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Localization-protected quantum order

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Closed quantum systems with quenched randomness exhibit many-body localized regimes wherein they do not equilibrate, even though prepared with macroscopic amounts of energy above their ground states. We show that such localized systems can order, in that individual many-body eigenstates can break symmetries or display topological order in the infinite-volume limit. Indeed, isolated localized quantum systems can order *even* at energy densities where the corresponding thermally equilibrated system is disordered, i.e., localization protects order. In addition, localized systems can move between ordered and disordered localized phases via nonthermodynamic transitions in the properties of the many-body eigenstates. We give evidence that such transitions may proceed via localized critical points. We note that localization provides protection against decoherence that may allow experimental manipulation of macroscopic quantum states. We also identify a "spectral transition" involving a sharp change in the spectral statistics of the many-body Hamiltonian.

I. INTRODUCTION

Our current understanding of the phases of quantum matter in equilibrium is built largely on the traditional Landau framework of broken symmetries[1] and the more recent framework, still in rapid evolution, of topological order and allied classifications.[2–7] There are interesting exceptions to these in the presence of quenched randomness such as Anderson localization,[8] which is firmly established in studies of noninteracting particles.[9] Recently, the work of Basko, Aleiner, and Altshuler[10] and others[11–17] has added to these a previously conjectured[8] extension of Anderson localization to closed, quantum interacting systems, the phenomenon now known as many-body localization (MBL).

To understand the nature of many-body localization, it is useful to first refer to the eigenstate thermalization hypothesis (ETH).[18–20] The ETH, when true, applies to the exact many-body eigenstates of the Hamiltonian of a closed, isolated quantum system, in the limit of many degrees of freedom. The ETH postulates that for a large class of quantum systems, the probability operator (a.k.a. the reduced density matrix) for any subsystem is, in any exact many-body eigenstate of the full system, equal to the equilibrium Boltzmann-Gibbs distribution at the temperature set by the energy density of the eigenstate. This occurs because the remainder of the full system successfully acts as a thermal bath for the subsystem in question.

The many-body localization phase transition is an *eigenstate phase transition* from the thermal phase where the exact many-body eigenstates obey the ETH to the localized phase where the eigenstates violate the ETH; the latter fail to be a heat bath that thermally equilibrates its subsystems.[8,10,12,15] Thus, this is a dynamic, but not thermodynamic, phase transition from the thermal phase where the system does thermally equilibrate under the dynamics due to its own Hamiltonian to the "glassy" localized phase where the isolated quantum system can remain far from thermal equilibrium forever. In

a related diagnostic, MBL eigenstates generally display an "area law" entanglement entropy,[21] unlike thermal eigenstates where the entanglement entropy is generally a "volume law" that reproduces the thermodynamic entropy.

In this work, we examine the highly excited eigenstates, defined as eigenstates with a macroscopic energy above the ground state, of MBL systems and point out that they come in many flavors, and may be classified in terms of broken symmetries, topological order, and/or criticality, very much as in the usual account of phases and phase transitions in equilibrium systems. We note that in the presence of many-body localization, equilibrium constraints on order can be evaded: symmetry breaking can occur in highly excited states of one-dimensional systems, and topological order can arise *even in the absence of a bulk gap*. Instead, it is the localization that "protects" the order. We suggest that there can even arise continuous phase transitions between distinct MBL phases, which proceed via MBL critical points. Finally, we point out the existence of yet another kind of transition within the MBL phase: a "spectral transition," which does not involve a change in the properties of the eigenstates, but instead a change in the spectral statistics of the system's Hamiltonian. We emphasize that unlike standard discussions of quantum phase transitions, our discussion is not about ground states or low-lying excited states, but is about highly excited eigenstates at energies that would correspond to nonzero (even infinite) temperature if the system could thermalize at these energies. This might be useful for experiments, in particular for experiments designed to exploit topological order. Note that here we only work with spin models that have a bounded spectrum. However, MBL is expected to occur even for particles moving in continuous space, at least for certain one-dimensional systems.[17]

This paper is structured as follows: We first consider three equivalent "integrable"[22] one-dimensional systems: the Majorana chain, *p*-wave superconducting chain, and transverse-field Ising chain.[23] The eigenstates of the disordered

versions of these Hamiltonians are localized and can be described in terms of noninteracting localized fermions. Reference 10 showed that this localization is robust to weak fermion-fermion interactions, and we assume in this paper that this is indeed true. The MBL eigenstates can break symmetries, and a robust notion of topological order can be defined for them even without a gap. We also show how by tuning parameters of the Hamiltonian, we can drive an eigenstate phase transition from one MBL phase to another. In the Ising chain, the transition is of the symmetry-breaking type, from a disordered paramagnet to a phase where the eigenstates are feline (Schrödinger cat) states with long-range spin-glass (SG) order. In the Majorana and Dirac fermion systems, the transition is between MBL states with and without topological order. We further argue that the critical point separating these two distinct MBL phases can itself be MBL. We close by discussing extensions of these ideas to higher dimensions. We emphasize again that the ideas discussed here apply only to closed quantum systems, i.e., it is essential that the system not be put in contact with an external thermal bath.

