NONLINEAR INSTABILITY IN A SEMICLASSICAL PROBLEM

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Abstract. We consider a nonlinear evolution problem with an asymptotic parameter and construct examples in which the linearized operator has spectrum uniformly bounded away from $\mathrm{Re}\, z \geq 0$ (that is, the problem is spectrally stable), yet the nonlinear evolution blows up in short times for arbitrarily small initial data.

We interpret the results in terms of semiclassical pseudospectrum of the linearized operator: despite having the spectrum in $\mathrm{Re}\, z < -\gamma_0 < 0$, the resolvent of the linearized operator grows very quickly in parts of the region $\mathrm{Re}\, z > 0$. We also illustrate the results numerically.

1. Introduction

For a large class of nonlinear evolutions the size of the resolvent has been proposed as an explanation of instability for spectrally stable problems. Celebrated examples include the plane Couette flow, plane Poiseuille flow and plane flow – see Trefethen-Embree [6, Chapter 20] for discussion and references. Motivated by this we consider the mathematical question of evolution involving a small parameter h (in fluid dynamics problem we can think of h as the reciprocal of the Reynolds number) in which the linearized operator has the spectrum lying in $\mathrm{Re}\, z < -\gamma_0 < 0$, uniformly in h, yet the the solutions of the nonlinear equation blow up at time $O(1)$ for data of size $O(\exp(-c/h)$.

We know of one rigorous example of such a phenomenon given by Sandstede-Scheel [11]. They considered $u_t = u_{xx} + u_x + u^3$ on $[0, \ell]$ with Dirichlet boundary conditions, and showed that blow up occurs with arbitrarily small initial data as $\ell \to \infty$. In that problem $h = 1/\ell$. The paper [11] is our starting point and we use its maximum principle approach to obtain results for suitable operators in any dimension. In addition, we emphasize the connection with the semiclassical pseudospectrum and provide some numerical comparisons.

We consider a semiclassical nonlinear evolution equation

$$hu_t = P(x, hD)u + u^3\,, \quad x \in \mathbb{R}^d\,, \;\; t \geq 0\,. \tag{1.1}$$

where $P(x, hD)$ is the following semiclassical differential operator

$$P(x, hD) := -\Big((hD)^2 + V(x)\Big) + ih\langle \nabla\rho, D\rangle + \mu\,, \quad D_j := \frac{1}{i}\partial_j \quad D^2 = -\Delta. \tag{1.2}$$

Here, $V \in C^\infty$ is a potential function with

$$\begin{cases} V(x) \geq C\langle x\rangle^k \text{ on } |x| \geq M, & |\partial^\alpha V(x)| \leq C_\alpha \langle x\rangle^k, \\ V(x_0) = 0 \text{ for some } x_0 \in \mathbb{R}^d, & V(x) \geq 0, \end{cases} \tag{1.3}$$

for some $C, C_\alpha, k, M > 0$. Also, $\rho \in C^\infty$ has the properties

$$|\partial^2_{ij}\rho| \leq C_{ij}\langle x\rangle^k, \quad |\nabla\rho|^2 \geq 4(\mu+\gamma_0), \tag{1.4}$$

for some $C_{ij}, \gamma_0, N > 0$. Finally, $\mu > 0$.

Remark: All of our results hold for weaker assumptions on the growth of V and ρ, however (1.3) and (1.4) are convenient for our purposes.

We will show in section 2 that for $V(x)$ and ρ as in (1.3) and (1.4) respectively, the linearized problem is spectrally stable, that is, the spectrum is bounded away from $\operatorname{Re} z \geq 0$ uniformly in h. Yet, we also show that (1.1) has an unstable equilibrium at $u \equiv 0$ for all potentials $V(x)$ satisfying (1.3) and all ρ satisfying (1.4). Specifically, we show

Theorem 1. *Fix $\mu > 0$. Then, for each*

$$0 < h < h_0,$$

where h_0 is small enough, there exists

$$u_0 \in C^\infty_c(\mathbb{R}^n), \ u_0 \geq 0, \ \|u_0\|_{C^p} \leq \exp\left(-\frac{1}{Ch}\right), \ p = 0, 1, \dots,$$

such that the solution to (1.1) *with $u(x,0) = u_0(x)$, satisfies*

$$\|u(x,t)\|_{L^\infty} \longrightarrow \infty, \ \ t \longrightarrow T,$$

where

$$T = O(1).$$

A nice example for which our assumptions hold is (1.2) with $x \in \mathbb{R}$, $V(x) = x^2$, and $\langle\nabla\rho, D\rangle = D_x$. That is

$$P_1(x, hD) := -\left((hD_x)^2 + x^2\right) + ihD_x + \mu, \quad x \in \mathbb{R} \tag{1.5}$$

