This is a preprint of: “Global Phase Portraits of Z2-Symmetric Planar Polynomial Hamiltonian Systems of Degree Three with a Nilpotent Saddle at the Origin”, Montserrat Corbera, Clàudia Valls, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, vol. 30(1), 2050006:1–30, 2020.
DOI: [10.1142/S0218127420500066]

GLOBAL PHASE PORTRAITS OF $\mathbb{Z}_2$-SYMMETRIC PLANAR POLYNOMIAL HAMILTONIAN SYSTEMS OF DEGREE 3 WITH A NILPOTENT SADDLE AT THE ORIGIN

MONTSERRAT CORBERA AND CLAUDIA VALLS

Abstract. We characterize the phase portraits in the Poincaré disk of all planar polynomial Hamiltonian systems of degree 3 with a nilpotent saddle at the origin and $\mathbb{Z}_2$-symmetric with $(x, y) \mapsto (-x, y)$.

1. Introduction and statement of the results

In this paper we study the global phase portraits of a class of $\mathbb{Z}_2$-symmetric planar polynomial Hamiltonian systems of degree three with a nilpotent saddle at the origin. We recall that a *planar polynomial Hamiltonian system* is a system of the form

$$x' = H_y, \quad y' = -H_x \tag{1}$$

where $H(x, y)$ is a real polynomial in the variables x and y. Here the prime denotes derivative with respect to the independent variable t. We say that system (1) has *degree* d if the maximum of the degrees of H_y and H_x is d. In this paper we will focus in the case in which $d = 3$.

Let $p \in \mathbb{R}^2$ be a singular point of a polynomial differential system in $\mathbb{R}^2$. Without loss of generality we can assume that p is at the origin of coordinates. We say that p is a *nilpotent saddle* if after a linear change of variables and a rescaling of the time (if necessary) the system can be written in the form

$$x' = y + P(x, y), \quad y' = Q(x, y)$$

where $P(x, y)$ and $Q(x, y)$ are real analytic functions without constant and linear terms, defined in a neighborhood of the origin. In this paper we will consider the case in which P and Q are polynomials of degree three.

The global phase portraits in the Poincaré disk of all planar polynomial Hamiltonian vector fields with degree three having a nilpotent center at the origin have been provided in several studies (see for instance [2, 3]). However,

2010 *Mathematics Subject Classification.* 34C05, 34C07, 34C08.
Key words and phrases. Polynomial Hamiltonian systems, nilpotent saddle, phase portrait, Poincaré compactification.

no global phase portraits in the Poincaré disk are given for planar polynomial Hamiltonian vector fields having a nilpotent saddle at the origin. This is mainly due to the fact that there is a very important additional difficulty caused by the possible saddle connections which normally are very difficult to detect.

In this paper we want to fill in this gap and provide the global phase portraits in the Poincaré disk of all reversible and equivariant planar polynomial differential systems of degree three that are symmetric with respect to the y-axis and with a nilpotent saddle at the origin. Let $X\colon U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field associated to system (1). We say that system (1) is *reversible* with respect to the y-axis if it satisfies

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} X(x,y) = -X(-x,y), \tag{2}$$

and we say that it is *equivariant* with respect to the y-axis if it satisfies

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} X(x,y) = X(-x,y). \tag{3}$$

Other classes of polynomial vector fields in $\mathbb{R}^2$ with a $\mathbb{Z}_2$-symmetry have been studied by several authors (see for instance [5, 6, 7, 8]).

The main tool in this paper will be the Poincaré compactification of polynomial vector fields. The Poincaré compactification that we shall use for describing the global phase portraits of our Hamiltonian systems is standard. For all the definitions and results on the Poincaré compactification see Chapter 5 of [4]. We say that two vector fields on the Poincaré disk are topologically equivalent if there exists a homeomorphism from one into the other which sends orbits to orbits preserving or reversing the direction of the flow. Our main result is the following one.

Theorem 1. *A Hamiltonian planar polynomial vector field of degree three with a nilpotent saddle at the origin and reversible with respect to the y-axis, after a linear change of variables and a rescaling of its independent variable can be written as one of the following classes:*

(I) $x' = y + by^3$, $y' = x^3$ *with* $b \in \mathbb{R}$*;*
(II) $x' = y + sx^2y + by^3$, $y' = x^3 - sxy^2$ *with* $b \in \mathbb{R}$*;*
(III) $x' = y + y^2 + by^3$, $y' = x^3$ *with* $b \in \mathbb{R}$*;*
(IV) $x' = y + y^2 + sx^2y + by^3$, $y' = ax^3 - sxy^2$ *with* $a > 0$ *and* $b \in \mathbb{R}$*;*
(V) $x' = y + \frac{x^2}{\sqrt{2}}$, $y' = ax^3 - \sqrt{2}xy$ *with* $a > -1$*;*
(VI) $x' = y + \frac{x^2}{\sqrt{2}} + sy^3$, $y' = ax^3 - \sqrt{2}xy$ *with* $a > -1$*;*
(VII) $x' = y + \frac{x^2}{\sqrt{2}} + sy^2 + by^3$, $y' = ax^3 - \sqrt{2}xy$ *with* $a > -1$ *and* $b \in \mathbb{R}$*;*
(VIII) $x' = y + \frac{x^2}{\sqrt{2}} + sx^2y + by^2 + cy^3$, $y' = ax^3 - \sqrt{2}xy - sxy^2$ *with* $a > -1$ *and* $b, c \in \mathbb{R}$*,*

with $s \in \{-1, 1\}$, $a \neq 0$ in systems (V) and $a^2 + b^2 \neq 0$ in systems (VII). Moreover, the global phase portraits of these families that have at most one infinite singular point, are topologically equivalent to the following ones of Figure 1:

- 1.1–1.3 *for systems* (I);
- 1.1–1.7 *for systems* (II);
- 1.1, 1.3, *and* 1.8–1.12 *for systems* (III);
- 1.1, 1.3, 1.5, *and* 1.7–1.33 *for systems* (IV);
- 1.2 *and* 1.34 *for systems* (V);
- 1.1–1.3, 1.13, 1.35, *and* 1.36 *for systems* (VI);
- 1.1–1.3, 1.8–1.18, 1.25, *and* 1.35–1.45 *for systems* (VII);
- 1.1–1.33, *and* 1.35–1.148 *for systems (VIII).*

Note that the conditions $a \neq 0$ in system (V) and $a^2 + b^2 \neq 0$ in system (VII) are due to the fact that we are studying systems with degree three and otherwise in any of these cases, the corresponding system would be quadratic.

The proof of Theorem 1 is given in sections 3–19.

2. Preliminary results

A vector field is said to have the *finite sectorial decomposition property* at a singular point q if either q is a center, a focus or a node, or it has a neighborhood consisting of a finite union of parabolic, hyperbolic or elliptic sectors. We note that all the isolated singular points of a polynomial differential system satisfy the finite vectorial decomposition property.

Theorem 2 (Poincaré Formula)**.** *Let q be an isolated singular point having the finite sectorial decomposition property. Let e, h an p denote the number of elliptic, hyperbolic and parabolic sectors of q, respectively. Then the index of q is $(e - h)/2 + 1$.*

The indices of a saddle, a center and a cusp are $-1, 1$ and 0, respectively.

Theorem 3 (Poincaré–Hopf Theorem)**.** *For every vector field on the sphere $\mathbb{S}^2$ with a finite number of singular points, the sum of the indices of these singular points is* 2.

Nilpotent singular points of Hamiltonian planar polynomial vector fields are either saddles, centers, or cusps (for more details see Chapter Theorem 3.5 of [4] and take into account that Hamiltonian systems cannot have foci).

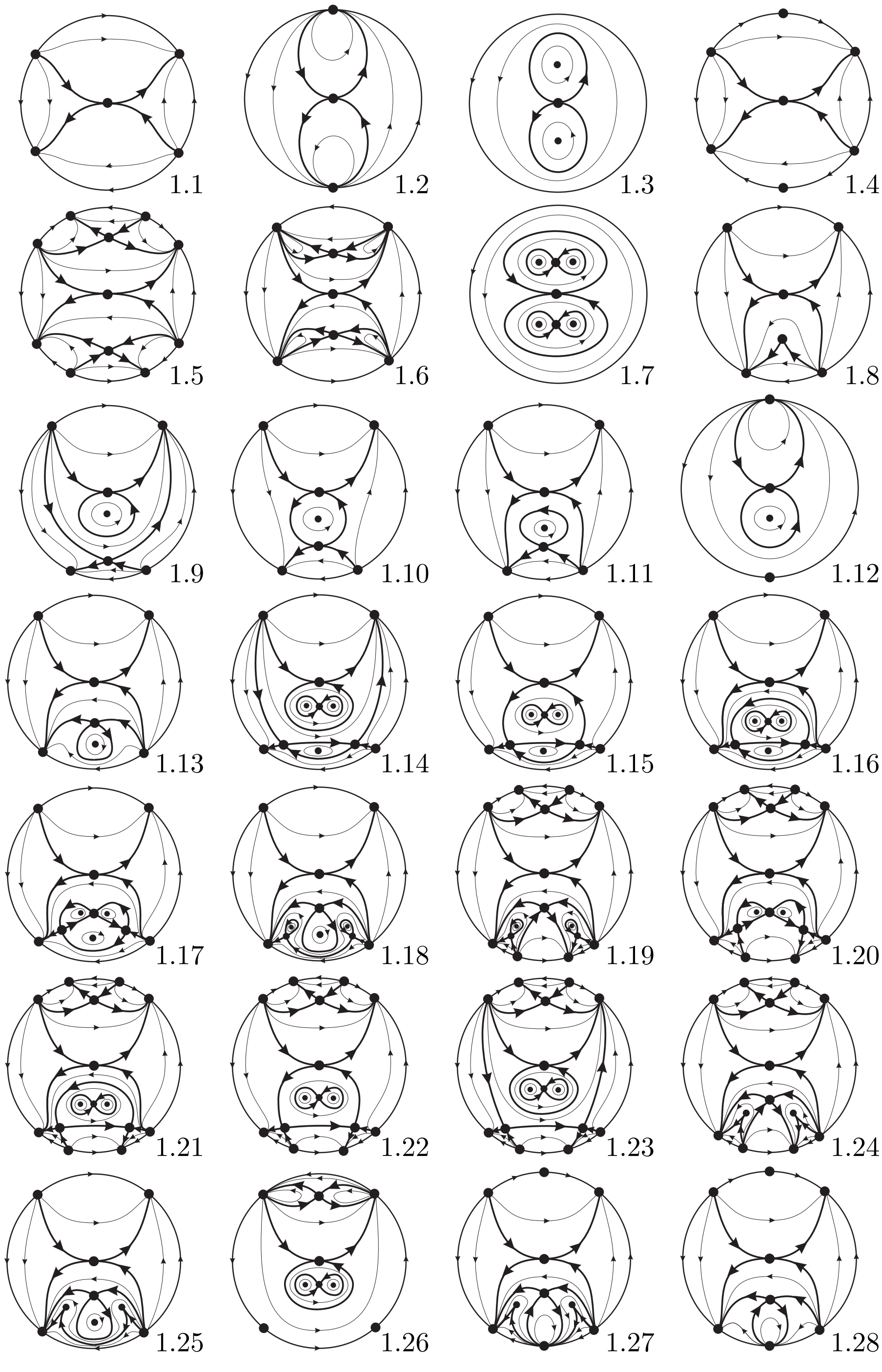
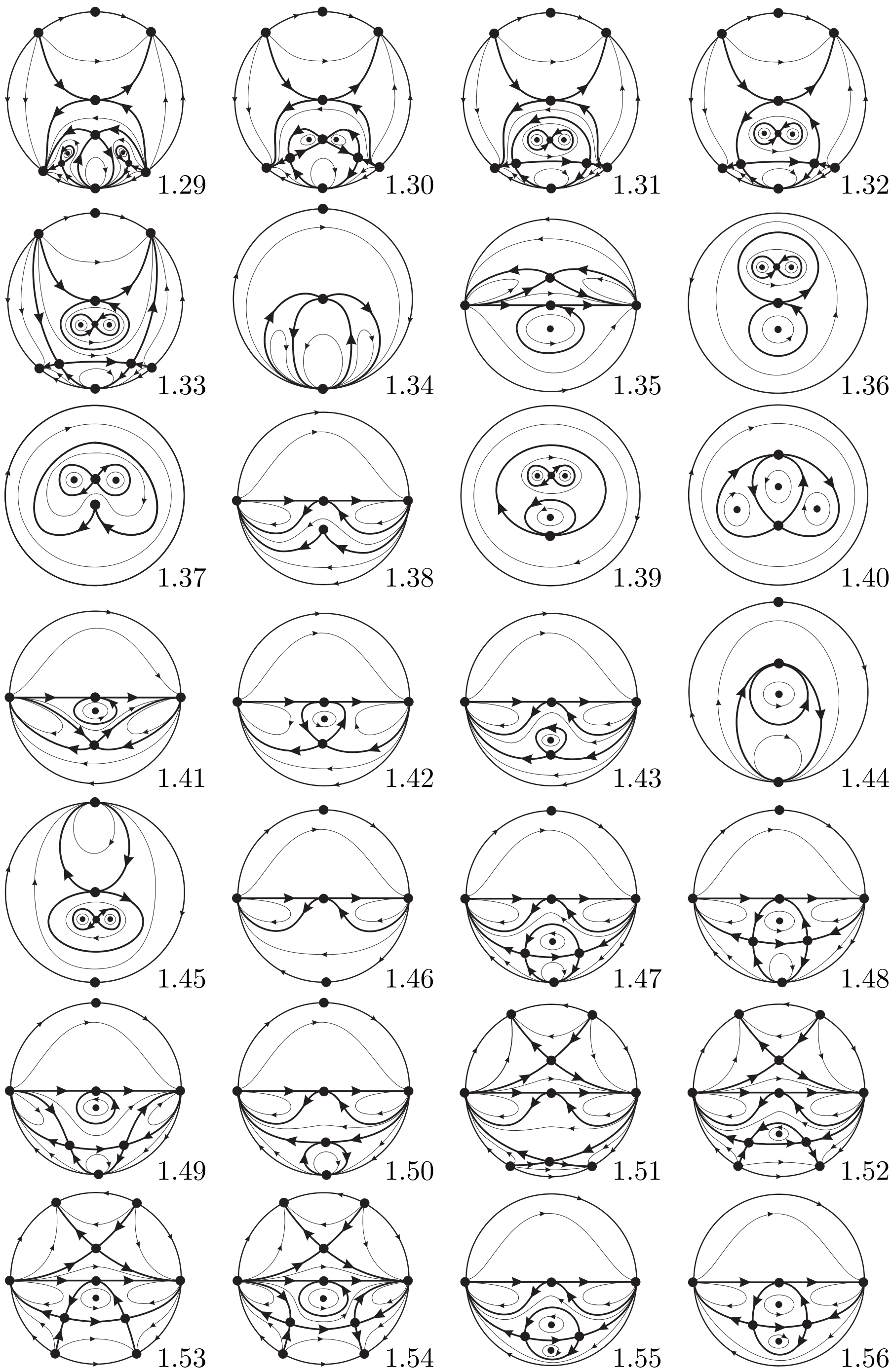


Figure 1. Phase portraits for the Hamiltonian planar polynomial vector fields of degree three with a nilpotent saddle at the origin and reversible with respect to the y-axis.



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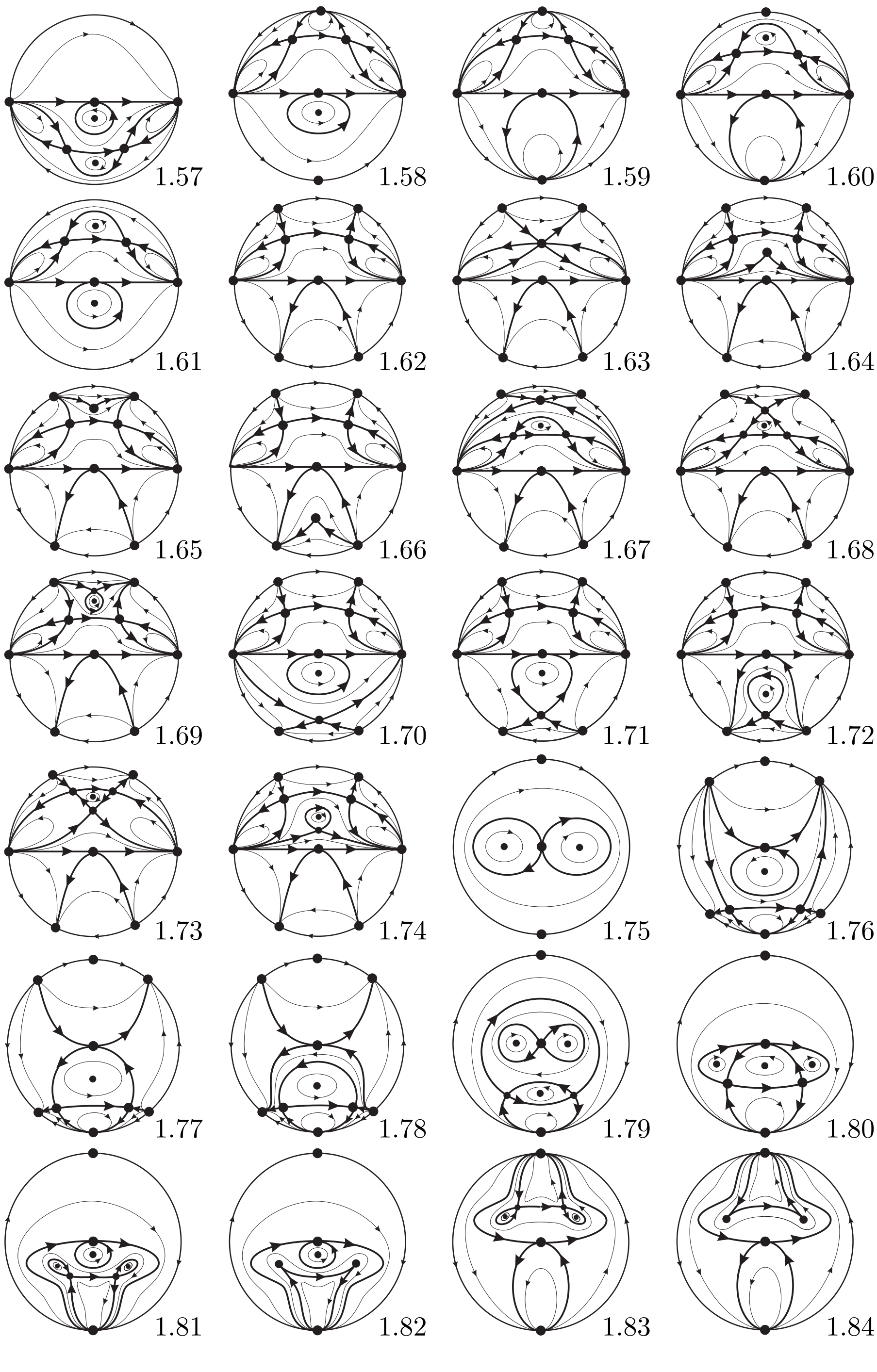