II. ONE-DIMENSIONAL SYSTEMS

The three noninteracting one-dimensional models studied in this paper (to which we will add interactions perturbatively) are the transverse-field Ising, Majorana, and Dirac fermion chains of finite length L sites, with open ends:

$$H_{\text{Ising}} = -\sum_{i=1}^{L-1} J_i \sigma_i^z \sigma_{i+1}^z - \sum_{i=1}^{L} h_i \sigma_i^x, \tag{1}$$

$$H_{\text{Majorana}} = -\sum_{j=1}^{L-1} i J_j b_j a_{j+1} - \sum_{j=1}^{L} i h_j a_j b_j, \tag{2}$$

$$H_{\text{Dirac}} = -\sum_{j=1}^{L-1} J_j (c_j^\dagger c_{j+1}^\dagger + c_j^\dagger c_{j+1} + \text{H.c.})$$
$$- \sum_{j=1}^{L} h_j (1 - 2c_j^\dagger c_j), \tag{3}$$

where the $\sigma^{x,z}$ are Pauli matrices, the a,b are (self-adjoint) Majorana fermion operators, and the $c,c^\dagger$ are conventional Dirac fermion operators. The parameters J_i and h_i are drawn from distributions $P(J)$ and $P(h)$ with means $\overline{J} > 0$ and $\overline{h} > 0$ and variances δ_J^2 and δ_h^2. The precise details of the distribution are unimportant for our present purposes, but it is vital that at least one of the variances be nonzero. For specificity, take the distributions to be log-normal, so all J_i and h_i are positive. We consider the disorder strengths δ^2 to be parameters in our analysis. These three Hamiltonians are related to one another by the duality transformations[23,24]

$$a_k = c_k + c_k^\dagger = \left(\prod_{j<k} \sigma_j^x\right) \sigma_k^z,$$
$$\tag{4}$$
$$b_k = -i(c_k^\dagger - c_k) = \left(\prod_{j<k} \sigma_j^x\right) \sigma_k^y.$$

The above Hamiltonians all possess a global Z_2 symmetry, implemented by the operator $\hat{P}$, which takes the form

$$\hat{P} = \prod_{j=1}^{L} \sigma_j^x = \prod_{j=1}^{L} i a_j b_j = \prod_{j=1}^{L} (1 - 2c_j^\dagger c_j). \tag{5}$$

In the ordered phase (typical $h/J < 1$), there is an "edge-mode" operator $\hat{O}^\dagger$, whose commutator with the Hamiltonian is exponentially small $[(h/J)^L \ll 1]$ in L. In the Majorana language,[23]

$$\hat{O}^\dagger = a_1 + i b_L + \frac{h_1}{J_1} a_2 + i \frac{h_{L-1}}{J_{L-1}} b_{L-1} + \frac{h_1 h_2}{J_1 J_2} a_3$$
$$+ i \frac{h_{L-1} h_{L-2}}{J_{L-1} J_{L-2}} b_{L-2} + \dots . \tag{6}$$

This operator creates a Dirac fermion which is *bilocalized* near the two edges, and has an energy which is exponentially small in the system size $E \sim \exp[-L \ln(J/h)]$. The existence of this quasi-zero-energy edge mode can be considered a diagnostic for topological order. Since $\hat{O}$ and $\hat{P}$ anticommute, acting on a state with $\hat{O}$ or $\hat{O}^\dagger$ flips its eigenvalue under $\hat{P}$.

A. Localization in the ordered phase

To discuss localization in the noninteracting model, it is convenient to use the Ising spin formulation (1). The results apply equally to the Majorana and Dirac fermion chains.

In the ordered phase $\overline{J} - \sqrt{\delta_J^2} \gg \overline{h}$, the ground state is a ferromagnet, with long-range order of σ^z. It is even under $\hat{P}$, consisting of a linear combination of states with the average z magnetization "up" and "down." Let us call it $|0,+\rangle$. It is a "feline" (Schrödinger cat) state consisting of a coherent linear combination of two "macroscopically" different states. If we let $\hat{O}$ operate on this ground state, this makes the other "ground state" $|0,-\rangle = \hat{O}|0,+\rangle$, which is also feline, is odd under $\hat{P}$, and is higher in energy by an amount that is exponentially small in L.