It is easy to see (and will be described in Section 2) that

$$\begin{aligned}\operatorname{Spec}(P_1(x,hD)) &= \{\mu - 1/4 - h(2n+1) \ : \ n = 0, 1, 2\cdots\} \\ &\subset \{z \ : \ \operatorname{Re} z \leq \mu - 1/4\}.\end{aligned}$$

For $\mu > \frac{1}{4}$ the spectrum intersects the right half plane and thus instability of the linear problem follows. We are interested in the range $0 < \mu < \frac{1}{4}$, where we will relate the instability of $u = 0$ to the presence of pseudospectrum in the right half plane.

For more about (1.5) see [6, Chapter 12]. In particular, Cossu-Chomaz [1] relate it to the linearized Ginzburg-Landau equation and analyze resolvent of (1.5) and the norm of the semigroup $e^{P_1(x,hD)t}$ numerically.

The operator (1.5) is also closely related to the advection-diffusion operators mentioned above, $-D_y^2 + iD_y = \partial_y^2 + \partial_y$, on $[0, \ell]$, with, say, Dirichlet boundary conditions; see [6, Chapter 12] for a discussion and references. When rescaled using $x = y/\ell$, $h = 1/l$ the operator becomes the semiclassical operator $-(hD)^2 + ihD_x$ on $[0, 1]$. When the domain is extended to $\mathbb{R}$, the potential x^2 is added to $(hD_x)^2$ to produce a confinement similar to a boundary.

We relate the blow-up of solutions to (1.1) to the presence of pseudospectrum of (1.2) in the right half plane. However, because estimates on semigroups for (1.1) with quasimode initial data are poor, we are unable to exhibit blow-up starting from a quasimode. Instead, we present a simple and explicit construction of quasimodes for $P(x, hD)$ (for a more general setting see [5]). We then use these quasimodes as initial data in numerical simulations and observe that, although in some cases the ansatz solution blows up more quickly, the solutions with quasimode initial data behave similarly to what is expected from a pure eigenvalue for (1.2) with positive real part.