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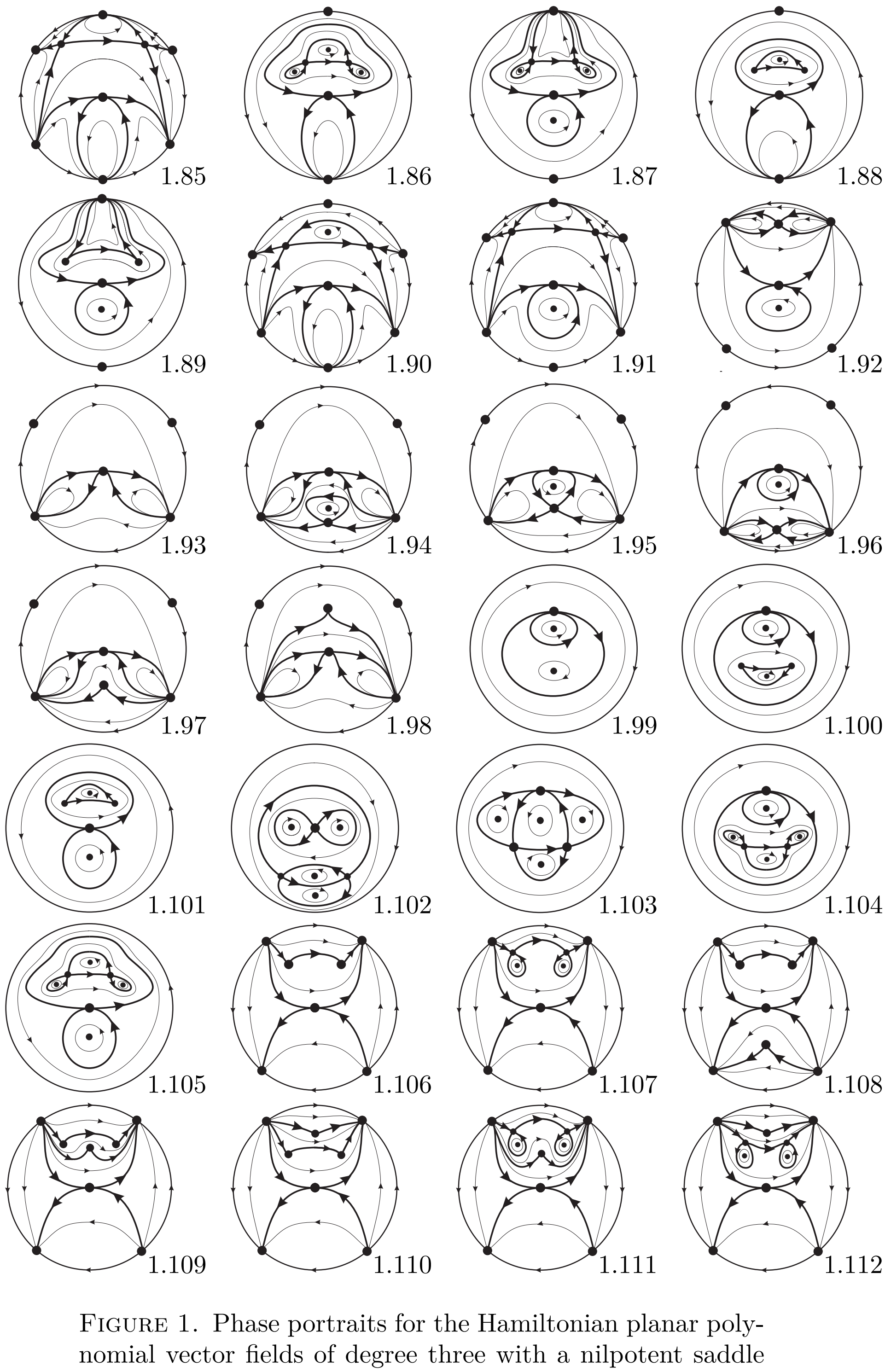

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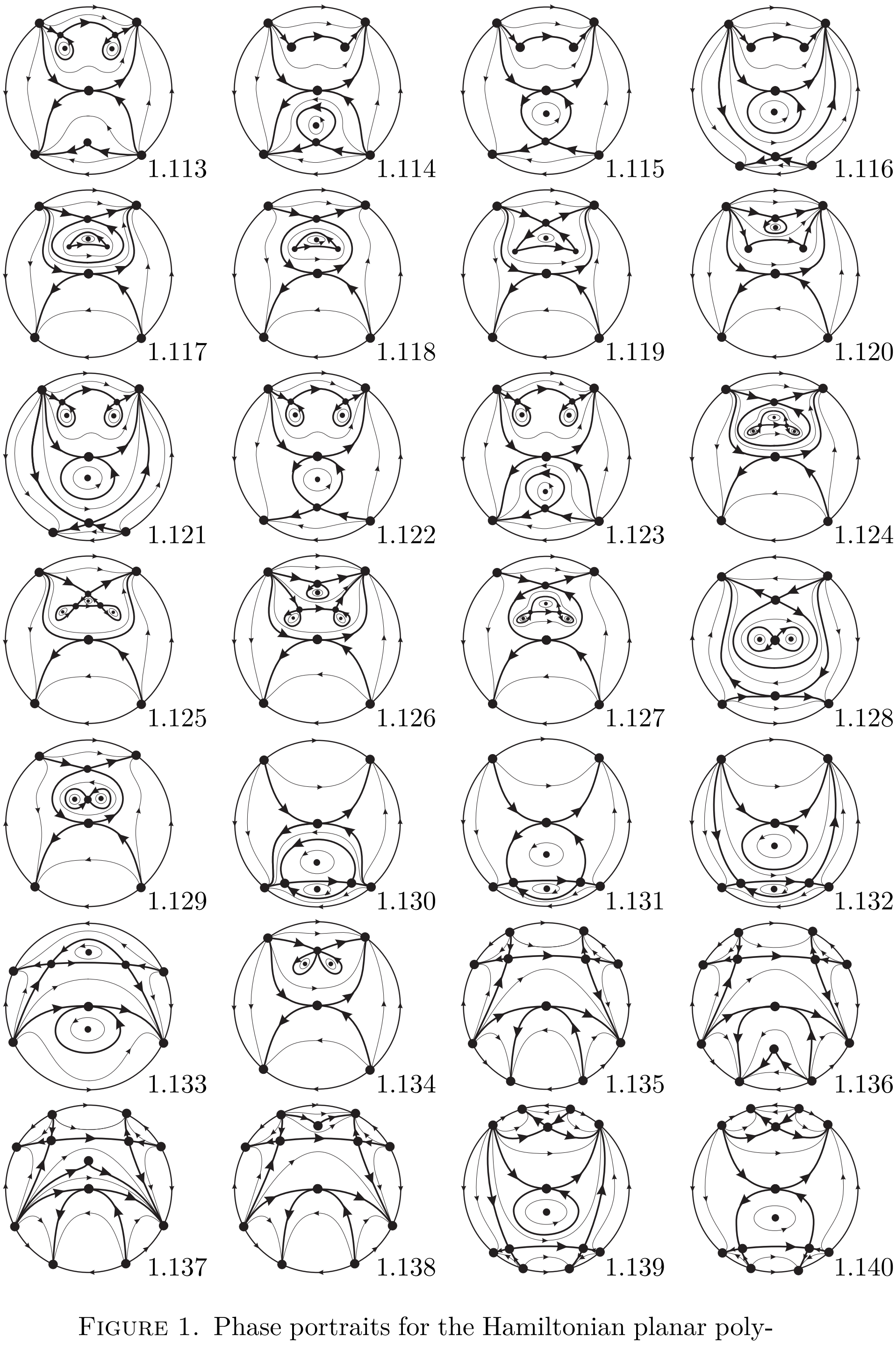

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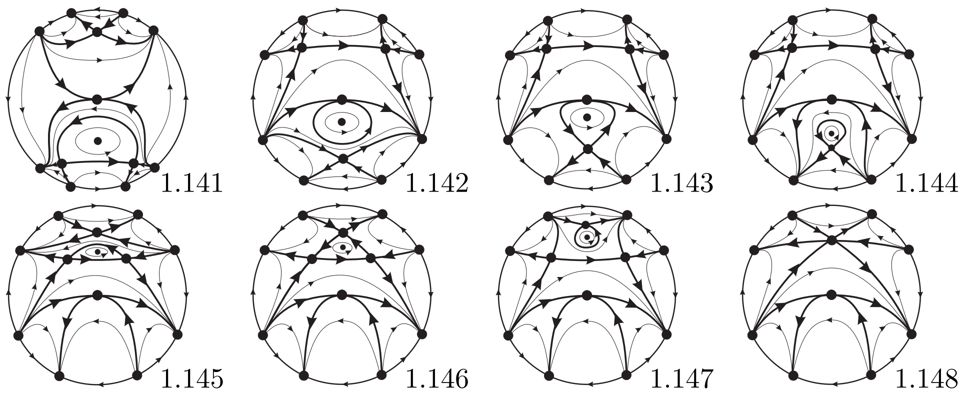

FIGURE 1. Phase portraits for the Hamiltonian planar polynomial vector fields of degree three with a nilpotent saddle at the origin and reversible with respect to the y-axis.

3. Proof of the normal form in Theorem 1

Without loss of generality we can assume that a cubic planar Hamiltonian system with a nilpotent point at the origin is given by

$$
\begin{aligned}
x' &= y + b_2x^2 + a_2x^3 + 2b_3xy + 2a_3x^2y + 3b_4y^2 + 3a_4xy^2 + 4a_5y^3,\\
y' &= -3b_1x^2 - 4a_1x^3 - 2b_2xy - 3a_2x^2y - b_3y^2 - 2a_3xy^2 - a_4y^3.
\end{aligned} \tag{4}
$$

which corresponds to equation

$$x' = \frac{\partial H}{\partial y}, \quad y' = -\frac{\partial H}{\partial x},$$

where

$$
\begin{aligned}
H(x,y) = {} & y^2/2 + a_1x^4 + a_2x^3y + a_3x^2y^2 + a_4xy^3 + a_5y^4 + b_1x^3 + b_2x^2y\\
& + b_3xy^2 + b_4y^3.
\end{aligned}
$$

By hypothesis, systems (4) must have the symmetry $(x, y, t) \mapsto (-x, y, -t)$. Imposing this condition using (3) we get to the contradiction $2y = 0$, so only condition (2) is possible. Imposing it, we must have that $a_2 = a_4 = b_1 = b_3 = 0$. Hence systems (4) become

$$
\begin{aligned}
x' &= y + b_2x^2 + 2a_3x^2y + 3b_4y^2 + 4a_5y^3,\\
y' &= -4a_1x^3 - 2b_2xy - 2a_3xy^2.
\end{aligned} \tag{5}
$$

Since systems (5) must have a saddle at the origin, by Theorem 3.5 of [4] we must have $a_1 < b_2^2/2$. Therefore we obtain (5) with $a_1 < b_2^2/2$. Now we provide the normal form to system (5) to transform it into other systems with less parameters.

Case 1. Assume $b_2 \neq 0$ and $a_3 \neq 0$. By the change of coordinates and reparametrization of time of the form

$$X \to \alpha x, \quad Y \to \beta y, \quad \tau \to \gamma t, \tag{6}$$

with $\gamma = b_2/\sqrt{|a_3|}$, $\alpha = \beta\gamma$ and $\beta = b_2\sqrt{2}/\gamma^2$ and we get system (VIII) with $s = 1$ when $a_3 > 0$ and system (VIII) with $s = -1$ when $a_3 < 0$.

Case 2. Assume $b_2 \neq 0$, $a_3 = 0$ and $b_2 b_4 \neq 0$. By the change of coordinates and reparametrization of time as in (6) with $\gamma = 2^{1/4}\sqrt{|b_2/b_4|}/\sqrt{3}$, $\beta = b_2\sqrt{2}/\gamma^2$ and $\alpha = \beta\gamma$ we get systems (VII) with $s = 1$ if $b_2 b_4 > 0$ and systems (VII) with $s = -1$ if $b_2 b_4 < 0$.

Case 3. Assume $b_2 \neq 0$, $a_3 = b_4 = 0$ and $a_5 \neq 0$. By the change of coordinates and reparametrization of time as in (6) with $\gamma = (b^2/(2|a_5|))^{1/4}$, $\beta = b_2\sqrt{2}/\gamma^2$ and $\alpha = \beta\gamma$ we get systems (VI) with $s = 1$ if $a_5 > 0$ and systems (VI) with $s = -1$ if $a_5 < 0$.

Case 4. Assume $b_2 \neq 0$, $a_3 = b_4 = a_5 = 0$. By the change of coordinates and reparametrization of time as in (6) with $\gamma = 1$, $\beta = b_2\sqrt{2}$ and $\alpha = \beta$ we get systems (V).

Case 5. Assume $b_2 = 0$, $b_4 \neq 0$ and $a_3 \neq 0$. The change of coordinates and reparametrization of time as in (6) with $\gamma = \sqrt{2|a_3|}/(3b_4)$, $\beta = 3b_4$ and $\alpha = \beta\gamma$, we get systems (IV) with $s = 1$ if $a_3 > 0$ and systems (IV) with $s = -1$ if $a_3 < 0$.

Case 6. Assume $b_2 = 0$, $b_4 \neq 0$ and $a_3 = 0$. The change of coordinates and reparametrization of time as in (6) with $\gamma = (-4a_1/(9b_4^2))^{1/4}$, $\beta = 3b_4$ and $\alpha = \beta\gamma$, we get systems (III).

Case 7. Assume $b_2 = 0$, $b_4 = 0$ and $a_3 \neq 0$. The change of coordinates and reparametrization of time as in (6) with $\beta = |a_3|/\sqrt{-a_1}$, $\gamma = \sqrt{2|a_3|}/\beta$ and $\alpha = \beta\gamma$, we get systems (II) with $s = 1$ if $a_3 > 0$ and systems (II) with $s = -1$ if $a_3 < 0$.

Case 8. Finally, if $b_2 = 0$, $b_4 = 0$ and $a_3 = 0$. The change of coordinates and reparametrization of time as in (6) with $\gamma = ((-4a_1)/\beta^2)^{1/4}$, $\beta = 1$ and $\alpha = \beta\gamma$, we get systems (I).

In short, we have proved the first part (or the normal form part) of Theorem 1.

4. Global phase portrait of system (I)

Consider system (I)

$$x' = y + by^3, \quad y' = x^3$$

with $b \in \mathbb{R}$.

In the local chart U_1 system (I)

$$u' = 1 - bu^4 - u^2v^2, \quad v' = -uv(bu^2 + v^2).$$

So, either there are no infinite singular points in the local chart U_1 (when $b \leq 0$) or there are two when $b > 0$, namely, $(\pm b^{-1/4}, 0)$. Computing the eigenvalues of the Jacobian matrix at these points we get that $(b^{-1/4}, 0)$ is an attracting node and $(-b^{-1/4}, 0)$ is a repelling node.

On the local chart U_2 system (I) becomes

$$u' = b - u^4 + v^2, \quad v' = -u^3v.$$

The origin is a singular point if and only if $b = 0$. If $b = 0$ it is linearly zero, and using blow-up techniques we get that it is the union of two elliptic (one stable and one unstable) and four parabolic (two stable and two unstable) sectors.

The finite singular points are on $x = 0$ and $1 + by^2 = 0$. So, if $b \geq 0$ there are no finite singular points among the origin and if $b < 0$ there are two more finite singular points which are $(0, \pm\sqrt{\frac{-1}{b}})$. They are both nilpotent. Using Theorem 3.5 in [4] and that the system is Hamiltonian we conclude that they are both centers.

Gluing all the information on the finite and infinite singular points we get that the global phase portraits are topologically equivalent to the following ones of Figure 1: 1.1 when $b > 0$, 1.2 when $b = 0$, and 1.3 when $b < 0$.

5. Global phase portrait of system (II)

Consider system (II)

$$x' = y + sx^2y + by^3, \quad y' = x^3 - sxy^2 = x(x^2 - sy^2)$$

with $b \in \mathbb{R}$ and $s \in \{-1; 1\}$.

We first study the infinite singular point. In the local chart U_1 systems (II) become

$$u' = 1 - u^2(2s + bu^2 + v^2), \quad v' = -uv(s + bu^2 + v^2).$$

The infinite singular points satisfy $v = 0$ and $1 - 2su^2 - bu^4 = 0$. If $b = 0$, there are two infinite singular points if $s = 1$ and zero if $s = -1$. On the other hand if $b \neq 0$, either there are zero infinite singular points (when $b < -1$ and $s = 1$, or $b < 0$ and $s = -1$), or two (when $b > 0$, or when $b = -1$ and $s = 1$), or four (when $b \in (-1, 0)$ and $s = 1$). When $s = -1$ and $b > 0$, or $s = 1$ and $b \geq 0$, the two infinite singular points are nodes (one stable and one unstable). When $s = 1$ and $b = -1$ the two infinite singular points are linearly zero. Applying the blow up technique we conclude that they are two elliptic (one stable and one unstable) and four parabolic (two stable and two unstable) sectors. Finally, when $s = 1$ and $b \in (-1, 0)$ the four infinite singular points are nodes (two stable and two unstable).

In the local chart U_2 systems (II) become

$$u' = b + 2su^2 - u^4 + v^2, \quad v' = vu(s - u^2).$$

The origin of the local chart U_2 is a singular point if and only if $b = 0$. In this case it is linearly zero. Applying blow up techniques (see for instance [1] for details on this technique) we get that the origin of the local chart U_2 is formed by two elliptic (one stable and one unstable) and four parabolic (two stable and two unstable) sectors if $s = -1$ and by two hyperbolic sectors (one stable and one unstable) if $s = 1$.

Now we study the finite singular points. When $b \geq 0$ there are no finite singular points among the origin. If $b < 0$ the finite singular points different from the origin and with $x = 0$ are of the form $(0, \pm\sqrt{-1/b})$ which are always real because $b < 0$. In this case computing the eigenvalues of the Jacobian matrix at these points and using that the system is Hamiltonian we conclude that they are both saddles if $s = 1$ and are both centers if $s = -1$. The finite singular points with $x \neq 0$ are of the form $x = \pm\sqrt{sy^2}$. So if $s = -1$ there are no finite singular points with $x \neq 0$ and if $s = 1$ and $b < -1$ there are four singular points which are of the form

$$\Big(\pm\frac{1}{\sqrt{-1-b}}, \frac{1}{\sqrt{-1-b}}\Big), \quad \Big(\pm\frac{1}{\sqrt{-1-b}}, -\frac{1}{\sqrt{-1-b}}\Big).$$

Computing the eigenvalues of the Jacobian matrix at these points and taking into account that the system is Hamiltonian we conclude that they are all centers.

The saddles $(0, \pm\sqrt{-1/b})$ (which exist when $b < 0$ and $s = 1$) are not connected with the saddle at the origin because the Hamiltonian evaluated on the saddles is never zero. On the other hand, they are connected among them due to the symmetry.