The normal modes in the ordered phase are fermion operators, which anticommute with each other and with $\hat{P}$. The action of these normal modes on the spin state is straightforward: they create domain walls (with respect to the ferromagnetic ground state). Domain walls sit on bonds, and are created by the self-adjoint operators

$$d_{k+1/2} = \left(\prod_{j<k} \sigma_j^x\right)$$
$$= \left(\prod_{j<k} i a_j b_j\right) = \left(\prod_{j<k} (1 - 2c_j^\dagger c_j)\right). \tag{7}$$

For this integrable Hamiltonian, the properties of an eigenstate may be extracted from the behavior of the normal modes, which create domain walls. In the classical limit $h = 0$, the normal modes are trivially localized. Away from the classical limit, they hop under the action of the transverse fields h_i, and see a spatial potential J_i. In the clean limit, the normal modes (and hence the domain walls) are delocalized over the chain. However, it is well known that noninteracting fermions hopping in a one-dimensional random potential always

experience localization.[9] Therefore, in the presence of nonvanishing disorder in J_i, the normal modes (and hence the domain walls) must necessarily be localized. In the strong disorder limit, when $\delta_J^2 \gg \overline{h}^2$, the single-domain-wall eigenstates are exponentially localized, with localization length

$$\xi_{\text{loc}} \sim \frac{1}{\ln \left| \delta_J^2 / \overline{h}^2 \right|}. \tag{8}$$

This follows straightforwardly from perturbation theory in small h/δ_J. Localization of the domain walls in each many-body eigenstate implies that each eigenstate has long-range spin-glass order,[25] i.e., the correlation function $\langle \sigma_0^z \sigma_r^z \rangle$ within the eigenstate is nonzero for large r, with its sign set by how many domain walls are present and localized between sites 0 and r. Observe that this spin-glass order breaks the global Z_2 symmetry in these highly excited localized eigenstates of this one-dimensional system, although at thermal equilibrium such discrete symmetry breaking at nonzero temperatures is forbidden. This eigenstate spin-glass ordering is also discussed in Pekker *et al.*[26]

Just as this model in the ordered phase has two nearly degenerate ground states $|0,\pm\rangle$ that are each feline, the excited eigenstates also come in nearly degenerate feline pairs $|n,\pm\rangle$, produced by adding some particular set of localized domain walls n to each of the two ground states. The states $|n,\pm\rangle$ are orthogonal eigenstates of the global spin-flip operator $\hat{P}$, with eigenvalue ± 1. From these two eigenstates we can make the state $|n,\uparrow\rangle \equiv (|n,+\rangle + |n,-\rangle)/\sqrt{2}$ which has some particular pattern of local z magnetizations dictated by the localized domain walls. Each eigenstate $|n,\pm\rangle$ is a (feline) coherent linear combination of this spin-glass state $|n,\uparrow\rangle$ with the opposite state under the global spin-flip operation $|n,\downarrow\rangle = \hat{P}|n,\uparrow\rangle$. However, the states $|n,\pm\rangle$ are eigenstates of the Hamiltonian even for a finite-sized system, unlike the spin-glass superpositions $|n,\uparrow\rangle$ and $|n,\downarrow\rangle$. The energy splitting between $|n,\pm\rangle$ is exponentially small in system size, and thus the energy uncertainty in the symmetry-breaking states $|n,\uparrow\rangle$ and $|n,\downarrow\rangle$ vanishes exponentially with the number of spins in the infinite-volume limit. We note in passing that the existence of spin-glass order in the eigenstates is consistent with exponential localization of the local fluctuations of conserved quantities, a hallmark of the MBL phase.

B. Robustness of localization to weak interactions

We assume we are working with highly excited states, where the localized domain walls are dense and frequently overlap (the typical domain-wall separation can be less than a localization length).[27] We then add to the Hamiltonian weak short-range interactions between these domain walls, and ask whether they cause a breakdown of localization. Weak interactions between domain walls may be introduced in the spin model by adding, e.g., a $\lambda \sigma_i^x \sigma_{i+1}^x$ term to the Hamiltonian (1). Such a term preserves the global spin-flip symmetry of the problem, but upon fermionization (4) gives rise to a Hamiltonian containing local four fermion "interaction" terms. More general interaction terms may also be introduced in the spin model, but we assume that these terms commute with $\hat{P}$ and thus respect the Z_2 symmetry. The situation with weak
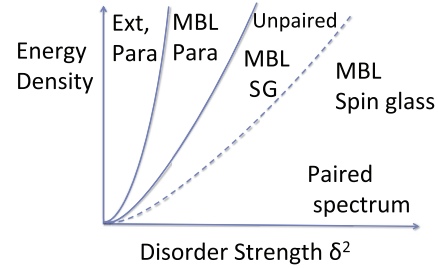

FIG. 1. (Color online) In the one-dimensional models we consider, eigenstate properties depend on the energy density, the typical value of h/J, the disorder strengths δ_J and δ_h, and the fermion-fermion interaction strength λ. This figure shows, schematically, a slice through the phase diagram, for typical $h/J < 1$ and nonzero λ. Extended (Ext) and many-body localized (MBL) phases are separated by the localization transition, ordered spin glass (SG) and paramagnetic (Para) phases are separated by an eigenstate phase transition, and regions with paired and unpaired spectra are separated by a spectral transition. Here and in Fig. 3, we choose the zero of energy density to be the ground state.

interactions is then essentially identical to that analyzed by Basko *et al.*,[10] who show that many-body localization is stable to weak nonzero interaction λ as long as the localization length of the single-particle states is finite.

