\Sigma_{H,p}}\psi_j^2 w^2(1-\kappa_\delta)^2 d\lambda_H\right)^{\frac{1}{2}}.$$

By Cauchy-Schwarz,

$$\begin{aligned} r^{\frac{n-1}{2}}\sum_{j=1}^{N(r)}\left(\int_{\Sigma_{H,p}}\psi_j^2(1-\kappa_\delta)^2 w^2 d\lambda_H\right)^{\frac{1}{2}} &\leq r^{\frac{n-1}{2}}(N(r))^{1/2}\left(\int_{\Sigma_{H,p}}\sum_{j=1}^{N(r)}\psi_j^2 w^2(1-\kappa_\delta)^2 d\lambda_H\right)^{\frac{1}{2}} \\ &\leq C\lambda_H(\mathrm{supp}(1-\kappa_\delta)1_{\Sigma_{H,p}})^{\frac{1}{2}} \\ &\leq C\delta^{1/2}, \end{aligned}$$

and this proves (2.8).

We dedicate the rest of the proof to proving that there exists $C_n > 0$ such that

$$\limsup_{r\to 0} F_r f_1 \leq C_n \int |w|\sqrt{f_1}d\sigma_{\Sigma_{H,p}}. \qquad (2.9)$$

Inserting (2.9) into (2.8) proves (2.5). Putting (2.4) together with (2.5) concludes the proof of the theorem. Note that for any positive function $\theta \in L^1(\Sigma_{H,p}, \sigma_{\Sigma_{H,p}})$

$$F_r\theta = \sum_{j=1}^{N(r)}\left(r^{n-1}\int_{\Sigma_{H,p}}\psi_j^2 w^2\theta\, d\sigma_{\Sigma_{H,p}}\right)^{\frac{1}{2}}.$$

By (2.7) and (2.6)

$$|F_r\theta| \le C_n N(r)^{\frac{1}{2}} r^{\frac{n-1}{2}} \left(\int_{\Sigma_{H,p}} \sum_{j=1}^{N(r)} \psi_j^2 w^2 \theta \, d\sigma_{\Sigma_{H,p}} \right)^{\frac{1}{2}} \le C\|\theta\|_{L^1}^{\frac{1}{2}} \tag{2.10}$$

where C is independent of r.

Next, suppose that $\theta \ge 0$ is a continuous function. Let $\xi_j \in \Sigma_{H,p}$ be so that $B_j = B(\xi_j, r)$ and note that for r small enough, $C_n^{-1} r^{n-1} \le \sigma_{\Sigma_{H,p}}(B(\xi_j, r)) \le C_n r^{n-1}$, where C_n depends only on n. Using this, and that $\operatorname{supp}(\psi_j 1_{\Sigma_{H,p}}) \subset B_j$, the definition of $F_r\theta$ yields

$$F_r\theta \le C_n \int_{\Sigma_{H,p}} \sum_{j=1}^{N(r)} 1_{B_j} \left(\frac{1}{\sigma_{\Sigma_{H,p}}(B_j)} \int_{B_j} w^2 \theta d\sigma_{\Sigma_{H,p}} \right)^{1/2} d\sigma_{\Sigma_{H,p}}$$

Now, since ϕ_h is compactly microlocalized (see (1.6)), we may assume $\Sigma_{H,p}$ is compact. Then, since w and θ are continuous, they are uniformly continuous. In particular, for any $\varepsilon_0 > 0$ there exists r small enough so that for all $\xi \in \Sigma_{H,p}$ and $\rho \in B(\xi, r)$,

$$\left(\frac{1}{\sigma_{\Sigma_{H,p}}(B(\xi, r))} \int_{B(\xi,r)} w^2 \theta d\sigma_{\Sigma_{H,p}} \right)^{1/2} \le |w|(\rho)\sqrt{\theta}(\rho) + \frac{\varepsilon_0}{\operatorname{vol}(\Sigma_{H,p})}.$$

Thus,

$$F_r\theta \le C_n \int_{\Sigma_{H,p}} |w|\sqrt{\theta} d\sigma_{\Sigma_{H,p}} + C_n \varepsilon_0.$$

Next, let $\{\theta_m\}_m$ be a sequence of continuous positive functions with $\theta_m \to f_1$ in L^1. We may assume by taking a subsequence that $\theta_m \to f_1$ a.e. Fix $\varepsilon_0 > 0$ and let $M_0 > 0$ be so that $\|f_1 - \theta_m\|_{L^1}^{1/2} \le \varepsilon_0$ for all $m \ge M_0$. Since $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$, for all $m \ge M_0$

$$|F_r f_1| \le F_r|f_1 - \theta_m| + F_r\theta_m \le C\|f_1 - \theta_m\|_{L^1}^{1/2} + F_r\theta_m \le C\varepsilon_0 + F_r\theta_m. \tag{2.11}$$

Now,

$$\int_{\Sigma_{H,p}} |w|\sqrt{\theta_m} d\sigma_{\Sigma_{H,p}} = \int_{\Sigma_{H,p}} |w|\sqrt{\max(\theta_m, 1)} d\sigma_{\Sigma_{H,p}} + \int_{\Sigma_{H,p}} 1_{\{\theta_m \ge 1\}} |w|(\sqrt{\theta_m} - 1) d\sigma_{\Sigma_{H,p}}.$$

Observe next that $\max(\theta_m, 1) \to \max(f_1, 1)$ a.e. and by the dominated convergence theorem,

$$\int_{\Sigma_{H,p}} |w|\sqrt{\max(\theta_m, 1)} d\sigma_{\Sigma_{H,p}} \to \int_{\Sigma_{H,p}} |w|\sqrt{\max(f_1, 1)} d\sigma_{\Sigma_{H,p}}.$$

Also,

$$\left| \int_{\Sigma_{H,p}} 1_{\{\theta_m \ge 1\}} |w|(\sqrt{\theta_m} - \sqrt{f_1}) d\sigma_{\Sigma_{H,p}} \right| = \left| \int_{\Sigma_{H,p}} 1_{\{\theta_m \ge 1\}} |w| \frac{\theta_m - f_1}{\sqrt{\theta_m} + \sqrt{f_1}} d\sigma_{\Sigma_{H,p}} \right| \le C\|\theta_m - f_1\|_{L^1}.$$

This proves that $\int_{\Sigma_{H,p}} |w|\sqrt{\theta_m} d\sigma_{\Sigma_{H,p}} \to \int_{\Sigma_{H,p}} |w|\sqrt{f_1} d\sigma_{\Sigma_{H,p}}$. Therefore, there exists $M_1 > 0$ so that for $m \geq M_1$,

$$\int_{\Sigma_{H,p}} |w|\sqrt{\theta_m} d\sigma_{\Sigma_{H,p}} \leq \int_{\Sigma_{H,p}} |w|\sqrt{f_1} d\sigma_{\Sigma_{H,p}} + \varepsilon_0.$$

Let $m \geq \max(M_0, M_1)$ and choose r small enough so that $F_r\theta_m \leq C_n \int_{\Sigma_{H,p}} |w|\sqrt{\theta_m} d\sigma_{\Sigma_{H,p}} + \varepsilon_0$. Then, (2.11) yields that for some $C_n > 0$

$$F_r f_1 \leq C\varepsilon_0 + F_r\theta_m \leq C_n \int_{\Sigma_{H,p}} |w|\sqrt{f_1} d\sigma_{\Sigma_{H,p}} + C_n\varepsilon_0.$$

This proves (2.9) as claimed, and hence concludes the proof of the theorem. □

2.3. **Proof of Theorem 6 for any** $A \subset H$**.** We first sketch the steps necessary to pass to $A \subset H$. We break the integral into two pieces. First, in an h-independent neigborhood of the conormal bundle N^*H, we approximate 1_A by an (h-independent) smooth function and apply the theorem on all of H. Then, to estimate the piece bounded away from N^*H, we approximate 1_A by a smooth function depending badly on h. We are then able to perform integration by parts to estimate contributions away from ∂A and a simple volume bound near ∂A. In order to handle the boundary of H itself, we extend H to a larger closed embedded submanifold $\tilde{H} \subset M$ so that $H^o \Subset \tilde{H}^o$ is an open subset.

Let $A \subset H$ be a subset with $\dim_{\text{box}}(\partial A) < n - k - \frac{1}{2}$ and indicator function 1_A. Extend H to $\tilde{H}$ another closed, embedded submanifold of codimension k so that H^o is compactly contained in the interior $\tilde{H}^o$. We will actually apply Theorem 6 to $\tilde{H}$ and $w \in C_c^\infty(\tilde{H}^o)$. Since $C_c^\infty(\tilde{H}^o)$ is dense in $L^2(\tilde{H}^o)$, for any $\delta > 0$, we can find a positive function $\psi_A \in C_c^\infty(\tilde{H}^o)$ with

$$\|\psi_A - 1_A\|_{L^2(\tilde{H})} \leq \delta.$$

For any $\varepsilon > 0$ and $w \in C_c^\infty(\tilde{H}^o)$,

$$\Big|\int_{\tilde{H}} 1_A\, w\, \phi_h d\sigma_{\tilde{H}}\Big| \leq \Big|\int_{\tilde{H}} 1_A Op_h(\beta_\varepsilon)(w\,\phi_h) d\sigma_{\tilde{H}}\Big| + \Big|\langle(1 - Op_h(\beta_\varepsilon))(w\phi_h), 1_A\rangle_{\tilde{H}}\Big|.$$

We claim that if $A \subset H$ has boundary satisfying $\dim_{\text{box}}(\partial A) < n - k - \frac{1}{2}$ Then, for all $\delta > 0$ and $\epsilon > 0$,

$$\|(1 - Op_h(\beta_\varepsilon))^* 1_A\|_{L^2(\tilde{H})} = O_{\varepsilon,\delta}(h^{\frac{1}{4}+\delta}). \tag{2.12}$$

We postpone the proof of (2.12) until the end. Assuming that (2.12) holds, the upper bound for eigenfunctions of Laplace-like operators $\|\phi_h\|_{L^2(\tilde{H})} \leq Ch^{-\frac{k-1}{2}-\frac{1}{4}}$ [BGT07, Theorem 3], [Tac10, Theorem 1.7] together with Cauchy-Schwarz give

$$
\begin{aligned}
&h^{\frac{k-1}{2}}\Big|\int_H 1_A\, w\, \phi_h d\sigma_H\Big| \leq \\
&\leq h^{\frac{k-1}{2}}\Big|\int_{\tilde{H}} 1_A Op_h(\beta_\varepsilon)(w\,\phi_h)d\sigma_H\Big| + h^{\frac{k-1}{2}}\|w\phi_h\|_{L^2(\tilde{H})}\|Op_h(1-\beta_\varepsilon)^*1_A\|_{L^2(\tilde{H})}\\
&\leq h^{\frac{k-1}{2}}\Big|\int_{\tilde{H}}(1_A-\psi_A)Op_h(\beta_\varepsilon)(w\phi_h)d\sigma_{\tilde{H}}\Big| + h^{\frac{k-1}{2}}\Big|\int_{\tilde{H}}\psi_A Op_h(\beta_\varepsilon)(w\phi_h)d\sigma_{\tilde{H}}\Big| + o_\varepsilon(1)\\
&=: T_{1,h}+T_{2,h}+o_\varepsilon(1). \qquad (2.13)
\end{aligned}
$$

Remark 2. Note that in fact, when $k>1$, the estimates from [BGT07, Tac10] show that $\|\phi_h\|_{L^2(\tilde{H})}\leq Ch^{-\frac{k-1}{2}-\frac{1}{4}}$ can be improved to $\|\phi_h\|_{L^2(\tilde{H})}\leq Ch^{-\frac{k-1}{2}}$ if $k>2$ and $\|\phi_h\|_{L^2(\tilde{H})}\leq Ch^{-\frac{k-1}{2}}(\log h^{-1})^{1/2}$ if $k=2$. Therefore, when $k>1$, we may allow A with boundary having higher box dimension than the upper bound requested.

Next, note that $\|Op_h(\beta_\varepsilon)(w\,\phi_h)\|_{L^2(\tilde{H})}=O(h^{\frac{1-k}{2}})$ and apply Cauchy–Schwarz to obtain

$$
T_{1,h}\leq \|1_A-\psi_A\|_{L^2(\tilde{H})}h^{\frac{k-1}{2}}\|Op_h(\beta_\varepsilon)(w\,\phi_h)\|_{L^2(\tilde{H})}\leq C\delta,
$$

for some $C>0$. Finally, to bound the second term in (2.13) we note that

$$
T_{2,h}=h^{\frac{k-1}{2}}\left|\int_{\tilde{H}}Op_h(\beta_\varepsilon)(\psi_A\, w\,\phi_h)d\sigma_{\tilde{H}}\right| + o(1) = h^{\frac{k-1}{2}}\left|\int_{\tilde{H}}\psi_A\, w\,\phi_h\, d\sigma_{\tilde{H}}\right| + o(1),
$$

and that by Theorem 6 with $A=\tilde{H}$ and $w\in C_c^\infty(\tilde{H}^o)$ there exists $C_{n,k}>0$ for which

$$
\begin{aligned}
\limsup_{h\to 0} T_{2,h} \leq C_{n,k}\int_{\Sigma_{\tilde{H}}}\psi_A\,|w|\sqrt{f|H_pr_H|^{-1}}\,d\sigma_{SN^*\tilde{H}} \leq \\
C_{n,k}\int_{\pi_H^{-1}(A)}|w|\sqrt{f|H_pr_H|^{-1}}\,d\sigma_{\Sigma_{H,p}} + C\delta\|f\|_{L^1(\tilde{H})}\|w\|_{L^\infty(\tilde{H})}.
\end{aligned}
$$

The last equality follows from Cauchy-Schwarz and the bound $\|\psi_A-1_A\|_{L^2(\tilde{H})}\leq\delta$. This gives the stated result provided (2.12) holds. We proceed to prove (2.12).

To prove (2.12) we first introduce a cut-off function $\chi_h\in C_c^\infty(\tilde{H}^o)$ so that $(1-\chi_h)1_A$ is smooth and close to 1_A and χ_h is 1 in a neighborhood of ∂A. For this, fix $0<\delta<1$ and cover ∂A by $(n-k)$-dimensional cubes $Q_{i,h}\subset\tilde{H}^o$, with $1\leq i\leq N(h)$, and side length h^δ with disjoint interiors. This can by done so that

$$
\limsup_{h\to 0^+}\frac{\log N(h)}{\delta\log h^{-1}}=\dim_{\text{box}}(\partial A).
$$

We decompose

$$
\begin{aligned}
\|(1-Op_h(\beta_\varepsilon))^*1_A\|_{L^2(\tilde{H})} = \|(1-Op_h(\beta_\varepsilon))^*(1-\chi_h)1_A\|_{L^2(\tilde{H})}\\
+\|(1-Op_h(\beta_\varepsilon))^*\chi_h1_A\|_{L^2(\tilde{H})}. \qquad (2.14)
\end{aligned}
$$

We bound $\|(1-Op_h(\beta_\varepsilon))^*\chi_h1_A\|_{L^2(\tilde{H})}$ using that $1-Op_h(\beta_\varepsilon)$ is L^2-bounded and that χ_h1_A has compact support. We proceed to bound $\|\chi_h1_A\|_{L^2(\tilde{H})}$. Cover each cube

$Q_{i,h}$ by 2^{n-k} open balls $B_{i,h}$ of radius h^δ. Let $\chi_{i,h} \in C_c^\infty(B_{i,h};[0,1])$ be a partition of unity near ∂A subordinate to $B_{i,h}$ and define $\chi_h = \sum_{i=1}^{N(h)} \chi_{i,h}$. Then,

$$\chi_h \equiv 1 \text{ in a neighborhood of } \partial A, \qquad \operatorname{supp}\chi_h \subset \{x \in H:\ d(x,\partial A) \leq 2h^\delta\},$$
$$|\partial_x^\alpha \chi_h| = O_\alpha(h^{-|\alpha|\delta}). \tag{2.15}$$

Moreover, since the volume of each cube $Q_{i,h}$ is $h^{\delta(n-k)}$, there is $C>0$ so that

$$\|\chi_h\|^2_{L^2(\tilde H)} \leq CN(h)h^{\delta(n-k)} \leq Ch^{\delta(n-k-\dim_{\text{box}}(\partial A))}.$$

It follows that

$$\|(1-Op_h(\beta_\varepsilon))^*\chi_h 1_A\|_{L^2(\tilde H)} = O\Big(h^{\frac{\delta}{2}(n-k-\dim_{\text{box}}(\partial A))}\Big). \tag{2.16}$$

On the other hand, the function $(1-\chi_h)1_A = (1-\chi_h)$ satisfies the bounds (2.15). In particular, putting $\psi_h = 1-\chi_h$ in Lemma 12 below, for $\delta < 1$,

$$\|(1-Op_h(\beta_\varepsilon))^*(1-\chi_h)1_A\|_{L^\infty(\tilde H)} = O_\epsilon(h^\infty). \tag{2.17}$$

Combining (2.16) and (2.17) into (2.14), and taking $0<\delta<1$ sufficiently close to 1, proves (2.12) as claimed.

