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TOPOLOGICAL ENTROPY OF CONTINUOUS SELF–MAPS ON CLOSED SURFACES

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Abstract. The objective of the present work is to present sufficient conditions for having positive topological entropy for continuous self–maps defined on a closed surface by using the action of this map on the homological groups of the closed surface.

1. Introduction

Along this work by a *closed surface* we denote a connected compact surface with or without boundary, orientable or not. More precisely, *an orientable connected compact surface without boundary of genus $g \geq 0$, $\mathbb{M}_g$,* is homeomorphic to the sphere if $g = 0$, to the torus if $g = 1$, or to the connected sum of g copies of the torus if $g \geq 2$. *An orientable connected compact surface with boundary of genus $g \geq 0$, $\mathbb{M}_{g,b}$,* is homeomorphic to $\mathbb{M}_g$ minus a finite number $b > 0$ of open discs having pairwise disjoint closure. In what follows $\mathbb{M}_{g,0} = \mathbb{M}_g$.

A non–orientable connected compact surface without boundary of genus $g \geq 1$, $\mathbb{N}_g$, is homeomorphic to the real projective plane if $g = 1$, or to the connected sum of g copies of the real projective plane if $g > 1$. *A non–orientable connected compact surface with boundary of genus $g \geq 1$, $\mathbb{N}_{g,b}$,* is homeomorphic to $\mathbb{N}_g$ minus a finite number $b > 0$ of open discs having pairwise disjoint closure. In what follows $\mathbb{N}_{g,0} = \mathbb{N}_g$.

Let $f : \mathbb{X} \to \mathbb{X}$ be a continuous map on a closed surface $\mathbb{X}$. A point $x \in \mathbb{X}$ is periodic of period n if $f^n(x) = x$ and $f^k(x) \neq x$ for $k = 1, \ldots, n-1$.

The *topological entropy* of a continuous map $f : \mathbb{X} \to \mathbb{X}$ denoted by $h(f)$ is a non–negative real number (possibly infinite) which measures how much f mixes up the phase space of $\mathbb{X}$. When $h(f)$ is positive the dynamics of the system is said to be *complicated* and the positivity of $h(f)$ is used as a measure of the so called *topological chaos*.

Here we introduce the topological entropy using the definition of Bowen [4].

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Since it is possible to embedded any surface orientable or not in $\mathbb{R}^4$ by the Whitney immersion theorem see [11], we consider the distance between two points of $\mathbb{X}$ as the distance of these two points in $\mathbb{R}^4$. Now, we define the distance d_n on G by

$$d_n(x,y) = \max_{0\le i\le n} d(f^i(x), f^i(y)), \quad \forall x,y \in G.$$

A finite set S is called (n,ε)*–separated with respect to* f if for different points $x,y \in S$ we have $d_n(x,y) > \varepsilon$. We denote by S_n the maximal cardinality of an (n,ε)–separated set. Define

$$h(f,\varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log S_n.$$

Then

$$h(f) = \lim_{\varepsilon\to 0} h(f,\varepsilon)$$

is the *topological entropy* of f.

We have chosen the definition by Bowen because, probably it is the shorter one. The classical definition was due to Adler, Konheim and McAndrew [1]. See for instance the book of Hasselblatt and Katok [7] and [3] for other equivalent definitions and properties of the topological entropy. See [1, 2, 12] for more details on the topological entropy.

Our main results are the following.

Theorem 1. *Let $\mathbb{M}_g$ be an orientable connected compact surface without boundary of genus g. Then the following statements hold.*

(a) *If the degree $d \notin \{-1,0,1\}$, then the topological entropy of f is positive.*
(b) *If the degree $d \in \{-1,0,1\}$ and the number of roots of the characteristic polynomial of f_{*1} equal to ± 1 or 0 taking into account their multiplicities is not even, then the topological entropy of f is positive.*

Theorem 2. *Let $\mathbb{M}_{g,b}$, $b>0$, be an orientable connected compact surface with boundary of genus g. Then the following statements hold.*

(a) *If $2g+b-1$ is even and the number of roots of the characteristic polynomial of f_{*1} equal to ± 1 or 0 taking into account their multiplicities is not even, then the topological entropy of f is positive.*
(b) *If $2g+b-1$ is odd and the number of roots of the characteristic polynomial of f_{*1} equal to ± 1 or 0 taking into account their multiplicities is not odd, then the topological entropy of f is positive.*

Theorem 3. *Let $\mathbb{N}_{g,b}$, $b\ge 0$, be a non–orientable connected compact surface with boundary of genus g. Then the following statements hold.*

(a) *If $g+b-1$ is even and the number of roots of the characteristic polynomial of f_{*1} equal to ± 1 or 0 taking into account their multiplicities is not even, then the topological entropy of f is positive.*

(b) *If* $g+b-1$ *is odd and the number of roots of the characteristic polynomial of* f_{*1} *equal to* ± 1 *or* 0 *taking into account their multiplicities is not odd, then the topological entropy of* f *is positive.*