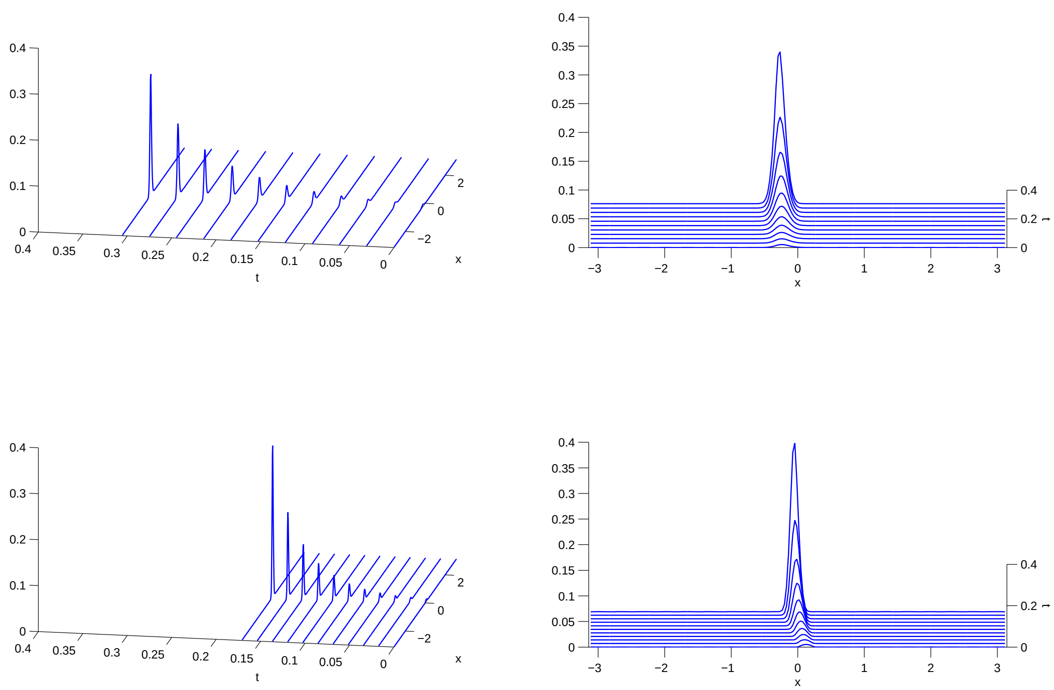

FIGURE 1. The plot shows numerical simulations of the evolution of (1.1) with $h = 1/193$ and two initial data. The evolution with initial data a real valued $O(h^3)$ error quasimode with eigenvalue $z = \frac{1}{16}$ is shown in the top two graphs and that with the ansatz constructed in the proof of Lemma 3 as initial data is shown in the bottom two graphs. We observe that, when the initial data is a quasimode, blowup occurs in time ≈ 0.3, while for ansatz initial data, blow-up occurs in time ≈ 0.175. However, as would be expected from eigenfunction initial data, we see that the solution with quasimode initial data exhibits little transport to the left. On the other hand, the ansatz transports left significantly.

The paper is organized as follows. In Section 2 we review the definitions of spectra and pseudospectra and discuss them for our class of operators. In Section 3 we give a construction of quasimodes for one dimensional problems. Although the results are known, (see [3],[5],[12]) a self-contained presentation is useful since we need the quasimodes for our numerical experiments. Also, there is no reference in which analytic potentials (for which quasimodes have $O(\exp(-c/h))$ accuracy) is treated by elementary methods in one dimension. Section 4 is devoted to the proof of Theorem 1 using heat equation methods. Finally, in Section 5 we report on some numerical experiments which suggest that quasimode initial data gives more natural blow-up and that blow-up occurs at complex energies.

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2. Spectrum and Pseudospectrum

We do not use the results of this section to prove Theorem 1. Instead, we present them to emphasize the connection of the size of the resolvent with instability. We believe that instability based on quasimodes would be more natural and allow for proof of instability at complex energies. We illustrate this with numerics in Section 5.

To describe the spectrum of $P(x, hD)$, we observe that

$$e^{\frac{\rho(x)}{2h}} P(x, hD) e^{-\frac{\rho(x)}{2h}} = -\left[(hD)^2 + V(x) + \frac{1}{4}|\nabla\rho|^2 + \frac{h}{2}\Delta\rho\right] + \mu.$$

Thus, the spectrum of $P(x, hD)$ is given by that of a Schrödinger operator with potential $V(x) + \frac{1}{4}|\nabla\rho|^2 + \frac{h}{2}\Delta\rho$. Since $V(x)$ and ρ have the properties given in (1.3) and (1.4) respectively, $P(x, hD)$ has a discrete spectrum that has real part bounded above by $-\gamma_0$ (see for instance [13, Section 6.3]).

We now examine the pseudospectral properties of (1.2).

Definition. Let $Q(x, hD)$ be a second order semiclassical differential operator. Then, $z \in \Lambda(Q)$ if and only if $\exists\, u(h) \in H^2(\mathbb{R}^d)$ such that $\|u\|_{L^2} = 1$ and

$$\|(Q(x, hD) - z)\, u(h)\|_{L^2} = O(h^\infty).$$

We say z is in the semiclassical *pseudospectrum* of Q if $z \in \overline{\Lambda(Q)}$.

Remark. We note that for $z \in \Lambda(Q)$, $\|Q(x, hD) - z)^{-1}\| \geq h^{-N}/C_N$, for any N. This relates our definition to the more standard defintions of pseudospectra in terms of the resolvent. For discussion and generalizations see Dencker [4] and Pravda-Starov [10].