□

Lemma 12. *Suppose that $\psi_h \in C_c^\infty(\tilde H^o)$ satisfies* (2.15) *for some $0<\delta<1$. Then,*

$$\|(1-Op_h(\beta_\varepsilon))^*\psi_h\|_{L^\infty(\tilde H)} = O_{\varepsilon,\delta}(h^\infty).$$

Proof. We work in Fermi normal coordinates (x',x'') so that x' is a coordinate on $\tilde H$. Integrating by parts with

$$L_{x'} := \frac{1}{|y'-x'|^2+|\xi'|^2}\left(\sum_{j=1}^n \xi_j' hD_{y_j'} + \sum_{j=1}^n (x_j'-y_j')hD_{\xi_j'}\right),$$

relation (2.15) gives

$$\begin{aligned}
&[(1-Op_h(\beta_\varepsilon))^*\psi_h](x') = \\
&\quad = \frac{1}{(2\pi h)^{n-k}}\iint e^{\frac{i}{h}\langle x'-y',\xi'\rangle}(1-\beta_\varepsilon(y',\xi'))(\psi_h(y'))dy'd\xi' \\
&\quad = \frac{1}{(2\pi h)^{n-k}}\iint e^{\frac{i}{h}\langle x'-y',\xi'\rangle}(L_{x'}^*)^N\big[(1-\beta_\varepsilon(y',\xi'))\psi_h(y')\big]dy'd\xi' \\
&\quad = O_{\varepsilon,N}\big(h^{k-n+N(1-\delta)}\big)
\end{aligned}$$

□

2.4. **Localizing near bicharacteristics: Proof of Proposition 10.** Throughout the proof of Proposition 10 we will need the following lemma. Since it is a local result, we state it for functions and operators acting on $\mathbb{R}^n$. We write $(x_1, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}$ for coordinates in $\mathbb{R}^n$ and $(\xi_1, \tilde{\xi})$ for the dual coordinates.

Lemma 13. *Let $\kappa = \kappa(x_1, \tilde{x}, \tilde{\xi})$ be a smooth function with compact support and fix $\rho_0 \in T^*\mathbb{R}^n$ with*

$$p(\rho_0) \neq 0 \quad or \quad \partial_{\xi_1} p(\rho_0) \neq 0.$$

Then, there exists $C_0, T_0 > 0$ and a neighborhood V of ρ_0 so that for all $0 < T < T_0$ the following holds. Let U be a neighborhood of $\operatorname{supp} \kappa$ and $b \in C_c^\infty(T^*\mathbb{R}^n)$ with

$$\bigcup_{|t|<T} \varphi_t(\{p = 0\} \cap U) \subset \{b \equiv 1\}. \tag{2.18}$$

Let $\chi \in C_c^\infty(V)$, $\tilde{\chi} \in C_c^\infty(T^*\mathbb{R}^n)$ with $\tilde{\chi} \equiv 1$ in a neighborhood of $\operatorname{supp} \chi$, and $q = q(x_1) \in C^\infty(\mathbb{R}; S^\infty(T^*\mathbb{R}^{n-1}))$. Then, there exists $C > 0$ so that the following hold. If $p(\rho_0) \neq 0$, then

$$\|Op_h(q)Op_h(\kappa)Op_h(\chi)\phi_h\|_{L^\infty_{x_1} L^2_{\tilde{x}}} \leq C\|Op_h(\tilde{\chi})P\phi_h\|_{L^2_x} + O(h^\infty)\|\phi_h\|_{L^2_x}.$$

If $p(\rho_0) = 0$, then

$$\begin{aligned}
\sup_{|x_1| < \frac{T}{3}|\partial_{\xi_1} p(\rho_0)|} &\|Op_h(q)Op_h(\kappa)Op_h(\chi)\phi_h(x_1, \cdot)\|_{L^2_{\tilde{x}}} \leq \\
&\leq 4T^{-\frac{1}{2}}|\partial_{\xi_1} p(\rho_0)|^{-\frac{1}{2}}\|Op_h(b)Op_h(\chi)Op_h(q)\phi_h\|_{L^2_x} \\
&\quad + C_0 T^{\frac{1}{2}} h^{-1}\|Op_h(b)Op_h(p)Op_h(\chi)Op_h(q)\phi_h\|_{L^2_x} + Ch^{-1}\|Op_h(\tilde{\chi})P\phi_h\| \\
&\quad + Ch^{1/2}\|Op_h(\tilde{\chi})\phi_h\|_{L^2_x} + O(h^\infty)\|\phi_h\|_{L^2_x}.
\end{aligned}$$

The proof of Lemma 13 is very similar to that of [Gal17, Lemma 4.3], although some alterations are needed. For the sake of completeness we include the proof.

Proof. First, suppose $\rho_0 \in T^*M$ is so that $p(\rho_0) \neq 0$. Then, there exists a neighborhood $U \subset T^*\mathbb{R}^n$ of ρ_0 with $U \subset \{p \neq 0\}$. One can then carry an elliptic parametrix construction so that

$$Op_h(q\,\kappa\,\chi)\phi_h = Op_h(\tilde{e})Op_h(\tilde{\chi})Op_h(p)\phi_h + O_{\Psi^{-\infty}}(h^\infty)\phi_h, \tag{2.19}$$

for all χ supported in U and some suitable $\tilde{e}$. Therefore,

$$\|Op_h(q\,\kappa\,\chi)\phi_h(0, x')\|_{L^2_{\tilde{x}}} \leq C\|Op_h(\tilde{\chi})P\phi_h\|_{L^2_x} + O(h^\infty)\|\phi_h\|_{L^2_x},$$

as claimed. We may assume from now on that

$$\rho_0 \in \{\partial_{\xi_1} p \neq 0\} \cap \{p = 0\}.$$

By the implicit function theorem, for $\tilde{\chi}$ supported sufficiently close to ρ_0, and $\operatorname{supp} \chi \subset \{\tilde{\chi} \equiv 1\}$

$$p(x, \xi)\tilde{\chi}(x, \xi) = e(x, \xi)(\xi_1 - a(x, \tilde{\xi}))$$

with $e(x, \xi)$ elliptic on $\operatorname{supp} \chi$ and $\xi = (\xi_1, \tilde{\xi})$. In particular,

$$Op_h(p)Op_h(\chi) = Op_h(e)(hD_{x_1} - Op_h(a)))Op_h(\chi) + hOp_h(R)Op_h(\chi).$$

Therefore,

$$(hD_{x_1} - Op_h(a))w = f,$$

where we have set

$$w := Op_h(\chi)Op_h(q)\phi_h,$$
$$f := [Op_h(e)^{-1}Op_h(p)Op_h(\chi)Op_h(q) + hOp_h(R_1)Op_h(\chi)Op_h(q)]\phi_h + O(h^\infty)$$

and $Op_h(e)^{-1}$ denotes a microlocal parametrix for $Op_h(e)$ near $\operatorname{supp}\chi$. Defining

$$A(t, s, \tilde{x}, hD_{\tilde{x}}) := -\int_s^t a(x_1, \tilde{x}, hD_{\tilde{x}})dx_1,$$

we obtain that for all $s, t \in \mathbb{R}$

$$w(s, \tilde{x}) = e^{-\frac{i}{h}A(t,s,\tilde{x},hD_{\tilde{x}})}w(t, \tilde{x}) - \frac{i}{h}\int_s^t e^{-\frac{i}{h}A(x_1,s,\tilde{x},hD_{\tilde{x}})}f(x_1, \tilde{x})dx_1.$$

Let $\delta > 0$ be such that

$$\delta \leq \frac{T}{3}|\partial_{\xi_1}p(\rho_0)| \leq \frac{T}{2}\inf\left\{|\partial_{\xi_1}p(x, \xi)| : \ (x, \xi) \in \operatorname{supp}\chi\right\} \tag{2.20}$$

and $\Phi \in C_c^\infty(\mathbb{R}; [0, 2\delta^{-1}])$ with $\operatorname{supp}\Phi \subset [0, \delta]$ and $\int_{\mathbb{R}}\Phi = 1$. Then, integrating in t,

$$w(s, \tilde{x}) = \int_{\mathbb{R}}\Phi(t)e^{-\frac{i}{h}A(t,s,\tilde{x},hD_{\tilde{x}})}w(t, \tilde{x})dt - \frac{i}{h}\int_{\mathbb{R}}\Phi(t)\int_s^t e^{-\frac{i}{h}A(x_1,s,\tilde{x},hD_{\tilde{x}})}f(x_1, \tilde{x})dx_1dt.$$

Next, applying propagation of singularities, we claim that

$$\begin{aligned} Op_h(\kappa)w(x_1, \tilde{x}) = &\int_{\mathbb{R}}\Phi(t)Op_h(\kappa)e^{-\frac{i}{h}A(t,x_1,\tilde{x},hD_{\tilde{x}})}Op_h(b)w(t, \tilde{x})dt \\ &- \frac{i}{h}\int_{\mathbb{R}}\Phi(t)\int_{x_1}^t Op_h(\kappa)e^{-\frac{i}{h}A(s,t,\tilde{x},hD_{\tilde{x}})}Op_h(b)f(s, \tilde{x})dsdt \\ &+ R_h(x_1, \tilde{x}) + O(h^\infty)\|\phi_h\|_{L^2}, \end{aligned} \tag{2.21}$$

with $\|R_h(x_1, \tilde{x})\|_{L^\infty_{x_1}L^2_{\tilde{x}}} = O(h^{-1}\|Op_h(\tilde{\chi})P\phi_h\|_{L^2_x})$. Indeed, (2.21) follows once we show that for any $v \in S^0(T^*M)$ supported on $\tilde{\chi} \equiv 1$ and $x_1 \in [0, \delta]$

$$\|Op_h(\kappa)e^{-\frac{i}{h}A(x_1,t,\tilde{x},hD_{\tilde{x}})}(1 - Op_h(b))Op_h(v)\phi_h\|_{L^2_x} \leq C\|Op_h(\tilde{\chi})P\phi_h\|_{L^2_x} + O(h^\infty)\|\phi_h\|_{L^2_x}. \tag{2.22}$$

Let $\chi_\varepsilon \in C_c^\infty(\mathbb{R}; [0, 1])$ be as in (2.1). By the same construction carried in (2.19) (which gives that ϕ_h is microlocalized on $\{p = 0\}$) we conclude

$$\|Op_h(\kappa)e^{-\frac{i}{h}A(x_1,t,\tilde{x},hD_{\tilde{x}})}(1 - Op_h(b))Op_h(v)(1 - Op_h(\chi_\varepsilon(p))\phi_h\|_{L^2_x} \leq C_\varepsilon\|Op_h(\tilde{\chi})P\phi_h\|_{L^2_x} + O(h^\infty)\|\phi_h\|_{L^2_x}. \tag{2.23}$$

Therefore, to prove (2.22) we need to estimate

$$\|Op_h(\kappa)e^{-\frac{i}{h}A(x_1,t,\tilde{x},hD_{\tilde{x}})}(1 - Op_h(b))Op_h(v)Op_h(\chi_\varepsilon(p))\phi_h\|_{L^2_x}.$$

Let $\tilde{\varphi}_t$ denote the Hamiltonian flow of $\tilde{p}(x,\xi)=\xi_1-a(x,\tilde{\xi})$. Then, for $(x,\xi)\in\{(x,\xi): |p(x,\xi)|\leq C\varepsilon^2\}$ and $|t|\leq 1$, we have $d(\varphi_t(x,\xi),\tilde{\varphi}_t(x,\xi))\leq C\varepsilon^2$. By (2.18), b is identically 1 in a neighborhood of

$$\bigcup_{|t|\leq T}\varphi_t(\{\operatorname{supp}\kappa\}\cap\{p=0\})$$

and thus for $\varepsilon>0$ small enough on

$$\bigcup_{|t|\leq 2T}\tilde{\varphi}_t(\{\operatorname{supp}\kappa\}\cap\{|p|\leq C\varepsilon^2\}).$$

In particular, since we assume that $\operatorname{supp}\Phi\subset[0,\delta]$ and δ satisfies (2.20), we have

$$\|\Phi(t)Op_h(\kappa)e^{-\frac{i}{h}A(t,s,\tilde{x},hD_{\tilde{x}})}(1-Op_h(b))Op_h(a)Op_h(\chi_\varepsilon(p))\phi_h\|_{L^2_x}=O_\varepsilon(h^\infty)\|\phi_h\|_{L^2_x}. \tag{2.24}$$

Together (2.23) and (2.24) give (2.22). In particular, we obtain (2.21) which, since

$$|\Phi(t)|\leq 2\delta^{-1},\qquad \text{and hence}\qquad \|\Phi\|_{L^2}\leq 2\delta^{-\frac{1}{2}},$$

implies that for $|x_1|\leq\delta$,

$$\begin{aligned}\|Op_h(\kappa)w(x_1,\cdot)\|_{L^2_{\tilde{x}}}\leq 2\delta^{-1/2}\|Op_h(b)w\|_{L^2_x}+C_0\delta^{1/2}h^{-1}\|Op_h(b)f\|_{L^2_x}\\ +Ch^{-1}\|Op_h(\tilde{\chi})P\phi_h\|_{L^2_x}+O(h^\infty)\|\phi_h\|_{L^2_x}.\end{aligned}$$

To see this, we start by applying Cauchy-Schwarz to the first term in (2.21) and use that $e^{-\frac{i}{h}A(t,x_1,\tilde{x},hD_{\tilde{x}})}$ is a unitary operator to get

$$\left\|\int_{\mathbb{R}}\Phi(t)Op_h(\kappa)e^{-\frac{i}{h}A(t,x_1,\tilde{x},hD_{\tilde{x}})}Op_h(b_0)w(t,\tilde{x})dt\right\|_{L^\infty_{x_1}L^2_{\tilde{x}}}\leq\|\Phi\|_2\,\|Op_h(\kappa)\|\|Op_h(b_0)w\|_{L^2_{t,\tilde{x}}}.$$

To bound the second term in (2.21) we apply Minkowski's integral inequality, use that the support of Φ is contained in $[0,\delta]$, and that $|x_1|<\delta$ to get

$$\begin{aligned}&\left\|\int_{\mathbb{R}}\Phi(t)\int_{x_1}^t Op_h(\kappa)e^{-\frac{i}{h}A(s,t,\tilde{x},hD_{\tilde{x}})}Op_h(b_0)f(s,\tilde{x})dsdt\right\|_{L^\infty_{x_1}L^2_{\tilde{x}}}\\ &\leq\left\|\int_{\mathbb{R}}\Phi(t)\left(\int_{\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}}\mathbf{1}_{[-\delta,\delta]}(s)Op_h(\kappa)e^{-\frac{i}{h}A(s,t,\tilde{x},hD_{\tilde{x}})}Op_h(b_0)f(s,\tilde{x})ds\right)^2d\tilde{x}\right)^{\frac{1}{2}}dt\right\|_{L^\infty_{x_1}}\\ &\leq\|\mathbf{1}_{[-\delta,\delta]}(s)\|_{L^2_s}\,\|Op_h(\kappa)\|\|Op_h(b_0)f\|_{L^2_{s,\tilde{x}}}.\end{aligned}$$

Now,

$$Op_h(q)Op_h(\kappa)Op_h(\chi)=Op_h(\kappa)Op_h(\chi)Op_h(q)+[Op_h(q),Op_h(\kappa)Op_h(\chi)].$$

Therefore, since

$$\|[Op_h(q),Op_h(\kappa)Op_h(\chi)]\phi_h(x_1,\cdot)\|_{L^2_{\tilde{x}}}\leq Ch^{\frac{1}{2}}\|Op_h(\tilde{\chi})\phi_h\|_{L^2_x}+O(h^\infty)\|\phi_h\|_{L^2_x},$$

we have the following L^2 bound for $|x_1| \le \delta = \frac{T}{3}|\partial_{\xi_1} p(\rho_0)|$

$$\|Op_h(\kappa)Op_h(\chi)Op_h(q)\phi_h(x_1,\cdot)\|_{L^2_{\bar{x}}} \le 2\delta^{-1/2}\|Op_h(b)w\|_{L^2_x} + C_0\delta^{1/2}h^{-1}\|Op_h(b)f\|_{L^2_x}$$
$$+Ch^{-1}\|Op_h(\tilde{\chi})P\phi_h\|_{L^2_x} + Ch^{\frac{1}{2}}\|Op_h(\tilde{\chi})\phi_h\|_{L^2_x} + O(h^\infty)\|\phi_h\|_{L^2_x} \quad (2.25)$$

finishing the proof. □

The proof of Proposition 10 hinges on Lemma 14 below. This lemma is dedicated to obtaining a gain in the bound for $\|Op_h(\beta_\varepsilon)Op_h(\chi)\phi_h\|_{L^2(H)}$ by localizing in phase space near bicharacteristics emanating from $\Sigma_{H,p}$. The key idea is that microlocalization near a family of bicharacteristics parametrized by H implies a quantitative gain in the $L^2(H)$ norm. By decomposing ϕ_h into many pieces microlocalized along well-chosen families of bicharacteristics, we are able to extract Proposition 10.

Let $\Xi : H \to \Sigma_{H,p}$ be a smooth section (i.e. $\Xi \in C^\infty$ and $\Xi(x) \in T^*_xM$); where we continue to write $\Sigma_{H,p} = \{p = 0\} \cap N^*H$. Let $\chi \in C^\infty_c(T^*M)$ supported near $\rho_0 \in \Sigma_{H,p}$.