In short gluing all the information on the finite and infinite singular points we get that the unique possible global phase portraits for system (II) are topologically equivalent to the following ones of Figure 1: 1.1 when $b > 0$ (either $s = 1$ or $s = -1$); 1.2 when $b = 0$ and $s = -1$; 1.3 when $b < 0$ and $s = -1$; 1.4 when $b = 0$ and $s = 1$; 1.5 when $b \in (-1, 0)$ and $s = 1$; 1.6 when $b = -1$ and $s = 1$; and, finally, 1.7 when $b < -1$ and $s = 1$.

6. Global phase portrait of system (III)

Consider system (III)

$$x' = y + y^2 + by^3, \quad y' = x^3$$

with $b \in \mathbb{R}$.

We first study the infinite singular points. In the local chart U_1 system (III) is

$$u' = 1 - bu^4 - u^2v(u+v), \quad v' = -uv(bu^2 + v(u+v)).$$

The infinite singular points in the local chart U_1 must satisfy $v = 0$ and $1 - bu^4 = 0$. So, either there are no infinite singular points in the local chart U_1 (when $b \leq 0$) or there are two when $b > 0$, namely, $(\pm b^{-1/4}, 0)$. Computing the eigenvalues of the Jacobian matrix at these points we get that $(b^{-1/4}, 0)$ is an attracting node and $(-b^{-1/4}, 0)$ is a repelling node.

On the local chart U_2 system (III) becomes

$$u' = b - u^4 + v + v^2, \quad v' = -u^3v.$$

The origin is a singular point if and only if $b = 0$. If $b = 0$ it is nilpotent. Using Theorem 3.5 in [4] together with blow-up techniques we get that it is the union of one hyperbolic and one elliptic sectors.

Now we study the finite singular points. The finite singular points are on $x = 0$ and $y(1 + y + by^2) = 0$. If $b = 0$ among the origin, there is a unique finite singular point $(0, -1)$. Computing the eigenvalues of the Jacobian matrix at this point we get that it is nilpotent. Using Theorem 3.5 in [4] and that the system is Hamiltonian we conclude that it is a center.

If $b \neq 0$, the finite singular points, whenever they exist are $(0, y_\pm) = \left(0, \frac{-1\pm\sqrt{1-4b}}{2b}\right)$.

If $b > 0$, they exist whenever $b \in (0, 1/4]$. If $b = 1/4$, both points coalesce in the finite singular point $(0, -2)$ which is linearly zero. Using blow up techniques we get that it is the union of two hyperbolic sectors (cusp). If $b \in (0, 1/4)$ both singular points $(0, y_\pm)$ are nilpotent. Using Theorem 3.5 in [4] and that the system is Hamiltonian we get that $(0, y_+)$ is a center and $(0, y_-)$ is a saddle. We note that the possible saddle connection between the saddle $(0, y_-)$ and the origin is when $b = 2/9$.

Finally, if $b < 0$ the two finite singular points $(0, y_\pm)$ exist and are again nilpotent. Using Theorem 3.5 in [4] and that the system is Hamiltonian we get that they are both centers.

In short we have the following: If $b > 1/4$ the unique global phase portrait is topologically equivalent to 1.1 of Figure 1. If $b = 1/4$ the unique global phase portrait is topologically equivalent to 1.8 of Figure 1. If $b \in (0, 1/4)$ we get three possible different global phase portraits: 1.9 attained for example when $b = 11/90$, 1.10 which is the saddle connection attained when $b = 2/9$, and 1.11 attained for example when $b = 23/100$. If $b = 0$ the unique global phase portrait is topologically equivalent to 1.12 of Figure 1 and if $b < 0$ the unique global phase portrait is topologically equivalent to 1.3 of Figure 1.

7. Global phase portrait of system (IV)

Consider system (IV)

$$x' = y + y^2 + sx^2y + by^3, \quad y' = ax^3 - sxy^2 = x(ax^2 - y^2),$$

with $a > 0$, $b \in \mathbb{R}$ and $s \in \{-1; 1\}$.

We first study the infinite singular points. In the local chart U_1 system (IV) becomes

$$u' = a - u^2(2s + bu^2 + v(u + v)), \quad v' = -uv(s + bu^2 + v(u + v)).$$

When $v = 0$ the infinite singular points on the local chart U_1 satisfy $a-2su^2-bu^4 = 0$. If $b = 0$, there are two infinite singular points if $s = 1$ and zero if $s = -1$. In this case the infinite singular points are $(\pm\sqrt{a}/2, 0)$. Computing the eigenvalues of the Jacobian matrix at these points we get that $(\sqrt{a}/2, 0)$ is an attracting node and $(-\sqrt{a}/2, 0)$ is a repelling node. On the other hand if $b \neq 0$, either there are zero (when $b < 0$ and $s = -1$, or $b < -1/a$ and $s = 1$), or two (when $b > 0$, or $b = -1/a$ and $s = 1$), or four (when $b \in (-1/a, 0)$ and $s = 1$). In the case $b > 0$, the infinite singular points are $(\pm\sqrt{\frac{-s+\sqrt{ab+1}}{b}}, 0)$ and are a stable node and an unstable node, respectively. If $b = -1/a$ and $s = 1$, the infinite singular points are $(\pm\sqrt{a}, 0)$ and they are nilpotent. Using Theorem 3.5 in [4] and blow up techniques we see that they are both formed by one hyperbolic and one elliptic sector. In case $b \in (-1/a, 0)$ and $s = 1$ there are four infinite singular points $(u_\pm^{(1)}, 0) = (\pm\sqrt{(-1 + \sqrt{1 + ab})/b}, 0)$ and $(u_\pm^{(2)}, 0) = (\pm\sqrt{(-1 - \sqrt{1 + ab})/b}, 0)$. Computing the eigenvalues of the Jacobian matrix at these points we get that $(u_+^{(1)}, 0)$ and $(u_-^{(2)}, 0)$ are attracting nodes and $(u_-^{(1)}, 0)$ and $(u_+^{(2)}, 0)$ are repelling nodes.

On the local chart U_2 we get

$$u' = b - au^4 + 2su^2 + v + v^2, \quad v' = uv(s - au^2).$$

The origin $(0, 0)$ is an infinite singular point if and only if $b = 0$. It is nilpotent. Taking into account Theorem 3.5 in [4] and applying blow-ups we get that the origin of U_2 is formed by one elliptic (stable), one hyperbolic (unstable) and two parabolic (one stable and one unstable) sectors.

Now we study the finite singular points.

We consider separately the cases $s = 1$ and $s = -1$.

Subcase $s = 1$. We first study the finite singular points at $x = 0$. If $b = 0$ the unique singular point is $(0, -1)$. Computing the eigenvalues of the Jacobian matrix at this point we get that it is a saddle. If $b = 1/4$ then the singular point is $(0, -2)$. It is nilpotent and using Theorem 3.5 in [4] we get that it is a cusp. If $b > 1/4$ there are no finite singular points on $x = 0$. Finally, if $b \in (-\infty, 0) \cup (0, 1/4)$ there are two finite singular points $(0, y_\pm) = (0, \frac{-1\pm\sqrt{1-4b}}{2b})$. Computing the eigenvalues of the Jacobian matrix at these points we get that $(0, y_+)$ is a saddle and $(0, y_-)$ is a center if $b \in (0, 1/4)$ and a saddle if $b < 0$. The saddle $(0, y_+)$ cannot be connected with the origin. When $b < 0$, the saddle $(0, y_-)$ cannot be connected with the origin and both saddles $(0, y_+)$ and $(0, y_-)$ are also not connected.

If $x \neq 0$ the singular points are $(\pm\bar{y}/\sqrt{a}, \bar{y})$ with $\bar{y}$ satisfying $1 + \bar{y} + \frac{1+ab}{a}\bar{y}^2 = 0$. Hence, if $b = -1/a$ the two finite singular points $(\mp 1/\sqrt{a}, -1)$ are centers. If $b = \frac{a-4}{4a}$ the two singular points $(\pm\frac{2}{\sqrt{a}}, -2)$ are both nilpotent. Using Theorem 3.5 in [4] we get that they are both cusps. If $b > \frac{a-4}{4a}$ there are no finite singular points with $x \neq 0$ and if $b < \frac{a-4}{4a}$ there are the four finite singular points $(\pm\bar{y}_\pm/\sqrt{a}, \bar{y}_\pm)$ with

$$\bar{y}_\pm = \frac{-a \pm \sqrt{a^2(1-4b) - 4a}}{2(1+ab)}.$$

Computing the eigenvalues of the Jacobian matrix at these points we get: $(\pm\bar{y}_+/\sqrt{a}, \bar{y}_+)$ are centers and $(\pm\bar{y}_-/\sqrt{a}, \bar{y}_-)$ are centers for $b < -1/a$ and saddles for $b > -1/a$.

The saddles $(\pm\frac{\bar{y}_-}{\sqrt{a}}, \bar{y}_-)$ can be connected with the origin if and only if $b = \frac{2a-9}{9a}$ and they are connected one with each other by symmetry. The saddles $(\pm\frac{\bar{y}_-}{\sqrt{a}}, \bar{y}_-)$ are never connected with the saddle $(0, y_-)$ (whenever it exists) and may be connected with the saddle $(0, y_+)$ along the curve $b = b(a)$ which is a real root of the polynomial

(7)
$$\begin{aligned}&4 + 12a - 3a^2 - 36b - 102ab + 78a^2b - 9a^3b + 81b^2 + 180ab^2 - 534a^2b^2\\ &+ 162a^3b^2 - 9a^4b^2 + 162ab^3 + 1080a^2b^3 - 864a^3b^3 + 108a^4b^3 + 81a^2b^4\\ &+ 1440a^3b^4 - 432a^4b^4 + 576a^4b^5.\end{aligned}$$

Taking into account the information on the finite and infinite singular points together with the possible saddle connections and separating into different regions, we have the following possible global phase portraits of Figure 1 (whenever there is more than one possible phase portrait, in parenthesis we give values of the parameters where they are realized): 1.1 if $a > 0$ and $b > 1/4$; 1.8 if $a > 0$ and $b = 1/4$; 1.13 if $a \in (0, 4)$ and $b \in (0, 1/4)$ or $a \geq 4$ and $\frac{a-4}{4a} < b < \frac{1}{4}$; and 1.5 if $a \in (0, 4)$ and $\frac{a-4}{4a} < b < 0$.

If $a > 4$ and $0 < b < \frac{a-4}{4a}$ they are 1.14 ($a = 6$, $b = 1/25$); 1.15 ($a = 6$, $b = 1/18$. Here there is a saddle connection); 1.16 ($a = 6$, $b = 3/50$); 1.17 ($a = 6$, $b \approx 0.06681525$. Here $b = b(a)$ is a real solution of (7) and corresponds to a saddle connection); and 1.18 ($a = 6$, $b = 7/100$).

If $a \in (0,4)$ and $b \in (-\frac{1}{a}, \frac{a-4}{4a})$, they are 1.19 ($a=3$, $b=-1/10$); 1.20 ($a=3$, $b \approx -0.10277429$. Here $b=b(a)$ is a real solution of (7) and corresponds to a saddle connection); 1.21 ($a=3$, $b=-27/250$); 1.22 ($a=3$, $b=-1/9$. Here there is a saddle connection); and 1.23 ($a=3$, $b=-1/5$). If $a \geq 4$ and $b \in (-\frac{1}{a}, 0)$, they are 1.23.

If $a>0$ and $b<-\frac{1}{a}$, they are 1.7. If $a \in (0,4)$ and $b=\frac{a-4}{4a}$, they are 1.24. If $a>4$ and $b=\frac{a-4}{4a}$, they are 1.25. If $a>0$ and $b=-\frac{1}{a}$, they are 1.26. If $a=4$ and $b=0$ they are 1.27. If $a \in (0,4)$ and $b=0$ they are 1.28.

If $a>4$ and $b=0$, they are 1.29 ($a=21/5$); 1.30 ($a=\frac{2(3+2\sqrt{3})}{3}$. Here a is a real solution of (7) with $b=0$ and corresponds to a saddle connection); 1.31 ($a=22/5$); 1.32 ($a=9/2$. Here there is a saddle connection); and 1.33 ($a=5$).

Subcase $s=-1$. The unique finite singular points among the origin are located on $x=0$. If $b=0$ there is the singular point $(0,-1)$ which is a center. If $b=1/4$ then the singular point is $(0,2)$ which is nilpotent. Using Theorem 3.5 in [4] we get that it is a cusp. If $b>1/4$ there are no finite singular points on $x=0$. If $b<1/4$ then there are two finite singular points $(0,y_\pm)=(0,\frac{-1\pm\sqrt{1-4b}}{2b})$. Computing the eigenvalues of the Jacobian matrix at these points we have that $(0,y_+)$ is a center and $(0,y_-)$ is a saddle if $b>0$ and a center if $b<0$. The saddle $(0,y_-)$ can be connected with the origin if and only if $b=2/9$.

In short, again writing in parenthesis values of parameters where more than one possible phase portrait is realized in a region, we have the following global phase portraits of Figure 1: 1.3 if $b<0$; 1.12 if $b=0$; 1.1 if $b>1/4$; 1.8 if $b=1/4$; and if $b \in (0,1/4)$ they are: 1.9 ($a=1$, $b=15/72$), 1.10 ($a=1$, $b=2/9$. Here we have the saddle connection), and 1.11 ($a=1$, $b=23/100$).

8. Global phase portrait of system (V)

Consider system (V)

$$x' = y + \frac{x^2}{\sqrt{2}}, \quad y' = ax^3 - \sqrt{2}xy = x(ax^2 - \sqrt{2}y)$$

with $a>-1$ and $a \neq 0$.

In the local chart U_1 system (V) becomes

$$u' = a - \frac{1}{2}uv(3\sqrt{2}+2uv), \quad v' = -\frac{1}{2}v^2(\sqrt{2}+2uv).$$

Since $a \neq 0$ there are no infinite singular points.

In U_2 system (V) becomes

$$u' = -au^4 + \frac{3u^2v}{\sqrt{2}} + v^2, \quad v' = v(-au^3 + \sqrt{2}uv).$$

The origin is a singular point which is linearly zero. Using blow-up techniques we get that: it is the union of two elliptic (one stable and one unstable) and four parabolic (two stable and two unstable) sectors if $a > 0$; and it is the union of three elliptic (two stable and one unstable), one hyperbolic (unstable) and two parabolic (one stable and one unstable) sectors, if $a < 0$.

The unique finite singular point is the origin. Taking into account the information on the infinite singular points we get that the global phase portraits of system (V) are topologically equivalent to 1.2 of Figure 1 if $a > 0$ and to 1.34 of Figure 1 if $a < 0$.

9. Global phase portrait of system (VI)

Consider system (VI)

$$x' = y + \frac{x^2}{\sqrt{2}} + sy^3, \quad y' = ax^3 - \sqrt{2}xy = x(ax^2 - \sqrt{2}y),$$

with $a > -1$ and $s \in \{-1; 1\}$.

In the local chart U_1 system (VI) is

$$u' = a - \frac{1}{2}u(2su^3 + v(3\sqrt{2} + 2uv)), \quad v' = -\frac{v}{2}(2su^3 + v(\sqrt{2} + 2uv)).$$

The infinite singular points in the local chart U_1 must satisfy $v = 0$ and $a - su^4 = 0$. So, either there are no infinite singular points in the local chart U_1 (when $as < 0$), or there is one (when $a = 0$), or there are two when $as > 0$, namely, $(\pm(a/s)^{-1/4}, 0)$. In the case $a = 0$, the unique singular point is the origin which is linearly zero. Using blow-up techniques we get that it is the union of two elliptic (one stable and one unstable) and four parabolic (two stable and two unstable) sectors. When $as > 0$, computing the eigenvalues of the Jacobian matrix at $(\pm(a/s)^{-1/4}, 0)$ we get that $((a/s)^{-1/4}, 0)$ is an attracting node if $s = 1$ and a repelling node if $s = 1$, and $(-(a/s)^{-1/4}, 0)$ is a repelling node if $s = 1$ and an attracting node if $s = -1$.

We now show that the origin of the local chart U_2 is not a singular point. Indeed, on the local chart U_2 system (VI) becomes

$$u' = s + v^2 - au^4 + \frac{3}{\sqrt{2}}u^2v, \quad v' = uv(\sqrt{2}v - au^2),$$

and the origin is not a singular point.

Now we compute the finite singular points. If $x = 0$ the unique singular point is the origin if $s = 1$. On the other hand, if $s = -1$ then besides the origin on $x = 0$ we have the singular points $(0, 1)$ and $(0, -1)$ which are a saddle and a center, respectively. Moreover, the saddle $(0,1)$ is never connected with the saddle at origin.

If $a=0$ or $as>0$ there are no finite singular points with $x\neq 0$. If $as<0$ we get the singular points

$$\Big(\pm 2^{1/4}\Big(\frac{1+a}{a^3}\Big)^{1/4}, \sqrt{\frac{1+a}{a}}\Big) \quad \text{if } a>0,$$

and

$$\Big(\pm 2^{1/4}\Big(-\frac{1+a}{a^3}\Big)^{1/4}, -\sqrt{-\frac{1+a}{a}}\Big) \quad \text{if } a\in(-1,0),$$

which are two centers.

In short, gluing all the information on the finite and infinite singular points we get the following global phase portraits of Figure 1: 1.2 if $a=0$, $s=1$; 1.1 if $a>0, s=1$; 1.3 if $a\in(-1,0)$, $s=1$; 1.35 if $a=0$, $s=-1$; 1.36 if $a>0$, $s=-1$; and 1.13 if $a\in(-1,0)$, $s=-1$.

10. Global phase portrait of system (VII)

Consider system (VII)

$$x'=y+\frac{x^2}{\sqrt{2}}+sy^2+by^3, \quad y'=ax^3-\sqrt{2}xy=x(ax^2-\sqrt{2}y),$$

with $a>-1$, $b\in\mathbb{R}$, $s\in\{-1;1\}$ and $a^2+b^2\neq 0$.

We first study the infinite singular points. In the local chart U_1 if $(a,b)\neq(0,0)$, system (VII) becomes

$$u'=a-\frac{1}{2}u(2bu^3+v(3\sqrt{2}+2u(su+v))), \quad v'=-\frac{1}{2}v(2bu^3+v(\sqrt{2}+2u(su+v))).$$

The infinite singular points in the local chart U_1 must satisfy $v=0$ and $a-bu^4=0$. So, either there are no infinite singular points in the local chart U_1 (when either $b=0$ and $a\neq 0$, or $ab<0$), or there is one (when $a=0$ and $b\neq 0$) which is the origin, or there are two when $ab>0$, namely, $(\pm(a/b)^{-1/4},0)$. When $a=0$ and $b\neq 0$, the origin is linearly zero. Applying blow up techniques we obtain that it is the union of two elliptic (one stable and one unstable) and four parabolic sectors (two stable and two unstable). When $ab>0$, computing the eigenvalues of the Jacobian matrix at the points $(\pm(a/b)^{-1/4},0)$ we get that $((a/b)^{-1/4},0)$ is an attracting node if $b>0$ and a repelling node if $b<0$, and $(-(a/b)^{-1/4},0)$ is a repelling node if $b>0$ and an attracting node if $b<0$.