The criterion for $z \in \Lambda(Q)$ is based on Hörmander's bracket condition (see Zworski [12] and Dencker-Sjöstrand-Zworski [5]):

$$Q(x_0, \xi_0) = z \text{ and } \{\mathrm{Re}\, Q, \mathrm{Im}\, Q\}(x_0, \xi_0) < 0, \tag{2.1}$$

then $z \in \Lambda(Q)$. We use this condition to show that the pseudospectrum of $P(x, hD)$ nontrivially interesects the right half plane. Specifically,

Lemma 1. *For $P(x, hD)$ given by (1.2), $\overline{\Lambda(P(x, hD))} \bigcap \{\operatorname{Im} z = 0\} = (-\infty, \mu]$.*

Proof. First, observe that

$$P(x, \xi) = -|\xi|^2 + i\langle \nabla\rho, \xi\rangle - V(x) + \mu \quad \text{ and } \{\operatorname{Re} P, \operatorname{Im} P\} = -2\langle \partial^2\rho\xi, \xi\rangle + \langle \nabla V, \nabla\rho\rangle.$$

We have assumed $\operatorname{Im} z = 0$. Therefore, we need only show that, for a dense subset $U \subset (-\infty, \mu]$, $y \in U$ implies that there exists x such that (2.1) holds for the symbol $P(x, \xi)$, at $(x, 0)$ with $z = y$.

We proceed by contradiction. Suppose there is no such U. Then, there exists $O \subset [0, \infty)$ open such that for all $x \in V^{-1}(O)$, $\langle \nabla V, \nabla\rho\rangle(x, 0) \geq 0$. Let $\varphi_t := \exp(ti\langle \nabla\rho, D\rangle)$ be the integral flow of $i\langle \nabla\rho, D\rangle$ and $x_0 \in \mathbb{R}^d$ have $V(x_0) = 0$. Define $f(t) := V(\varphi_t(x_0))$. Then $\partial_t f = \langle \nabla V(\varphi_t(x_0)), \nabla\rho(\varphi_t(x_0))\rangle$.

Suppose that $\varphi_t(x_0)$ escapes every compact set as $|t|$ increases. Then (1.4) implies that $f(t) \to \infty$ as $|t|$ increases. Let $w \in O$ and $t_0 := \inf\{t \in \mathbb{R} : f(t) = w\}$. Then t_0 is finite since $w \geq f(0)$ and $f(t) \to \infty$. Together, $f(t_0) = w \in O$ and $f^{-1}(O)$ open imply the existance of $\delta > 0$ such that for $t \in (t_0 - \delta, t_0 + \delta)$, $f(t) \in O$. But, $f(t) \in O$ implies $f'(t) \geq 0$. Therefore, $f(t) \leq w$ for $t \in (t_0 - \delta, t_0)$ and thus, since $f(t) \to \infty$, there exists $t < t_0$ such that $f(t) = w$, a contradiction.

We have shown that there is a dense subset $U \subset (-\infty, \mu]$ with $U \subset \Lambda(P)$. Hence $(-\infty, \mu] \subset \overline{\Lambda(P)}$. Next, observe that $\sup \operatorname{Re} P(x, \xi) = \mu$ and thus, $\overline{\Lambda(P(x, hD))} \bigcap \{\operatorname{Im} z = 0\} = (-\infty, \mu]$ as desired.

To finish the proof, we need only show that $\varphi_t(x_0)$ escapes every compact set. Suppose the flow at x_0 exists for all $t \in \mathbb{R}$. Define $h(t) := \rho(\varphi_t(x_0))$. Then $\partial_t h = |\nabla\rho|^2 \geq c > 0$ and we have that $h \to \pm\infty$ as $t \to \pm\infty$. But, $\rho \in C^\infty$ and is therefore bounded on every compact set. Thus, $\varphi_t(x_0)$ escapes every compact set as $t \to \pm\infty$.

Now, suppose the flow at x_0 is not global. Then, $\varphi_t(x_0)$ is an integral curve of $i\langle \nabla\rho, D\rangle$ with t domain a proper subset of $\mathbb{R}$. Thus, as proved in [9, Lemma 17.10], $\varphi_t(x_0)$ escapes every compact set. □

Putting this together with our discussion of the spectrum of $P(x, hD)$, we have that for $0 < \mu$ and ρ as in (1.4), although $\operatorname{Spec}(P)$ is bounded away from $\operatorname{Re} z \geq 0$, $\overline{\Lambda(P)}$ nontrivially intersects $\operatorname{Re} z \geq 0$.