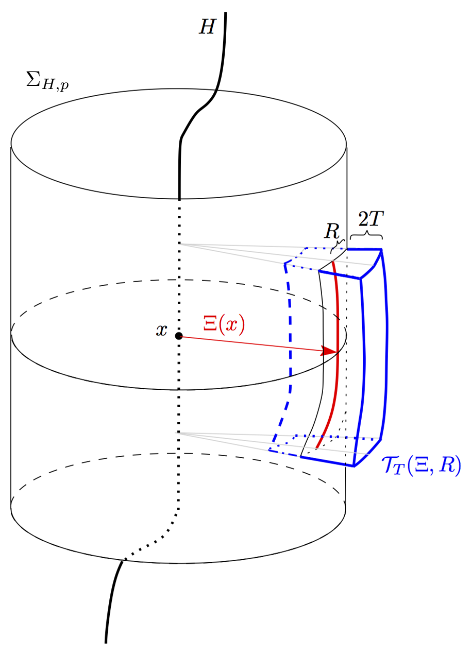

Figure 2. We show a schematic of $\Xi(x)$, $\Sigma_{H,p}$, and $\mathcal{T}_T(\Xi, R)$ for H a curve and $d = 3$.

We choose Fermi coordinates with respect to H, $(x_1, \bar{x}, x')$, so that $H = \{(x_1, \bar{x}) = 0\}$ and, making additional rotation in $(x_1, \bar{x})$ if necessary, so that

$$|H_p r_H(\rho_0)| = \partial_{\xi_1} p(\rho_0) \neq 0.$$

Moreover, note that for u supported near x_0 we have $\|u\|_{L^2_x} \le 2\|u\|_{L^2(M)}$.

For each $(0, x') \in H$ in the projection of $\operatorname{supp}\chi$ onto H define a vector-valued function $a(x_1; x') \in C^\infty(\mathbb{R}^{n-k+1}; \mathbb{R}^n)$ so that $\xi - a(x_1; x')$ vanishes on the bicharacteristic emanating from $((0,x'),\Xi((0,x'))))$. This is possible since we have chosen coordinates so that

$$\partial_{\xi_1}p(\rho_0)\neq 0,$$

and hence the bicharacteristic emanating from $((0,x'),\Xi((0,x'))))$ may be written locally as

$$\gamma_{x'}:(-T_\chi,T_\chi)\to T^*M,\qquad\qquad \gamma_{x'}(x_1)=(x(x_1;x'),a(x_1;x')) \tag{2.26}$$

where $T_\chi>0$ is small enough, and x,a are smooth functions depending on χ. Indeed, if we write $\gamma_{x'}(t)=(x(t),\xi(t))$, we have that $\frac{d}{dt}x_1(t)=\partial_{\xi_1}p(\gamma_{x'}(t))$ which allows us to use the inverse function theorem to locally write $t=t(x_1)$ as a function of x_1.

To exploit the construction of the function a we further localize in phase space on tubes of small radius R that cover $\operatorname{supp}(\chi 1_{\Sigma_{H,p}})$. We define the tubes

$$\mathcal{T}_T(\Xi,R):=\bigcup_{|t|\leq 2T}\varphi_t(\{(x,\xi)\in\Sigma_{H,p}:\ d((x,\xi),(x,\Xi(x)))<R\}), \tag{2.27}$$

where $d((x,\xi),(x,\Xi(x)))$ describes the distance in $\Sigma_{H,p}\cap T^*_xM$ between the points (x,ξ) and $(x,\Xi(x))$ (see Figure 2 for a schematic picture of these objects).

The spirit of the following result is similar to that of [Gal17, Lemma 5.2]. Lemma 14 is dedicated to showing that microlocalizing with χ supported on $\mathcal{T}_T(\Xi,R)$ gives an R^{k-1} gain in the bound for $\|Op_h(\beta_\varepsilon w)Op_h(\chi)\phi_h\|_{L^2(H)}$.

Lemma 14. *There exist $C_{n,k}>0$ depending only on n and k and $c>0$ depending only on (M,g,H) so that the following holds. Let $\chi\in C_c^\infty(T^*M)$ supported sufficiently close to $\rho_0\in\Sigma_{H,p}$ satisfy*

$$H_p\chi\equiv 0,\ \ \text{on } \Lambda_{H,T},$$

where $0<T\leq T_\chi$ and T_χ is defined in (2.26). Let $\Xi:H\to\Sigma_{H,p}$ be a smooth section. Then for all $0<R<c$ and $w\in C_c^\infty(H^o)$ if

$$\operatorname{supp}(\chi 1_{\Lambda_{H,T}})\subset\mathcal{T}_T(\Xi,R), \tag{2.28}$$

then

$$\lim_{\varepsilon\to 0}\limsup_{h\to 0}h^{k-1}\|Op_h(\beta_\varepsilon w)Op_h(\chi)\phi_h\|^2_{L^2(H)}\leq\, C_{n,k}\,\frac{R^{k-1}}{T|H_pr_H(\rho_0)|}\int_{\Lambda_{H,T}}\chi^2\tilde{w}^2d\mu, \tag{2.29}$$

where $\tilde{w}\in C_c^\infty(T^*M)$ is any extension of w for which $H_p\tilde{w}\equiv 0$ on $\Lambda_{H,T}$. In addition, if the assumption in (2.28) is not enforced, then (2.29) holds with $R=1$ and $C_{n,k}$ replaced by a constant depending on (M,g,H,p).

Proof. In what follows we write $\bar{x}$ for the normal coordinates to H that are not x_1. With this notation $x=(x_1,\bar{x},x')$. As before, let $\iota_{w,\varepsilon}\in C_c^\infty(H)$ with

$$\iota_{w,\varepsilon}(x')\equiv 1\quad\text{for}\quad x'\in\operatorname{supp}w,\qquad\qquad \lim_{\varepsilon\to 0}\iota_{w,\varepsilon}=1_{\operatorname{supp}w}.$$

Define also

$$\kappa_\varepsilon(x,\xi)=\beta_\varepsilon(x',\xi')\chi_\varepsilon(|(x_1,\bar{x})|)\iota_{w,\varepsilon}(x').$$

and $\tilde{w}\in C_c^\infty(T^*M)$ with

$$H_p\tilde{w}=0,\quad\ \text{on } \Lambda_{H,T},\qquad \tilde{w}|_{\Sigma_{H,p}}=w.$$

Since ϕ_h is compactly microlocalized (see (1.6)), we have $\|\phi_h\|_{L^2(H)} \leq Ch^{-\frac{k}{2}}$, we bound

$$\|Op_h(\beta_\varepsilon w)Op_h(\chi)\phi_h\|_{L^2(H)} \leq \|Op_h(\kappa_\varepsilon \tilde{w}\chi)\phi_h\|_{L^2(H)} + O_\varepsilon(h^{\frac{2-k}{2}}) = \|v_h\|_{L^2(H)} + O_\varepsilon(h^{\frac{2-k}{2}}).$$

for

$$v_h := e^{-\frac{i}{h}\langle \bar{x}\,,\,\bar{a}(x_1;x')\rangle} Op_h(\kappa_\varepsilon \tilde{w}\chi)\phi_h,$$

where $\bar{a}(x_1;x') = (a_2(x_1,x'),\dots,a_k(x_1,x'))$ and a is defined in (2.26). The reason for working with this function v_h is that

$$e^{-\frac{i}{h}\langle \bar{x}\,,\,\bar{a}(x_1;x')\rangle}(hD_{x_i})^\ell v_h = (hD_{x_i} - a_i)^\ell (Op_h(\kappa_\varepsilon \tilde{w}\chi)\phi_h),$$

for $i = 2,\dots,k$, and this will allow to obtain a gain in the L^2-norm bound, since, as we will see below, $\sup_{T_\delta(\Xi,R)\cap\Lambda_{H,T}} \max_i |\xi_i - a_i(x_1,x')| \leq 3R$. We bound $\|v_h\|_{L^2(H)}$ using the version of the Sobolev Embedding Theorem given in [Gal17, Lemma 5.1] which states that if $\ell > (k-1)/2$, there exists $C_{\ell,k} > 0$ depending only on ℓ and k so that for all $\alpha > 0$

$$\|v_h(x_1,\cdot,x')\|_{L^\infty_{\bar{x}}} \leq C_{\ell,k} h^{1-k}\left(\alpha^{k-1}\|v_h(x_1,\cdot,x')\|^2_{L^2_{\bar{x}}} + \alpha^{k-1-2\ell}\sum_{i=2}^{k}\|(hD_{x_i})^\ell v_h(x_1,\cdot,x')\|^2_{L^2_{\bar{x}}}\right),$$

for all x_1, x'. Now, for all $x_1, \bar{x}$, integrate in x' to get

$$\|v_h(x_1,\bar{x},\cdot)\|^2_{L^2_{x'}} \leq C_{\ell,k}h^{1-k}\left(\alpha^{k-1}\|v_h(x_1,\cdot)\|^2_{L^2_{\bar{x},x'}} + \alpha^{k-1-2\ell}\sum_{i=2}^{k}\|(hD_{x_i})^\ell v_h(x_1,\cdot)\|^2_{L^2_{\bar{x},x'}}\right).$$

In particular, setting $(x_1,\bar{x}) = (0,0)$ on the left hand side we get

$$\|v_h\|^2_{L^2(H)} \leq C_{\ell,k}h^{1-k}\left(\alpha^{k-1}\|v_h(0,\cdot)\|^2_{L^2_{\bar{x},x'}} + \alpha^{k-1-2\ell}\sum_{i=2}^{k}\|(hD_{x_i})^\ell v_h(0,\cdot)\|^2_{L^2_{\bar{x},x'}}\right). \tag{2.30}$$

We will end up choosing $\alpha = R$ and $\ell = k$.

Remark 3. Note that when $k = 1$ (i.e. in the case of H is a hypersurface), estimates on the derivatives are not necessary. In particular, since H acts as a single space-like hypersurface in the energy estimates of Lemma 13, we cannot hope to gain additional powers of R in the $L^2(H)$ norm from better control on derivatives along H.

By (1.8) we may assume, without loss of generality, that $\partial_{\xi_1}p \neq 0$ on $\operatorname{supp}\kappa_\varepsilon \cap \{p = 0\}$. We choose Fermi coordinates with respect to H so that

$$|H_p r_H(\rho_0)| = \partial_{\xi_1}p(\rho_0) \neq 0 \qquad \text{or} \qquad p(\rho_0)\neq 0.$$

Moreover, in these coordinates $\|u\|_{L^2_x} \leq 2\|u\|_{L^2(M)}$. Hence, we will apply Lemma 13 with $\kappa = \kappa_\varepsilon$ and χ (here we shrink the support of χ if necessary). In order to apply the lemma, we note that

$$\operatorname{supp}\kappa_\varepsilon \cap \{p = 0\} \subset \{(x,\xi):\ |x_1| \leq 3\varepsilon,\ |\xi'| \leq 3\varepsilon,\ p = 0\},$$

and define $b_\varepsilon \in C_c^\infty(T^*M;[0,1])$ so that

$$\begin{aligned} &\bullet\ b_\varepsilon \equiv 1 \quad \text{on} \quad \bigcup_{|t|\leq T/3} \varphi_t(\{(x,\xi):\ |(x_1,\bar{x})| \leq 3\varepsilon,\ |\xi'| \leq 3\varepsilon,\ p=0\}),\\ &\bullet\ \operatorname{supp} b_\varepsilon \subset \bigcup_{|t|\leq T/2} \varphi_t(\{(x,\xi):\ |(x_1,\bar{x})| \leq 4\varepsilon,\ |\xi'| \leq 4\varepsilon, |p| \leq 2\varepsilon\}). \end{aligned} \tag{2.31}$$

We apply Lemma 13 with $\kappa = \kappa_\varepsilon$ and χ (here we shrink the support of χ if necessary). Next, let $\tilde{\iota}_{w,\varepsilon}$ be an extension of $\iota_{w,\varepsilon}$ off of $\Sigma_{H,p}$ so that $H_p \iota_{w,\varepsilon} \equiv 0$ in a neighborhood of $b_\varepsilon \equiv 1$. We do this as in Lemma 11 using that H_p is transverse to $\Sigma_{H,p}$ to solve the initial value problem.

Next, we choose q to obtain a gain in the $L^2(H)$ restriction norm related to R. Let

$$T_{\rho_0} := T|\partial_{\xi_1} p(\rho_0)|.$$

Applying Lemma 13 with $\kappa = \kappa_\varepsilon$, χ, $b = \tilde{\iota}_{w,\varepsilon} b_\varepsilon$, and $q = 1$, we have

$$\begin{aligned} \|v_h(0,\cdot)\|_{L^2_{\bar{x},x'}} \leq 8T_{\rho_0}^{-\frac12} \|Op_h(\tilde{\iota}_{w,\varepsilon} b_\varepsilon \tilde{w}\chi)\phi_h\|_{L^2(M)} \\ + C_0 T^{\frac12} h^{-1} \|Op_h(\tilde{\iota}_{w,\varepsilon} b_\varepsilon) P Op_h(\tilde{w}\chi)\phi_h\|_{L^2(M)} + o_{\varepsilon,T}(1) \end{aligned}$$

with $C_0 > 0$ independent of T. Here we have used that in our coordinates $\|u\|_{L^2_x} \leq 2\|u\|_{L^2(M)}$.

Let ℓ with $2\ell > k-1$ and define

$$Q_i = (hD_{x_i} - a_i)^\ell \qquad \text{and} \qquad Q_i = Op_h(q_i).$$

In particular, $q_i = (\xi_i - a_i)^\ell + O(h)$. Then, Lemma 13 gives that there exists $C_0 > 0$ independent of T so that

$$\begin{aligned} \|(hD_{x_i})^\ell v_h(0,\cdot)\|_{L^2_{\bar{x},x'}} \leq 128 T_{\rho_0}^{-\frac12} \|Op(\tilde{\iota}_{w,\varepsilon} b_\varepsilon) Op_h(\tilde{w}\chi) Q_i \phi_h\|_{L^2(M)} \\ + C_0 T^{\frac12} h^{-1} \|Op_h(\tilde{\iota}_{w,\varepsilon} b_\varepsilon) P Op(\tilde{w}\chi) Q_i \phi_h\|_{L^2(M)} + o_{\varepsilon,T}(1). \end{aligned}$$

Applying (2.30) gives that for any $\alpha > 0$

$$\begin{aligned} &h^{k-1} \|Op_h(\beta_\varepsilon w) Op_h(\chi)\phi_h\|^2_{L^2(H)} \\ &\leq C_{\ell,k} \alpha^{k-1} \Big(T_{\rho_0}^{-1} \|Op_h(\tilde{\iota}_{w,\varepsilon} b_\varepsilon \tilde{w}\chi)\phi_h\|^2_{L^2(M)} + h^{-2} C_0^2 T \|Op_h(\tilde{\iota}_{w,\varepsilon} b_\varepsilon) P Op_h(\tilde{w}\chi)\phi_h\|^2_{L^2(M)} \Big) \\ &\quad + C_{\ell,k} \alpha^{k-2\ell-1} \sum_{i=2}^k T_{\rho_0}^{-1} \|Op_h(\tilde{\iota}_{w,\varepsilon} b_\varepsilon) Op_h(\tilde{w}\chi) Q_i \phi_h\|^2_{L^2(M)} \\ &\quad + C_{\ell,k} \alpha^{k-2\ell-1} h^{-2} \sum_{i=2}^k C_0^2 T \|Op_h(\tilde{\iota}_{w,\varepsilon} b_\varepsilon) P Op_h(\tilde{w}\chi) Q_i \phi_h\|^2_{L^2(M)} + o_{\varepsilon,T}(1). \end{aligned} \tag{2.32}$$

Now, we use that $H_p(\tilde{w}\chi) = 0$, $P\phi_h = o(h)$, and

$$P Op_h(\tilde{w}\chi)\phi_h = Op_h(\tilde{w}\chi) P\phi_h + \frac{h}{i} Op_h(H_p(\tilde{w}\chi))\phi_h + O_{L^2}(h^2).$$

In particular, since μ is the defect measure associated to $\{\phi_h\}$, we obtain

$$\limsup_{h\to 0} h^{k-1}\|Op_h(\beta_\varepsilon)Op_h(\chi w)\phi_h\|^2_{L^2(H)} \leq$$
$$C_{\ell,k}\alpha^{k-1}\int_{T^*M} \tilde{\iota}^2_{w,\varepsilon} b^2_\varepsilon (T^{-1}_{\rho_0}\chi^2 + C_0^2 T|H_p(\tilde{w}\chi)|^2)d\mu$$
$$+ C_{\ell,k}\alpha^{k-2\ell-1}\sum_{i=2}^k \int_{T^*M} \tilde{\iota}^2_{w,\varepsilon} b^2_\varepsilon (T^{-1}_{\rho_0}\chi^2 q_i^2 + C_0^2 T|H_p(\tilde{w}\chi q_i)|^2)d\mu.$$

Next, we observe that by (2.31) and the fact that $0 \leq b^2_\varepsilon \leq 1$, we have

$$\lim_{\varepsilon\to 0} \tilde{\iota}^2_{w,\varepsilon} b^2_\varepsilon \leq \tilde{w}^2 1_{\operatorname{supp}\tilde{w}}.$$