In the local chart U_2 for $(a,b)\neq(0,0)$ we get

$$u'=b-au^4+sv+\frac{3u^2v}{\sqrt{2}}+v^2, \quad v'=v(-au^3+\sqrt{2}uv)$$

The origin $(0,0)$ is not an infinite singular point if $b \neq 0$. If $b = 0$ it is nilpotent and by Theorem 3.5 in [4] together with blow-up techniques we get that it is the union of one hyperbolic, one elliptic and two parabolic sectors.

Now we study the finite singular points. We consider two different subcases.

Subcase $s = 1$. If $b = 0$ the singular point is $(0,-1)$. Computing the eigenvalues of the Jacobian matrix at this point we get that it is a center. If $b = 1/4$ the singular point is $(0,-2)$. It is nilpotent and using Theorem 3.5 in [4] we get that it is a cusp. If $b > 1/4$ there are no finite singular points on $x = 0$. Finally, if $b < 1/4$ then there are the two finite singular points $(0, y_\pm) = (0, \frac{-1\pm\sqrt{1-4b}}{2b})$. Computing the eigenvalues of the Jacobian matrix at these points we get that $(0, y_+)$ is a center and $(0, y_-)$ is a saddle. The saddle $(0, y_-)$ and the origin can be connected if and only if $b = 2/9$.

Now we study the singular points with $x \neq 0$. If $a = 0$ there are no finite singular points on $x \neq 0$. If $a \neq 0$ the singular points on $x \neq 0$, whenever they exist, are of the form

$$(\pm\bar{x}_\pm, \bar{y}_\pm), \quad \bar{x}_\pm = \sqrt{\frac{\sqrt{2}y_\pm}{a}}, \quad \bar{y}_\pm = \frac{-a \pm \sqrt{a^2 - 4ab(a+1)}}{2ab}.$$

Studying when these finite singular points are defined and in these cases computing the eigenvalues of the Jacobian matrix at these points we get: if $a \in (-1,0)$ and $b > 0$ we have the two centers $(\pm\bar{x}_+, \bar{y}_+)$ and when $a > 0$ and $b < 0$ we have the two centers $(\pm\bar{x}_-, \bar{y}_-)$. For any other value of the parameters $(a,b) \in (-1,\infty)\times\mathbb{R}$ with $a^2+b^2 \neq 0$, there are no finite singular points on $x \neq 0$.

Taking into account the information on the finite and infinite singular points together with the possible saddle connections and separating into different regions, we have the following possible global phase portraits of Figure 1 (whenever there is more than one possible phase portrait, in parenthesis we give values of the parameters where they are realized): 1.3 if $b > 1/4$ and $a \in (-1,0)$; 1.2 if $b > 1/4$ and $a = 0$; 1.1 if $b > 1/4$ and $a > 0$; 1.37 if $b = 1/4$ and $a \in (-1,0)$; 1.38 if $b = 1/4$ and $a = 0$; and 1.8 if $b = 1/4$ and $a > 0$.

If $b \in (0,1/4)$ and $a \in (-1,0)$ they are 1.39 ($a = -1/2$, $b = 1/10$) and 1.40 ($a = -1/2$, $b = 2/9$. Here we have a saddle connection). If $b \in (0,1/4)$ and $a = 0$ they are 1.41 ($b = 1/10$); 1.42 ($b = 2/9$. Here we have a saddle connection); and 1.43 ($b = 23/100$). If $b \in (0,1/4)$ and $a > 0$ they are 1.9 ($a = 1/2$, $b = 1/10$); 1.10 ($a = 1/2$, $b = 2/9$. Here we have a saddle connection); and 1.11 ($a = 1/2$, $b = 23/100$).

If $b = 0$ and $a \in (-1, 0)$ they are 1.44. If $b = 0$ and $a > 0$ they are 1.12. If $b < 0$ and $a \in (-1, 0)$ they are 1.13. If $b < 0$ and $a = 0$ they are 1.35 and if $b < 0$ and $a > 0$ they are 1.36.

Subcase $s = -1$. First we study the finite singular points at $x = 0$. If $b = 0$ the singular point is $(0, 1)$. Computing the eigenvalues of the Jacobian matrix at this point we get that it is a saddle. If $b = 1/4$ then the singular point is $(0, 2)$ which is nilpotent. Using Theorem 3.5 in [4] we get that it is a cusp. If $b > 1/4$ there are no finite singular points on $x = 0$. Finally, if $b < 1/4$ and $b \neq 0$ then there are the two finite singular points $(0, y_\pm) = (0, \frac{1\pm\sqrt{1-4b}}{2b})$. Computing the eigenvalues of the Jacobian matrix at these points we get that $(0, y_+)$ is a center and $(0, y_-)$ is a saddle.

Now we study the singular points with $x \neq 0$. If $a = 0$ there are no finite singular points on $x \neq 0$. If $a \neq 0$ and $b = 0$, there are the two singular points

$$\left(\pm\frac{\sqrt{2\sqrt{1+a}}}{a}, 1 + \frac{1}{a}\right),$$

where are centers. If $a \neq 0$ and $b \neq 0$ the singular points on $x \neq 0$, whenever they exist, are of the form

$$(\pm\bar{x}_\pm, \bar{y}_\pm), \quad \bar{x}_\pm = \sqrt{\frac{\sqrt{2}y_\pm}{a}}, \quad \bar{y}_\pm = \frac{a \pm \sqrt{a^2 - 4ab(a+1)}}{2ab}.$$

Studying when these finite singular points are defined and in these cases computing the eigenvalues of the Jacobian matrix at these points (using Theorem 3.5 in [4] whenever needed), we get: if $a \in (-1, 0)$, $b \in (a/(4(1 + a)), 0)$ we have that $(\pm\bar{x}_+, \bar{y}_+)$ are centers and $(\pm\bar{x}_-, \bar{y}_-)$ are saddles; if $a \in (-1, 0)$ and $b > 0$ we have the two centers $(\pm\bar{x}_+, \bar{y}_+)$; if $a > 0$ and $b < 0$ we have the two centers $(\pm\bar{x}_-, \bar{y}_-)$; if if $a > 0$ and $b \in (0, a/(4(1 + a))$ we have the two saddles $(\pm\bar{x}_+, \bar{y}_+)$ and the two centers $(\pm\bar{x}_-, \bar{y}_-)$; if $a > 0$ and $b = a/(4(1 + a))$ we have the two cusps $(\pm\bar{x}_+, \bar{y}_+)$; and for any other value of the parameters $(a, b) \in (-1, \infty) \times \mathbb{R}$ with $a^2 + b^2 \neq 0$, there are no finite singular points on $x \neq 0$.

Now we study the possible saddle connections. The saddle $(0, y_-)$ and the saddle at the origin are never connected. The saddles $(\pm\bar{x}_+, \bar{y}_+)$ (whenever they exist) are connected one with each other by symmetry and the saddles $(\pm\bar{x}_-, \bar{y}_-)$ (whenever they exist) are also connected one with each other by symmetry. Moreover, the possible saddle connection between the saddles $(\pm\bar{x}_+, \bar{y}_+)$ and the origin, or between the saddles $(\pm\bar{x}_-, \bar{y}_-)$ and the origin is given on the curve $b = 2a/(9(1 + a))$. On the other hand, the possible saddle connection with the saddles $(\pm\bar{x}_+, \bar{y}_+)$ and the saddle $(0, y_-)$, or with the saddles $(\pm\bar{x}_-, \bar{y}_-)$ and the saddle $(0, y_-)$ is given on the curve $b = b(a)$

which is a real root of the polynomial

$$
\begin{aligned}
&-12(3a^2+3a+1)a^4 - 2(864a^4+1728a^3+1212a^2+348a+25)a^2b^2 \\
&+36(2a+1)^3(8a^2+8a-1)ab^3 + 6(2a+1)(36a^2+36a+11)a^3b \\
&+81(2a+1)^4b^4.
\end{aligned}
\tag{8}
$$

Taking into account the information on the finite and infinite singular points together with the possible saddle connections and separating into different regions, we have the following possible global phase portraits of Figure 1 (whenever there is more than one possible phase portrait, in parenthesis we give values of the parameters where they are realized): 1.3 if $b > 1/4$ and $a \in (-1, 0)$; 1.2 if $b > 1/4$ and $a = 0$; 1.1 if $b > 1/4$ and $a > 0$; 1.36 if $b \in (0, 1/4)$ and $a \in (-1, 0)$, or $b < 0$ and $a > 0$; 1.35 if $b \in (0, 1/4)$ and $a = 0$, or $b < 0$ and $a = 0$; 1.13 if $b \in (a/(4(1+a)), 1/4)$ and $a > 0$, or $b < a/(4(1+a))$ and $a \in (-1, 0)$; 1.25 if $b = a/(4(1+a))$ and $a \neq 0$; and 1.45 if $b = 0$ and $a \neq 0$.

If $b \in (0, a/(4(1+a)))$ and $a > 0$, or $b \in (a/(4(1+a)), 0)$ and $a \in (-1, 0)$ they are 1.14 ($a = 2$, $b = 7/50$); 1.15 ($a = 2$, $b = 2a/(9(1+a)) = 4/27$. Here there is a saddle connection); 1.16 ($a = 2$, $b = 3/20$); 1.17 ($a = 2$, $b \approx 0.15045562$. Here $b = b(a)$ is a real solution of equation (8) and so there is a saddle connection); and 1.18 ($a = 2$, $b = 4/25$).

Finally, if $b = 1/4$ and $a \in (-1, 0)$ they are 1.37; if $b = 1/4$ and $a = 0$ they are 1.38; and if $b = 1/4$ and $a > 0$ they are 1.8.

11. Global phase portrait of system (VIII): infinite singular points

Consider system (VIII)

$$
x' = y + \frac{x^2}{\sqrt{2}} + sx^2y + by^2 + cy^3, \quad y' = ax^3 - \sqrt{2}xy - sxy^2
$$

with $a > -1$, $b, c \in \mathbb{R}$ and $s \in \{-1; 1\}$.

We study in this section the infinite singular points.

Lemma 4. *On the local chart U_1 we have:*

(a) *no singular points if $cs > 0$ and $as < 0$, or $c = 0$ and $as < 0$, or $ac < -1$ and $as > 0$;*
(b) *one singular point (the origin) if $a = 0$ and $cs \geq 0$. It is the union of two elliptic (one stable and one unstable) and four parabolic sectors (two stable and two unstable);*
(c) *two singular points which are an attracting node and a repelling node if $c > 0$ and $a > 0$, or $c < 0$ and $a < 0$, or $c = 0$ and $as > 0$;*

(d) *two singular points which are the union of one elliptic and one hyperbolic sector if* $cs < 0$, $as > 0$ *with* $c = -1/a$ *and* $b \neq -3s/(\sqrt{2}a)$, *or the union of two elliptic sectors (one stable and one unstable) and four parabolic sectors (two stable and two unstable) if* $cs < 0$, $as > 0$ *with* $c = -1/a$ *and* $b = -3s/(\sqrt{2}a)$;
(e) *three singular points which are an attracting node, a repelling node and the origin which is formed by two elliptic (one stable and one unstable) and four parabolic (two stable and two unstable) sectors if* $cs < 0$ *and* $a = 0$;
(f) *four singular points which are two repelling and two attracting nodes if* $ac + 1 > 0$, $cs < 0$ *and* $as > 0$.

The origin of the local chart U_2 *is an infinite singular point if and only if* $c = 0$. *If* $b \neq 0$ *it is the union of one hyperbolic and one elliptic sectors. If* $b = 0$ *it is the union of two hyperbolic sectors (one stable and one unstable) when* $s = 1$ *and the union of two elliptic (one stable and one unstable) and four parabolic sectors (two stable and two unstable) when* $s = -1$.

Proof. On the local chart U_1 system (VIII) becomes

$$\begin{aligned} u' &= a - \frac{1}{2}u(2cu^3 + 3\sqrt{2}v + 2bu^2v + 4su + 2uv^2), \\ v' &= -\frac{1}{2}v(2cu^3 + \sqrt{2}v + 2bu^2v + 2su + 2uv^2). \end{aligned}$$

When $v = 0$ the infinite singular points on the local chart U_1 satisfy $a - 2su^2 - cu^4 = 0$.

If $cs > 0$ and $as < 0$, or $c = 0$ and $as < 0$, or $ac < -1$ and $as > 0$ there are no infinite singular points in the local chart U_1. This proves statement (a).

If $a = 0$ and $cs \geq 0$ the unique finite singular point is the origin, which is linearly zero. Applying blow-up techniques we get that it is the union of two elliptic (one stable and one unstable) and four parabolic sectors (two stable and two unstable). This proves statement (b).

If $c > 0$ and $a > 0$ then there are two infinite singular points on the local chart U_1 which are $(u_\pm^{(2)}, 0) = (\pm\sqrt{\frac{-s}{c} + \frac{\sqrt{ac+1}}{c}}, 0)$. Computing the eigenvalues of the Jacobian matrix at these points we get that $(u_+^{(2)}, 0)$ is an attracting node and $(u_-^{(2)}, 0)$ is a repelling node.

If $c < 0$ and $a < 0$ then there exits two infinite singular points in the local chart U_1 which are $(u_\pm^{(1)}, 0) = (\pm\sqrt{\frac{-s}{c} - \frac{\sqrt{ac+1}}{c}}, 0)$. Computing the eigenvalues of the Jacobian matrix at these points we get that $(u_+^{(1)}, 0)$ is a repelling node and $(u_-^{(1)}, 0)$ is an attracting node.

If $c = 0$ and $as > 0$ there are two infinite singular points on the local chart U_1 that are $(\pm u^{(0)}, 0) = \Big(\pm \sqrt{\frac{a}{2s}}, 0\Big)$. Computing the eigenvalues of the Jacobian matrix at these points we get that $(u^{(0)}, 0)$ is an attracting node and $(-u^{(0)}, 0)$ is a repelling node when $s = 1$, while $(u^{(0)}, 0)$ is a repelling node and $(-u^{(0)}, 0)$ is an attracting node when $s = -1$. Statement (c) is proved.

If $cs < 0$, $as > 0$ with $c = -1/a$ then there are two infinite singular points in the local chart U_1, which are $(\pm\sqrt{as}, 0)$. They are nilpotent if $b \neq -3s/(\sqrt{2}a)$ and are linearly zero if $b = -3s/(\sqrt{2}a)$. In the first case, applying Theorem 3.5 in [4] together with blow-up techniques we obtain that both are the union of one elliptic and one hyperbolic sector. In the second case, applying again blow-up techniques we conclude that they are the union of two elliptic sectors (one stable and one unstable) and four parabolic sectors (two stable and two unstable). This proves statement (d).

If $cs < 0$ and $a = 0$ then there are three infinite singular points which are the origin $(0, 0)$ and the points $(\pm u^{(1)}, 0)$ if $s = 1$, or $(\pm u^{(2)}, 0)$ if $s = -1$. The origin of the local chart U_1 it is linearly zero. Computing the eigenvalues of the Jacobian matrix at this point we get that it is formed by two elliptic (one stable and one unstable) and four parabolic (two stable and two unstable) sectors. On the other hand, computing the eigenvalues of the Jacobian matrix at the points $(\pm u^{(1)}, 0)$ we get that: $(u^{(1)}, 0)$ is a repelling node and $(-u^{(1)}, 0)$ is an attracting node when $s = 1$, while $(u^{(1)}, 0)$ is an attracting node and $(-u^{(1)}, 0)$ is a repelling node when $s = -1$. Statement (e) is proved.

Finally, when $ac + 1 > 0$, $cs < 0$ and $as > 0$ there are the four singular points $(u^{(1)}_{\pm}, 0)$ and $(u^{(2)}_{\pm}, 0)$. Computing the eigenvalues of the Jacobian matrix at these points we get that $(u^{(1)}_{-}, 0)$ and $(u^{(2)}_{+}, 0)$ are attracting nodes while $(u^{(1)}_{+}, 0)$ and $(u^{(2)}_{-}, 0)$ are repelling nodes. This concludes the proof of the lemma concerning the local chart U_1.

On the local chart U_2 system (VIII) becomes

$$u' = c + bv + 2su^2 + v^2 + \frac{3}{\sqrt{2}}u^2v - au^4, \quad v' = -uv(-s - \sqrt{2}v + au^2).$$

The origin $(0, 0)$ is an infinite singular point if and only if $c = 0$. If $b \neq 0$ it is nilpotent and by Theorem 3.5 in [4] together with blow-up techniques we get that it is the union of one hyperbolic and one elliptic sectors. If $b = 0$ it is linearly zero, using blow-up techniques we get that it is the union of two hyperbolic sectors (one stable and one unstable) when $s = 1$ and the union of two elliptic (one stable and one unstable) and four parabolic sectors (two stable and two unstable) when $s = -1$. The proof of the lemma is complete. □

12. Global phase portraits of system (VIII) when $a = 0$

We compute the finite singular points. On $x = 0$ the finite singular points are $(0, -1/b)$ if $c = 0$ (with $b \neq 0$), and if $c \neq 0$ they are

$$p^{\pm} = (0, y_{\pm}) = \left(0, \frac{-b \pm \sqrt{b^2 - 4c}}{2c}\right)$$

which exist for $c < c_2 = c_2(b) := b^2/4$. On the other hand, if $x \neq 0$ the finite singular points are

$$q^{\pm} = \left(\pm\,\bar{x}, -\frac{\sqrt{2}}{s}\right) = \left(\pm\,\sqrt{2}\sqrt{\frac{s(\sqrt{2}b - s) - 2c}{s}}, -\frac{\sqrt{2}}{s}\right)$$

which exist for $c < c_1 = c_1(b) := (s(\sqrt{2}b - s))/2$. For $c = c_1(b)$ the singular points $q^{\pm}$ become $p^{\pm}$. The type of these finite singular points will be studied later.

Now we investigate the possible saddle connections. The possible saddle connections between the saddles $p^{\pm}$ (whenever they exist) and the origin occurs when

$$c = c(b) = 2b^2/9. \tag{9}$$

Due to the symmetry of the problem the saddles $q^{\pm}$ are always connected. We study the possible saddle connections between the saddles $q^{\pm}$ and the origin. Doing so, we get the curve

$$c = c(b) = \frac{(2\sqrt{2}b - 3s)s}{3}. \tag{10}$$

Furthermore, the possible saddle connection between the saddles $p^{\pm}$ and the saddles $q^{\pm}$ (whenever they exist) is along the curve $c = c(b)$ which is a real solution of the cubic

$$36c^3 - 12s\sqrt{2}bc^2 + 36c^2 - 10b^2c + 18s\sqrt{2}bc + 9c - 4s\sqrt{2}b^3 - 2b^2 = 0. \tag{11}$$

Moreover, when $c = 0$ the point $(0, -1/b)$ (whenever it is a saddle) cannot be connected with the saddle at the origin and can be connected with the points $q^{\pm}$ (whenever they are saddles) if and only if $s = -1$ and $b = 1/(2\sqrt{2})$, but this will not happen. Finally, we observe that when p^+ and p^- are saddles, then $y_+ > 0$ and $y_- < 0$ and so they are never connected.