For the specific case, $V(x) = |x|^2$, and $\nabla\rho$ constant with $|\nabla\rho| = 1$, the above argument gives us that

$$\operatorname{spec}(P(x, hD)) = \left\{-(2n + 1)h + (\mu - \frac{1}{4}) : n \geq 0\right\}.$$

In addition, the pseudospectrum is given by,

$$\overline{\Lambda(P(x, hD))} = \left\{z : \operatorname{Re} z \leq -(\operatorname{Im} z)^2 + \mu\right\}.$$

We see that for $\mu > \frac{1}{4}$, the spectrum interesects the right half plane and so instability of $u \equiv 0$ is a classical result. However, for $0 < \mu < \frac{1}{4}$, the spectrum is bounded away from the $\operatorname{Re} z \geq 0$ and only the pseudospectrum enters the right half plane. Yet, in the regime $0 < \mu < \frac{1}{4}$, we will show that $u \equiv 0$ is unstable and, moreover, for arbitrarily small initial data, the solution blows up in finite time.

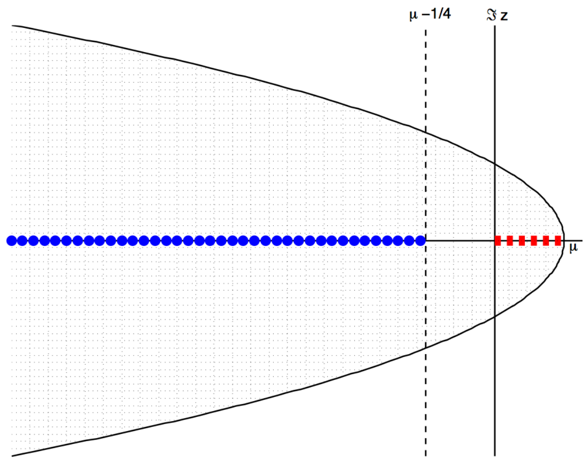

Figure 2. We see that the spectrum of (1.5) (blue dots) is bounded away from $\operatorname{Re} z = 0$, while the pseudospectrum (shaded region) enters the right half plane. The region for which we prove blow-up corresponds to the dashed red line.

3. One Dimensional Quasimodes

We proceed by constructing quasimodes for operators in the one dimensional case with $i\langle \nabla\rho, D\rangle = \partial_x$. We implement WKB expansion for the quasimode following the method used in [3]. Let

$$P(x, hD) := -(hD_x)^2 + ihD_x + V\,, \tag{3.1}$$

where $V \in C^\infty$ and V may be complex.

Remark. The following theorem is a special case of general theorems about quasimodes [5, Theorems 2 and 2′]. For the reader′s convenience we present a direct proof in the spirit of Davies [3].

Theorem 2. *Suppose that* $P(x, hD)$ *is given by* (3.1) *and that*

$$z = -\xi_0^2 + i\xi_0 + V(x_0)\,,$$

where x_0 *satisfies the condition that*

$$\operatorname{Re} V'(x_0) > 0\,.$$

There exists an $h-$*dependent function* $\varphi \in C_c^\infty(\mathbb{R})$, *such that* $\|\varphi\|_{L^2} = 1$ *and*

$$\|(P(x, hD) - z)\varphi\|_{L^2} = O(h^\infty).$$

In addition φ *is microlocalized to* (x_0, ξ_0) *in the sense that for every* $g \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ *vanishing in a neighbourhood of* (x_0, ξ_0),

$$\|g(x, hD)\varphi\|_{L^2} = O(h^\infty)\,.$$

When V is real analytic than we can find φ such that

$$\|(P(x, hD) - z)\varphi\|_{L^2} \leq C \exp(-1/Ch)\,.$$

Proof. Let $\chi \in C_c^\infty(\mathbb{R})$ with $\chi(x) = 1$ if $|x| < \delta/2$ and $\chi(x) = 0$ if $|x| > \delta$ where δ will be determined below. Define $f := \exp(i\psi/h)a(x)$ where

$$a(x) = \sum_{m=0}^{N-2} a_m(x)h^m.$$

Finally, let $g(x_0 + x) := \chi(x)f(x)$ for all $x \in \mathbb{R}$.

We will find appropriate a_m and ψ in what follows. First, by a simple computation

$$(P(x, hD) - z)\, f = \left(\sum_{m=0}^{N} h^m \phi_m\right) e^{i\psi/h}$$

where ϕ_m are inductively defined by

$$\phi_m \quad := \quad (-(\psi')^2 + i\psi' + V - z)a_m + i\psi'' a_{m-1} + (2i\psi' + 1)a'_{m-1} + a''_{m-2},$$

where we use the convention that $a_m \equiv 0$ for $m > N - 2$ or $m < 0$. Now, we set $\phi_m = 0$ for $0 \leq m \leq N - 1$. Given that δ is small enough, this will enable us to determine all a_m as well as ψ.