Sending $\varepsilon \to 0$ and using $H_p(\tilde{w}\chi) = 0$ on $\Lambda_{H,T}$ (together with $\mu(T^*M) = 1$ to apply the dominated convergence theorem) we have

$$\lim_{\varepsilon\to 0}\limsup_{h\to 0} h^{k-1}\|Op_h(\beta_\varepsilon w)Op_h(\chi)\phi_h\|^2_{L^2(H)} \leq C_{\ell,k}\alpha^{k-1}T^{-1}_{\rho_0}\int_{\Lambda_{H,T}} \chi^2\tilde{w}^2 d\mu$$
$$+ C_{\ell,k}\alpha^{k-2\ell-1}\sum_{i=2}^k \int_{\Lambda_{H,T}} \chi^2\tilde{w}^2(T^{-1}_{\rho_0}q_i^2 + C_0^2 T|H_p q_i|^2)d\mu. \tag{2.33}$$

Next, assume that $\operatorname{supp}(\chi 1_{\Lambda_{H,T}}) \subset \mathcal{T}_T(\Xi, R)$. By [Gal17, Lemma 3.1], where G_t is used to denote $\exp(tH_p) = \varphi_t$,

$$\sup_{\mathcal{T}_T(\Xi,R)\cap\Lambda_{H,T}} \max_i |\xi_i - a_i(x_1, x')| \leq 3R. \tag{2.34}$$

Hence, since $H_p(\xi_i - a_i(x_1, x')) = 0$ on $\gamma_{x'}$,

$$\sup_{\mathcal{T}_T(\Xi,R)\cap\Lambda_{H,T}} |H_p q_i| \leq CR^\ell.$$

Furthermore,

$$\sup_{\mathcal{T}_T(\Xi,R)\cap\Lambda_{H,T}} |q_i| \leq (1 + C\delta)R^\ell + O(R^{2l})$$

Thus, taking T small enough, we obtain from (2.33) that

$$\lim_{\varepsilon\to 0}\limsup_{h\to 0} h^{k-1}\|Op_h(\beta_\varepsilon w)Op_h(\chi)\phi_h\|^2_{L^2(H)}$$
$$\leq C_{\ell,k}T^{-1}_{\rho_0}\int_{\Lambda_{H,T}} \chi^2\tilde{w}^2(\alpha^{k-1} + \alpha^{k-2\ell-1}R^{2\ell})d\mu.$$

Choosing $\alpha = R$ and fixing $\ell = k$ gives (2.29). □

Remark 4. To see that the conclusion in Remark 1 holds, observe that the estimate in (2.33) holds for $H_1 = H_1(h)$ as long as $\Sigma_{H_1,p}$ and $\Sigma_{H,p}$ are $o(1)$ close. That is, as long as

$$\sup\{d_s(\rho, \rho_1) : \rho \in \Sigma_{H,p},\ \rho_1 \in \Sigma_{H_1,p}\} = o(1),$$

where we continue to write d_s for the distance induced by the Sasaki metric. In particular, it is enough that H and H_1 are $o(1)$ close in the C^1 topology. That is, if in local coordinates $H = \{(x', 0)\}$, then $H_1(h) = \{(x', f_h(x')\}$ with

$$\|f_h\|_{L^\infty} + \|\nabla f_h\|_{L^\infty} = o(1).$$

Indeed, the same arguments apply to H_1 with $\iota_{w,\varepsilon}$ and $\tilde{w}_\varepsilon$ adapted to H_1. Then, when evaluating the limits in (2.33), the fact that $\Sigma_{H_1,p}$ and $\Sigma_{H,p}$ are $o(1)$ close implies that the right hand side converges as claimed to an integral over $\Lambda_{H,T}$.

We now present the proof of Proposition 10.

2.4.1. **Proof of Proposition 10.** Let $\chi \in C_c^\infty(T^*M)$ so that $H_p\chi \equiv 0$ on $\Lambda_{H,T}$ for some $T > 0$. Also, fix $w \in C_c^\infty(H)$.

For all $\delta > 0$, we can find (x_j, r_j) and (Ξ_j, R_j) with $j = 1, \dots K(\delta)$ so that if we set

$$U_j := \{(x, \xi) :\ x \in B(x_j, r_j),\ \xi \in B(\Xi_j(x), R_j)\} \subset \Sigma_{H,p} \qquad \text{and} \qquad \mathcal{U} = \bigcup_{j=1}^K U_j,$$

where $B(x_j, r_j) \subset H$ and $B(\Xi_j(x), R_j) \subset \{\xi \in N_x^*H :\ p(x, \xi) = 0\}$ are balls of radius r_j and R_j respectively, then

$$\operatorname{supp}(\chi 1_{\Sigma_{H,p}}) \ \subset\ \mathcal{U},$$

and

$$\sum_{j=1}^K \sigma_{\Sigma_{H,p}}(U_j) \ \leq\ \sigma_{\Sigma_{H,p}}\big(\operatorname{supp}(\chi 1_{\Sigma_{H,p}})\big) + \delta. \tag{2.35}$$

Let $\tilde{\chi}_j$ be a partition of unity for $\mathcal{U}$ subordinate to $\{U_j\}$. Apply Lemma 11 to obtain the flow invariant extensions

$$\chi_j \in C_c^\infty(T^*M; [0, 1])$$

so that

(1) $H_p\chi_j \equiv 0$ on $\Lambda_{H,T}$,
(2) $(\operatorname{supp} \chi_j 1_{\Lambda_{H,T}}) \subset \bigcup_{|t|<T} \varphi_t(U_j) \subset \mathcal{T}_T(\Xi_j, R_j)$,
(3) $\{x :\ (x, \xi) \in (\operatorname{supp} \chi_j 1_{T_H^*M})\} \subset B(x_j, r_j)$,
(4) $\sum_{j=1}^K \chi_j \equiv 1$ on $\bigcup_{|t|<T} \varphi_t(\mathcal{U})$,
(5) $0 \leq \sum_{j=1}^K \chi_j \leq 1$ on $\Lambda_{H,T}$.

Note that, since $H_p\chi \equiv 0$ on $\Lambda_{H,T}$, we have

$$\operatorname{supp}(\chi 1_{\Lambda_{H,T}}) = \bigcup_{|t|<T} G^t(\operatorname{supp} \chi 1_{\Sigma_{H,p}}) \subset \bigcup_{|t|<T} G^t(\mathcal{U}).$$

Therefore,

$$\operatorname{supp}\Big(1 - \sum_{j=1}^K \chi_j\Big) \cap \operatorname{supp}(\chi 1_{\Lambda_{H,T}}) = \emptyset.$$

By Lemma 14, we conclude

$$\lim_{\varepsilon\to 0}\limsup_{h\to 0} h^{\frac{k-1}{2}}\left|\int_H Op_h(\beta_\varepsilon w)\Big[Op_h\Big(1-\sum_{j=1}^K \chi_j\Big)Op_h(\chi)\phi_h\Big]d\sigma_H\right| = 0.$$

We then have

$$\lim_{\varepsilon\to 0}\limsup_{h\to 0} h^{\frac{k-1}{2}}\left|\int_H Op_h(\beta_\varepsilon w)[Op_h(\chi)\phi_h]d\sigma_H\right| =$$
$$= \lim_{\varepsilon\to 0}\limsup_{h\to 0} h^{\frac{k-1}{2}}\left|\int_H Op_h(\beta_\varepsilon w)\Big[Op_h\Big(\sum_{j=1}^K \chi_j\Big)Op_h(\chi)\phi_h\Big]d\sigma_H\right|.$$

Now, to recover the spatial localization we introduce $\psi_j \in C_c^\infty(H)$ with $\operatorname{supp}\psi_j \subset B(x_j, 2r_j)$ and

$$\psi_j(x')\chi_j(0,x',\xi) = \chi_j(0,x',\xi), \qquad (x',\xi)\in T_H^*M.$$

Then,

$$\|Op_h(\chi_j)\phi_h\|_{L^2(H)} = \|\psi_j Op_h(\chi_j)\phi_h\|_{L^2(H)} + O(h^{\frac{2-k}{2}}).$$

In fact, on $\mathbb{R}^d$ with the standard quantization, we have $[(1-\psi_j)Op_h(\chi_j)\phi_h]|_H = 0$. Hence, the above estimate follows from the fact that quantizations differ by $O_{L^2\to L^2}(h)$ together with the standard restriction estimate for compactly microlocalized functions [Zwo12, Lemma 7.10].

In what follows we bound $\|Op_h(\beta_\varepsilon)[Op_h(\chi_j\chi)\phi_h]\|_{L^2(H)}$ using Lemma 14 applied to $\chi_j\chi$. This can be done since $H_p(\chi\chi_j)\equiv 0$ on $\Lambda_{H,T}$. Lemma 14 yields that there exists $C_{n,k}>0$ depending only on k and $\rho_j \in (B(x_j,3r_j)\times B(\Xi(x_j),3R_j))\cap\Sigma_{H,p}$ so that, for any $\tilde{w}\in C_c^\infty(T^*M)$ extension of w with $H_p\tilde{w}\equiv 0$ on $\Lambda_{H,T}$, and $T_{\rho_j} := T|\partial_{\xi_1}p(\rho_j)|$,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\limsup_{h\to 0} h^{\frac{k-1}{2}}\left|\int_H Op_h(\beta_\varepsilon w)[Op_h(\chi)\phi_h]d\sigma_H\right| \\
&\qquad\le \lim_{\varepsilon\to 0}\limsup_{h\to 0} h^{\frac{k-1}{2}}\sum_{j=1}^K \|1_{\operatorname{supp}\psi_j}\|_{L^2(H)}\|Op_h(\beta_\varepsilon w)[Op_h(\chi_j\chi)\phi_h]\|_{L^2(H)} \\
&\qquad\le C_{n,k}\sum_{j=1}^K \|1_{\operatorname{supp}\psi_j}\|_{L^2(H)}\left(T_{\rho_j}^{-1}R_j^{k-1}\int_{\Lambda_{H,T}}\chi_j^2\chi^2\tilde{w}^2 d\mu\right)^{1/2} \\
&\qquad\le C_{n,k}\sum_{j=1}^K r_j^{\frac{n-k}{2}}R_j^{\frac{k-1}{2}}\left(T_{\rho_j}^{-1}\int_{\Lambda_{H,T}}\chi_j^2\chi^2\tilde{w}^2 d\mu\right)^{1/2} \\
&\qquad\le C_{n,k}\left(\sum_{j=1}^K r_j^{n-k}R_j^{k-1}\right)^{1/2}\left(T_{\rho_j}^{-1}\int_{\Lambda_{H,T}}\sum_{j=1}^K\chi_j^2\chi^2\tilde{w}^2 d\mu\right)^{1/2}.
\end{aligned} \tag{2.36}$$

Now, note that

$$\sigma_{\Sigma_{H,p}}(U_j) = c_k c_n r_j^{n-k}R_j^{k-1} + O(r_j^{n-k+1}R_j^{k-1} + r_j^{n-k}R_j^k)$$

Thus, for r_j, R_j small enough

$$\sum_{j=1}^{K} r_j^{n-k} R_j^{k-1} \;\le\; c_{n,k} \sum_{j=1}^{K} \sigma_{\Sigma_{H,p}}(U_j) \le c_{n,k}\left[\sigma_{\Sigma_{H,p}}\big(\operatorname{supp}\chi 1_{\Sigma_{H,p}}\big)+\delta\right] \tag{2.37}$$

where we use (2.35) in the last inequality.

Next, observe that by continuity of $|H_p r_H|^{-1}$ on $\Sigma_{H,p}$, as $r_j, R_j \to 0$,

$$\sum_{j=1}^{K} \chi^2\chi_j^2 1_{U_j} \sup_{U_j} |H_p r_H|^{-1} \to \chi^2 |H_p r_H|^{-1}$$

pointwise and the dominated convergence theorem implies

$$T_{\rho_j}^{-1} \int_{\Lambda_{H,T}} \sum_{j=1}^{K} \chi_j^2 \chi^2 \tilde{w}^2 d\mu \to \int_{\Sigma_{H,p}} \chi^2 w^2 |H_p r_H|^{-1} d\mu_H. \tag{2.38}$$

Using (2.37) and (2.38) in (2.36), yields

$$\lim_{\varepsilon\to 0}\limsup_{h\to 0} h^{\frac{k-1}{2}} \left|\int_H Op_h(\beta_\varepsilon w)[Op_h(\chi)\phi_h] d\sigma_H\right|$$
$$\le C_{n,k} c_{n,k}^{1/2} \Big[\sigma_{\Sigma_{H,p}}\big(\operatorname{supp}\chi 1_{\Sigma_{H,p}}\big)+\delta\Big]^{1/2} \left(\int_{\Sigma_{H,p}} \chi^2 w^2 |H_p r_H|^{-1} d\mu_H + \delta\right)^{1/2}.$$

Since $\delta > 0$ is arbitrary, this completes the proof of the proposition. ∎

3. Proof of Theorem 3

While Theorem 3 is only stated for Laplace eigenfunctions, in this proof we work with operators P as in Theorem 7 and ϕ_h compactly microlocalized quasimodes (see (1.6) and (1.7)). When the codimension of H is equal to 1 and $\Sigma_{H,p}$ is compact we can include an estimate on the normal derivate in all of our results. In particular, for ν a unit normal to H, we may **replace all instances** of $\int_A \phi_h d\sigma_H$ with

$$\Big|\int_A \phi_h d\sigma_H\Big| + \Big|\int_A hD_\nu \phi_h d\sigma_H\Big|.$$

To see this, observe that if ϕ_h is a quasimode for $P\in \Psi^\infty(M)$ with real p so that H is conormally transverse for p and $\{\phi_h\}$ is compactly microlocalized, then letting D_ν denote a vector field which agrees with the normal derivative on H and is extended smoothly to M we obtain

$$hD_\nu P\phi_h = o_{L^2}(h\|\phi_h\|_{L^2}).$$

In particular,

$$PhD_\nu\phi_h + [hD_\nu, P]\phi_h = o_{L^2}(h\|\phi_h\|_{L^2}). \tag{3.1}$$

Let $\chi \in S^0(T^*M)$ have $\chi \equiv 1$ in a neighborhood of N^*H and

$$\operatorname{supp}\chi \subset \Big\{(x,\xi)\in T^*M : |\langle \nu(x),\xi\rangle| > \frac{|\xi|}{2}\Big\}.$$

Then, there exists $E \in \Psi^\infty(M)$ so that

$$Op_h(\chi)[hD_\nu, P] = hEhD_\nu$$

and in particular, applying $Op(\chi)$ to (3.1) we find

$$(Op_h(\chi)P + hE)hD_\nu\phi_h = o_{L^2}(h\|\phi_h\|_{L^2}).$$

Now, $\sigma(Op_h(\chi)P+hE) = \chi\, p$. Therefore, since $\chi \equiv 1$ in a neighborhood of N^*H and H is conormally transverse (see (1.8)) for p, H is conormally transverse for $\chi(x,\xi)p(x,\xi)$. Thus, Theorem 6 applies and gives

$$\limsup_{h\to 0^+}\left|\int_A whD_\nu\phi_h d\sigma_H\right| \le C_{n,k}\int_{\pi_H^{-1}(A)}|w|\sqrt{\tilde{f}|H_p r_H|^{-1}}d\sigma_{\Sigma_{H,p}},$$

where

$$\tilde{\mu}_{H,\chi p} = \tilde{f}d\sigma_{\Sigma_{H,p}} + \tilde{\lambda}_H$$

with $\tilde{\lambda}_H \perp \sigma_{\Sigma_{H,p}}$ and $\tilde{\mu}$ is the defect measure for $hD_\nu\phi_h$. It is straightforward to see that

$$\tilde{\mu} = |\langle \nu(x), \xi\rangle|^2\, \mu,$$

and hence (for $t_0 > 0$ chosen small enough)

$$\tilde{\mu}_{H,\chi p} = |\langle \nu(x),\xi\rangle|^2 \mu_{H,p} = |\langle \nu(x),\xi\rangle|^2(f d\sigma_{\Sigma_{H,p}} + \lambda_H).$$

In particular,

$$\begin{aligned}\limsup_{h\to 0^+}\left|\int_A whD_\nu\phi_h d\sigma_H\right| &\le C_{n,k}\int_{\pi_H^{-1}(A)}|w|\sqrt{f|H_p r_H|^{-1}}|\langle \nu(x),\xi\rangle| d\sigma_{\Sigma_{H,p}}\\ &\le \tilde{C}\int_{\pi_H^{-1}(A)}|w|\sqrt{f|H_p r_H|^{-1}}d\sigma_{\Sigma_{H,p}},\end{aligned}$$

since $\Sigma_{H,p}$ is compact and f is supported on $\Sigma_{H,p}$.

Remark 5. Note that the constant $\tilde{C}$ now depends on $\sup_{\Sigma_{H,p}} |\langle \nu(x),\xi\rangle|$.

This proves that the analog of Theorem 6 holds for $hD_\nu\phi_h$. One can then obtain an analog of Theorem 7 for $hD_\nu\phi_h$, which in turn implies Theorem 3.