If the disorder is weak, the localization will be destroyed by strong enough interaction, making the extended, thermal phase at weak disorder and high-energy density, as shown in Fig. 1. In terms of the Ising model, in the extended phase the spin-spin correlations within a thermal eigenstate are short range. If we assume the eigenstate phase transitions are continuous (not first order), then there must be two transitions, as indicated in Fig. 1. Starting in the extended thermal phase and increasing the disorder, first we reach the localization transition. Once we enter the localized phase, the spin-spin correlations in the eigenstates can start increasing to longer range than they are at thermal equilibrium. The transition to the MBL spin-glass phase is when the spin correlations develop long-range order.

C. Spectral transition in the ordered phase

In the MBL spin-glass phase, all eigenstates come in parity-related pairs $|n,\pm\rangle$ which differ only by their occupation of the single-particle edge mode created by $O^\dagger$. The energy of the edge mode is exponentially small in the chain length $E_O \sim \exp(-L/\xi)$, where ξ is the localization length of the edge mode. Thus, the level splitting within such a pair is $E_O \sim \exp(-L/\xi)$. Meanwhile, the typical many-body level spacing is of order $\delta E \sim \exp(-sL)$, where s is the thermodynamic entropy per site that would result if the system equilibrated at the given energy density. Thus, tuning ξ or s leads to a spectral transition. For strong disorder and thus small localization length ξ and/or for low-energy density and thus low entropy s, the spectrum is "paired." Here, the edge-mode level splitting is less than the typical level spacing so the many-body spectrum at large L consists of nearly degenerate doublets with Poisson interdoublet level-spacing statistics. The level spacing within each doublet is exponentially small in system size compared to the typical level spacing between doublets. Meanwhile, for weaker disorder and/or higher entropy, the spectrum is
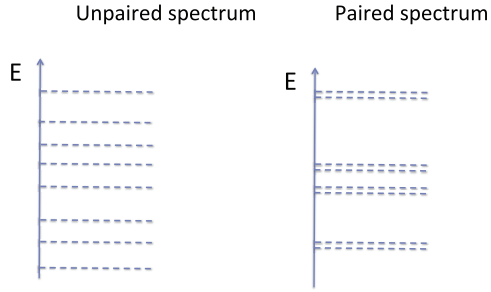
FIG. 2. (Color online) An illustration of the spectral transition between paired and unpaired many-body spectra, driven for example by tuning disorder strength.

unpaired. In this unpaired regime, the energy of the single-particle edge mode is exponentially larger than the typical many-body level spacing, and as a result we have Poisson statistics for all of the individual many-body energy levels. The Poisson (or paired Poisson) level statistics themselves are also a diagnostic for MBL since in the thermal phase, the level statistics are instead those of a random matrix ensemble, for example, the Gaussian orthogonal ensemble (GOE) for the models we are presently considering.

The spectral transition is illustrated in Fig. 2. It does not involve a change in the symmetry or topological order of the eigenstates, but rather involves a change in the many-body spectral statistics of the Hamiltonian. It may be detected in numerics by examining the ratio of two consecutive gaps $r = \min(\delta_n, \delta_{n+1})/\max(\delta_n, \delta_{n+1})$, where δ_n is the energy gap between the nth and $(n + 1)$th many-body eigenstates. This ratio will have average value $(2 \ln 2 - 1)$ in the "unpaired" localized regime,[12] whereas the average value in the paired regime will be exponentially small in system size (zero in the infinite-volume limit). Thus, the ratio of two consecutive gaps provides a sharp diagnostic for the spectral transition.

D. Magnetic response

One curious feature of these pairs $|n, \pm\rangle$ of feline eigenstates is that the states $|n, \uparrow\rangle$ have a typical z magnetization m that scales as $\sim \sqrt{L}$. A longitudinal magnetic field B added to the Hamiltonian thus acts on the Hilbert space of a pair (in the $B = 0$ eigenstate basis) as

$$H_{\text{eff}} \sim \begin{pmatrix} E & B\sqrt{L} \\ B\sqrt{L} & E + \bar{h}e^{-L/\xi} \end{pmatrix}. \qquad (9)$$