Observe that, using the condition, $\phi_0 = 0$, we obtain

$$\psi'^2 - i\psi' = V - z.$$

Now, letting $z = -\xi_0^2 + i\xi_0 + V(x_0)$, we have a complex eikonal equation

$$\left(\psi' - \frac{i}{2}\right)^2 = V(x_0 + x) - V(x_0) + \left(\xi_0 - \frac{i}{2}\right)^2.$$

Then, letting $\tilde{\psi} = \psi - \frac{i}{2}x$, we have

$$\begin{aligned}
\tilde{\psi}(x) \quad &:= \quad \int_0^x \left(V(x_0 + t) - V(x_0) + \left(\xi_0 - \frac{i}{2}\right)^2\right)^{1/2} dt \\
&= \quad \int_0^x \left(\xi_0 - \frac{i}{2}\right)\left(1 + \left(\xi_0 - \frac{i}{2}\right)^{-2} (V(x_0 + t) - V(x_0))\right)^{1/2} dt \\
&= \quad \left(\xi_0 - \frac{i}{2}\right)\left(x + \frac{1}{4}\left(\xi_0 - \frac{i}{2}\right)^{-2} V'(x_0)x^2 + O(x^3)\right)
\end{aligned}$$

and hence

$$\psi = \xi_0 x + \frac{\left(\xi_0 + \frac{i}{2}\right)}{4\left(\xi_0^2 + \frac{1}{4}\right)} V'(x_0)x^2 + O(x^3).$$

Now, we have assumed that $\operatorname{Re} V'(x_0) > 0$. Therefore there exists $\gamma > 0$ such that

$$\gamma x^2 \leq \operatorname{Im} \psi(x) \leq 3\gamma x^2$$

for all small enough x and h. Also, for x and h small enough

$$\theta := (2i\psi' + 1)^{-1}$$

satisifies $|\theta(x)| \leq \beta$. We choose $\delta > 0$ small enough so that these conditions both hold for $0 < h < \delta^2$ and $|x| < \delta$.

The condition $\phi_{m+1} = 0$, implies

$$a_m' = -\theta(i\psi'' a_m + a_{m-1}'')$$

with the convention that $a_{-1} \equiv 0$ and initial conditions,

$$a_0(0) = 1, \quad a_m(0) = 0, \ m > 0.$$

Putting $G(x) := \int_0^x i\psi''(y)\theta(y)dy$ we obtain $a_0 = \exp(-G(x))$ and

$$a_{m+1}(x) := -e^{-G(x)} \int_0^x e^{G(y)}\theta(y)a_m''(y)dy, \ m > 0. \tag{3.2}$$

Before proceeding to show exponential error for V analytic, we show $O(h^N)$ error for arbitrary V. To complete the proof of $O(h^N)$ quasimodes, we need to estimate

$$\|(P(x, hD) - z)g\|_{L^2} / \|g\|_{L^2}.$$

Let C denote various positive constants that are independent of h and x. Then,

$$\begin{aligned} \|g\|_{L^2}^2 &\geq \int_{-\delta/2}^{\delta/2} |f(x)|^2 dx \geq \int_{-\delta/2}^{\delta/2} e^{-6\gamma x^2 h^{-1} - C} dx \\ &= \int_{-\delta h^{-1/2}/2}^{\delta h^{-1/2}/2} e^{-6\gamma t^2 - C} h^{1/2} dt \geq \int_{-1/2}^{1/2} e^{-6\gamma t^2 - C} h^{1/2} dt = Ch^{1/2}. \end{aligned} \tag{3.3}$$

Next, we compute

$$\begin{aligned} \|(P(x, hD) - z)\, g\|_{L^2} &= \|h^2 f\chi'' + 2h^2 f'\chi' + hf\chi' + \chi(P(x, hD)f - zf)\|_{L^2} \\ &\leq h^2\|f\chi''\|_{L^2} + 2h^2\|f'\chi'\|_{L^2} + h\|f\chi'\|_{L^2} + \|h^N \chi \phi_N e^{i\psi/h}\|_{L^2}. \end{aligned} \tag{3.4}$$