We consider the cases $s = 1$ and $s = -1$.

Subcase $s = 1$. In view of the infinite singular points, we consider the cases $c = 0$, $c < 0$ and $c > 0$ separately.

Subcase $c = 0$. If $b = 0$ there are no finite singular points among the origin. Now we study the finite singular points different from the origin. If $b \neq 0$ there are three finite singular points $(0, -1/b)$, $q^{\pm}$ if $b > 1/\sqrt{2}$ and one finite singular point $(0, -1/b)$ if $b \leq 1/\sqrt{2}$. Computing the eigenvalues of the Jacobian matrix at the singular point $(0, -1/b)$ and using Theorem 3.15 in [4] we get that it is a center if $b > 1/\sqrt{2}$, a nilpotent saddle if $b = 1/\sqrt{2}$ and a saddle if $b < 1/\sqrt{2}$. Moreover, when $b > 1/\sqrt{2}$, computing the eigenvalues of the Jacobian matrix at the singular points $q^{\pm}$ we get that they are both saddles.

Taking into account the information on the finite and infinite singular points together with the possible saddle connections and separating into different regions, we have the following possible global phase portraits of Figure 1 (whenever there is more than one possible phase portrait, in parenthesis we give values of the parameters where they are realized): 1.46 if $b = 0$. For $b > 1/\sqrt{2}$ they are: 1.47 ($b = 4/(3\sqrt{2})$); 1.48 ($b = 3/(2\sqrt{2})$ and corresponds to the saddle connection in (10)); and 1.49 ($b = (3 + 2\sqrt{2})/(2\sqrt{2})$). Finally, when $b \leq 1/\sqrt{2}$ with $b \neq 0$ they are 1.50.

Subcase $c < 0$. We define the two regions with $(b, c) \in \mathbb{R} \times \mathbb{R}^-$:

$$
\begin{aligned}
S_1 &= \{b < 1/\sqrt{2}, c \in [c_1, 0)\},\\
S_2 &= \{b < 1/\sqrt{2}, c < c_1\} \cup \{b \geq 1/\sqrt{2}, c < 0\}.
\end{aligned}
$$

We study the finite singular points different from the origin. In the region S_1, there exist two finite singular points $p^{\pm}$. Computing the eigenvalues of the Jacobian matrix at these points and using Theorem 3.5 in [4] (in the case $c = c_1$) we get that they are both saddles. In the region S_2 we have the four finite singular points p^+ (which is a center) and p^-, $q^{\pm}$ which are saddles.

Taking into account the information on the finite and infinite singular points together with the possible saddle connections and separating into different regions, we have the following possible global phase portraits of Figure 1 (whenever there is more than one possible phase portrait, in parenthesis we give values of the parameters where they are realized): 1.51 in region S_1. In region S_2 they are: 1.52 ($b = -4$, $c = -4$); 1.53 ($b = 1/2$, $c = (2\sqrt{2}b - 3)/3$ and is the saddle connection in (10)); and 1.54 ($b = 2$, $c = -2$).

Subcase $c > 0$. We define the regions with $(b, c) \in \mathbb{R} \times \mathbb{R}^+$:

$$R_1 = \{0 < b \leq 1/\sqrt{2}, c \in (0, c_2)\} \cup \{1/\sqrt{2} < b < \sqrt{2}, c \in (c_1, c_2)\}$$
$$R_2 = \{b > \sqrt{2}, c \in (c_1, c_2)\} \cup \{b < 0, c \in (0, c_2)\},$$
$$R_3 = \{b > 1/\sqrt{2}, c \in (0, c_1)\}, \quad R_4 = \{c > c_2\}$$
$$L_1 = \{1/\sqrt{2} < b < \sqrt{2}, c = c_1\}, \quad L_2 = \{b > \sqrt{2}, c = c_1\},$$
$$L_3 = \{b > 0, b \neq \sqrt{2}, c = c_2\} \cup \{b < 0, c = c_2\}, \quad P = (\sqrt{2}, c_2) = (\sqrt{2}, 1/2).$$

We study the finite singular points different from the origin. In the region R_1, there exist two finite singular points $p^\pm$. Computing the eigenvalues of the Jacobian matrix at these points we get that p^+ is a saddle and p^- is a center. In the region R_2, there exist again the two finite singular points $p^\pm$. Computing the eigenvalues of the Jacobian matrix at these points we get that p^+ is a center and p^- is a saddle. In the region R_3 we have four finite singular points: $p^\pm$ which are centers, and $q^\pm$ which are saddles. In the region R_4 there are no finite singular points. In the region L_1 we have two finite singular points: p^+ which is a nilpotent saddle and p^- which is a center. In the region L_2 we have two finite singular points: p^+ which is a center and p^- which is a nilpotent saddle. Finally, in the region L_3 we have a unique finite singular point $p^- = p^+$ which is a cusp. Finally, in the region P we get that the unique finite singular point $(0, -\sqrt{2})$ is linearly zero. Applying blow-up techniques we conclude that it is the union of two hyperbolic sectors (one stable and one unstable).

Taking into account the information on the finite and infinite singular points together with the possible saddle connections and separating into different regions, we have the following possible global phase portraits of Figure 1 (whenever there is more than one possible phase portrait, in parenthesis we give values of the parameters where they are realized): 1.35 in region R_1. In region R_2 they are: 1.41 ($b = 3, c = 19/10$); 1.42 ($b = 3$, $c = 2$. Here there is the saddle connection given in (9)); 1.43 ($b = 3$, $c = 43/20$); and 1.35 ($b = -3$, $c = 1/2$). In region R_3 they are: 1.55 ($b = 1, c = 1/10$); 1.56 ($b = 3/2$, $c = -1 + \sqrt{2}$. Here there is the saddle connection given in (10)); and 1.57 ($b = 3/2$, $c = 1/10$). In regions R_4, L_1, and L_2 they are 1.2, 1.35, and 1.43, respectively. Finally in the regions L_3, or P they are 1.38.

Subcase $s = -1$. Again, in view of the infinite singular points, we consider the cases $c = 0$, $c < 0$ and $c > 0$ separately.

Subcase $c = 0$. We study the finite singular points different from the origin. If $b > 0$, or $b \in (-1/\sqrt{2}, 0)$, then there are three finite singular points: $(0, -1/b)$ and $q^\pm$. Computing the eigenvalues of the Jacobian matrix at the singular point $(0, -1/b)$ and using Theorem 3.15 in [4] we get that it is a center and $q^\pm$ are two saddles. If $b \leq -1/\sqrt{2}$ then there is a unique finite singular point $(0, -1/b)$ which is a saddle. Finally, if $b = 0$ the unique finite singular points are $q^{\pm}$ which are saddles.

Taking into account the possible saddle connections we get that the unique global phase portraits are topologically equivalent to the following ones of Figure 1: 1.58 for $b > 0$; 1.59 for $b = 0$; 1.60 for $b \in (-1/\sqrt{2}, 0)$; and 1.50 for $b \leq -1/\sqrt{2}$.

Subcase $c < 0$. We define the following regions with $(b, c) \in \mathbb{R} \times \mathbb{R}^-$

$$S_1 = \{b \leq -1/\sqrt{2}\} \cup \{b > -1/\sqrt{2}, c \leq c_1\},$$
$$S_2 = \{b > -1/\sqrt{2}, c \in (c_1, 0)\}.$$

We study the finite singular points different from the origin. In the region S_1, there exist the two finite singular points $p^{\pm}$. Computing the eigenvalues of the Jacobian matrix at these points we get that p^+ is a center and p^- is a saddle (this saddle is a nilpotent saddle when $c = c_1$). In the region S_2 there are the four finite singular points: $p^{\pm}$ which are centers and $q^{\pm}$ which are saddles. Taking into account the information in the finite and infinite singular points together with the possible saddle connections studied above, we conclude that the unique global phase portraits are topologically equivalent to the following ones of Figure 1: 1.35 in the region S_1 and 1.61 in the region S_2.

Subcase $c > 0$. If $c > 0$ we define the regions. We note that $(b, c) \in \mathbb{R} \times \mathbb{R}^+$:

$$R_1 = \{c = c_2, b \neq 0, b \neq -\sqrt{2}\}, \quad R_2 = \{c > c_2\},$$
$$P = (-\sqrt{2}, c_1) = (-\sqrt{2}, 1/2), \quad R_3 = \{b < -1/\sqrt{2}, c \in (0, c_1]\} \setminus P$$
$$R_4 = \{b \in (-\sqrt{2}, -1/\sqrt{2}), c \in (c_1, c_2)\} \cup \{b \in [-1/\sqrt{2}, 0), c \in (0, c_2)\},$$
$$R_5 = \{b > 0, c \in (0, c_2)\} \cup \{b < -\sqrt{2}, c \in (c_1, c_2)\}.$$

We study the finite singular points different from the origin. In the region R_1 there are the three finite singular points $p^- = p^+$ and $q^{\pm}$. Computing the eigenvalues of the Jacobian matrix at this point and using Theorem 3.5 in [4] we conclude that p^- is a cusp and $q^{\pm}$ are saddles. In the region R_2 there are the two saddles $q^{\pm}$. In the region R_3 we have the two saddles $p^{\pm}$ (they are nilpotent when $c = c_1$). In the region R_4 we have the four finite singular points p^- which is a center, p^+ and $q^{\pm}$ which are all saddles. In the region R_5 we have the singular point p^+ which is a center and the singular points p^- and $q^{\pm}$ which are all saddles. Note that the point P is the linearly zero point $(0, \sqrt{2})$ which is formed by six hyperbolic sectors (three stable and three unstable).

Taking into account the information on the finite and infinite singular points together with the possible saddle connections and separating into different regions, we have the following possible global phase portraits of

Figure 1 (whenever there is more than one possible phase portrait, in parenthesis we give values of the parameters where they are realized): 1.62 in region R_2; 1.63 in region P; and 1.51 in region R_3. In the region R_1 they are: 1.64 ($b = -3$, $c = 9/4$), 1.65 ($b = -6/5$, $c = 9/25$), and 1.66 ($b = 1$, $c = 1/4$). We recall the phase portraits in region R_1 correspond to the cases in which the center and the saddle $(0, y_\pm)$ coalesce.

In the region R_4 there are three possible global phase portraits: 1.67 ($b = -6/5$, $c = 71/200$); 1.68 ($b = -6/5$, $c \approx 0.35793523$. Here $c = c(b)$ is a real solution of the cubic equation given in (11) and so there is a saddle connection); and 1.69 ($b = -6/5$, $c = 359/1000$). Finally, in the region R_5 there are six possible global phase portraits: 1.70 ($b = 1$, $c = 1/10$); 1.71 ($b = 1$, $c = 2/9$. This corresponds to the saddle connection given in (9)); 1.72 ($b = 1$ and $c = 6/25$); 1.52 ($b = -3$, $c = 17/10$); 1.73 ($b = -3$, $c \approx 1.85771681$. Here $c = c(b)$ is a real solution of the cubic equation given in (11) and so there is a saddle connection); and 1.74 ($b = -3$, $c = 21/10$).

13. Global phase portraits of system (VIII) when $c = 0$, $a \neq 0$

We compute the finite singular points different from the origin. If $b = 0$ there are no finite singular points with $x = 0$ different from the origin. If $b \neq 0$ there is a unique finite singular point different with $x = 0$ which is $(0, -1/b)$. Computing the eigenvalues of the Jacobian matrix at this point we get that if $b < s/\sqrt{2}$ it is a saddle, if $b > s/\sqrt{2}$ it is a center and if $b = s/\sqrt{2}$ it is nilpotent. Using Theorem 3.5 in [4] we get that if $a > 1$ it is a center, if $a < 1$ it is a saddle, and if $a = 1$ it is a saddle when $s = 1$ and a center when $s = -1$.

Now we study the finite singular points with $x \neq 0$. These are (whenever they exist) $(\bar{x}_\pm, \bar{y}_\pm)$ and $(-\bar{x}_\pm, \bar{y}_\pm)$ where

$$\bar{x}_\pm = \sqrt{\frac{\bar{y}_\pm(\sqrt{2} + s\bar{y}_\pm)}{a}}, \; \bar{y}_\pm = \frac{-2ab - 3\sqrt{2}s \pm \sqrt{(2ab + 3\sqrt{2}s)^2 - 16(1 + a)}}{4}.$$

The total number of solutions $(\pm\bar{x}_\pm, \bar{y}_\pm)$ can change when $(2ab + 3\sqrt{2}s)^2 - 16(1 + a) = 0$ (in which case the solutions $\bar{y}_\pm$ coincide), or when $\bar{y}_\pm = 0$, or when $\bar{y}_\pm = -\sqrt{2}s$ (in the last two cases $\bar{x}_\pm = 0$). The first condition leads to the values of b equal to $b_\pm$ where

$$b_\pm = b_\pm(a) = \frac{-3\sqrt{2}as \pm 4\sqrt{a^2(1 + a)}}{2a^2}.$$

The second condition is never possible while the third one is possible if and only if $b = 1/\sqrt{2}s$. Moreover $\bar{x}_\pm$ evaluated at $y = \bar{y}\pm$ with $b = b_+$ is not defined for $a \in (-1, 0) \cup (0, +\infty)$ when $s = -1$ and for $a \in (-1, 0) \cup (0, 1)$ when $s = 1$, while $\bar{x}_\pm$ evaluated at $y = \bar{y}\pm$ with $b = b_-$ is not defined for $a > 1$ when $s = 1$. So in these cases the number of real solutions $(\pm\bar{x}_\pm, \bar{y}_\pm)$

does not change on the curves $b = b_\pm$. The type of the singular points $(\pm\bar{x}_\pm, \bar{y}_\pm)$ will be studied later.

The saddle $(0, -1/b)$ (whenever it exists) and the origin are never connected while the saddles $(\bar{x}_+, \bar{y}_+)$ (respectively $(\bar{x}_-, \bar{y}_-)$) and the saddles $(-\bar{x}_+, \bar{y}_+)$ (respectively $(-\bar{x}_-, \bar{y}_-)$), whenever they exist, are connected by symmetry. The connection between the saddles $(\pm\bar{x}_-, \bar{y}_-)$ and the saddle at the origin occurs on the curve

$$b = b(a) = \frac{-3s + 3\sqrt{1+a}}{\sqrt{2}a}, \tag{12}$$

while the connection between the saddles $(\pm\bar{x}_+, \bar{y}_+)$ and the saddle at the origin occurs on the curve

$$b = b(a) = \frac{-3s - 3\sqrt{1+a}}{\sqrt{2}a}. \tag{13}$$

We will see that the points $(\pm\bar{x}_+, \bar{y}_+)$ and $(\pm\bar{x}_-, \bar{y}_-)$ are never simultaneously four saddles and so we do not need to study the possible saddle connections among them. Finally, the connection between the saddles $(\pm\bar{x}_\pm, \bar{y}_\pm)$ and the saddle $(0, -1/b)$ occurs on the curve which is a real root with $b = b_1(a) > 0$ of the polynomial

$$-8as - 16\sqrt{2}ab - 3s(-1-4a+8a^2)b^2 + 2\sqrt{2}(3+10a+12a^2)b^3 + 2as(1+3a+3a^2)b^4. \tag{14}$$

We study the cases $s = 1$ and $s = -1$ separately.

Subcase $s = 1$. Taking into account the information on the finite and infinite singular points we distinguish between $b = 0$; $a > 0$ and $b \neq 0$; and $a < 0$ and $b \neq 0$.

Subcase $b = 0$. If $a > 0$ there are no finite singular points among the origin. If $a \in (-1, 0)$, besides the origin, there are the two finite singular points $(\pm\bar{x}_+, \bar{y}_+)$ which are two centers. The unique possible global phase portraits are topologically equivalent to 1.4 in Figure 1 when $a > 0$ and 1.75 in Figure 1 when $a \in (-1, 0)$.

Subcase $a > 0$ and $b \neq 0$. We consider the following regions for $(a, b) \in \mathbb{R}^+ \times (\mathbb{R} \setminus \{0\})$:

$$S_1 = \{a > 1, b \geq 1/\sqrt{2}\} \cup \{a \in (0, 1], b > 1/\sqrt{2}\},$$

$$S_2 = \{a > 1, b_+ < b < 1/\sqrt{2}\}, \quad S_3 = \{a > 1, b = b_+\} \cup \{b = b_-\},$$

$$S_4 = \{a > 1, b_- < b < b_+\} \cup \{a \in (0, 1], b_- < b \leq 1/\sqrt{2}\}, \quad S_5 = \{b < b_-\}.$$

If $(a, b) \in S_1$ then there are two unique finite singular points with $x \neq 0$ which are $(\pm\bar{x}_-, \bar{y}_-)$ that are both saddles. If $(a, b) \in S_2$ the four finite singular points with $x \neq 0$ exist, being $(\pm\bar{x}_+, \bar{y}_+)$ two centers and $(\pm\bar{x}_-, \bar{y}_-)$

two saddles. If $(a,b) \in S_3$ we have $\bar{y}_- = \bar{y}_+$, and the two finite singular points (with $x \neq 0$) $(\pm\bar{x}_-, \bar{y}_-) = (\pm\bar{x}_+, \bar{y}_+)$ become nilpotent. Using Theorem 3.5 in [4] we obtain that they are both cusps. If $(a,b) \in S_4$ there are no finite singular points with $x \neq 0$. If $(a,b) \in S_5$ the four finite singular points with $x \neq 0$ exist, being $(\pm\bar{x}_+, \bar{y}_+)$ two saddles and $(\pm\bar{x}_-, \bar{y}_-)$ two centers.

Taking into account the information on the finite and infinite singular points together with the possible saddle connections and separating into different regions, we have the following possible global phase portraits of Figure 1 (whenever there is more than one possible phase portrait, in parenthesis we give values of the parameters where they are realized): 1.27 in the region S_3 and 1.28 in the region S_4.

In the region S_1 they are: 1.76 ($a = 2$, $b = 1$); 1.77 ($a = 2$, $b = 3(-1 + \sqrt{3})/(2\sqrt{2})$. Here $b = b(a)$ is as in (12) and so there is a saddle connection); and 1.78 ($a = 2$, $b = 3/4$).

In the region S_2 they are: 1.33 ($a = 4$, $b = 69/100$); 1.32 ($a = 4$, $b = 3(\sqrt{5} - 1)/(4\sqrt{2})$. Here $b = b(a)$ is as in (12) and so there is a saddle connection); 1.31 ($a = 4$, $b = 63/100$); 1.30 ($a = 4$, $b \approx 0.59902710$. Here $b = b_1(a)$ is a real root with $b > 0$ of the polynomial in (14) and so it corresponds to a saddle connection); and 1.29 ($a = 4$, $b = 59/100$).

In the region S_5 they are: 1.29 ($a = 2$, $b = -71/25$); 1.30 ($a = 2$, $b \approx -2.89367324$. Here $b = b_1(a)$ is a real root with $b < 0$ of the polynomial in (14) and so it corresponds to a saddle connection); 1.31 ($a = 2$, $b = -579/200$); 1.32 ($a = 2$, $b = -3(1 + \sqrt{3})/(2\sqrt{2})$. Here $b = b(a)$ is as in (13) and so there is a saddle connection); and 1.33 ($a = 2$, $b = -3$).