It is easy to verify that the energy eigenstates each have an adiabatic magnetic susceptibility $\chi = (\partial m/\partial B)_{B=0}$ that is exponentially large in L, but is of opposite sign between the two paired eigenstates. The feline nature of the eigenstates is thus destroyed by an exponentially small (in L) longitudinal field. Interestingly, this makes this system a possibly very sensitive magnetic field sensor. If a state with a nonzero z magnetization is prepared, it will be localized and stable as long as the magnitude of the longitudinal field B is larger than $\sim e^{-L/\xi}$. However, once the magnitude of the longitudinal field drops below this level, then the system's many-body eigenstates, although still localized, are feline and the magnetization will decrease. The main caveat to

this apparently exponentially fine-in-L field sensitivity is that the resulting magnetization dynamics is exponentially slow in L, and the utility of this idea will be limited by the decoherence rate of any real system. We note, however, that, in our three-dimensional world, three-dimensional MBL systems where only the boundary is coupled to the environment should be protected from environmental decoherence since the effects of the environment should only propagate into the system up to one localization length.

E. Localization in the paramagnetic phase

We now turn our attention to the disordered regime $\bar{h} - \sqrt{\sigma_h^2} \gg \bar{J}$. In this regime, the (Ising) ground state has all spins aligned on average along the x axis, and the elementary excitations are spin flips. A similar analysis to above leads one to conclude that all spin flips are localized in the noninteracting model, as long as $\delta_h^2 \neq 0$. Localization is robust against weak enough interactions so long as the localization length ξ is finite. Thus, the eigenstates of the disordered Ising paramagnet can also be MBL. However, these eigenstates do not break Z_2 symmetry and do not come in pairs. Likewise, there is no edge mode in the fermion language, and no topological order.

These results can also be understood in terms of the well-known self-duality of the one-dimensional Ising model,[24] which swaps h and J. In particular, it follows from self-duality that if the ordered phase is MBL, the disordered phase is also MBL. However, the spectrum in the paramagnetic phase is not paired. This is because the pair of parity-related MBL eigenstates $|n, \pm\rangle$ on the ordered side differ by the edge mode $\hat{O}$, and the self-duality acts nonlocally on the edge mode, mapping the parity related-pair of ordered states to disordered states with different boundary conditions. Thus, the disordered side (with specified boundary conditions) has states that do not come in pairs and do not have edge modes, and are separated from their counterparts on the ordered side by a phase transition.

F. Phase transition

In the absence of interactions ($\lambda = 0$) there is a continuous transition between localized phases, which proceeds via an infinite-randomness critical point *even* for highly excited states. We begin by discussing this noninteracting picture, before examining the effect of interactions. The critical regime is treated using the strong disorder renormalization group (SDRG).[28] The SDRG proceeds by sequentially identifying the strongest bond or field in the Hamiltonian, diagonalizing it, and determining the coupling to the rest of the system perturbatively. This procedure can be shown to be asymptotically exact because the system flows to strong disorder.

When looking for ground states,[28] after diagonalizing a particular bond or field, we should truncate to the lower-energy subspace. However, this method can be easily generalized to obtain excited states by sometimes truncating into the higher-energy subspace instead.[16] It can be verified that this does not change the flow equations for $|J_i|$ or $|h_i|$, but merely introduces some extra minus signs (some renormalized bonds become antiferromagnetic and some renormalized fields point along $-x$). We thus recover the flow equations[28] for the probability

distributions $P(h)$ and $P(J)$ as flow equations for $P(|h|)$ and $P(|J|)$. The flow[28] is to strong disorder, so the transition should proceed via an infinite-randomness critical point, for all excited states.

In either phase, we have already argued that the noninteracting system is localized. However, at criticality ($\varepsilon = 0$), the spectrum of the noninteracting Hamiltonian contains states with all localization lengths and the single-fermion states in the limit of zero energy have a diverging localization length. Nonetheless, the critical point of the noninteracting Hamiltonian (1,2,3) is localized in the following sense. Almost all single-fermion wave functions are localized, with only the limiting zero-energy states having infinite localization length. The entanglement entropy (within a many-body eigenstate) of a subregion is thus subextensive, unlike a thermal state, which has extensive entanglement entropy.[21] In fact, since the RG flow for $P(|J|), P(|h|)$ is the same in the excited eigenstates as for the ground state, the entanglement entropy is also the same in the excited eigenstates as in the ground state. As determined in Ref. [29], the entanglement entropy of a subregion at the infinite-randomness critical point has a leading term that scales as $\ln L$, whereas a thermal state would have an entanglement entropy that scales as L. Thus, the (noninteracting) critical point violates the ETH, and can be considered localized.