Thus, we need to estimate each of the norms. Note that χ' and χ'' have support in $\{x : \delta/2 \leq |x| \leq \delta\}$. Thus, we have

$$\|f\chi''\|_{L_2}^2 \leq C_3 \int_{\delta/2 \leq |x| \leq \delta} e^{-2\gamma x^2 h^{-1} + C} dx \leq Ce^{-\gamma\delta^2/2h}. \tag{3.5}$$

Similarly,

$$\|f'\chi'\|_{L^2}^2 \leq Ce^{-\gamma\delta^2/2h}, \quad \|f\chi'\|_{L^2}^2 \leq Ce^{-\gamma\delta^2/2h}. \tag{3.6}$$

Next, observe that

$$\begin{aligned} \|h^N \chi\phi_N e^{i\psi/h}\|_{L^2}^2 &\leq h^{2N}\|\phi_N\|_{L^\infty}^2 \int_{-\delta}^{\delta} e^{-2\gamma x^2 h^{-1} + C} dx \leq Ch^{2N}\|\phi_N\|_{L^\infty}^2 \int_{-\delta h^{-1/2}}^{\delta h^{-1/2}} e^{-2\gamma x^2 + C} h^{1/2} dx \\ &\leq Ch^{2N}\|\phi_N\|_{L^\infty}^2 \int_{-\infty}^{\infty} e^{-2\gamma x^2 + C} h^{1/2} dx \leq C\left(h^N\|\phi_N\|_{L^\infty}\right)^2 h^{1/2}. \end{aligned} \tag{3.7}$$

Now, $|\phi_m| \leq c_m$ on $\{x : |x| \leq \delta\}$, uniformly for $h \leq \delta^2$. Therefore, combining (3.4) with inequalities (3.3), (3.5), (3.6), and (3.7), gives $O(h^N)$ quasimodes for arbitrary N. We then normalize to obtain φ.

We will now assume that $V(x)$ is real analytic and prove exponential smallness of the error.

Lemma 2. *Let $I = [-\delta, \delta]$ where δ is a small constant. Suppose that τ_0, τ_1, τ_2, and d_0 are holomorphic functions of $z \in \Omega$ and $|\tau_2| \geq \frac{1}{C}$ for some $C > 0$ where Ω is a neighborhood of I in $\mathbb{C}$.*

If d_m is defined inductively by

$$d_{m+1}(z) = \int_0^z \tau_2(\zeta)d_m''(\zeta) + \tau_1(\zeta)d_m'(\zeta) + \tau_0(\zeta)d_m(\zeta)dy. \tag{3.8}$$

Then for some $C_1 > 0$, $C_2 > 0$ and $[-\delta, \delta] \subset \tilde{\Omega} \subset \Omega$,

$$\sup_{\tilde{\Omega}} |\partial^p d_m| \leq C_2^{p+1} p! C_1^{m+1} m^m. \tag{3.9}$$

Proof. Using integration by parts, we obtain that

$$d_{m+1}(z) = \tau_2(z)d_m'(z) - \tau_2(0)d_m'(0) + (\tau_1(z) - \tau_2'(z))d_m(z) + \int_0^z (\tau_2''(\zeta) - \tau_1'(\zeta) + \tau_0(\zeta))d_m(\zeta)d\zeta.$$

Then, since τ_2 is holomorphic in Ω and $\inf_\Omega |\tau_2| \geq \frac{1}{C}$ for some $C > 0$, we make the conformal change of variables $z \to w$ where

$$\frac{dw}{dz} = \tau_2(z)^{-1}.$$

Then, letting $b_m = d_m(z(w))$, we have

$$\partial_w^p(b_{m+1})(w) = \partial_w^{p+1} b_m(w) - \delta_{p0}(\partial_w b_m)(0) + \partial_w^p(\rho_0 b_m)(w) + \partial_w^p \int_0^{z(w)} \rho_1(\zeta) b_m(\zeta) d\zeta$$

where $\rho_0 = (\tau_1 - \tau_2')(z(w))$ and $\rho_1 = ((\tau_2'' - \tau_1' + \tau_0)\tau_2)(z(w))$. Put $\Omega_w := \{w : z(w) \in \Omega\}$. Then, since the change of variables was conformal, and τ_i, $i = 0, 1, 2$, are holomorphic, we have that there exists $C_\rho > 0$ such that

$$|\partial_w^p \rho_i|_{\Omega_w} \leq C_\rho^{p+1} p^p \quad \text{for } i = 0, 1$$

where we define $|f|_\Omega := \sup_\Omega |f|$ for a function f defined on Ω.