Subcase $a \in (-1, 0)$ and $b \neq 0$. We have the following regions for $(a,b) \in (-1,0) \times (\mathbb{R} \setminus \{0\})$:

$$
\begin{aligned}
&R_1 = \{1/\sqrt{2} < b < b_-\}, \quad L_1 = \{b = b_-\},\\
&R_2 = \{b > b_-\}, \quad R_3 = \{0 < b \leq 1/\sqrt{2}\} \cup \{b < 0\}.
\end{aligned}
$$

If $(a,b) \in R_1$ the four finite singular points with $x \neq 0$ exist, being $(\pm\bar{x}_+, \bar{y}_+)$ two centers and $(\pm\bar{x}_-, \bar{y}_-)$ two saddles. If $(a,b) \in L_1$ we have $\bar{y}_+ = \bar{y}_-$, and the two finite singular points (with $x \neq 0$) $(\pm\bar{x}_-, \bar{y}_-) = (\pm\bar{x}_+, \bar{y}_+)$ become nilpotent. Using Theorem 3.5 in [4] we obtain that they are both cusps. If $(a,b) \in R_2$ there are no finite singular points with $x \neq 0$. If $(a,b) \in R_3$ there are two unique finite singular points with $x \neq 0$ which are $(\pm\bar{x}_+, \bar{y}_+)$ and are both centers.

Taking into account the information on the finite and infinite singular points together with the possible saddle connections and separating into different regions, we have the following possible global phase portraits of Figure 1 (whenever there is more than one possible phase portrait, in parenthesis we give values of the parameters where they are realized): 1.44 in R_2

and 1.45 in R_3. In the region R_1 they are: 1.79 ($a = -2/5$, $b = 9/10$); 1.80 ($a = -2/5$, $b = -3(\sqrt{15} - 5)/(2\sqrt{2})$. Here $b = b(a)$ is as in (12) and so there is a saddle connection); and 1.81 ($a = -2/5$, $b = 13/10$). In the region L_1 the cusps correspond to the case in which the saddles and the centers in the region R_1 coalesce. So, the unique global phase portraits are topologically equivalent to 1.82.

Subcase $s = -1$. Again, we distinguish between the cases $b = 0$; $a > 0$ and $b \neq 0$; and $a < 0$ and $b \neq 0$.

Subcase $b = 0$. If $a \in (-1, 0)$ there are two finite singular points besides the origin which are two saddles. If $a \in (0, 1/8)$ then there are four singular points: $(\pm\bar{x}_+, \bar{y}_+)$ which are saddles and $(\pm\bar{x}_-, \bar{y}_-)$ which are centers. If $a = 1/8$ there are two finite singular points besides the origin which are two cusps (it corresponds to the fact that the saddles and the centers when $a \in (0, 1/8)$ coalesce). Finally, if $a > 1/8$ there are no more finite singular points among the origin. Taking all this into account together with the possible saddle connections we conclude that the possible global phase portraits are: 1.83 when $a \in (0, 1/8)$; 1.84 when $a = 1/8$; 1.2 when $a > 1/8$; and 1.85 when $a \in (-1, 0)$.

Subcase $a > 0$ and $b \neq 0$. We have the following regions for $(a, b) \in \mathbb{R}^+ \times (\mathbb{R} \setminus \{0\})$:

$$
\begin{aligned}
&S_1 = \{b < -1/\sqrt{2}\} \cup \{a \in (0,1), b = -1/\sqrt{2}\},\\
&N_1 = \{a \in (0,1), b = b_-\}, \quad S_2 = \{a \in (0,1), -1/\sqrt{2} < b < b_-\},\\
&S_3 = \{a \in (0,1), b > b_-\} \cup \{a \geq 1, b \geq -1/\sqrt{2}\}.
\end{aligned}
$$

If $(a, b) \in S_1$ there are two unique finite singular points (centers) with $x \neq 0$ which are $(\pm\bar{x}_-, \bar{y}_-)$. If $(a, b) \in N_1$ there are two finite singular points (cusps) with $x \neq 0$ which are $(\pm\bar{x}_-, \bar{y}_-) = (\pm\bar{x}_+, \bar{y}_+)$. If $(a, b) \in S_2$ the four finite singular points with $x \neq 0$ exist being $(\pm\bar{x}_+, \bar{y}_+)$ two saddles and $(\pm\bar{x}_-, \bar{y}_-)$ two centers. If $(a, b) \in S_3$ there are no finite singular points with $x \neq 0$.

Taking into account the information on the finite and infinite singular points together with the information of the possible saddle connections we have the following possible global phase portraits of Figure 1 (whenever there is more than one possible phase portrait, in parenthesis we give values of the parameters when they are realized): 1.45 in the region S_1; 1.86 ($a = 1/2$, $b = -67/100$) and 1.87 ($a = 1/10$, $b = 1/10$) in the region S_2; 1.88 ($a = 1/2$, $b = b_-$) and 1.89 ($a = 1/10$, $b = b_-$) in the region N_1 (they correspond to the cases in which the saddles and the centers in the region S_2 coalesce); and 1.12 in the region S_3.

Subcase $a \in (-1,0)$ and $b \neq 0$. We consider the following regions for $(a,b) \in (-1,0) \times (\mathbb{R} \setminus \{0\})$:

$$R_1 = \{b < b_-\}, \quad R_2 = \{b = b_-\},$$
$$R_3 = \{b_- < b \leq -1/\sqrt{2}\}, \quad R_4 = \{-1/\sqrt{2} < b < 0\} \cup \{b > 0\}.$$

If $(a,b) \in R_1$ the four finite singular points with $x \neq 0$ exist, being $(\pm\bar{x}_+, \bar{y}_+)$ two centers and $(\pm\bar{x}_-, \bar{y}_-)$ two saddles. If $(a,b) \in R_2$ we have $\bar{y}_- = \bar{y}_+$, and the two finite singular points (with $x \neq 0$) $(\pm\bar{x}_-, \bar{y}_-) = (\pm\bar{x}_+, \bar{y}_+)$ become nilpotent. Using Theorem 3.5 in [4] we obtain that they are both cusps. If $(a,b) \in R_3$ there are no finite singular points with $x \neq 0$. If $(a,b) \in R_4$ there are two unique finite singular points with $x \neq 0$ which are $(\pm\bar{x}_+, \bar{y}_+)$ that are both saddles.

Taking into account the information on the finite and infinite singular points together with the possible saddle connections and separating into different regions, we have the following possible global phase portraits of Figure 1 (whenever there is more than one possible phase portrait, in parenthesis we give values of the parameters where they are realized): 1.27 in R_2 and 1.28 in R_3. In the region R_4 they are: 1.90 ($a = -1/2$, $b = -1/2$) and 1.91 of Figure 1 ($a = -1/2$, $b = 1$).

In the region R_1 they are: 1.29 ($a = -9/10$, $b = -307/100$); 1.30 ($a = -9/10$, $b = -(10 + \sqrt{10})/(3\sqrt{2})$. Here $b = b(a)$ is as in (12) and thus corresponds to a saddle connection); 1.31 ($a = -9/10$, $b = -16/5$); 1.32 ($a = -9/10$, $b \approx -3.31071911$. Here $b = b_1(a)$ is a real root with $b < 0$ of of the polynomial in (14) and thus corresponds to a saddle connection); and 1.33 ($a = -9/10$, $b = -4$).

14. Global phase portraits of system (VIII) when $ac = -1$, $as > 0$, $cs < 0$

The finite singular points (whenever they exist) are $p^{\pm} = (0, y_{\pm})$ and $(\pm\bar{x}, \bar{y})$ where

$$y_{\pm} = \frac{ab}{2} \pm \frac{1}{2}\sqrt{a(4 + ab^2)}, \quad \bar{y} = \frac{-2(1+a)s}{3\sqrt{2} + 2abs}$$

and

$$\bar{x} = 2\sqrt{\frac{-(1+a)(\sqrt{2}ab + s(2-a))}{a(3\sqrt{2} + 2abs)^2}}.$$

The total number of solutions $p^{\pm}$ can change when $a(4 + ab^2) = 0$ that is $b = \pm\frac{2}{\sqrt{-a}}$ with $a \in (-1,0)$ and $s = -1$. The total number of solutions $(\pm\overline{x}, \overline{y})$ can change when either $3\sqrt{2} + 2abs = 0$, that is, $b = -3/(\sqrt{2}as)$ (in which case $\overline{x}$ and $\overline{y}$ disappear), or $3\sqrt{2} + 2abs \neq 0$ and $\sqrt{2}ab + s(2-a) = 0$, that is, $b = (a-2)s/(\sqrt{2}a)$ (in which case $\overline{x} = 0$). Note that $(a-2)s/(\sqrt{2}a) = -3/(\sqrt{2}as)$ if and only if $a = -1$, and so it is never possible.

The saddles p^- and p^+ whenever they exist cannot be connected one with each other. The saddles $p^\pm$ and the origin can be connected if and only if $s = -1$ and along the curve $b = b_2(a) = \pm 3/(\sqrt{-2a})$. On the other hand the points $(\pm\bar{x}, \bar{y})$ are never saddles.

We distinguish between the cases: $s = 1$ and $s = -1$.

Subcase $s = 1$**.** In this case, taking into account that $a > 0$ we have the following regions for $(a, b) \in \mathbb{R}^+ \times \mathbb{R}$:

$$S_1 = \{b < -3/(\sqrt{2}a)\} \cup \{-3/(\sqrt{2}a) < b < (a-2)/(\sqrt{2}a)\},$$
$$L_1 = \{b = -3/(\sqrt{2}a)\}, \quad S_2 = \{b \geq (a-2)/(\sqrt{2}a)\}.$$

We study the finite singular points different from the origin. If $(a, b) \in S_1$ then there exist the four finite singular points $p^\pm$ which are two saddles and $(\pm\bar{x}, \bar{y})$ which are two centers. If $(a, b) \in L_1$, then there exist the two finite singular points $p^\pm$ which are two saddles. In the region S_2 there are the two finite singular points p^- (which is a center) and p^+ (which is a saddle).

Taking into account the information on the finite and infinite singular points and the fact that there are no saddle connections in this case we conclude that the global phase portraits are topologically equivalent to the following ones of Figure 1: 1.26 in the region S_1; 1.6 in the region L_1; and 1.92 in the region S_2.

Subcase $s = -1$**.** In this case $a < 0$, i.e. $a \in (-1, 0)$. We consider the following regions for $(a, b) \in (-1, 0) \times \mathbb{R}$:

$$R_1 = \{b > 2/\sqrt{-a}\} \cup \{(2-a)/(\sqrt{2}a) < b < -2/\sqrt{-a}\}$$
$$M_1 = \{b = 2/\sqrt{-a}\} \cup \{b = -2/\sqrt{-a}\}$$
$$R_2 = \{-2/\sqrt{-a} < b < 2/\sqrt{-a}\}, \quad M_2 = \{b = 3/(\sqrt{2}a)\},$$
$$R_3 = \{b < (2-a)/(\sqrt{2}a), b \neq 3/(\sqrt{2}a)\}.$$

We study the finite singular points different from the origin. If $(a, b) \in R_1$ then there exist the two singular points p^+, which is a center and p^-, which is a saddle. If $(a, b) \in M_1$, there exist the unique finite singular $p^- = p^+$ which is a cusp. In the region R_2 there are no finite singular points. In the region M_2 there exist the two singular points $p^\pm$ which are two saddles and in the region R_3 there are the four singular points $p^\pm$ which are saddles and $(\pm\bar{x}, \bar{y})$ which are centers.

Taking into account the information on the finite and infinite singular points together with the possible saddle connections and separating into different regions, we have the following possible global phase portraits of Figure 1 (whenever there is more than one possible phase portrait, in parenthesis we give values of the parameters where they are realized): 1.93 in R_2, 1.6 in M_2, and 1.26 in R_3. In the region R_1 they are: 1.94 ($a = -1/2$,

$b = 29/10$); 1.95 ($a = -1/2$, $b = 3$. Here we have a saddle connection); 1.96 ($a = -1/2$, $b = 4$); and 1.92 ($a = -1/2$, $b = -3$). In the region M_1 they are 1.97 when $b = 2/\sqrt{-a}$) and 1.98 when $b = -2/\sqrt{-a}$. Note that these global phase portraits correspond to the ones in the region R_1 which the saddle and the center coalesce.

15. Global phase portraits of system (VIII) when $ac \neq 0$ and $ac \neq -1$: finite singular points on $x = 0$

We study the case $ac \neq 0$ with either $ac < -1$ and $as > 0$ or $cs > 0$ and $as < 0$. In this section we focus on the finite singular points on $x = 0$.

If $b^2 - 4c < 0$ there are no finite singular points on $x = 0$.

If $b^2 = 4c$ there is a unique finite singular point on $x = 0$ which is $(0, -2/b)$. Computing the eigenvalues of the Jacobian matrix at this point we get that if $b \neq \sqrt{2}s$ it is nilpotent and if $b = \sqrt{2}s$ it is linearly zero. In the first case using Theorem 3.5 in [4] we get that it is a cusp and if $b = \sqrt{2}s$ then using blow-up techniques we conclude that it is the union of two hyperbolic sectors (one stable and one unstable) if $s = 1$ and is the union of six hyperbolic sectors (3 stable and 3 unstable) if $s = -1$.

If $b^2 > 4c$ we get that there are two finite singular points with $x = 0$ which are $p^{\pm} = (0, y_{\pm}) = \left(0, \frac{-b \pm \sqrt{b^2 - 4c}}{2c}\right)$. We introduce the notation

$$c_1 = c_1(b) = \frac{1}{2}(-1 + \sqrt{2}bs), \quad c_2 = c_2(b) = \frac{b^2}{4}$$

We distinguish between the cases $s = 1$ and $s = 1$.

Subcase $s = 1$. In this case computing the eigenvalues of the Jacobian matrix at the points $p^{\pm}$ we get the following regions

$$\begin{aligned}
R_1 &= \{(b,c) \in \mathbb{R}^2 : c_1 < c < c_2, b > \sqrt{2}\} \cup \{(b,c) \in \mathbb{R}^2 : 0 < c < c_2, b < 0\}, \\
&\quad \cup \{(b,c) \in \mathbb{R}^2 : c < c_1, b < 1/\sqrt{2}\} \cup \{(b,c) \in \mathbb{R}^2 c < 0, b \geq 1/\sqrt{2}\}, \\
R_2 &= \{(b,c) \in \mathbb{R}^2 : 0 < c < c_1, b > 1/\sqrt{2}\}, \\
R_3 &= \{(b,c) \in \mathbb{R}^2 : 0 < c < c_2, 0 < b \leq 1/\sqrt{2}\} \\
&\quad \cup \{(b,c) \in \mathbb{R}^2 : c_1 < c < c_2, 1/\sqrt{2} < b < \sqrt{2}\}, \\
R_4 &= \{(b,c) \in \mathbb{R}^2 : c_1 < c < 0, b < 1/\sqrt{2}\}, \\
L_1 &= \{c = c_1, b < \sqrt{2}\}, \quad L_2 = \{c = c_1, b > \sqrt{2}\}.
\end{aligned}$$

If $(b,c) \in R_1$ then p^+ is a center and p^- is a saddle. If $(b,c) \in R_2$, both $p^{\pm}$ are centers. If $(b,c) \in R_3$ then p^+ is a saddle and p^- is a center, if $(b,c) \in R_4$ both $p^{\pm}$ are saddles. If $(b,c) \in L_2$ then p^+ is a center and p^- is nilpotent. Using Theorem 3.15 in [4] we get that it is a saddle if $\sqrt{2} < b \leq 3/\sqrt{2}$

or $b > 3/\sqrt{2}$ and $a > 1/(2-\sqrt{2}b)$, and it is a center if $b > 3/\sqrt{2}$ and $a \leq 1/(2-\sqrt{2}b)$. On the other hand, if $(b,c) \in L_1$ then p^- is a center if $1/\sqrt{2} < b < \sqrt{2}$ and a saddle if $b < 1/\sqrt{2}$ (note that $b = 1/\sqrt{2}$ corresponds to $c = 0$). Moreover p^+ is nilpotent. Using Theorem 3.5 in [4] we get that it is a saddle if $a \leq 1/(2-\sqrt{2}b)$ and a center if $a > 1/(2-\sqrt{2}b)$.

Subcase $s = -1$. In this case computing the eigenvalues of the Jacobian matrix at the points $p^\pm$ we get the following regions

$$
\begin{aligned}
R_1 &= \{(b,c) \in \mathbb{R}^2 : c_1 < c < c_2, b < -\sqrt{2}\} \cup \{(b,c) \in \mathbb{R}^2 : c < 0, b \leq -1/\sqrt{2}\},\\
&\quad \cup \{(b,c) \in \mathbb{R}^2 : c < c_1, b > -1/\sqrt{2}\} \cup \{(b,c) \in \mathbb{R}^2 : 0 < c < c_2, b > 0\},\\
R_2 &= \{(b,c) \in \mathbb{R}^2 : 0 < c < c_1, b < -1/\sqrt{2}\},\\
R_3 &= \{(b,c) \in \mathbb{R}^2 : c_1 < c < c_2, -\sqrt{2} < b \leq -1/\sqrt{2}\}\\
&\quad \cup \{(b,c) \in \mathbb{R}^2 : 0 < c < c_2, -1/\sqrt{2} < b < 0\},\\
R_4 &= \{(b,c) \in \mathbb{R}^2 : c_1 < c < 0, b > -1/\sqrt{2}\},\\
L_1 &= \{c = c_1, b < -\sqrt{2}\}, \quad L_2 = \{c = c_1, -\sqrt{2} < b < -\sqrt{2}\},\\
L_3 &= \{c = c_1, b > -1/\sqrt{2}\}.
\end{aligned}
$$

If $(b,c) \in R_1$ then p^- is a saddle and p^+ is a center. If $(b,c) \in R_2$, both $p^\pm$ are both saddles. If $(b,c) \in R_3$ then p^- is a center and p^+ is a saddle, if $(b,c) \in R_4$ both $p^\pm$ are both centers. If $(b,c) \in L_1$ then p^- is a saddle and p^+ is nilpotent. Using Theorem 3.15 in [4] we get that it is a saddle if $-3/\sqrt{2} \leq b < -\sqrt{2}$ or $b < -3/\sqrt{2}$ and $a \geq 1/(2+\sqrt{2}b)$, and it is a center if $b < -3/\sqrt{2}$ and $a < 1/(2+\sqrt{2}b)$. On the other hand, if $(b,c) \in L_2$ then p^+ is is a saddle and p^- is nilpotent. Using Theorem 3.5 in [4] we get that it is a saddle if $a < 1/(2+\sqrt{2}b)$ and a center if $a \geq 1/(2+\sqrt{2}b)$. Finally, if $(a,b) \in L_3$ then p^+ is a center and p^- is nilpotent. Using again Theorem 3.5 in [4] we get that it is a saddle if $a < 1/(2+\sqrt{2}b)$ and a center if $a \geq 1/(2+\sqrt{2}b)$.