4. Proof of Theorem 2

We prove Theorem 2 by contradiction. Suppose that there exists a sequence $\{\phi_{h_m}\}$ and $c > 0$ such that

$$\left|\int_A \phi_{h_m} d\sigma_H\right| \ge c h_m^{\frac{1-k}{2}}. \tag{4.1}$$

Then, we may extract a subsequence (still writing it as ϕ_{h_m}) with defect measure μ. Let μ_H be the induced measure on SN^*H and λ_H be the measure on SN^*H with $\lambda_H \perp \sigma_{SN^*H}$ and so that

$$\mu_H = f\,\sigma_{SN^*H} + \lambda_H,$$

for $f \in L^1(SN^*H, \sigma_{SN^*H})$. Then,

$$\int_{\pi_H^{-1}(A)} \sqrt{f}\, d\sigma_{SN^*H} = \int_{\mathcal{R}_H \cap \pi_H^{-1}(A)} \sqrt{f}\, d\sigma_{SN^*H} + \int_{\mathcal{R}_H^c \cap \pi_H^{-1}(A)} \sqrt{f}\, d\sigma_{SN^*H}$$
$$= \int_{\mathcal{R}_H^c \cap \pi_H^{-1}(A)} \sqrt{f}\, d\sigma_{SN^*H}, \tag{4.2}$$

where the last equality follows from the fact that $\sigma_{SN^*H}(\mathcal{R}_H \cap \pi_H^{-1}(A)) = 0$. Also, since $\lambda_H \perp \sigma_{SN^*H}$, there exist $V, W \subset SN^*H$ so that $\lambda_H(W) = \sigma_{SN^*H}(V) = 0$ and $SN^*H = V \cup W$. Next, we use that Lemma 15 below gives $\mu_H(\mathcal{R}_H^c) = 0$. It follows that

$$\int_{\mathcal{R}_H^c \cap \pi_H^{-1}(A)} \sqrt{f}\, d\sigma_{SN^*H} \leq \left(\int_{\mathcal{R}_H^c \cap \pi_H^{-1}(A)} f\, d\sigma_{SN^*H} \right)^{\frac{1}{2}} = \mu_H(\mathcal{R}_H^c \cap \pi_H^{-1}(A) \cap W)^{\frac{1}{2}} = 0. \tag{4.3}$$

Combining (4.2) and (4.3) gives $\int_{\pi_H^{-1}(A)} \sqrt{f}\, d\sigma_{SN^*H} = 0$, and so Theorem 7 gives a contradiction to (4.1).

□

Lemma 15. *Let $H \subset M$ and suppose that $\{\phi_h\}$ is a sequence of eigenfunctions with defect measure μ. Then,*

$$\mu_H(\mathcal{R}_H) = \mu_H(SN^*H).$$

Proof. Let $B \subset SN^*H$ be an open set and for $\delta > 0$ define

$$B_{2\delta} := \bigcup_{-2\delta < t < 2\delta} G^t(B).$$

Observe that the triple (S^*M, μ, G^t) forms a measure preserving dynamical system. The Poincaré Recurrence Theorem [BS02, Lemma 4.2.1, 4.2.2] implies that for μ-a.e. $\rho \in B_{2\delta}$ there exist $t_n^\pm \to \pm\infty$ so that $G^{t_n^\pm}(\rho) \in B_{2\delta}$. By the definition of $B_{2\delta}$, there exists $s_n^\pm$ with $|s_n^\pm - t_n^\pm| < 2\delta$ such that $G^{s_n^\pm}(\rho) \in B$. In particular, for μ-a.e. $\rho \in B_{2\delta}$,

$$\bigcap_{T>0} \overline{\bigcup_{t \geq T} G^t(\rho) \cap B} \neq \emptyset, \qquad \text{and} \qquad \bigcap_{T>0} \overline{\bigcup_{t \geq T} G^{-t}(\rho) \cap B} \neq \emptyset. \tag{4.4}$$

We have used that the sets $\overline{\cup_{t \geq T} G^{\pm t}(\rho) \cap B}$ are non-empty, compact, and nested as T grows.

We next show that (4.4) holds for μ_H-a.e. point in B. To do so, suppose the opposite. Then, there exists $A \subset B$ with $\mu_H(A) > 0$ so that for each $\rho \in A$, there exists $T > 0$ with

$$\bigcup_{t \geq T} G^t(\rho) \cap B = \emptyset \qquad \text{or} \qquad \bigcup_{t \geq T} G^{-t}(\rho) \cap B = \emptyset. \tag{4.5}$$

We relate μ and μ_H using [CGT18, Lemma 6] which gives

$$\mu|_{B_{2\delta}} = \mu_H dt.$$

Then, if we let

$$A_\delta := \bigcup_{-\delta < t < \delta} G^t(A),$$

we have

$$\mu(A_\delta) = 2\delta \cdot \mu_{_H}(A) > 0.$$

Then $A_\delta \subset B_{2\delta}$, and for all $\rho \in A_\delta$ there exists $T > 0$ so that (4.5) holds. Since this implies that (4.4) does not hold for a subset of $B_{2\delta}$ of positive μ measure, we have arrived at a contradiction. Thus (4.4) holds for $\mu_{_H}$ a.e. point in B.

To finish the argument, let $\{B_k\}$ be a countable basis for the topology on SN^*H. Then for each k there is a subset $\tilde{B}_k \subset B_k$ of full $\mu_{_H}$ measure so that for *every* $\rho \in \tilde{B}_k$ relation (4.4) holds with $B = B_k$.

Let $X_k := \tilde{B}_k \cup (SN^*H \setminus B_k)$. Next, note that $\cap_k X_k \subset \mathcal{R}_H$. Indeed, if $\rho \in \cap_k X_k$ and $\mathcal{U} \subset SN^*H$ is an open neighborhood of ρ, then there exists ℓ so that $\rho \in B_\ell \subset \mathcal{U}$. In particular, since $\rho \in X_\ell$, we know that $\rho \in \tilde{B}_\ell$ and so $\bigcap_{T>0} \overline{\bigcup_{t \geq T} G^t(\rho) \cap B_\ell} \neq \emptyset$. We conclude that ρ returns infinitely oftern to $\mathcal{U}$.

Noting that $X_k = \tilde{B}_k \cup (SN^*H \setminus B_k)$ has full $\mu_{_H}$ measure, we conclude that $\cap_k X_k \subset \mathcal{R}_H$ has full measure and thus $\mu_{_H}(\mathcal{R}_H \cap SN^*H) = \mu_{_H}(SN^*H)$ as claimed. ☐

5. Recurrence: Proof of Theorem 4

This section is dedicated to the proof of Theorem 4. Recall that $\mathcal{L}_H$ is defined in (1.4) and denotes the loop set. In Section 5.1 we prove the theorem for assumptions A, showing that $\sigma_{_{SN^*H}}(\mathcal{L}_H) = 0$. We then use the fact that for $H = \{x\}$ a point, $\sigma_{_{SN^*H}}(\mathcal{L}_H) = 0$ to prove case B, In Section 5.2 we present a tool for proving that $\sigma_{_{SN^*H}}(\mathcal{R}_H \cap A) = 0$ for $A \subset SN^*H$. In particular, we prove that it suffices to show that $t \mapsto \mathrm{vol}(G^t(A))$ is integrable either for positive times or for negative ones. In Section 5.3 we show that for manifolds with Anosov flow we have $\sigma_{_{SN^*H}}(\mathcal{R}_H) = \sigma_{_{SN^*H}}(\mathcal{R}_H \cap \mathcal{A}_H)$, where $\mathcal{A}_H$ is the set of points in SN^*H at which the tangent space to SN^*H splits into a direct sum of stable and unbounded directions. A similar statement is proved for (M, g) with no focal points, but with $\mathcal{N}_H$ instead of $\mathcal{A}_H$. In Section 5.4 we prove Theorem 4 for assumptions C, D, E and F, by taking advantage of the fact when (M, g) has Anosov flow we have some control on the structure of $\mathcal{A}_H$ and, in some cases, on the integrability of $t \mapsto \mathrm{vol}(G^t(\mathcal{A}_H))$.

5.1. **Proof of parts A and B.**

In this section we prove that $\sigma_{_{SN^*H}}(\mathcal{R}_H) = 0$ for (M, g) and H satisfying the assumptions in parts A and B in Theorem 4.

Proof of part A. For this part we assume that (M, g) has no conjugate points and H has codimension $k > \frac{n+1}{2}$. The strategy of the proof is to show that the set $\{\rho \in SN^*H : \ \exists t > 0 \text{ s.t. } G^t(\rho) \in SN^*H\}$ has dimension strictly smaller than $n - 1 = \dim SN^*H$, and hence has measure zero. We prove this using the implicit function theorem together with the fact that, since (M, g) has no conjugate points, we can control the rank of the exponential map.

Note that, since (M, g) has no conjugate points, for each point $x \in M$ the exponential map $\exp_x : T_xM \to M$ has no critical points. In particular, if we define the map

$$\psi^x : \mathbb{R} \times SN^*_xH \to M, \qquad \psi^x(t, \xi) = \pi G^t(x, \xi),$$

with $\pi : T^*M \to M$ the canonical projection we have for all $(t,\xi) \in \mathbb{R} \times SN^*_xH$

$$\operatorname{rank}(d\psi^x)_{(t,\xi)} = n - \dim H.$$

Remark 6. Indeed, note that the fact that $\exp_x : T_xM \to M$ has no critical points implies that $T^*_xM \setminus \{0\} \ni (x,\xi) \mapsto \pi G^1(x,\xi) = \pi G^{|\xi|}(x,\xi/|\xi|)$ has no critical points.

This implies that if we define

$$\psi : \mathbb{R} \times SN^*H \to M, \qquad \psi(t,\rho) = \pi G^t(\rho),$$

then its differential

$$(d\psi)_{(t,\rho)} : T_{(t,\rho)}(\mathbb{R} \times SN^*H) \to T_{\pi G^t(\rho)}M$$

has

$$\operatorname{rank}(d\psi)_{(t,\rho)} \geq n - \dim H = k,$$

for all $(t,\rho) \in \mathbb{R} \times SN^*H$. Note that $\psi^{-1}(H) = \{(t,\rho) \in \mathbb{R} \times SN^*H : \; G^t(\rho) \in S^*_HM\}$.

Let

$$\begin{gathered} f_i \in C^\infty(M;\mathbb{R}), \qquad F = (f_1, \dots, f_k) : M \to \mathbb{R}^k, \\ F^{-1}(0) = H, \qquad \{df_i\}_{i=1}^k \text{ linearly independent on } H. \end{gathered} \tag{5.1}$$

The composition $F \circ \psi : \mathbb{R} \times SN^*H \to \mathbb{R}^k$ satisfies $(F \circ \psi)^{-1}(0) = \psi^{-1}(H)$. Note that since $\operatorname{rank}(d\psi)_{(t,\rho)} \geq k$, we have

$$\operatorname{rank}(d(F \circ \psi)_{(t,\rho)}) \geq \operatorname{rank}(dF)_{\psi(t,\rho)} + \operatorname{rank}(d\psi)_{(t,\rho)} - \dim M \geq 2k - n$$

for $(t,\rho) \in (F \circ \psi)^{-1}(0)$. Since by assumption $k > \frac{n+1}{2}$, we have

$$\operatorname{rank}(d(F \circ \psi)_{(t,\rho)}) \geq 2.$$

Moreover, since the geodesic flow is transverse to H along N^*H, $d(F \circ \psi)_{(t,\rho)}\partial_t \neq 0$ whenever $G^t(\rho) \in SN^*H$. Indeed, suppose that $G^t(\rho) \in SN^*H$ and $d(F \circ \psi)_{(t,\rho)}\partial_t = 0$. Observe that $d\psi_{(t,\rho)}(\partial_t) = d\pi H_p(G^t(\rho))$, so that $d(F \circ \psi)_{(t,\rho)}(\partial_t) = d\pi H_p(G^t(\rho))(F)$ and, lifting F to a function on T^*M independent of the fiber variable, $d(F \circ \psi)_{(t,\rho)}(\partial_t) = H_p(G^t t(\rho))F \neq 0$ by the assumption that $\{(df_j)_x : \; j = 1, \dots k\}$ define H and $G^t(\rho) \in SN^*H$.

Applying the implicit function theorem, we see that given $(t_0,\rho_0) \in \psi^{-1}(H)$ with $G^{t_0}(\rho_0) \in SN^*H$, there exists a neighborhood U of (t_0,ρ_0), an open neighborhood $V \subset \mathbb{R}^\ell$ of 0 for some $\ell \leq n-2$, and smooth functions $s : SN^*H \to \mathbb{R}$, $f : V \to SN^*H$ with $s(\rho_0) = t_0$, $f(0) = \rho_0$, so that

$$U \cap \psi^{-1}(H) = \{\big(s(f(q)), f(q)\big) : \; q \in V\}.$$

In particular, since $\dim V < n - 1 = \dim(SN^*H)$,

$$\sigma_{SN^*H}\Big(\rho \in SN^*H : \;\; \text{there exists } t \text{ such that } (t,\rho) \in U \text{ and } G^t(\rho) \in SN^*H\Big) = 0.$$

In particular, by compactness of $[0,j]$, for any $j > 0$,

$$\sigma_{SN^*H}\Big(\rho \in SN^*H : \;\; \text{there exists } t \in [0,j] \text{ such that } G^t(\rho) \in SN^*H\Big) = 0.$$

Taking the union over $j > 0$ we find

$$\sigma_{SN^*H}(\mathcal{L}_H) = 0.$$

In particular, since $\mathcal{L}_H \supset \mathcal{R}_H$, this implies that $\sigma_{_{SN^*H}}(\mathcal{R}_H) = 0$. □

Proof of part B. Now, suppose that (M, g) has no conjugate points and $K \subset M$ is a geodesic sphere. Then there exists $p \in M$ and $t \in \mathbb{R}$ so that $K = H_t := \pi G^t(SN^*H)$ for $H = \{p\}$. Applying the result in Part A gives that $\sigma_{_{SN^*H}}(\mathcal{R}_H) = 0$. In particular, by Lemma 16 below we conclude $\sigma_{_{SN^*H_t}}(\mathcal{R}_{H_t}) = 0$ as claimed.

□

Lemma 16. *Suppose that $H \subset M$ is a submanifold and for $t \in \mathbb{R}$ define $H_t := \pi G^t(SN^*H)$. Then, for any $t \in \mathbb{R}$ so that H_t is a smooth submanifold of M having codimension* 1

$$\sigma_{\Sigma_{H,p}}(\mathcal{R}_H) = 0 \qquad \textit{if and only if} \qquad \sigma_{SN^*H_t}(\mathcal{R}_{H_t}) = 0.$$

Proof. First, observe that if $H \subset M$ is a submanifold, then for $t \in \mathbb{R}$ and $H_t := \pi G^t(SN^*H)$, we have

$$SN^*H_t = G^t(SN^*H) \sqcup G^{-t}(SN^*H)$$

whenever H_t is a smooth submanifold of M. To see this, observe that since H_t has codimension 1, for each $x_0 \in H_t$, there are exactly two elements in $SN^*_{x_0}H_t$ and hence these elements are given by

$$G^t(x, \xi) \qquad \text{and} \qquad G^{-t}(x, -\xi)$$

for some $(x, \xi) \in SN^*H$. Note that $\mathcal{R}_{H_t} = G^t(\mathcal{R}_H) \cup G^{-t}(\mathcal{R}_H)$. Therefore, since $G^{\pm t} : SN^*H \to SN^*H_t$ is a diffeomorphism onto its image, $\sigma_{_{SN^*H}}(\mathcal{R}_H) = 0$ if and only if $\sigma_{SN^*H_t}(\mathcal{R}_{H_t}) = 0$. □

5.2. A tool for proving that $\sigma_{_{SN^*H}}(\mathcal{R}_H) = 0$.

Given $X \subset S^*M$ submanifold, we write $\text{vol}(X)$ for the volume induced by the Sasaki metric on X (see (1.3)). This section is dedicated to showing that $\sigma_{_{SN^*H}}(\mathcal{R}_H \cap A) = 0$ whenever the map $t \mapsto \text{vol}(G^t(A))$ is integrable either on $(0, \infty)$ or on $(-\infty, 0)$. We will later use that the integrability of this function can always be established if (M, g) has Anosov flow and A is a set of points in SN^*H at which the tangent to SN^*H space is either stable or unstable.

We start with a lemma where we prove that for any $\rho \in SN^*H$ the tangent space $T_\rho(SN^*H)$ has no component in the direction of $\mathbb{R}H_p$ with $p = |\xi|_{g(x)}$.