We now discuss whether localization can survive interactions at the critical point. The key question is whether the presence of a subextensive number of critical modes can cause the rest of the degrees of freedom to delocalize, by mediating resonances between distant near-degenerate localized modes. In the supplement of Ref. [16], it was determined that the mediated interactions fall off exponentially with distance, whereas the typical level spacing decreases as a power law of distance. Thus, the mediated interaction between distant near-degenerate modes should be weaker than the level splitting, and mediated interactions should be unable to delocalize the formerly localized degrees of freedom. This argument suggests that the critical point separating two MBL phases can itself be MBL. A slice through the phase diagram is presented in Fig. [1]. For more detail on the phase transition within the MBL regime, see Ref. [26].

G. Topological order in the fermion language

Translating to the Majorana and Dirac fermion languages, we again conclude that all bulk modes are localized, with localization lengths equal to those calculated above. The bulk normal modes are Dirac fermions. Meanwhile, the nearly degenerate pairs of excited eigenstates are states where the single-particle edge mode created by $\hat{O}^{\dagger}$ is either occupied or unoccupied. Thus, the eigenstates of (2,3) have topological order. The existence of topological order follows trivially in these noninteracting models because the edge mode does not interact with anything in the bulk, and is bilocalized as two Majorana modes, one near each end of the chain. In the clean system, the addition of arbitrarily weak interactions destroys the topological order at any nonzero temperature since the Majorana end modes can couple through the thermally excited and delocalized domain walls. However, in the disordered system, we will show that many-body localization protects the topological order.

H. Localization protects topological order

We now demonstrate that the edge Majoranas remain localized in the presence of interactions, even in the absence of a gap. We assume that the interaction is local and commutes with $\hat{P}$. It therefore conserves fermion number modulo 2, and is thus a product of an even number of Majorana operators. Before turning on interactions, there are an odd number of quasi-zero-energy Majorana modes localized at each edge. We can not turn this into an even number of Majoranas by acting with an even number of Majorana operators, so the edge Majorana must survive the addition of interactions. The Majorana can not disappear because of hybridization with localized bulk modes because the bulk modes are Dirac. The only way to make the topological order disappear is to couple the two Majoranas at either edge. However, the interaction can not do this because it is short range, and the Majorana can not be passed from one bulk Dirac mode to another, until it reaches the other Majorana at the opposite end of the chain, *because the bulk is MBL*, and does not allow energy or particles to propagate. Thus, MBL in the bulk protects topological order in highly excited localized states, just as a bulk gap can protect topological order in ground states. This is in sharp contrast to the clean (non-MBL) system, where the Majorana edge modes can hybridize with each other through delocalized bulk modes in any excited state. The localization protection of the edge Majoranas in quantum states other than ground states might be useful for experiments designed to exploit topological order. In particular, it might be useful for experiments designed to detect Majorana fermions in quantum wires.[30]

III. $d \geqslant 2$-DIMENSIONAL SYSTEMS

The ideas discussed above have a straightforward extension to the Ising model in more than one dimension. A major difference is that in two or more dimensions, thermodynamic Z_2 symmetry breaking persists to nonzero excitation-energy densities even in the extended (thermal) phase of an Ising ferromagnet. As a result, the phase diagram contains another type of phase transition: the usual thermodynamic phase transition between states with and without ferromagnetic order. A slice through the ($d > 1$)-dimensional phase diagram is presented in Fig. [3].

As in Fig. [1], we assume that the eigenstate phase transitions are continuous (not first order). This again implies that a localized paramagnetic phase exists between the extended (thermal) paramagnetic phase and the localized spin glass in the higher-energy regime above the ferromagnetic phases. Moving across this phase by increasing disorder, the spin-spin correlation length grows continuously from the finite thermal value of the extended phase and diverges at the transition to the spin glass. We note that this symmetry-breaking paramagnet-to-spin-glass phase transition in the localized Ising model in $d \geqslant 2$ is also governed by an infinite-randomness fixed point[31,32] which again should extend to finite-energy density states and lead to subthermal entanglement. We believe that localization persists here too, although a more detailed analysis is desirable. Assuming the symmetry-breaking transition in the localized phase out of the paramagnet is indeed in the infinite-randomness universality class, there should be
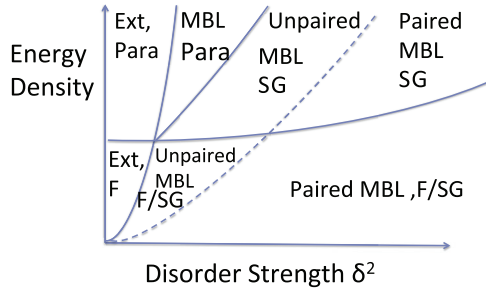
FIG. 3. (Color online) A (schematic) slice through the $(d > 1)$-dimensional phase diagram at nonzero λ and typical $h/J < 1$. In addition to the eigenstate phase transitions and spectral transition present in one dimension (Fig. 1), now there is also the usual thermodynamic phase transition between phases with and without ferromagnetic order, i.e., net spontaneous magnetization. The presence of ferromagnetic order is labeled by F.

localized domain walls in the ordered phase just across the transition. The higher-energy eigenstates with these domain walls present are spin-glass states. This is why we believe the spin-glass phase always exists between the paramagnet and the ferromagnet in the localized regime, as shown in Fig. 3. We also note that unlike in $d = 1$, the nearest-neighbor transverse-field Ising model in $d \geqslant 2$ *is* an interacting system on its own.