16. Global phase portraits of system (VIII) when $ac \neq 0$ and $ac \neq -1$: preliminaries on finite singular points with $x \neq 0$

We continue determining the finite singular points of systems (VIII) with $x \neq 0$. We first introduce four auxiliary lemmas.

Auxiliary lemmas.

Lemma 5. *There exist at most four finite singular points for system (VIII) with $x \neq 0$ which are, whenever they exist,*

$$
\left(\pm\sqrt{\frac{\bar{y}_+(\sqrt{2}+s\bar{y}_+)}{a}}, \bar{y}_+\right) \quad \textit{and} \quad \left(\pm\sqrt{\frac{\bar{y}_-(\sqrt{2}+s\bar{y}_-)}{a}}, \bar{y}_-\right) \tag{15}
$$

with

$$\bar{y}_{\pm} = -\frac{2ab + 3\sqrt{2}s \pm \sqrt{(2ab+3\sqrt{2}s)^2 - 16(1+a)(ac+1)}}{4(1+ac)}. \tag{16}$$

Proof. We compute the Gröebner basis for the polynomials in system (VIII) and we obtain six polynomials $\{p_1, \dots, p_6\}$ where

$$p_2 = \frac{1}{2}xy\big(2(1+a)+(3\sqrt{2}+2ab)y+2(1+ac)y^2\big), \quad p_6 = -x\big(ax^2-\sqrt{2}y-sy^2\big).$$

From $p_6 = 0$ we get

$$x^2 = \frac{y(\sqrt{2}+sy)}{a}. \tag{17}$$

Recall that $a \neq 0$. Moreover since $x \neq 0$ we must have, in particular, $y \neq 0$. Now solving $p_2 = 0$ with respect to y we get $y = \bar{y}_{\pm}$ as in (16). Substituting these values of y and the corresponding x given in (17) into the Gröebner basis we get $p_i = 0$ for $i = 1, \dots, 6$ and so we have at most two solutions in y and two solutions in x for each solution in y. In short, there are at most four solutions of system (VIII) with $x \neq 0$ and they are, whenever they exist, the ones in the statement of the lemma. This concludes the proof of the lemma. □

Lemma 6. *The finite singular points with $x \neq 0$ are either elementary or nilpotent. They are nilpotent if and only if*

$$c = c_0 := \frac{2a^2b^2 + 6\sqrt{2}abs + 1 - 8a}{8a(1+a)}$$

and in this case, if they exist, they are two cusps.

Proof. We compute the Gröebner basis for the polynomials $\dot{x}, \dot{y}$ and the determinant of the linear part of system (VIII) and we get a set of 22 polynomials. Recall that $xy \neq 0$ (because if $y = 0$ it follows easily from system (VIII) that $x = 0$). One of the polynomials in the Gröebner basis can be written as $-16a(1+a)^2xy(c-c_0)(c-c_1)$. Hence we must study the cases $c = c_1$ and $c = c_0$. If $c = c_1$, substituting it into the Gröebner basis we get a solution of the form $y = -s\sqrt{2} = -\sqrt{2}/s$. This is not possible because then $x = 0$ (see (17)). On the other hand, if $c = c_0$, substituting it into the Gröebner basis we get the following cases:

(i) $b = -s(1-2a)/(\sqrt{2}a)$;
(ii) $b = -s(5+2a)/(\sqrt{2}a)$;
(iii) $b = -3s/(\sqrt{2}a)$;
(iv) $y = -4s(1+a)/(3\sqrt{2}+2asb)$ with $b \neq -3s/(\sqrt{2}a)$.

We consider each case (i)–(iv) separately. Note that in statement (iv) we have that $y = \bar{y}_+ = \bar{y}_-$ given in (16) (because $c = c_0$).

Substituting the value of b in case (i) into the Gröebner basis we get the solution $y = -s\sqrt{2}$ which is not possible because then $x = 0$ (see Lemma 5). On the other hand, substituting the value of b in case (ii) into the Gröebner basis we get the solutions $y = s\sqrt{2}$ and $x = \pm 2/\sqrt{as}$. Both solutions are included in case (iv) (whenever b is as in (ii)) so we will study them later when we study the general case (iv). Substituting the value of b in (iii) into the Gröebner basis we get the solution $x = y = 0$ which is not possible. Finally, introducing the value of y in (iv) into the Gröebner basis we get a polynomial quadratic in x. Solving it we obtain $x = \pm\overline{x}$ with

$$\overline{x} = \sqrt{s\frac{8(1+a)(-1+2a-\sqrt{2}abs)}{a(3\sqrt{2}+2sab)^2}}. \tag{18}$$

Note that $\overline{x}$ is precisely the value of the component x in the solutions given in (15) with $\bar{y}_+ = \bar{y}_-$ because $c = c_0$ and so the four solutions become two (whenever $\overline{x}$ exists).

Substituting these two values of x into the Gröebner basis we get that it is identically zero. Now taking $c = c_0$, and computing the eigenvalues of the Jacobian matrix at the points $(\pm\overline{x}, y)$ with y as in (iv) and $\overline{x}$ as in (18) we get that it is not the 2×2-zero matrix. Hence both points $(\pm\overline{x}, y)$ are nilpotent. Applying Theorem 3.5 in [4] we conclude that both of them are cusps. To conclude the proof of the lemma we just note that taking $c = c_0$ and solving the polynomials $\dot{x} = \dot{y} = 0$ we get exactly two solutions with $x \neq 0$ (which are $(\pm\overline{x}, y)$ and two solutions with $x = 0$. This concludes the proof of the lemma. □

Lemma 7. *In the case $ac + 1 > 0$, $cs < 0$ and $as > 0$ there cannot be two centers with $x = 0$.*

Proof. Note that when $s = 1$ the regions in which we can have two centers with $x = 0$ are: $0 < c < c_1$, $b > 1/\sqrt{2}$, or $c = c_1$, $b > 3/\sqrt{2}$, $a \leq 1/(2-\sqrt{2}b)$, or $c = c_1$, $1/\sqrt{2} < b < \sqrt{2}$ and $a > 1/(2-\sqrt{2}b)$. In the three cases we have that $c > 0$ which is not possible because then $cs > 0$.

When $s = -1$, the regions in which we can have two centers with $x = 0$ are: $c_1 < c < 0$, $b > -1/\sqrt{2}$, or $c = c_1$, $b > -1/\sqrt{2}$ and $a \geq 1/(2+\sqrt{2}b)$. In both cases we have that $c < 0$ which is again not possible because then $cs > 0$. This concludes the proof of the lemma. □

Now we investigate the possible saddle connections.

Saddle connections. The saddles p^+ and p^- (whenever they exist) cannot be connected. The saddles $p^\pm$ (whenever they exist) and the origin can be connected along the surface $c = c(a, b)$ in the parameter space with

$$c = \frac{2b^2}{9}. \tag{19}$$

The saddles that are not on $x = 0$, that is, the saddles $(\pm\bar{x}_\pm, \bar{y}_\pm)$ with $\bar{x}_\pm \neq 0$ (whenever they exist) are connected by symmetry and they are connected with the saddle at the origin along the surface $c = c(a, b)$ in the parameter space with

$$c = \frac{2ab^2 + 6\sqrt{2}bs - 9}{9(1+a)}. \tag{20}$$

The saddles $(\pm\bar{x}_+, \bar{y}_+)$ and the saddles $(\pm\bar{x}_-, \bar{y}_-)$ can be connected (whenever they exist) if and only if $c = c_0$ (in which case they become two cusps).

Finally, the saddles $(\pm\bar{x}_\pm, \bar{y}_\pm)$ with $\bar{x}_\pm \neq 0$ and the saddles $p^\pm$ (whenever they exist) can be connected on a surface $c = c(a, b)$ in the parameter space which is any real solution (whenever it exists) of the equation

$$(2b - \sqrt{2}s(1+2c))\big(C_1(a,b,c) + 2\sqrt{2}bsC_2(a,b,c)\big) = 0, \tag{21}$$

where $C_1 = C_1(a, b, c)$ is

$$\begin{aligned}
C_1 = {} & -64a^2b^4 + 24ab^6 + 96a^2b^6 - 192a^3b^6 + 16a^2b^8 + 48a^3b^8 + 48a^4b^8 + 576a^2b^2c \\
& - 252ab^4c - 1328a^2b^4c + 1632a^3b^4c + 120ab^6c + 264a^2b^6c + 144a^3b^6c - 1248a^4b^6c \\
& + 48a^3b^8c + 144a^4b^8c + 144a^5b^8c - 1296a^2c^2 + 792ab^2c^2 + 5760a^2b^2c^2 - 2880a^3b^2c^2 \\
& - 1152ab^4c^2 - 3684a^2b^4c^2 - 6816a^3b^4c^2 + 8544a^4b^4c^2 + 456a^2b^6c^2 + 936a^3b^6c^2 \\
& + 144a^4b^6c^2 - 2592a^5b^6c^2 + 48a^4b^8c^2 + 144a^5b^8c^2 + 144a^6b^8c^2 - 648ac^3 \\
& - 7776a^2c^3 - 2592a^3c^3 + 18b^2c^3 + 3456ab^2c^3 + 13608a^2b^2c^3 + 26208a^3b^2c^3 \\
& - 17280a^4b^2c^3 - 516ab^4c^3 - 4772a^2b^4c^3 - 10272a^3b^4c^3 - 2448a^4b^4c^3 \\
& + 13824a^5b^4c^3 - 264a^3b^6c^3 - 1392a^4b^6c^3 - 2592a^5b^6c^3 - 1728a^6b^6c^3 - 81c^4 \\
& - 3240ac^4 - 17496a^2c^4 - 12960a^3c^4 - 1296a^4c^4 + 90b^2c^4 + 3672ab^2c^4 \\
& + 16560a^2b^2c^4 + 27360a^3b^2c^4 - 4320a^4b^2c^4 - 23040a^5b^2c^4 + 200a^2b^4c^4 \\
& + 2784a^3b^4c^4 + 9696a^4b^4c^4 + 13824a^5b^4c^4 + 6912a^6b^4c^4 - 324c^5 - 5184ac^5 \\
& - 18144a^2c^5 - 20736a^3c^5 - 5184a^4c^5 + 144ab^2c^5 - 288a^2b^2c^5 - 6336a^3b^2c^5 \\
& - 19584a^4b^2c^5 - 23040a^5b^2c^5 - 9216a^6b^2c^5 - 324c^6 - 2592ac^6 - 7776a^2c^6 \\
& - 10368a^3c^6 - 5184a^4c^6
\end{aligned}$$

and

$$\begin{aligned}
C_2 = {} & -64a^2b^4 + 24ab^6 + 80a^2b^6 + 96a^3b^6 + 576a^2b^2c - 252ab^4c - 840a^2b^4c \\
& - 1392a^3b^4c + 88a^2b^6c + 288a^3b^6c + 336a^4b^6c - 1296a^2c^2 + 792ab^2c^2 \\
& + 2448a^2b^2c^2 + 6624a^3b^2c^2 + 72ab^4c^2 - 708a^2b^4c^2 - 2808a^3b^4c^2 - 4416a^4b^4c^2 \\
& + 120a^3b^6c^2 + 384a^4b^6c^2 + 432a^5b^6c^2 - 648ac^3 - 1296a^2c^3 - 10368a^3c^3 \\
& + 18b^2c^3 - 312ab^2c^3 + 1944a^2b^2c^3 + 10512a^3b^2c^3 + 16608a^4b^2c^3 - 296a^2b^4c^3 \\
& - 2100a^3b^4c^3 - 4800a^4b^4c^3 - 3888a^5b^4c^3 - 81c^4 - 108ac^4 - 2592a^2c^4 \\
& - 16848a^3c^4 - 15984a^4c^4 + 24ab^2c^4 + 1944a^2b^2c^4 + 9504a^3b^2c^4 + 17376a^4b^2c^4 \\
& + 10368a^5b^2c^4 + 54c^5 + 216ac^5 - 1728a^2c^5 - 9504a^3c^5 - 14688a^4c^5 - 6912a^5c^5.
\end{aligned}$$

17. Global phase portraits of system (VIII) when $ac < -1$ and $as > 0$, or $cs > 0$ and $as < 0$

By Theorems 2 and 3, since there are no singular points at infinity and the saddle at the origin have total index -2 on the Poincaré sphere, the remaining finite singular points on the Poincaré sphere have to have total index 4. Moreover, studying the finite singular points with $x = 0$, we have the following possibilities: (i) there are no singular points in $x = 0$, (ii) a center and a saddle, (iii) two centers, (iv) two saddles, (v) a cusp. In all this chapter whenever there is more than one possible phase portrait, in parenthesis we give values of the parameters where they are realized.

In case (i) the total index of the singular points with $x = 0$ in the Poincaré sphere is -2. Hence, the remaining singular points with $x \neq 0$ have to have total index 2. Hence, in view of Lemmas 5 and 6 and taking into account the symmetry of the system, they can only be two centers. The global phase portraits of the systems in case (i) are topologically equivalent to 1.3 in Figure 1 and are realized whenever $b^2 < 4c$.

In case (ii) the total index of the singular points with $x = 0$ in the Poincaré sphere is -2. Hence, the remaining singular points with $x \neq 0$ have to have total index 4. Hence, in view of Lemma 6 and taking into account the symmetry of the system, they can only be two centers. Taking into account the possible saddle connections we have that the possible global phase portraits are topologically equivalent to the following ones of Figure 1: 1.39 ($s = 1$, $a = -4/5$, $b = 3$, $c = 17/10$); 1.40 ($s = 1$, $a = -4/5$, $b = 3$, $c = 2$. Here c is as in (19) and so there is a saddle connection); and 1.36 ($s = 1$, $a = -4/5$, $b = 1/2$, $c = 1/20$).

In case (iii) the total index of the singular points with $x = 0$ in the Poincaré sphere is 2. Hence, the remaining singular points with $x \neq 0$ have to have total index 0. Hence, in view of Lemmas 5 and 6 and taking into account the symmetry of the system, they can only be: (iii.1) no points, (iii.2) two cusps (that in view of Lemma 6 it only happens when $c = c_0$), (iii.3) two centers and two saddles. Taking into account the possible saddle connections we conclude that the global phase portraits are topologically equivalent to the following ones of Figure 1: for case (iii.1) to 1.99 ($s = 1$, $a = -4/5$, $b = 3$, $c = 3/2$) and 1.3 ($s = -1$, $a = 1$, $b = 3$, $c = -1/2$); for case (iii.2) to 1.100 ($s = 1$, $a = -2/5$, $b = 2$, $c = c_0$) and to 1.101 ($s = -1$, $a = 1/2$, $b = -1/2$, $c = c_0$); for case (iii.3) to 1.102 ($s = 1$, $a = -2/5$, $b = 2$, $c = 9/10$), 1.103 ($s = 1$, $a = -2/5$, $b = 2$, $c = (60\sqrt{2} - 61)/27$. Here c is as in (20) and so we have a saddle connection), 1.104 ($s = 1$, $a = -2/5$, $b = 2$, $c = 4/5$), and 1.105 ($s = -1$, $a = 1/4$, $b = 18/25$, $c = -1$).

In case (iv) the total index of the singular points with $x = 0$ in the Poincaré sphere is 6. Hence, the remaining singular points with $x \neq 0$ have to have total index 8. Hence, in view of Lemmas 5 and 6 and taking into account the symmetry of the system, they can only be four centers. Taking into account that there are no saddle connections, we conclude that the global phase portraits are topologically equivalent to 1.7 in Figure 1 which is realized for example when $s = 1$, $a = 1$, $b = -3$ and $c = -2$.

In case (v) the total index of the singular points with $x = 0$ in the Poincaré sphere is -2. Hence the remaining singular points with $x \neq 0$ have to have total index 4. Hence, in view of Lemmas 5 and 6 and taking into account the symmetry of the system, they can only be two centers. The global phase portraits of the systems in case (v) are topologically equivalent to 1.37 in Figure 1 and it is realized whenever $b^2 = 4c$.

18. Global phase portraits of system (VIII) when $ac > 0$

By Theorem 2, the two nodes at infinity and the saddle at the origin have total index 1 on the Poincaré sphere. Then by Theorem 3 the remaining finite singular points on the Poincaré sphere have to have total index 0. Moreover, studying the finite singular points with $x = 0$ in the case $ac > 0$, we have the following possibilities: (i) there are no points, (ii) a cusp (whenever $c = b^2/4$ and $b \neq -\sqrt{2}$ when $s = -1$), (iii) a center and a saddle p, (iv) two saddles p_1 and p_2, (v) two centers; and (vi) a point which is the union of six hyperbolic sectors (whenever $b = -\sqrt{2}$, $c = b^2/4$ and $s = -1$). In all the chapter whenever there is more than one possible phase portrait, we give in parenthesis values of the parameters where they are realized.

In case (i) the total index of the infinite singular points and the finite singular points with $x = 0$ in the Poincaré sphere is 1. Hence, the remaining finite singular points with $x \neq 0$ have to have total index 0. Hence, in view of Lemma 6 and taking into account the symmetry of the system, they can only be: (i.1) do not exist, (i.2) two cusps, (i.3) two centers and two saddles p_3, p_4. Taking into account the fact that there are no saddle connections in this region, we conclude that the global phase portraits are topologically equivalent to the following ones of Figure 1: 1.1 in case (i.1) ($s = 1$, $a = 1/10$, $b = 1/2$, $c = 4$), to 1.106 in case (i.2) ($s = -1$, $a = 1/10$, $b = 0$, $c = c_0$), and to 1.107 in case (i.3) ($s = -1$, $a = 1/10$, $b = -1$, $c = 1$).

In case (ii) the total index of the infinite singular points and the finite singular points with $x = 0$ in the Poincaré sphere is 1. Hence, the remaining finite singular points with $x \neq 0$ have to have total index 0. So, in view of Lemma 6 and taking into account the symmetry of the system, they can only be: (ii.1) do not exist, (ii.2) two cusps, (ii.3) two centers and two saddles p_3, p_4. We recall that here $c = b^2/4$ and that we exclude the case $b = -\sqrt{2}$ with $s = -1$. Taking into account that there are no saddle connections, the global phase portraits are topologically equivalent to the following ones of Figure 1: for case (ii.1) to 1.8 ($s = 1$, $a = 1/2$, $b = 1$); for case (ii.2) (we must have $s=-1$ and $a=1/(8+6\sqrt{2}b+2b^2)$) to 1.108 ($b=1$), 1.109 ($b=-3$), and 1.110 ($b=-1/2$); and for case (ii.3) to 1.111 ($s=-1$, $a=1/2$, $b=-2$), 1.112 ($s=-1$, $a=1/2$, $b=-1$), and 1.113 ($s=-1$, $a=3/100$, $b=1$).

In case (iii) the total index of the infinite singular points and the finite singular points with $x=0$ in the Poincaré sphere is 1. Hence, the remaining finite singular points with $x\neq 0$ have to have total index 0. Hence, in view of Lemma 6 and taking into account the symmetry of the system, they can only be: (iii.1) do not exist, (iii.2) two cusps, (iii.3) two centers and two saddles p_3 and p_4.