2. Lefschetz zeta functions for surfaces

Let f be a continuous self–map defined on $\mathbb{M}_{g,b}$ or $\mathbb{N}_{g,b}$, respectively. For a closed surface the homological groups with coefficients in $\mathbb{Q}$ are linear vector spaces over $\mathbb{Q}$. We recall the homological spaces of $\mathbb{M}_{g,b}$ with coefficients in $\mathbb{Q}$, i.e.

$$H_k(\mathbb{M}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \overset{n_k}{\cdots} \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = 2g$ if $b = 0$, $n_1 = 2g + b - 1$ if $b > 0$, $n_2 = 1$ if $b = 0$, and $n_2 = 0$ if $b > 0$; and the induced linear maps $f_{*k} : H_k(\mathbb{M}_{g,b}, \mathbb{Q}) \to H_k(\mathbb{M}_{g,b}, \mathbb{Q})$ by f on the homological group $H_k(\mathbb{M}_{g,b}, \mathbb{Q})$ are $f_{*0} = (1)$, $f_{*2} = (d)$ where d is the *degree* of the map f if $b = 0$, $f_{*2} = (0)$ if $b > 0$, and $f_{*1} = A$ where A is an $n_1 \times n_1$ integral matrix (see for additional details [10, 13]).

We recall that the homological groups of $\mathbb{N}_{g,b}$ with coefficients in $\mathbb{Q}$, i.e.

$$H_k(\mathbb{N}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \overset{n_k}{\cdots} \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = g + b - 1$ and $n_2 = 0$; and the induced linear maps are $f_{*0} = (1)$ and $f_{*1} = A$ where A is an $n_1 \times n_1$ integral matrix (see again for additional details [10, 13]).

Let $f : \mathbb{X} \to \mathbb{X}$ be a continuous map and let $\mathbb{X}$ be either $\mathbb{M}_{g,b}$ or $\mathbb{N}_{g,b}$. Then the *Lefschetz number* of f is defined by

$$L(f) = \operatorname{trace}(f_{*0}) - \operatorname{trace}(f_{*1}) + \operatorname{trace}(f_{*2}).$$

We shall use the Lefschetz numbers of the iterates of f, i.e. $L(f^n)$. In order to study the whole sequence $\{L(f^n)\}_{n \geq 1}$ it is defined the formal *Lefschetz zeta function* of f as

$$Z_f(t) = \exp\left(\sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n \right).$$

The Lefschetz zeta function is in fact a generating function for the sequence of the Lefschetz numbers $L(f^n)$.

From the work of Franks in [6] we have for a continuous self–map of a closed surface that its Lefschetz zeta function is the rational function

$$Z_f(t) = \frac{\det(I - t f_{*1})}{\det(I - t f_{*0}) \det(I - t f_{*2})}, \tag{1}$$

where in $I-tf_{*k}$ the I denotes the $n_k\times n_k$ identity matrix, and $\det(I-tf_{*2})=1$ if $f_{*2}=(0)$. Then for a continuous map $f:\mathbb{M}_{g,b}\to\mathbb{M}_{g,b}$ we have

$$Z_f(t)=\begin{cases}\dfrac{\det(I-tA)}{(1-t)(1-dt)} & \text{if } b=0,\\[2mm] \dfrac{\det(I-tA)}{1-t} & \text{if } b>0,\end{cases}$$

and for a continuous map $f:\mathbb{N}_{g,b}\to\mathbb{N}_{g,b}$ we have

$$Z_f(t)=\frac{\det(I-tA)}{1-t}.$$

3. Basic results

In this section we present the main result stated in Theorem 7 for proving Theorems 1, 2 and 3. Since its proof is short and important for this work we provide it here.

For a polynomial $H(t)$ we define $H^*(t)$ by

$$H(t)=(1-t)^{\alpha}(1+t)^{\beta}t^{\gamma}H^*(t),$$

where α, β and γ are non-negative integers such that $1-t$, $1+t$ and t do not divide $H^*(t)$.

The *spectral radii* of the maps f_{*k} are denoted $\mathrm{sp}(f_{*k})$, and they are equal to the largest modulus of all the eigenvalues of the linear map f_{*k}. The *spectral radius of* f_* is

$$\mathrm{sp}(f_*)=\max_{k=0,\ldots,m}\mathrm{sp}(f_{*k}).$$

The next result is due to Manning [9].

Theorem 4. *Let $f:X\to X$ be a continuous map on a closed surface $\mathbb{X}$. Then $\log\max\{1,\mathrm{sp}(f_{*1})\}\le h(f)$.*

Lemma 5. *Let $f:\mathbb{X}\to\mathbb{X}$ be a continuous map and let $\mathbb{X}$ be a closed surface. If the topological entropy of f is zero, then all the eigenvalues of the induced homomorphism f_{*1} are zero or root of unity.*

Proof. Since the topological entropy is zero, by Theorem 4 we have $\mathrm{sp}(f_{*1})=1$. So, all the eigenvalues of f_{*1} have modulus in the interval $[0,1]$ and at least one of them is 1. Then the characteristic polynomial of f_{*1} is of the form $t^m p(t)$, where m is a non–negative integer, positive if the zero is an eigenvalue. And $p(t)$ is a polynomial with integer coefficients and whose independent term a_0 is non–zero. Since the product of all non-zeros eigenvalues of f_{*1} is the integer a_0 and, these eigenvalues have modulus in $(0,1]$, we have that any of these eigenvalues can have modulus smaller than one, otherwise we are in contradiction with the fact a_0 is an integer. In short, all the non–zero eigenvalues have modulus one, and consequently $a_0=1$.