Proposition 17. *Let (M, g) be a Riemannian manifold, and let $H \subset M$ be a submanifold. For all $\rho \in SN^*H$ let $\pi_{H_p} : T_\rho(S^*M) \to \mathbb{R}H_p$ be the orthogonal projection map, where H_p is the Hamiltonian vector field associated to $p(x, \xi) = |\xi|_{g(x)}$. Then,*

$$\pi_{H_p}(T_\rho(SN^*H)) = \{0\}.$$

Proof. Let (x', x'') be Fermi coordinates near H where we identify H with $\{(x', x'') : x'' = 0\}$. Writing (ξ', ξ'') for the associated cotangent coordinates,

$$N^*H = \Big\{(x', 0, 0, \xi'') : \ x' \in H, \ \xi'' \in \mathbb{R}^k\Big\}.$$

This implies that, if $\rho = (x', 0, 0, \xi'') \in N^*H$, then

$$T_\rho(N^*H) = \{\langle v, \partial_{x'}\rangle + \langle w, \partial_{\xi''}\rangle : v \in \mathbb{R}^{n-k},\ w \in \mathbb{R}^k\},$$

while, for $(x,\xi) \in SN^*H$, $p(x,\xi) = |\xi|_{g(x)} = |\xi''|$ and hence

$$(H_p)_{(x,\xi)} = \langle \xi'', \partial_{x''}\rangle \qquad (x,\xi) \in SN^*H.$$

Now, $\partial_{x''}$ is orthogonal to $\partial_{x'}$. Thus, since $\partial_{\xi''}$ is vertical and H_p is horizontal $\mathbb{R}H_p$ is orthogonal to TSN^*H. □

Lemma 18. *Let $A \subset SN^*H$.*

$$\text{If} \int_0^\infty \mathrm{vol}(G^t(A))dt < \infty, \quad \text{then} \quad \sigma_{SN^*H}(\mathcal{L}_H^{-\infty} \cap A) = 0. \tag{5.2}$$

$$\text{If} \int_{-\infty}^0 \mathrm{vol}(G^t(A))dt < \infty, \quad \text{then} \quad \sigma_{SN^*H}(\mathcal{L}_H^{+\infty} \cap A) = 0. \tag{5.3}$$

In particular, either assumption implies that $\sigma_{SN^*H}(\mathcal{R}_H \cap A) = 0$.

Proof. Suppose (5.2) holds. From now on, given $\rho \in SN^*H$ and $t \in \mathbb{R}$, we adopt the notation

$$J_t(\rho) := dG^t|_{T_\rho(SN^*H)} : T_\rho(SN^*H) \to dG^t(T_\rho(SN^*H)). \tag{5.4}$$

Note that

$$\int_A |\det J_t(\rho)|\, d\sigma_{SN^*H}(\rho) = \mathrm{vol}(G^t(A)). \tag{5.5}$$

We claim that there exist constants $C, \delta > 0$ so that for any Borel set $A \subset SN^*H$ and $T \in \mathbb{R}$,

$$\sigma_{SN^*H}\left(\bigcup_{t=T}^{T+\delta} G^t(A) \cap SN^*H\right) \le C \int_A |\det J_T(\rho)|\, d\sigma_{SN^*H}(\rho) = C\,\mathrm{vol}(G^T(A)). \tag{5.6}$$

We postpone the proof of claim (5.6) until the end. Assuming (5.6) for now, we note that since $t \mapsto G^t$ is a smooth group, for $\delta > 0$ small enough and $t \in [T, T+\delta]$,

$$|\det J_t(\rho)| \le 2|\det J_T(\rho)|. \tag{5.7}$$

Hence,

$$\begin{aligned}
\sum_{n>0} \sigma_{SN^*H}\big(\rho \in A :\ G^{-t}(\rho) \in SN^*H,\ \text{for some } t \in [n\delta, (n+1)\delta]\big) \le \\
\le C \sum_{n>0} \int_A |\det J_{n\delta}(\rho)|\, d\sigma_{SN^*H}(\rho) \\
\le 2C\delta^{-1} \int_0^\infty \int_A |\det J_t(\rho)|\, d\sigma_{SN^*H}(\rho) dt < \infty.
\end{aligned}$$

Therefore, by the Borel–Cantelli Lemma,

$$\sigma_{SN^*H}\big(\rho \in A :\ G^{-t}(\rho) \in SN^*H \text{ for infinitely many } t \in [0,\infty)\big) = 0$$

and in particular, $\sigma_{SN^*H}(\mathcal{L}_H^{-\infty} \cap A) = 0$. The case of (5.3) is identical.

In order to finish the proof of the lemma we need to establish the claim in (5.6). We proceed to do this. Fix $\varepsilon > 0$. Let $\{A_{i,\varepsilon}\}_{i=1}^{N(\varepsilon)}$ be a partition of $A \subset SN^*H$ into sets of radius less than ε.

Fix coordinates, y on $A_{i,\varepsilon}$. Then there exists $\rho_i \in A_{i,\varepsilon}$ so that for all $\rho \in A_{i,\varepsilon}$,

$$\begin{aligned} G^t(\rho) &= G^t(\rho_i)) + dG^t(y(\rho) - y_i(\rho)) + O(\varepsilon^2) \\ &= G^t(\rho_i) + dG^t(\pi_i(y(\rho) - y_i(\rho))) + O(\varepsilon^2) \end{aligned}$$

where $\pi_i : T_{\rho_i}(T^*M) \to T_{\rho_i}(SN^*H)$ is the projection operator. In the last line, we use that since $y, y_i \in SN^*H$, $y - y_i = \mathbf{v}d(\rho, \rho_i) + O(d(\rho, \rho_i)^2)$ where $\mathbf{v} \in T_{\rho_i}SN^*H$.

Therefore, using (5.5)

$$\sigma_{SN^*H}\Big(\bigcup_{t=T}^{T+\delta} G^t(A_{i,\varepsilon})\Big) \leq$$
$$\sup_{t\in[T,T+\delta]} |\det J_t(\rho_i)| \cdot \sigma_{SN^*H}(A_{i,\varepsilon})(1 + O(\varepsilon)) \sup_{\rho\in A_{i,\varepsilon}} \#\{t \in [T, T + \delta] : G^t(\rho) \in SN^*H\}.$$

Now, Proposition 17 together with the compactness of SN^*H give that for $\delta > 0$ small enough and all $\rho \in A$,

$$\#\{t \in [T, T + \delta] :\ G^t(\rho) \in SN^*H\} \leq 1.$$

In particular,

$$\begin{aligned} \sigma_{SN^*H}\Big(\bigcup_{t\in[T,T+\delta]} G^t(A)\Big) &\leq \sum_i \sigma_{SN^*H}\Big(\bigcup_{t\in[T,T+\delta]} G^t(A_{i,\varepsilon})\Big) \\ &\leq \sum_{i,j} \sup_{t\in[T,T+\delta]} |\det J_t(\rho_i)| \cdot \sigma_{SN^*H}(A_{i,\varepsilon})(1 + O(\varepsilon)) \\ &\leq \sum_{i,j} 2|\det J_T(\rho_i)| \cdot \sigma_{SN^*H}(A_{i,\varepsilon})(1 + O(\varepsilon)) \end{aligned}$$

where in the last line we use (5.7).

Sending $\varepsilon \to 0$, since dG^t is continuous, the Dominated Convergence Theorem shows that

$$\sigma_{SN^*H}\Big(\bigcup_{t=T}^{T+\delta} G^t(A)\Big) \leq \int_A 2|\det J_T(\rho)|\, d\sigma_{SN^*H}. \tag{5.8}$$

as desired. $\square$

5.3. **Manifolds with no focal points or Anosov flow.** This section is dedicated to the proof of Theorem 8. We need a preliminary lemma.

Lemma 19. *Suppose that $\rho_0 \in SN^*H$ with $G^{t_0}(\rho_0) \in SN^*H$ for some $t_0 > 0$. If there exists $\mathbf{w} \in T_{\rho_0}SN^*H$ with $dG^{t_0}\mathbf{w} \notin T_{G^{t_0}\rho_0}SN^*H \oplus \mathbb{R}H_p$, then there exists $U_{t_0,\rho_0} \subset \mathbb{R} \times SN^*H$ a neighborhood of (t_0, ρ_0) such that*

$$\sigma_{SN^*H}\Big(\rho \in SN^*H :\ \text{there exists } t \text{ with } (t, \rho) \in U_{t_0,\rho_0} \text{ and } G^t(\rho) \in SN^*H\Big) = 0.$$

Proof. Define

$$\psi : \mathbb{R} \times SN^*H \to S^*M, \qquad \psi(t,\rho) = G^t(\rho),$$

so that

$$d\psi_{(t,\rho)}(\tau, w) = \tau H_p(G^t(\rho)) + dG^t_\rho w.$$

and let $f_1, \dots f_n \in C^\infty(S^*M; \mathbb{R})$ be defining functions for SN^*H near $G^{t_0}(\rho_0)$. In particular,

$$SN^*H = \bigcap_{i=1}^{n} \{f_i = 0\}, \qquad \{df_i\} \text{ are linearly independent on } SN^*H.$$

Finally, let $F \in C^\infty(S^*M; \mathbb{R}^n)$ be given by

$$F = (f_1, \dots, f_n).$$

Note that $G^t(\rho) \in SN^*H$ if and only if $(t,\rho) \in (F \circ \psi)^{-1}(0)$. Now, since $dG^{t_0}\mathbf{w} \notin T_{G^{t_0}\rho_0}(SN^*H) \oplus \mathbb{R}H_p$, Proposition 17 gives that the vectors

$$d(F \circ \psi)_{(t_0,\rho_0)}(0, \mathbf{w}) = dF_{G^{t_0}(\rho_0)}\left(d\psi_{(t_0,\rho_0)}(0,\mathbf{w})\right)$$

and

$$d(F \circ \psi)_{(t_0,\rho_0)}(\tau, 0) = dF_{G^{t_0}(\rho_0)}\left(d\psi_{(t_0,\rho_0)}(\tau,0)\right)$$

are linearly independent. We then have that

$$\operatorname{rank}(d(F \circ \psi)_{(t_0,\rho_0)}) \geq 2.$$

By the implicit function theorem, there is a neighborhood U of (t_0, ρ_0), a neighborhood $V \subset \mathbb{R}^\ell$ of 0 for some $\ell \leq n-2$, and smooth functions $s : SN^*H \to \mathbb{R}$, $\alpha : V \to SN^*H$ with $s(0) = t_0$, $\alpha(0) = \rho_0$, so that

$$U \cap \psi^{-1}(SN^*H) = \{(s(\alpha(q)), \alpha(q)) : \ q \in V\}.$$

In particular, since $\dim V < n-1 = \dim(SN^*H)$,

$$\sigma_{SN^*H}\Big(\rho \in SN^*H : \ \text{ there exists } t \text{ such that } (t,\rho) \in U,\, G^t(\rho) \in SN^*H\Big) = 0,$$

as claimed. □

Remark 7. In fact Lemma 19 shows that the points $\rho \in SN^*H$ near ρ_0 which loop at times near t_0 are contained in a smooth submanifold of dimension $< n-1$.

Since it will be used frequently in this section, we recall the definition (1.17) of an Anosov flow: For all $\rho \in S^*M$,

$$T_\rho(S^*M) = E_+(\rho) \oplus E_-(\rho) \oplus \mathbb{R}H_p.$$

where E_-, E_+ are stable and unstable directions as before. Moreover, there exists $C > 0$ so that for all $\rho \in S^*M$,

$$|dG^t(v)| \leq Ce^{-t/C}|v| \qquad \text{for } v \in E_+ \ \text{ and } t \to +\infty,$$
$$|dG^t(v)| \leq C\, e^{t/C}|v| \qquad \text{for } v \in E_- \ \text{ and } t \to -\infty.$$

Recall also the notation $N_+(\rho)$, $N_-(\rho)$ from (1.14), $\mathcal{S}_H$, $\mathcal{M}_H$ from (1.15) and $\mathcal{N}_H$, $\mathcal{A}_H$ from (1.16). Next we present a proposition in which we show that if (M, g)

has Anosov geodesic flow, then for any compact subset $K \subset SN^*H \backslash \mathcal{S}_H$ there is a decomposition of K, $K = K^+ \cup K^-$ and T sufficiently large such that if $\rho_0 \in K^\pm$ and $G^{t_0}(\rho_0) \in SN^*H$ with either $\mp t_0 > T$, then there exists $\mathbf{w} \in T_{\rho_0}SN^*H$ with $dG^{t_0}\mathbf{w} \notin T_{G^{t_0}\rho_0}SN^*H \oplus \mathbb{R}H_p$. This will allow us to later use Lemma 19 to prove Theorem 8. We define the following functions $m, m_\pm : SN^*H \to \{0, \dots, n-1\}$

$$m(\rho) := \dim(N_+(\rho) + N_-(\rho)), \qquad m_\pm(\rho) := \dim N_\pm(\rho) \tag{5.9}$$

We first show that continuity of $E_\pm(\rho)$ implies that m, $m_\pm$ are upper semicontinuous.

Lemma 20. *Let $m, m_\pm$ be as in* (5.9)*. Then m, $m_\pm$ are upper semicontinuous.*

Proof. We prove this for $m_+(\rho) = \dim(T_\rho SN^*H \cap E_+(\rho))$. Let $\rho \in SN^*H$ and $\rho_j \to \rho$. Suppose that $\limsup_j m_+(\rho_j) > m_+(\rho)$. Then, without loss, we may assume that

$$\dim(T_{\rho_j}SN^*H \cap E_+(\rho_j)) > \dim(T_\rho SN^*H \cap E_+(\rho))$$

for all j. In particular, there exist $\{v_{1,j}, \dots v_{m_+(\rho)+1,j}\} \in E_+(\rho_j) \cap T_{\rho_j}SN^*H$ with $\{v_{i,j}\}_{i=1}^{m_+(\rho)+1}$ orthonormal. Extracting a subsequence so that $v_{i,j} \underset{j\to\infty}{\longrightarrow} v_i \in T_\rho T^*M$, we have, by continuity of $E_+(\rho)$ and $T_\rho SN^*H$, that $v_i \in E_+(\rho)$ and $v_i \in T_\rho SN^*H$. In particular, $v_i \in T_\rho SN^*H \cap E_+(\rho)$ and $\{v_i\}_{i=1}^{m_+(\rho)+1}$ are orthonormal, contradicting the definition of $m_+(\rho)$. ◻

Proposition 21. *Suppose (M, g) has Anosov geodesic flow and let $K \subset SN^*H \backslash \mathcal{S}_H$ be a compact set. There exist positive constants $T, \varepsilon > 0$ so that if $\rho_0 \in K$, $|t_0| \geq T$, and*

$$G^{t_0}(\rho_0) \in \overline{B(\rho_0, \varepsilon)} \cap SN^*H,$$

then there is $\mathbf{w} \in T_{\rho_0}(SN^*H)$ with

$$dG^{t_0}(\mathbf{w}) \notin T_{G^{t_0}(\rho_0)}(SN^*H) \oplus \mathbb{R}H_p. \tag{5.10}$$

Proof. Throughout the proof of this proposition we will use the norm induced by the Sasaki metric on TT^*M. Note, however, that any inner product norm suffices. Let $\rho_0 \in K$. Since $T_{\rho_0}(SN^*H) \neq N_+(\rho_0) \oplus N_-(\rho_0)$, we may choose

$$\mathbf{u} \in T_{\rho_0}(SN^*H) \setminus (N_+(\rho_0) \oplus N_-(\rho_0)), \quad \|\mathbf{u}\| = 1.$$

Now, let $\mathbf{u}_+ \in E_+(\rho_0)$ and $\mathbf{u}_- \in E_-(\rho_0)$ be such that

$$\mathbf{u} = \mathbf{u}_+ + \mathbf{u}_-.$$

Without loss of generality, we assume that $\mathbf{u}_-$ is orthogonal to $N_-(\rho_0)$ and, since ρ_0 varies in a compact subset of $SN^*H \backslash \mathcal{A}_H$, we may assume uniformly for $\rho_0 \in K$ that

$$M^{-1}\|\mathbf{u}_+\| \leq \|\mathbf{u}_-\| \leq M\|\mathbf{u}_+\|.$$

Since $dG^t : E_-(\rho_0) \to E_-(G^t(\rho_0))$ is an isomorphism,

$$\dim\big(\mathbb{R}dG^t(\mathbf{u}_-) \oplus dG^t(N_-(\rho_0))\big) = 1 + \dim N_-(\rho_0).$$

Note that for m_- as in (5.9), m_- is upper semicontinuous and we may choose $\varepsilon > 0$ uniform in $\rho_0 \in SN^*H$, so that $\dim N_-(G^t(\rho_0)) \leq \dim N_-(\rho_0)$ for all t such that $G^t(\rho_0) \in B(\rho_0, \varepsilon)$. For such values of t we then have

$$\dim\big(\mathbb{R}dG^t(\mathbf{u}_-) \oplus dG^t(N_-(\rho_0))\big) \geq 1 + \dim N_-(G^t(\rho_0)). \tag{5.11}$$

Next, we note that $\text{span}\big(dG^t(\mathbf{u}_-),\quad dG^t(N_-(\rho_0))\big) \subset E_-(G^t(\rho_0))$. Also, note that if $dG^t(\mathbf{w}) \in E_-(G^t(\rho_0))\backslash N_-(G^t(\rho_0))$, then $dG^t(\mathbf{w}) \notin T_{G^t(\rho_0)}(SN^*H)$. In particular, relation (5.11) gives that there exists a linear combination

$$\mathbf{w_t} = a_t\,\mathbf{u}_- + \mathbf{e}_-(t),$$

with $\mathbf{e}_-(t) \in N_-(\rho_0)$, so that

$$\left\|\pi_{t,\rho_0}(dG^t\mathbf{w_t})\right\| = 1 = \left\|dG^t\mathbf{w_t}\right\|,$$

where $\pi_{t,\rho_0} : T_{G^t(\rho_0)}(S^*M) \to V_{t,\rho_0}$ is the orthogonal projection map onto a subspace V_{t,ρ_0} of $T_{G^t(\rho_0)}(S^*M)$ chosen so that $T_{G^t(\rho_0)}(S^*M) = V_{t,\rho_0} \oplus T_{G^t(\rho_0)}(SN^*H)$ is an orthogonal decomposition. If we had that $\mathbf{w_t}$ was a tangent vector in $T_{G^t(\rho_0)}(S^*M)$, then we would be done. However, since $\mathbf{u}_-$ is not necessarily in $T_{G^t(\rho_0)}(S^*M)$ we have to modify $\mathbf{w_t}$ a bit. Consider the vector

$$\tilde{\mathbf{w}}_\mathbf{t} = a_t\,\mathbf{u} + \mathbf{e}_-(t),$$

and note that $\tilde{\mathbf{w}}_\mathbf{t} \in T_{\rho_0}(SN^*H)$. Then,

$$dG^t(\tilde{\mathbf{w}}_\mathbf{t}) = dG^t(\mathbf{w_t}) + a_t\,dG^t(\mathbf{u}_+).$$