In case (iii.1) we need to investigate the possible saddle connection between the saddle p and the origin. Doing so, we get that they can be connected along the surface (19). Taking all the above into account and recalling that we know the region in the parameters space where condition (iii.1) holds (because we know the region in the parameter space where condition (iii) holds and we also know that none of the solutions with $x\neq 0$ exist), we conclude that the global phase portraits are topologically equivalent to the following ones of Figure 1: 1.9 ($s=1$, $a=1$, $b=3$, $c=17/10$); 1.10 ($s=1$, $a=1$, $b=3$, $c=2$. Here c is the value in (19)); 1.11 ($s=1$, $a=1$, $b=3$, $c=21/10$); and 1.13 ($s=1$, $a=1/4$, $b=1/2$, $c=1/20$).

In the case (iii.2) we need to investigate the possible saddle connections between the saddle p and the saddle at the origin, between the cusps and the origin, and between the cusps and the saddle p. The possible saddle connection between the saddle p and the saddle at the origin is along the surface in (19) and between the cusps p_1, p_2 and the saddle p is, whenever it exist, along the surface which is a solution of the equation in (21). Finally, the possible saddle connection between the cusps and the origin is along the surface in (20). Taking into account that we know the region in the parameters space where condition (iii.2) holds (because we know the region in the parameter space where condition (iii) holds and we also know that $c=c_0$), we conclude that the global phase portraits are topologically equivalent to the following ones of Figure 1: 1.25 ($s=1$, $a=1$, $b=-6$, $c=c_0$); 1.114 ($s=-1$, $a=1/10$, $b=449/2000$, $c=c_0$); 1.115 ($s=-1$, $a=1/10$, $b=\sqrt{2}(-135+66\sqrt{5})/79$, $c=c_0$. Here b is the solution of (19)); 1.116 ($s=-1$, $a=1/10$, $b=23/100$, $c=c_0$); 1.117 ($s=-1$, $a=1/2$, $b=-87/100$, $c=c_0$); 1.118 ($s=-1$, $a=1/2$, $b=-3\sqrt{2}/5$, $c=c_0$. Here b is the solution of (19)); 1.119 ($s=-1$, $a=5/2$, $b=-1.2733525026$, $c=c_0$. Here b is the real solution of the surface in (21)); and 1.120 ($s=-1$, $a=5/2$, $b=-319/250$, $c=c_0$).

In case (iii.3) we need to investigate the possible saddle connections. The connections between the saddles p_3, p_4 and the saddle at the origin is given along the surface $c=c(a,b)$ given in (20) and the connection between the saddle p and the origin is along the surface $c=c(a,b)$ given in (19). Finally, the possible connection between the saddle p and the saddles p_3, p_4 is along the surface $c = c(a, b)$ which is any real solution (whenever it exists) of the equation (21).

Taking all the above into account and recalling that we know the region in the parameters space where condition (iii.3) holds (because we know the region in the parameter space where condition (iii) holds and we also know that all the four solutions with $x \neq 0$ exist), we conclude that the global phase portraits are topologically equivalent to the following ones of Figure 1: 1.14 ($s = 1$, $a = 1$, $b = -11/2$, $c = 1/10$); 1.15 ($s = 1$, $a = 1$, $b = -11/2$, $c = (103 - 66\sqrt{2})/36$. Here c is a value in (20)); 1.16 ($s = 1$, $a = 1$, $b = -11/2$, $c = 269/1000$); 1.17 ($s = 1$, $a = 1$, $b = -11/2$, $c = 0.27064859$. Here c is the value of the real solution of (21)); 1.18 ($s = 1$, $a = 1$, $b = -11/2$, $c = 3/10$); 1.121 ($s = -1$, $a = 1/10$, $b = 1/10$, $c = 1/1000$); 1.122 ($s = -1$, $a = 1/10$, $b = 1/10$, $c = 2b^2/9$. Here c is the value in (19)); 1.123 ($s = -1$, $a = 1/10$, $b = 1/10$, $c = 23/10000$); 1.124 ($s = -1$, $a = 1/4$, $b = -6/5$, $c = 44/125$); 1.125 ($s = -1$, $a = 1/4$, $b = -6/5$, $c = 0.35787810$. Here c is the value of the real solution of (21)); 1.126 ($s = -1$, $a = 1/4$, $b = -6/5$, $c = 359/1000$); and 1.127 ($s = -1$, $a = 1/4$, $b = -1$, $c = 2/9$. Here c is the value of the real solution of (19)).

In case (iv) the total index of the infinite singular points and the finite singular points with $x = 0$ in the Poincaré sphere is -1. Hence, the remaining finite singular points with $x \neq 0$ have to have total index 2. Taking into account Lemma 6 and the symmetry of the system they must be two centers. The saddles p_1 and p_2 cannot be connected one with each other and they can be connected with the saddle at the origin along the surface in (20). Taking all the above into account and recalling that we know the region in the parameters space where condition (iv) holds, we conclude that the global phase portraits are topologically equivalent to the following ones of Figure 1: 1.128 ($s = 1$, $a = -1/2$, $b = -1$, $c = -1$) and 1.129 ($s = -1$, $a = 1$, $b = -3/2$, $c = 1/2$. Here c is the value in (20) and so there is a saddle connection).

In case (v) the total index of the infinite singular points and the finite singular points with $x = 0$ in the Poincaré sphere is 3. Hence, the remaining finite singular points with $x \neq 0$ have to have total index -2. Taking into account Lemma 6 and the symmetry of the system they must be two saddles p_3 and p_4. The saddles p_3 and p_4 are connected by symmetry. We recall that the possible saddle connection between the saddles p_3, p_4 and the saddle at the origin is along the surface in (20). Taking all the above into account and recalling that we know the region in the parameters space where condition (v) holds, we conclude that the global phase portraits are topologically equivalent to the following ones of Figure 1: 1.130 ($s = 1$, $a = 1$, $b = 4/5$, $c = 1/25$); 1.131 ($s = 1$, $a = 1$, $b = 1$, $c = (6\sqrt{2} - 7)/18$.

Here c is the value in (20)); 1.132 ($s = 1$, $a = 1$, $b = 1$, $c = 1/20$); and 1.133 ($s = -1$, $a = -1/2$, $b = -1/2$, $c = -1/10$)

In case (vi) we have $s = -1$, $b = -\sqrt{2}$ and $c = 1/2$. In this case we must have $a > 0$ (because $c > 0$). Among the origin and the finite singular point $(0, \sqrt{2})$ which is formed by the union of six hyperbolic sectors (3 stable and 3 unstable) we have two finite singular points with $x = 0$ that are two centers. The possible global phase portraits are topologically equivalent to 1.134 of Figure 1.

19. Global phase portraits of system (VIII) when $ac + 1 > 0$, $cs < 0$ and $as > 0$

By Theorems 2 and 3 the four nodes at infinity and the saddle at the origin have total index 6 on the Poincaré sphere, the remaining finite singular points on the Poincaré sphere have to have total index -4. Moreover, studying the finite singular points with $x = 0$ in the case $ac + 1 > 0$, $cs < 0$ and $as > 0$, and taking into account Lemma 7, we have the following possibilities: (i) there are no singular points in $x = 0$ (since $b^2 < 4c$ we must have $c > 0$ and so $s = -1$), (ii) a cusp (since $b^2 = 4c$ we must have $c > 0$ and so $s = -1$ and also $b \neq -\sqrt{2}$), (iii) a center and a saddle p, (iv) two saddles p_1, p_2; and (v) a point which is the union of six hyperbolic sectors (three stable and three unstable) whenever $s = -1$, $b = -\sqrt{2}$ and $c = b^2/4$. Again, in all the chapter whenever there is more than one possible phase portrait, we give in parenthesis values of the parameters where they are realized.

In case (i) the total index of the infinite singular points and the finite singular points with $x = 0$ in the Poincaré sphere is -6. Hence, the remaining finite singular points with $x \neq 0$ have to have total index -4. In view of Lemma 6 and taking into account the symmetry of the system, they can only be two saddles. The two saddles must be connected by symmetry. We have $s = -1$ and we see that there are no possible saddle connections between these saddles and the saddle at the origin. Taking all the above into account we conclude that the global phase portraits are topologically equivalent to 1.135 in Figure 1. The phase portrait is realized for example when $a = -1/2$, $b = -2$, $c = 3/2$ and $s = -1$.

In case (ii) the total index of the infinite singular points and the finite singular points with $x = 0$ in the Poincaré sphere is again 6. By Lemma 6 and taking into account the symmetry of the system the remaining finite singular points with $x \neq 0$ can only be two saddles. The two saddles must be connected by symmetry. We have $s = -1$ and we see that there are no possible saddle connections between these saddles and the saddle at the origin (neither with the cusp and the saddle at the origin). Taking all the above into account we conclude that the global phase portraits are topologically equivalent to the following ones of Figure 1: 1.136 ($a = -1/2$, $b = 2$,

$c = 1$); 1.137 ($a = -1/2$, $b = -2$, $c = 1$); and 1.138 ($a = -1/2$, $b = -1$, $c = 1/4$).

In case (iii) the total index of the infinite singular points and the finite singular points with $x = 0$ in the Poincaré sphere is again 6. Hence, the remaining finite singular points with $x \neq 0$ have to have total index -4. So, in view of Lemma 6 taking also into account the symmetry of the system, they can only be two saddles p_3, p_4. The two saddles p_3, p_4 must be connected by symmetry. The possible saddle connection between these saddles and the saddle at the origin is along the surface in the parameter space given in (20), and between the saddle p and the origin is along the surface $c = c(a, b)$ given in (19). Finally, the possible connections between the saddle p and the saddles p_3, p_4 is along the surface $c = c(a, b)$ in the parameter space which is any real solution (whenever it exists) of the equation (21).

Taking all the above into account and recalling that we know the region in the parameters space where condition (iii) holds, we conclude that the global phase portraits are topologically equivalent to the following ones of Figure (1): 1.139 ($s = 1$, $a = 1/2$, $b = -3/2$, $c = -17/10$); 1.140 ($s = 1$, $a = 1/2$, $b = -1/2$, $c = -(35 + 12\sqrt{2})/54$. Here c is the value in (20)); 1.141 ($s = 1$, $a = 1/2$, $b = -1/2$, $c = -9/10$); 1.142 ($s = -1$, $a = -1/2$, $b = 1$, $c = 1/10$); 1.143 ($s = -1$, $a = -1/2$, $b = 1$, $c = 2/9$. Here c is the value in (19)); 1.144 ($s = -1$, $a = -1/2$, $b = 1$, $c = 6/25$); 1.145 ($s = -1$, $a = -1/2$, $b = -1/2$, $c = 3/100$); 1.146 ($s = -1$, $a = -1/2$, $b = -1/2$, $c = 0.05793018$. Here c is the value in (21)); and 1.147 ($s = -1$, $a = -1/2$, $b = -1/2$, $c = 3/50$).

In case (iv) the total index of the infinite singular points and the finite singular points with $x = 0$ in the Poincaré sphere is 2. Hence, the remaining finite singular points with $x \neq 0$ have to have total index 0. Hence, in view of Lemma 6 and taking into account the symmetry of the system, they can only be: (iv.1) none; (iv.2) two cusps, (iv.3) two saddles (p_3, p_4) and two centers.

In case (iv.1) we see that there are no saddle connections between the saddles p_1 (or p_2) and the origin. Hence, taking also into account that we know the region in the parameters space where condition (iv.1) holds (since we know the region in the parameter space where condition (iv) holds and we also know that in this case there are no solutions with $x \neq 0$), we conclude that the global phase portraits are topologically equivalent to 1.5 of Figure 1 which can be realized for instance when $s = 1$, $a = 1/4$, $b = -1$ and $c = -1$.

In case (iv.2) we can prove that there are again no saddle connections between the saddles p_1 (or p_2) and the saddle at the origin. Taking also into account that we know the region in the parameters space where condition (iv.2) holds (since we know the region in the parameter space where condition (iv) holds and $c = c_0$), we conclude that the global phase portraits are topologically equivalent to 1.24 of Figure 1 which can be realized for example when $s = 1$, $a = 1$, $b = -1$ and $c = c_0$.

In case (iv.3) the possible saddle connection between the saddles p_1 (or p_2) and the saddle at the origin is along the surface in (20), between the saddles p_3, p_4 and the saddle at the origin is along the surface in (19), and between the saddles p_1, p_2 and the saddles p_3, p_4 is along the surface which is any real solution of the equation in (21). Taking also into account that we know the region in the parameters space where condition (iv.3) holds (because we know the region in the parameter space where condition (iv) holds and we also know that all the four solutions with $x \neq 0$ exist), we conclude that the global phase portraits are topologically equivalent to the following ones of Figure 1: 1.23 ($s = 1$, $a = 1$, $b = -1/5$, $c = -63/100$); 1.22 ($s = 1$ $a = 1$, $b = -1/5$, $c = -(223 + 30\sqrt{2})/450$. Here $c = c(a, b)$ is the real solution of the equation (19)); 1.21 ($s = 1$, $a = 1$, $b = -1/5$, $c = -117/200$); 1.20 ($s = 1$, $a = 1$, $b = -1/5$, $c = -0.54752598$. Here $c = c(a, b)$ is the real solution of the equation (21)); and 1.19 ($s = 1$, $a = 1$, $b = -1/5$, $c = -27/50$).

In case (v) we have $s = -1$, $b = -\sqrt{2}$, $c = 1/2$. In this case, since $a < 0$, the unique finite singular point among the origin is $(0, \sqrt{2})$ which is the formed by the union of six hyperbolic sectors (3 stable and 3 unstable). So, the global phase portraits are topologically equivalent to 1.148 of Figure (1) which is achieved for instance when $a = -1/2$.

20. Global phase portraits of system (VIII): final remarks

We note that system (VIII) provides all phase portraits in Figure 1 except 1.34, which is not possible because from Lemma 4 the origin of U_2 cannot be the union of one hyperbolic and three elliptic sectors.

In the rest of this section we explain the procedure that we have used in Sections 17, 18 and 19 to give all possible phase portraits. We do it for the case (iv.3) with $s = 1$ in Section 19 to illustrate the procedure and the other cases have been treated in a similar way.

In case (iv.3) with $s = 1$ in Section 19 there are four singular points at infinity so $ac + 1 > 0$, $c < 0$ and $a > 0$ (see Lemma 4), and there are two singular points with $x = 0$ (two saddles). So, from the results in Section 15 we get that

$$(a, b, c) \in \{c_1 < c < 0, b < 1/\sqrt{2}, a > -1\},$$

where c_1 is defined in Section 15. Let $b^{*1} = (a - 2)/(\sqrt{2}a)$. Joining the above two conditions we get that $(a, b, c) \in S_1$ with

$$S_1 = \left\{a > 0, b \leq b^{*1}, c_2 < c < 0\right\} \cup \left\{a > 0, b^{*1} < b < \frac{1}{\sqrt{2}}, c_1 < c < 0\right\},$$

where again c_1, and c_2 are defined in Section 15. The four singular points with $x \neq 0$ are defined when $(2ab + 3\sqrt{2}s)^2 - 16(1 + a)(ac + 1) > 0$ and $\bar{x}_\pm^2$ evaluated at $\bar{y}_\pm$ is positive. On the other hand, the determinant of the Jacobian matrix evaluated at $x = \bar{x}_\pm$ is

$$d_\pm = -2\bar{x}_\pm \left(2ab\bar{y}_\pm + 3ac\bar{y}_\pm^2 + a + 3\sqrt{2}s\bar{y}_\pm + 3\bar{y}_\pm^2 + 1\right).$$

So $(\pm\bar{x}_\pm, \bar{y}_\pm)$ are two saddles and two centers when either $d_+ > 0$ and $d_- < 0$, or $d_+ < 0$ and $d_- > 0$. Let $b^{*2} = (2a - 1)/(\sqrt{2}a)$. Computing the set where $(\pm\bar{x}_\pm, \bar{y}_\pm)$ are two saddles and two centers for $a > 0$ and $b < 1/\sqrt{2}$ we get that $(\pm\bar{x}_+, y_+)$ are saddles and $(\pm\bar{x}_-, y_-)$ are centers when $(a, b, c) \in S_2$ with

$$S_2 = \left\{a > 0, b < -\frac{3}{\sqrt{2}a}, c_2 < c < c_0\right\},$$

and $(\pm\bar{x}_+, y_+)$ are centers and $(\pm\bar{x}_-, y_-)$ are saddles when $(a, b, c) \in S_3$ with

$$\begin{aligned} S_3 \quad = \quad & \left\{0 < a \leq 1, \frac{-3}{\sqrt{2}a} < b < b^{*2}, c_2 < c < c_0\right\} \cup \\ & \left\{a > 1, -\frac{3}{\sqrt{2}a} < b < \frac{1}{\sqrt{2}}, c_2 < c < c_0\right\}. \end{aligned}$$

In short, the conditions of case (iv.3) in Section 19 with $s = 1$ are satisfied for

$$(a, b, c) \in S_0 = S_1 \cap (S_2 \cup S_3).$$

The projection onto the (a, b)-plane of the region S_0 is the set

$$A = \{0 < a \leq 1, b < b^{*2}\} \cup \{a > 1, b < 1/\sqrt{2}\}.$$

The values of c on the region S_0 are delimited by one of the surfaces $c = c_2$, $c = c_1$, $c = c_0$, and $c = 0$. Moreover the projection on the (a, b)-plane of the intersection of the surfaces $c = c_0$ with $c = 0$, and of $c = c_1$ with $c = 0$ are the curves

$$b = b^{*3} = -(-3\sqrt{2} \pm 4\sqrt{1 + a})/(2a) \quad \text{and} \quad b = 1/\sqrt{3},$$

respectively. The surface $c = c_2$ does not intersect $c = 0$. The projection on the (a, b)-plane of the intersection of the surfaces $c = c_0$ with $c = c_1$ and of $c = c_2$ with $c = c_1$ are the curves $b = b^{*2}$ and $b = b^{*1}$, respectively. Finally the projection on the (a, b)-plane of the intersection of the surfaces $c = c_0$ and $c = c_3$ is the curve $b = -3/(\sqrt{2}a)$. In a similar way, we compute the projection on the (a, b)-plane of the intersections between the surfaces $c = c_0$, $c = c_1$ and $c = c_2$ with the surfaces providing all possible saddle connections (given in (19), (20) and (21)); and of the intersections between these last surfaces.

All the above mentioned intersections give a set of curves that divide A into several regions. For each one of these regions we pick a point (a, b), and for this value of (a, b) we find the set $c \in I$ for which $(a, b, c) \in S_0$.

Moreover, we also compute the values of c on the surfaces $c = 0$, $c = c_0$, $c = c_1$, $c = c_2$ and the ones given in (19), (20) and (21). These last values divide I in several regions. Finally we pick a value of c in each one of these regions and draw the phase portrait for the selected triple (a, b, c).

Acknowledgements

The first author is partially supported by a FEDER-MINECO Grant MTM2016-77278-P. The second author is partially supported by FCT/Portugal through UID/MAT/04459/2013.

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