A. Topological order in $d \geqslant 2$

From our results thus far we can immediately draw some interesting conclusions about topologically ordered systems in $d \geqslant 2$ by simply dualizing the Ising model. This leads to MBL-protected topological order in higher dimensions, as we now discuss.

Let us begin in $d = 2$ where the "Ising model" of a system that exhibits topological order is the the Z_2 lattice gauge theory with matter, governed by the Hamiltonian[24,33,34]

$$-H = \sum_p K_p \prod_{l \in \partial p} \sigma_l^z + \sum_l \Gamma_l \sigma_l^x$$
$$+ \sum_l J_l \sigma_l^z \prod_{s \in \partial l} \tau_s^z + \sum_s \Gamma_s^M \tau_s^x \qquad (10)$$

supplemented by the constraint that we restrict its action to "gauge-invariant" states defined by

$$G_s |\psi\rangle = |\psi\rangle, \quad G_s = \tau_s^x \prod_{l:s \in \partial l} \sigma_l^x. \qquad (11)$$

In the above, the gauge (σ_l^i) and matter (τ_s^i) operators act in spin-$\frac{1}{2}$ Hilbert spaces that live on the links l and sites s of the square lattice with plaquettes p and ∂p and ∂l are the boundaries of the corresponding objects. On dualizing the Ising model, we get the Z_2 gauge theory *without* matter, obtained from (10) by dropping the matter degrees of freedom entirely. In this case, the parameters K_p and Γ_l are, numerically, the onsite fields and bond interactions of the dual Ising model.

Let us briefly review some salient facts about the non-random system. The paramagnetic and ferromagnetic ground states of the Ising model dualize, respectively, to the (topologically ordered) deconfined and (nontopological) confined

phases of the gauge theory, where the terminology refers to the energy needed to separate two test charges to infinity. For our purposes, it is more useful to consider the standard equal-time diagnostic, which can be evaluated in individual eigenstates, namely, the Wilson loop[24] for a contour C:

$$W[C] = \left\langle \prod_{l \in C} \sigma_l^z \right\rangle, \qquad (12)$$

which exhibits a perimeter/area (P/A) law decay, $\ln W[C] \propto -P/-A$, in the topological/nontopological phase. A useful picture of the ground states and excitations is obtained by thinking in the basis of eigenstates of σ_l^x. At $\Gamma \gg K$, the nontopological ground state has $\sigma_l^x = 1$ on all bonds. The elementary excitations are small loops of bonds where $\sigma_l^x = -1$, which we refer to as bonds with Z_2 "electric" flux. These loops are dual to domain walls above the ferromagnetic state in the Ising language. As we pass to the topological phase at $\Gamma \ll K$, the loops/domain walls proliferate and their condensation signals the transition. The elementary excitations in the topological phase are visons, plaquettes where $\prod_P \sigma_l^z = -1$.

Now, the translation is straightforward and we conclude that with sufficient randomness in the couplings there exist both MBL localized topological and nontopological phases in the Z_2 gauge theory without matter. (In Fig. 3, we can simply relabel the paramagnetic and spin-glass phases as topological and nontopological, respectively.) Of maximum interest is the MBL topological phase whose topological order is protected by localization and would not exist in its absence at nonzero energy densities (the dual of the thermal paramagnet phase in Fig. 3 is a phase where topological order is destroyed by thermal fluctuations). In the MBL topological phase, which is usefully described as a state with a finite density of localized visons, the eigenstates display a "spin-glass" version of the perimeter law in which the magnitude of $W[C]$ decays exponentially with perimeter but with a sign that depends on how many localized visons are encircled by the loop. By contrast, in the nontopological phase, $W[C]$ exhibits an area law.

This account of the Z_2 topological phase can be extended in two directions. First, one can include gauge charged matter, as in (10), which is known to leave the clean system topologically ordered at $T = 0$ when $J \ll \Gamma^M$. In the presence of sufficient randomness, we expect that the matter excitations will be localized and we will obtain an MBL-protected topological phase in the presence of dynamical matter at nonzero energy densities. However, this phase can no longer be diagnosed by examining the Wilson loop, which exhibits a perimeter law in all phases in the presence of matter. Instead, we turn to a version of the Fredenhagen-Marcu order parameter,[34] which measures the "line tension" of Z_2 electric fields specified by their distribution in an eigenstate. The version needed for our random setting is