Since if a polynomial has integer coefficients, constant term 1 and all of whose roots have modulus 1, then all of its roots are roots of unity, see [14], the lemma follows. ☐

The n–th cyclotomic polynomial is defined by

$$c_n(t) = \prod_k (w_k - t),$$

being $w_k = e^{2\pi i k/n}$ a primitive n–th root of unity and where k runs over all the relative primes $\leq n$. See [8] for the properties of these polynomials.

For a positive integer n the *Euler function* is $\varphi(n) = n \prod_{p|n, p \text{ prime}} \left(1 - \frac{1}{p}\right)$. It is known that the degree of the polynomial $c_n(t)$ is $\varphi(n)$. Note that $\varphi(n)$ is even for $n > 2$.

A proof of the next result can be found in [8].

Proposition 6. *Let ξ be a primitive n–th root of the unity and $P(t)$ a polynomial with rational coefficients. If $P(\xi) = 0$ then $c_n(t)|P(t)$.*

The proofs of our results are strongly based in the next theorem.

Theorem 7 (Theorem 3.2 of [5])**.** *Let $\mathbb{X}$ be a closed surface, $f : \mathbb{X} \to \mathbb{X}$ be a continuous self–map, and let $\mathcal{Z}_f(t) = P(t)/Q(t)$ be its Lefschetz zeta function. If $P^*(t)$ or $Q^*(t)$ has odd degree, then the topological entropy of f is positive.*

Proof. From the definitions of a polynomial H^* and of the Lefschetz zeta function we have

$$\mathcal{Z}_f(t) = \frac{P(t)}{Q(t)} = (1-t)^a(1+t)^b t^c \frac{P^*(t)}{Q^*(t)},$$

where a, b and c are integers.

Assume now that the topological entropy $h(f) = 0$. Then by Lemma 5 all the eigenvalues of the induced homomorphisms f_{*1}'s are zero or roots of unity. Therefore, by (1) all the roots of the polynomials $P^*(t)$ and $Q^*(t)$ are roots of the unity different from ± 1 and zero. Hence, by Proposition 6 the polynomials $P^*(t)$ and $Q^*(t)$ are product of cyclotomic polynomials different from $c_1(t) = 1 - t$ and $c_2(t) = 1 + t$. Consequently $P^*(t)$ and $Q^*(t)$ have even degree because all the cyclotomic polynomials which appear in them have even degree due to the fact that the Euler function $\varphi(n)$ for $n > 2$ only takes even values. But this is a contradiction with the assumption that $P^*(t)$ or $Q^*(t)$ has odd degree. ☐

4. Proof of Theorems 1, 2 and 3

Proof of Theorem 1. Since $\mathbb{M}_g$ is an orientable connected compact surface without boundary of genus g, then the Lefschetz zeta function of f is equal to

$$Z_f(t) = \frac{\det(I - tA)}{(1-t)(1-dt)},$$

where d is the degree of f and $2g$ is the dimension of the characteristic polynomial $\det(I - tA)$ of $f_{*1} = A$. Note here that if $d \notin \{-1, 0, 1\}$, then $Q^*(t) = 1 - dt$ and therefore by Theorem 7 statement (a) of Theorem 1 is proved.

Assume now that $d \in \{-1, 0, 1\}$. Note that in this case $Q(t) = (1-t)(1-dt)$ and $Q^*(t) = 1$. So, by Theorem 7 the main role will be play by the $2g$ degree polynomial $P(t) = \det(I - tA)$ where $f_{*1} = A$. Since $2g$ is even and the number of roots of the characteristic polynomial of f_{*1} equal to ± 1 or 0 taking into account their multiplicities is not even, then $P^*(t)$ has odd degree. Therefore, statement (b) of Theorem 1 follows by the application of Theorem 7.

□

Proof of Theorem 2. Note now, since $\mathbb{M}_{g,b}$ is an orientable connected compact surface with boundary $(b > 0)$ of genus g, then the Lefschetz zeta function of f is equal to

$$Z_f(t) = \frac{\det(I - tA)}{1 - t}$$

being $2g + b - 1$ the degree of the characteristic polynomial $\det(I - tA)$ of $f_{*1} = A$. Now the proof is similar to the statements (b) and (c) of Theorem 1. □

Proof of Theorem 3. For a non–orientable connected compact surface with or without boundary $(b \geq 0)$ of genus $g \geq 1$, the Lefschetz zeta function of f is equal to

$$Z_f(t) = \frac{\det(I - tA)}{1 - t}$$

being $g + b - 1$ the degree of the characteristic polynomial $\det(I - tA)$ of $f_{*1} = A$. Then the proof if this theorems follows in a similar way to the proof of statements (b) and (c) of Theorem 1. □

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