By the definition of Anosov geodesic flow (see (1.17)), for all $\delta > 0$, there exists $T = T(\delta) > 0$ so that

$$\|(dG^t|_{E_-})^{-1}\| \le \delta, \quad t \ge T.$$

Thus, since $\mathbf{w_t} \in E_-(\rho_0)$ and $\|\mathbf{w_t}\| \le \delta$, we have

$$|a_t| \le \delta\|\mathbf{u}_-\|^{-1}, \qquad t \ge T.$$

Observe next, [Ebe73a, Corollary 2.14] that there exists $B > 0$ uniform in TS^*M so that for $v \in E_+(\rho)$, and $t \ge 0$ $\|dG^tv\| \le B\|v\|$. In particular, choosing $\delta < \frac{1}{2B}\|\mathbf{u}_-\|\|\mathbf{u}_+\|^{-1}$, for $t > T(\delta, K)$,

$$\|\pi_{t,\rho_0}(dG^t\tilde{\mathbf{w}}_\mathbf{t})\| \ge \|\pi_{t,\rho_0}(dG^t\mathbf{w_t})\| - \|a_t\,\pi_{t,\rho_0}(dG^t\mathbf{u}_+)\| > \frac{1}{2}.$$

Hence, there exists $\varepsilon > 0$ and $T > 0$ (uniform for $\rho_0 \in K$) so that if $G^{t_0}(\rho_0) \in SN^*H \cap B(\rho_0, \varepsilon)$ for some t_0 with $|t_0| > T$, then there is $\mathbf{w} = \tilde{\mathbf{w}}_{t_0} \in T_{\rho_0}(SN^*H)$ so that

$$dG^{t_0}(\mathbf{w}) \notin T_{G^{t_0}(\rho_0)}(SN^*H) \oplus \mathbb{R}H_p. \tag{5.12}$$

□

We now show that for manifolds with Anosov geodesic flow the set of points in $\mathcal{R}_H \cap [\mathcal{S}_H \setminus \mathcal{M}_H]$ has measure zero.

Lemma 22. *Suppose that (M, g) has Anosov geodesic flow. Then*

$$\sigma_{SN^*H}\Big(\mathcal{L}_H^{-\infty} \cap \{\rho \in SN^*H :\ T_\rho(SN^*H) \subset E_+(\rho)\}\Big) = 0$$

and

$$\sigma_{SN^*H}\Big(\mathcal{L}_H^{+\infty} \cap \{\rho \in SN^*H :\ T_\rho(SN^*H) \subset E_-(\rho)\}\Big) = 0.$$

In particular, $\sigma_{SN^*H}\big(\mathcal{R}_H \cap [\mathcal{S}_\mathcal{H} \setminus \mathcal{M}_\mathcal{H}]\big) = 0.$

Proof. Observe that setting

$$A := \{\rho \in SN^*H : \ T_\rho(SN^*H) \subset E_+(\rho)\},$$

we have, using the definition of Anosov flow (1.17)),

$$|\det J_t(\rho)| \leq C^{n-1} e^{-(n-1)t/C}, \qquad t \geq 0,$$

fot $J_t(\rho)$ defined in (5.4). It follows that

$$\mathrm{vol}(G^t(A)) \leq C^{n-1} e^{-(n-1)t/C} \sigma_{SN^*H}(A),$$

and so

$$\int_0^\infty \mathrm{vol}(G^t(A)) dt < \infty.$$

Therefore, the proof is complete by Lemma 18. The E_- case is identical where we integrate backwards in time rather than forwards. □

In what follows we write

$$\mathcal{M}_H^\pm := \Big\{\rho \in SN^*H : \ \ N_\pm(\rho) \neq \{0\}\Big\},$$

and note that

$$\mathcal{N}_H = \mathcal{S}_H \cup (\mathcal{M}_H^+ \cap \mathcal{M}_H^-),$$

and

$$SN^*H \setminus \mathcal{N}_H = \big[SN^*H \setminus (\mathcal{S}_H \cup \mathcal{M}_H^+)\big] \bigcup \big[SN^*H \setminus (\mathcal{S}_H \cup \mathcal{M}_H^-)\big].$$

We now prove the analog of Proposition 21 for manifolds with no focal points.

Proposition 23. *Suppose (M, g) has no focal points and let $K \subset SN^*H \setminus (\mathcal{S}_H \cup \mathcal{M}_H^\pm)$ be a compact set. There exist positive constants $T, \varepsilon > 0$ so that if $\rho_0 \in K$, $\mp t_0 \geq T$, and*

$$G^{t_0}(\rho_0) \in \ \overline{B(\rho_0, \varepsilon)} \cap SN^*H,$$

then there is $\mathbf{w} \in T_{\rho_0}(SN^*H)$ with

$$dG^{t_0}(\mathbf{w}) \notin T_{G^{t_0}(\rho_0)}(SN^*H) \oplus \mathbb{R}H_p. \tag{5.13}$$

Proof. We prove the lemma for $K \subset SN^*H \setminus (\mathcal{S}_H \cup \mathcal{M}_H^-)$, the other case follows similarly after sending $t \to -\infty$ rather than $t \to \infty$.

Define $\mathcal{C}_+^\varepsilon(\rho) \subset T_\rho S^*M$ as the conic set of vectors forming at least an $\varepsilon > 0$ angle with $E_+(\rho)$. Since m is upper semicontinuous, E_+ is continuous, and $T_\rho SN^*H \neq N_+(\rho) + N_-(\rho)$, there exists $\varepsilon > 0$ so that $T_\rho SN^*H \cap \mathcal{C}_+^\varepsilon(\rho) \neq 0$ for all $\rho \in K$.

Next, let $\rho_0 \in K$. Since $N_-(\rho_0) = \{0\}$, the upper semicontinuity of m_- implies that $N_-(\rho) = \{0\}$ for all $\rho \in B(\rho_0, \varepsilon)$, after possibly shrinking ε. In particular, the continuity of E_- implies that there exists $\delta > 0$ so that for $\rho \in B(\rho_0, \varepsilon)$, the angle between $E_-(\rho)$ and $T_\rho SN^*H$ is larger than δ (after possibly shrinking ε).

We claim that for $w \in \mathcal{C}_+^\varepsilon(\rho_0) \setminus \{0\}$, there exists $T = T(\delta, \varepsilon)$ so that for $t \geq T$,

$$\mathrm{dist}\Big(\frac{dG^t w}{\|dG^t w\|}\ , \, E_-(G^t(\rho_0))\Big) \leq \delta. \tag{5.14}$$

The proof of (5.14) is postponed until the end.

To finish the argument we argue by contradiction. Suppose that for $t_0 \geq T$, we have $G^t(\rho_0) \in B(\rho_0, \varepsilon)$ and

$$dG^{t_0}(T_{\rho_0} SN^*H) = T_{G^{t_0}(\rho_0)} SN^*H.$$

Then, using that $T_{\rho_0} SN^*H \cap \mathcal{C}_+^\varepsilon(\rho_0) \neq 0$, we conclude from the claim in (5.14) applied to some $w \in T_{\rho_0} SN^*H \cap \mathcal{C}_+^\varepsilon(\rho_0) \backslash \{0\}$ that there exists $v \in E_-(G^t(\rho_0))$ so that the angle between v and $\frac{dG^t w}{\|dG^t w\|} \in T_{G^{t_0}(\rho_0)} SN^*H$ is smaller than δ. In particular, setting $\rho := G^{t_0}(\rho_0) \in B(\rho_0, \varepsilon)$ we conclude that the angle between $T_\rho SN^*H$ and $E_-(\rho)$ is smaller than δ. And this is a contradiction since $\rho \in B(\rho_0, \varepsilon)$. This concludes the proof of the proposition once we have (5.14).

It only remains to prove the claim in (5.14). Let $w \in C_+^\varepsilon(\rho_0) \backslash \{0\}$. Then we can write

$$w = \tilde{u}_+ + \tilde{v}$$

with $\tilde{u}_+ \in E_+(\rho_0)$ and $\tilde{v} \in \tilde{V}(\rho_0)$, where $\tilde{V}(\rho_0) \subset T_\rho S^*M$ denotes the collection of vertical vectors $\tilde{v} \in T_{\rho_0} SN^*H$ with $\langle \tilde{v}, H_p \rangle_{g_s} = 0$ where g_s is the metric induced on TT^*M be the Sasaki metric. Now, since $E_+(\rho_0) \cap \tilde{V}(\rho_0) = \{0\}$ [Ebe73a, see right before Proposition 2.7] and $E_+(\rho)$ is continuous, there exists $c_\varepsilon > 0$ depending only on $\varepsilon > 0$ small enough so that

$$c_\varepsilon \|\tilde{u}_+\| \leq \|w\| \leq \frac{1}{c_\varepsilon} \|\tilde{v}\|.$$

For any $e_t \in E_-(G^t(\rho_0))$ we decompose

$$\left\| \frac{dG^t w}{\|dG^t w\|} - e_t \right\| \leq \left\| \frac{dG^t \tilde{u}_+}{\|dG^t w\|} \right\| + \left\| \frac{dG^t \tilde{v}}{\|dG^t w\|} - \frac{dG^t \tilde{v}}{\|dG^t \tilde{v}\|} \right\| + \left\| \frac{dG^t \tilde{v}}{\|dG^t \tilde{v}\|} - e_t \right\|, \tag{5.15}$$

and find $e_t \in E_-(G^t(\rho_0))$ so that each term in the RHS has size smaller than $\delta/3$.

Note that since $\tilde{v}$ is vertical, the Jacobi field through $G^t(\rho)$ with initial conditions given by $J(0) = (dG^t \tilde{v})^h$ and $\dot{J}(0) = (dG^t \tilde{v})^v$, where $()^h$ and $()^v$ denote respectively the horizontal and vertical parts, has $J(-t) = 0$ and hence, by [Ebe73a, Remark 2.10], there exists $T_1 = T_1(\delta) > 0$ so that for $G^t \rho$ in a compact set,

$$\operatorname{dist}(dG^t \tilde{v} / \|dG^t \tilde{v}\|, E_-(G^t(\rho))) < \delta/3.$$

In particular, for all $t \geq T_1$, there exists $e_t \in E_-(G^t(\rho_0))$ so that

$$\left\| \frac{dG^t \tilde{v}}{\|dG^t \tilde{v}\|} - e_t \right\| \leq \frac{\delta}{3}. \tag{5.16}$$

Next, observe that by [Ebe73a, Remark 2.10], for all $\alpha > 0$, there exists $T_2 = T_2(\alpha)$ so that for all ρ, and $|t| \geq T_2$,

$$\|dG^{-t}|_{dG^t \tilde{V}(\rho)}\| \leq \alpha. \tag{5.17}$$

In particular, by (5.17), given $R > 0$ there exists $T_3 = T_3(R, \varepsilon) > 0$ so that for $|t| \geq T_3$ and $z \in C_+^\varepsilon(\rho_0) \backslash \{0\}$,

$$\|dG^t z\| \geq R \|z\|.$$

Furthermore, by [Ebe73a, Corollary 2.14], there exists $B > 0$ so that for all $t \geq 0$ and all $u \in E_+(\rho_0)$,

$$\|dG^t u\| \leq B\|u\|. \tag{5.18}$$

In particular, setting $R_{\delta,\varepsilon} := 3Bc_\varepsilon^{-1}\delta^{-1}$, and letting $|t| \geq T_3(R_{\delta,\varepsilon}, \varepsilon)$,

$$\left\| \frac{dG^t \tilde{u}_+}{\|dG^t w\|} \right\| \leq \frac{B\|\tilde{u}_+\|}{\|dG^t w\|} \leq \frac{B\|\tilde{u}_+\|}{R_{\delta,\varepsilon}\|w\|} \leq \frac{\delta}{3}. \tag{5.19}$$

On the other hand, for $|t| \geq T_3(R_{\delta,\varepsilon}, \varepsilon)$,

$$\left\| \frac{dG^t \tilde{v}}{\|dG^t \tilde{v}\|} - \frac{dG^t \tilde{v}}{\|dG^t w\|} \right\| = \frac{1}{\|dG^t w\|} \big|\|dG^t \tilde{v}\| - \|dG^t w\|\big| \leq \frac{\|dG^t \tilde{u}_+\|}{\|dG^t w\|} \leq \frac{\delta}{3}. \tag{5.20}$$

Taking $T = \max\Big(T_3(R_{\delta,\varepsilon}, \varepsilon), T_1(\delta)\Big)$ we conclude that the claim in (5.14) holds after combining (5.16),(5.19), and (5.20), into (5.15).

□

Now that we have introduced Propositions 17, 23, and 21, we are ready to present the proof of Theorem 8.

Proof of Theorem 8. We start with the case in which (M, g) has no focal points. Recall from Lemma 20 that $m, m_\pm$ from (5.9) are upper semicontinuous. In particular, the sets

$$SN^*H \backslash \mathcal{S}_H = \{\rho \in SN^*H : m(\rho) < n-1\} \quad \textit{and} \quad SN^*H \backslash \mathcal{M}_H^\pm = \{\rho \in SN^*H : m_\pm(\rho) < 1\}$$

are open, and hence $SN^*H \setminus (\mathcal{S}_H \cup \mathcal{M}_H^\pm)$ are open as well. Thus, there exist collections $\{K_\ell^\pm\}_\ell$ of compact sets

$$K_\ell^+ \subset \; SN^*H \setminus (\mathcal{S}_H \cup \mathcal{M}_H^+), \qquad K_\ell^+ \subset \; SN^*H \setminus (\mathcal{S}_H \cup \mathcal{M}_H^-)$$

with

$$\sigma_{SN^*H}(K_\ell^\pm) \;\uparrow\; \sigma_{SN^*H}(SN^*H \setminus \mathcal{S}_H \cup \mathcal{M}_H^\pm).$$

Since

$$SN^*H \setminus (\mathcal{S}_H \cup (\mathcal{M}_H^+ \cap \mathcal{M}_H^-)) = \big[SN^*H \setminus (\mathcal{S}_H \cup \mathcal{M}_H^+)\big] \bigcup \big[SN^*H \setminus (\mathcal{S}_H \cup \mathcal{M}_H^-)\big],$$

the proof of the theorem will follow once we prove that for any compact subset $K^\pm \subset SN^*H \setminus (\mathcal{S}_H \cup \mathcal{M}_H^\pm)$

$$\sigma_H(\mathcal{R}_H \cap K^\pm) = 0. \tag{5.21}$$

We then proceed to prove (5.21).

Let $T_\pm > 0$ and $\varepsilon > 0$ be the constants associated to $K^\pm$ given by Proposition 23. Since

$$\mathcal{R}_H \subset \Big[\bigcap_{m>0} \bigcup_{n \geq m} A_n^\varepsilon\Big] \bigcap \Big[\bigcap_{m>0} \bigcup_{n \geq m} A_{-n}^\varepsilon\Big],$$

with

$$A_n^\varepsilon := \Big\{\rho \in SN^*H : \;\; G^t(\rho) \in \overline{B(\rho, \varepsilon)} \;\; \text{for some} \;\; t \in [n, n+1]\Big\},$$

we have that (5.21) is a consequence of showing that

$$\sigma_{SN^*H}(A_n^\varepsilon \cap K^\pm) = 0, \tag{5.22}$$

for all n with $\mp n \geq T_\pm$.

To prove (5.22) let $\rho_0 \in A_n^\varepsilon \cap K$. Since $G^{t_0}(\rho_0) \in \overline{B(\rho_0, \varepsilon)}$ for some $t_0 \in [n, n+1]$, and $\mp t_0 \geq T$, Proposition 23 combined with Lemma 19 give that there exists $U_{t_0,\rho_0} \subset \mathbb{R} \times SN^*H$ a neighborhood of (t_0, ρ_0) for which

$$\sigma_{SN^*H}\Big(\rho \in SN^*H:\ G^t(\rho) \in SN^*H \quad \text{for some } \ (t,\rho) \in U_{t_0,\rho_0}\Big) = 0.$$

Since, $K^\pm$ is compact if A_n^ε is closed, $A_n^\varepsilon \cap K^\pm$ is compact and we can cover $[n, n+1] \times (K^\pm \cap A_n^\varepsilon)$ by finitely many such neighborhoods and in particular,

$$\sigma_{SN^*H}\Big(\rho \in SN^*H:\ G^t(\rho) \in SN^*H \quad \text{for some } \ (t,\rho) \in [n,n+1] \times (K^\pm \cap A_n^\varepsilon)\Big) = 0.$$

and hence $\sigma_{SN^*H}(A_n^\varepsilon \cap K^\pm) = 0$. Therefore, we have (5.22) provided we show that A_n^ε is closed

We dedicate the end of the proof to showing that A_n^ε is closed. To see this, let $\{\rho_j\} \subset A_n^\varepsilon$ with $\rho_j \to \rho \in SN^*H$. For each j let $t_j \in [n, n+1]$ be such that $G^{t_j}(\rho_j) \in \overline{B(\rho_j, \varepsilon)}$. By possibly taking a subsequence of times, we may assume that there exists $t \in [n, n+1]$ with the property that $t_j \to t$ as $j \to \infty$. In particular, we have that $G^{t_j}(\rho_j) \to G^t(\rho)$. Then, the triangle inequality

$$d(G^t(\rho), \rho) \leq \limsup_{j\to\infty} \big(d(\rho, \rho_j) + d(\rho_j, G^{t_j}(\rho_j)) + d(G^{t_j}(\rho_j), G^t(\rho))\big) \leq \varepsilon$$

shows that $\rho \in A_n^\varepsilon$, as claimed.