$$R(L) = \frac{\langle \tau_s^z (\prod_{l \in C} \sigma_l^z) \tau_{s'}^z \rangle \langle \tau_s^z (\prod_{l \in \check{C}} \sigma_l^z) \tau_{s'}^z \rangle}{\langle \prod_{l \in C \cup \check{C}} \sigma_l^z \rangle}, \qquad (13)$$

where the notation is explained in Fig. 4 and the expectation values are taken in a particular eigenstate of the Hamiltonian. Extending the discussion in Ref. 34 to the present case we propose that $R(L \to \infty) = 0$ in the topological phase, but not in the confined phase.
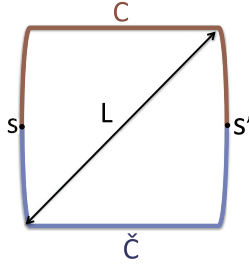
FIG. 4. (Color online) The contours used to define the Fredenhagen-Marcu order parameter for a translationally noninvariant system.

The second extension we propose involves two other common diagnostics of topological order in the clean system: a ground-state topological degeneracy of 4^g on closed manifolds of genus g and a topological entanglement entropy of $\ln 2$.[35] In the MBL Z_2 topologically ordered phase, we expect that the dominant finite-size effect in large systems of linear dimension L will arise from $O(e^{-L})$ tunneling between clusters of 4^g finite-energy density states which (roughly) exhibit the same pattern of localized excitations but differ in the presence or absence of visons threading the noncontractible loops on the manifold. Likewise, we expect individual eigenstates to exhibit an area law in entanglement and a topological constant piece of size $\ln 2$ which can be detected by means of the subtraction procedure outlined, e.g., in Refs. 36 and 37.

Dualizing the Ising model in $d > 2$ will yield a d-form gauge theory[38] with a topological phase with pointlike excitations and MBL stabilized topological order. Getting to a more conventional gauge theory in $d > 2$, however, requires some fresh thinking. For example, in $d = 3$ we need to consider the localization of stringlike excitations (vison loops) and one can not simply appeal to the body of results to date in this paper or elsewhere.

Finally, we would draw the reader's attention to Refs. 39 and 40 where the dynamical localization of excitations in Z_2 gauge theory with matter has been discussed in the language of the perturbed toric code relevant to quantum information storage.

IV. CONCLUSIONS AND OUTLOOK

Thus, we have shown that eigenstates of MBL systems come in many flavors, and may be classified in terms of broken symmetries, topological order, and criticality, just like extended states. Localization itself can protect order through the intuitive mechanism of localizing excitations that would disrupt it. We have also identified another kind of transition, the spectral transition, involving a change in the spectral statistics of the Hamiltonian. The protection of order and quantum coherence by localization might open the door to a new generation of quantum devices that are immune to environmental noise, and are not restricted to ground states or low-

energy states. It may also be useful for ongoing experiments attempting to observe Majorana fermions in quantum wires.[30]

We have focused, for pedagogical clarity, on broken Z_2 global symmetry and Z_2 topological order. Generalizations should be straightforward to other problems where the elementary excitations subversive of ordering can be localized at all energies by sufficiently random couplings. Immediate examples are $p \geqslant 3$ Z_p clock models in $d \geqslant 1$ and dual parafermionic systems in $d = 1$ (Ref. 23) and Z_p gauge theories in $d \geqslant 2$. Farther afield, we should flag other models with broken discrete symmetries and also topologically ordered systems with discrete gapped excitations such the Levin-Wen models that exhibit non-Abelian phases.[41] However, it is essential for our purposes that the system should not support long-range interactions, including those which might be mediated by Goldstone modes or gapless gauge excitations ("photons") that do not localize.[42] Thus, an extension of these ideas to systems with broken continuous symmetries and continuous gauge groups looks problematic. However, an extension to continuous symmetries might be possible if the Goldstone mode were gapped out by the Anderson-Higgs mechanism, or by placing the system on a Bethe lattice, where Goldstone bosons are absent.[43] We defer further consideration of these issues to future work.

While this work has focused on MBL eigenstates, an experimental construction of a system with a Hamiltonian that displays many-body localization will necessarily start with an initial state that is not an exact eigenstate of the Hamiltonian. We expect, based on analogies to the "diagonal ensemble" viewpoint advocated in Ref. 44, that for many initial states the density matrix at long times can be treated as being diagonal in the basis of energy eigenstates. However, a detailed understanding of the dynamical evolution of an initial superposition state (or mixed state) is an important topic for future work.

A final set of interesting open questions involves whether continuous phase transitions between distinct MBL phases necessarily proceed via a localized critical point. We have argued that for MBL phases in the one-dimensional random Ising model, phase transitions between MBL states proceed via an infinite-disorder critical point that is itself MBL. It remains to be determined to what extent this holds true for more complex models. Another open question involves the nature of the phase transition between extended (thermal) and MBL phases, which to our knowledge has not yet been determined in any system.

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