In the case that (M, g) has Anosov geodesic flow, we simply appeal to Proposition 21 in place of Proposition 23 to show that, for $K \subset SN^*H \setminus \mathcal{S}_H$ compact,

$$\sigma_{SN^*H}(K \cap \mathcal{R}_H) = 0.$$

and hence using that $SN^*H \setminus \mathcal{S}_H$ is open and approximating $SN^*H \setminus \mathcal{S}_H$ by compact sets, we see that $\sigma_{SN^*H}(\mathcal{R}_H \setminus \mathcal{S}_H) = 0$. Then, applying Lemma 22, $\sigma_{SN^*H}\big(\mathcal{R}_H \cap [\mathcal{S}_H \setminus \mathcal{M}_H]\big) = 0$ and the theorem follows. $\square$

5.4. **Proof of parts C, D, E and F.** In all of these cases (M, g) has Anosov flow (see (1.17)) for the definition.

Proof of part E. For this part we assume that (M, g) has Anosov geodesic flow, non-positive curvature, and H is totally geodesic.

We use that, since there are no parallel Jacobi fields on a manifold with non-positive curvature and Anosov geodesic flow [Ebe73b, Theorem 1 (6)], the spaces E_+ and E_- are nowhere horizontal. In particular, for any horizontal vector v^h, $\|dG^t v^h\| \to \infty$ for $t \to \pm\infty$. To take advantage of this, fix $\rho = (x, \xi) \in SN^*H$. Since H is totally geodesic, the horizontal lift v^h of any $v \in T_x H$ satisfies

$$v^h \in T_\rho(SN^*H).$$

On the other hand, $v^h \notin E_+(\rho) \cup E_-(\rho)$.

Suppose that H is $n-1$ dimensional. Then, we may choose linearly independent vectors $\{v_1, v_2, \dots, v_{n-1}\} \in T_xH$ and get

$$T_\rho(SN^*H) = \operatorname{span}\{v_1^h, v_2^h, \dots, v_{n-1}^h\}.$$

In particular, this yields that

$$T_\rho(SN^*H) \cap (E_+(\rho) \cup E_-(\rho)) = \emptyset.$$

Therefore,

$$\mathcal{S}_H = \emptyset,$$

and hence $\sigma_{SN^*H}(\mathcal{R}_H) = 0$.

To finish the proof we explain that it suffices to assume that H is $n-1$ dimensional. Note that since H is totally geodesic submanifold, $H_t := \pi(G^t(SN^*H))$ is also a totally geodesic submanifold. Now, for t small,

$$G^t : N^*H \to M$$

is an isometry, and in particular, H_t is an embedded submanifold of dimension $n-1$. Moreover, by Lemma 16, $\sigma_{SN^*H_t}(\mathcal{R}_{H_t}) = 0$ implies $\sigma_{SN^*H}(\mathcal{R}_H) = 0$. Therefore, it is enough to show that $\sigma_{SN^*H}(\mathcal{R}_H) = 0$ for every totally geodesic submanifold H of dimension $n-1$ which we have already done.

□

The proofs of Parts C, D, and F, rely on showing that in each of these settings one has that the set of points $\rho \in \mathcal{R}_H$ for which $T_\rho(SN^*H)$ is purely stable, or purely unstable, has full measure and applying Lemma 22.

Proof of part D. For this part we assume that (M, g) is a surface with Anosov geodesic flow. Theorem 8 implies

$$\sigma_{SN^*H}(\mathcal{R}_H) = \sigma_{SN^*H}\big(\mathcal{R}_H \cap \mathcal{S}_H \cap \mathcal{M}_H\big).$$

But, since $\dim M = 2$, we have $\dim SN^*H = 1$ and, since $E_+(\rho) \cap E_-(\rho) = \{0\}$, $\mathcal{M}_H = \emptyset$. Thus, $\sigma_{SN^*H}(\mathcal{R}_H) = 0$ as claimed. □

Proof of part F. For this part we assume that (M, g) has Anosov geodesic flow and H is a subset of a stable or unstable horosphere (see e.g. [Rug07, Chapter 4] or [KH95, Section 17.6, Theorem 6.2.8] for a definition a horosphere). The crucial fact is that a stable horosphere, H_+ has the property that $T_\rho SN^*H_+ \subset E_+(\rho)$ and an unstable horosphere, H_- has $T_\rho SN^*H_- \subset E_-(\rho)$. That $\sigma_{SN^*H}(\mathcal{R}_H) = 0$ then follows immediately from Lemma 22. □

Proof of part C. In part C, we claim that on a manifold of constant negative curvature, $\sigma_{SN^*H}(\mathcal{R}_H) = 0$ for all $H \subset M$. We start by showing that it suffices to assume that H is $n-1$ dimensional. Since the exponential map is a radial isometry, $H_t = \{\exp_x(t\xi) :\ (x, \xi) \in SN^*H\}$ is an embedded submanifold of dimension $n-1$ for small t. Moreover, by Lemma 16, $\sigma_{SN^*H_t}(\mathcal{R}_{H_t}) = 0$ implies $\sigma_{SN^*H}(\mathcal{R}_H) = 0$. Therefore, it is enough to show that $\sigma_{SN^*H}(\mathcal{R}_H) = 0$ for every submanifold H of dimension $n-1$.

We note that by Theorem 8 we have

$$\sigma_{SN^*H}(\mathcal{R}_H) = \sigma_{SN^*H}\left(\mathcal{R}_H \cap \mathcal{S}_H \cap \mathcal{M}_H\right).$$

Lemma 24. *Let (M,g) be a compact manifold with constant negative curvature and $H\subset M$ be a closed embedded hypersurface. Then*

$$\sigma_{SN^*H}(\mathcal{S}_H\cap\mathcal{M}_H)=0.$$

Note that this result combined with Theorem 8 yield that $\sigma_{SN^*H}(\mathcal{R}_H)=0$ finishing the proof of Part C.

□

The rest of this section is dedicated to the proof of Lemma 24. Since we may work locally to prove Lemma 24, we lift the hypersurface H to the universal cover $\mathbb{H}^n$. Hence, in this section we work with the hyperbolic space

$$\mathbb{H}^n=\Big\{(x_0,x_1,\dots,x_n)\in\mathbb{R}^{n+1}:\ x_0>0,\ x_0^2-\sum_{i=1}^n x_i^2=1\Big\}.$$

We endow $\mathbb{H}^n$ with the metric $g=dx_0^2-\sum_{i=1}^n dx_i^2$. To prove Lemma 24 we adopt the notation

$$\langle v,w\rangle_g=-v_0w_0+\sum_{i=1}^n v_iw_i$$

for the inner product induced by the metric g. We also write $\langle v,w\rangle=v_0w_0+\sum_{i=1}^n v_iw_i$ for the usual inner product in $\mathbb{R}^{n+1}$. With this notation the sphere bundle takes the form $S\mathbb{H}^n=\{(x,w):\ x\in\mathbb{H}^n, w\in\mathbb{R}^{n+1},\langle w,w\rangle_g=1,\langle x,w\rangle_g=0\}$, and its tangent space at $p=(x,w)$ can be decomposed into a direct sum $T_p(S\mathbb{H}^n)=E_+(p)\oplus E_-(p)\oplus\mathbb{R}X$ where the stable and stable fibers are $\tilde{E}_-(p)=\{(v,-v):\langle x,v\rangle_g=\langle w,v\rangle_g=0\}$ and $\tilde{E}_+(p)=\{(v,v):\langle x,v\rangle_g=\langle w,v\rangle_g=0\}$ and X is the generator of the geodesic flow. Since we work in the co-sphere bundle, we record the structure of the dual spaces. The co-sphere bundle is

$$S^*\mathbb{H}^n=\{(x,\xi):\ x\in\mathbb{H}^n,\xi\in\mathbb{R}^{n+1},\langle\xi,\xi\rangle_g=1,\langle x,\xi\rangle=0\},$$

and the tangent space at any $\rho=(x,\xi)\in S^*\mathbb{H}^n$ is

$$T_\rho(S^*\mathbb{H}^n)=\{(v_x,v_\xi):\ \langle x,v_x\rangle_g=\langle\xi,v_x\rangle+\langle x,v_\xi\rangle=\langle\xi,v_\xi\rangle_g=0\}.$$

We then have

$$T_\rho(S^*\mathbb{H}^n)=E_+(\rho)\oplus E_-(\rho)\oplus\mathbb{R}H_p,$$

where

$$E_+(\rho)=\{((v_0,v'),(v_0,-v')):\ \langle x,v\rangle_g=\langle\xi,v\rangle=0\}. \tag{5.23}$$

and

$$E_-(\rho)=\{((v_0,v'),(-v_0,v')):\ \langle x,v\rangle_g=\langle\xi,v\rangle=0\}. \tag{5.24}$$

Here, and in what follows, we adopt the notation (z_0,z',z_d) to represent a point in $\mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R}$.

Proof of Lemma 24. We assume that γ is a parametrization of $H\subset\mathbb{H}^n$ in a neighborhood $V\subset H$ of y. That is,

$$H\cap V=\{(\alpha(x'),x',\gamma(x')):x'\in\tilde{V}\},$$

for some $\tilde{V} \subset \mathbb{R}^{n-1}$ open, and where

$$\alpha(x') := \sqrt{1 + |x'|^2 + \gamma(x')^2}.$$

Using that $x_0 - \alpha(x')$ and $x_n - \gamma(x')$ are defining functions for H as a subset of $\mathbb{R}^{n+1}$ we find that

$$N^*H = \{(\alpha,\ x',\ \gamma,\ -\lambda f\alpha,\ \lambda(fx' - \partial\gamma),\ \lambda(f\gamma + 1)) :\ \lambda \in \mathbb{R}\},$$

where to shorten notation we write

$$f := \gamma - \langle x', \partial\gamma\rangle.$$

This yields that

$$SN^*H = \{(\alpha,\ x',\ \gamma,\ -\lambda f\alpha,\ \lambda(fx' - \partial\gamma),\ \lambda(f\gamma + 1))\},$$

where

$$\lambda := (1 + |\partial\gamma|^2 + f^2)^{-\frac{1}{2}}.$$

Therefore, given $\rho = (x, \xi) \in SN^*H$ we find

$$T_\rho(SN^*H) = \{(\langle\partial\alpha, w\rangle,\ w,\ \langle\partial\gamma, w\rangle,\ \langle A, w\rangle,\ \langle B, w\rangle,\ \langle C, w\rangle) :\ w \in \mathbb{R}^{n-1}\}, \quad (5.25)$$

where

$$A := -\partial(\lambda f\alpha), \qquad B := \partial(\lambda(fx' - \partial\gamma)), \qquad C := \partial(\lambda(f\gamma + 1)).$$

We assume without loss of generality that $y = (\alpha(0), 0, \gamma(0))$, where $\gamma(0) = 0$ and $\partial\gamma(0) = 0$. Note that, with

$$\gamma(x') = \frac{1}{2}\langle Qx', x'\rangle + O(|x'|^3),$$

where Q is an $(n-1) \times (n-1)$ symmetric matrix we have

$$\alpha = 1 + \frac{1}{2}|x'|^2 + O(|x'|^4), \qquad \partial\alpha = x' + O(|x'|^3),$$
$$f = -\frac{1}{2}\langle Qx', x'\rangle + O(|x'|^3), \qquad \partial f = -Qx' + O(|x'|^2),$$
$$\lambda = 1 - \frac{1}{2}|Qx'|^2 + O(|x'|^4), \qquad \partial\lambda = -\langle Q^2x', w\rangle + O(|x'|^3).$$

Now, suppose there exist two non-zero vectors

$$X_+ \in E_+(\rho) \cap T_\rho(SN^*H) \qquad \text{and} \qquad X_- \in E_-(\rho) \cap T_\rho(SN^*H).$$

Then, according to (5.25), (5.23) and (5.24) we have that there exist $w_+, w_- \in \mathbb{R}^{n-1}$ so that

$$X_\pm = (\langle\partial\alpha, w_\pm\rangle,\ w_\pm,\ \langle\partial\gamma, w_\pm\rangle,\ \langle A, w_\pm\rangle,\ \langle B, w_\pm\rangle,\ \langle C, w_\pm\rangle)$$

and satisfying

i) $\langle\partial\alpha, w_\pm\rangle = \pm\langle A, w_\pm\rangle$

ii) $w_\pm = \mp\langle B, w_\pm\rangle$

iii) $\langle\partial\gamma, w_\pm\rangle = \mp\langle C, w_\pm\rangle$

iv) $\langle x, X_\pm\rangle_g = 0$

v) $\langle \xi, X_\pm\rangle = 0.$

We proceed to showing that there cannot exist $w_\pm$ satisfying conditions $(i),(ii)$ and (iii) for all $\rho=(x,\xi)$ in a subset of SN^*H with positive measure on which $T_\rho(SN^*H)=N_+(\rho)\oplus N_-(\rho)$, $N_+(\rho)\neq\{0\}$, and $N_-(\rho)\neq\{0\}$. Indeed, conditions $(i),(ii)$ and (iii) read

i) $\langle x',w_\pm\rangle=\pm\langle Qx',w_\pm\rangle+O(|x'|^2)$
ii) $w_\pm=\pm Qw_\pm\pm(\partial^3\gamma(0)x')w_\pm+O(|x'|^2)$
iii) $\langle Qx',w_\pm\rangle=\pm\langle Q^2x',w_\pm\rangle+O(|x'|^2)$.

These equations imply that $w_\pm=\pm Qw_\pm$ and so $Q^2w_\pm=w_\pm$. Furthermore, we claim that we may assume that $\partial^3\gamma(0)=0$. Indeed, let $\rho\in SN^*H$ be such that $T_\rho(SN^*H)=N_+(\rho)\oplus N_-(\rho)$. Then, if $w\in T_\rho(SN^*H)$, we may decompose w it as $w=w_++w_-$ and use that condition (ii) gives $(\partial^3\gamma(0)x')w=0$. If we had that condition (ii) holds on a set of ρ's with positive measure, we must have that $\partial^3\gamma(0)=0$ since we just showed that condition (ii) should also hold for all $w\in T_\rho(SN^*H)$. We then work with

$$\gamma(x')=\frac{1}{2}\langle Qx',x'\rangle+O(|x'|^4).$$

From this we get the improved estimates

$$f=-\frac{1}{2}\langle Qx',x'\rangle+O(|x'|^4)\qquad\text{and}\qquad\partial f=-Qx'+O(|x'|^3).$$

We derive the contradiction from studying the second order terms in $w_\pm=\mp\langle B,w_\pm\rangle$. Indeed,

$$\langle B,w_\pm\rangle=\pm Qw_\pm+D(w_\pm)+O(|x'|^3),$$

where

$$D(w_\pm):=-\partial^4\gamma(0)x'^2w_\pm+\langle x',w_\pm\rangle(Qx'\mp x')-\frac{1}{2}\langle Qx',x'\mp Qx'\rangle w_\pm,$$

and where $\partial^4\gamma(0)x'^2w_\pm$ denotes the vector whose i-th entry is given by $(\partial^4\gamma(0)x'^2w)_k=\frac{1}{12}\partial_{ijkl}\gamma(0)x_kx_lw_j$. Since $D(w_\pm)$ is a second order term in x', equation $w_\pm=\mp\langle B,w_\pm\rangle$ gives that

$$D(w_\pm)=0.$$

To take advantage of this condition, we assume without loss of generality that

$$Q=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&\tilde Q\end{pmatrix},$$

where $\tilde Q$ is an $(n-3)\times(n-3)$ matrix, and that

$$w_+=(1,0,\dots,0)\qquad\text{and}\qquad w_-=(0,1,0\dots,0).$$

We now use that all the coordinates of the vectors $D(w_\pm)$ equal 0. Making the second coordinate of the vector $D(w_+)$ equal to 0 gives

$$-\frac{1}{12}\sum_{k,l=1}^n\partial_{21kl}\gamma(0)x_kx_l-2x_1x_2=0,$$

while setting the first coordinate of the vector $D(w_-)$ equal to 0 yields

$$-\frac{1}{12}\sum_{k,l=1}^{n}\partial_{12kl}\gamma(0)x_kx_l+2x_1x_2=0.$$

This concludes the proof since we cannot have the two relations holding simultaneously for x' in a subset of H that has positive measure.

□

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