We claim that for some $C_1 > 0$, $C_0 > C_\rho$,

$$|\partial_w^p b_m|_{\Omega_w} \leq C_0^{p+1} C_1^{m+1} (m + p)^{m+p}.$$

We prove the claim by induction. The holomorphy of b_0 gives us the base case. We now prove the inductive case.

By the inductive hypothesis, we have that

$$|\partial_w^{p+1} b_m|_{\Omega_w} \leq C_0^{p+2} C_1^{m+1} (m + p + 1)^{m+p+1}. \tag{3.10}$$

Similarly,

$$|b_m(0)|_{\Omega_w} \leq C_1^{m+1} m^m. \tag{3.11}$$

Next, we prove similar estimates for $\partial_w^p(\rho_0 b_m)$. By Leibniz rule, we have that

$$|\partial^p(\rho_0 b_m)|_{\Omega_w} = \left|\sum_{k=0}^{p}\frac{p!}{k!(p-k)!}\partial^k\rho_0\partial^{p-k}b_m\right|_{\Omega_w} \leq \sum_{k=0}^{p} C_0^{p+2}C_1^{m+1}r_{k,m,p} \tag{3.12}$$

where

$$r_{k,m,p} := \frac{p!}{k!(p-k)!}k^k(m+p-k)^{m+p-k}.$$

We claim that for $0 \leq k \leq \frac{p}{2}$, $r_{k,m,p} \geq r_{p-k,m,p}$. To see this, we write this inequality as

$$k^k(m+p-k)^{m+p-k} \geq (p-k)^{p-k}(m+k)^{m+k}, \quad \text{for } 0 \leq k \leq \frac{p}{2}.$$

Putting $x := \frac{k}{m}$ and $y = \frac{p-k}{m}$, the inequality is equivalent to

$$x^x(1+y)^{1+y} \geq y^y(1+x)^{1+x}, \quad \text{for } 0 \leq x \leq y$$

which follows from the monotonicity of the function $x \mapsto \left(\frac{1+x}{x}\right)^x(1+x)$.

Next, observe that, $0 \leq k < p-1$,

$$\begin{aligned}
\frac{r_{k+1,m,p}}{r_{k,m,p}} &= \frac{p-k}{k+1}\frac{(k+1)^{k+1}}{k^k}\frac{(m+p-k-1)^{m+p-k-1}}{(m+p-k)^{m+p-k}} \\
&= \frac{p-k}{m+p-k-1}\left(1+\frac{1}{k}\right)^k\left(1-\frac{1}{m+p-k}\right)^{m+p-k} \\
&\leq \frac{p-k}{m+p-k-1}e^{1-1+\frac{1}{2(m+p-k)}} \leq e^{\frac{1}{2(m+p-k)}},
\end{aligned}$$

where we use $\log(1-x) \leq -x + \frac{x^2}{2}$. Then, since for $0 \leq k \leq \frac{p}{2}$, $r_{k,m,p} \geq r_{p-k,m,p}$, we have

$$\begin{aligned}
|\partial^p(\rho_0 b_m)|_{\Omega_w} &\leq 2C_0^{p+2}C_1^{m+1}\sum_{k=0}^{\frac{p}{2}+1} r_{k,m,p} \leq 2C_0^{p+2}C_1^{m+1}\sum_{k=0}^{\frac{p}{2}+1} r_{0,m,p}\prod_{n=0}^{k-1}e^{\frac{1}{2(m+p-n)}} \\
&\leq 2C_0^{p+2}C_1^{m+1}\sum_{k=0}^{\frac{p}{2}+1} r_{0,m,p}e^{\frac{k}{2m+p}} \leq 2C_0^{p+2}C_1^{m+1}\sum_{k=0}^{\frac{p}{2}+1} r_{0,m,p}e^{\frac{p+2}{4m+2p}} \\
&\leq C_0^{p+2}C_1^{m+1}(p+2)r_{0,m,p}e^{\frac{1}{2}} \leq e^{\frac{1}{2}}C_0^{p+2}C_1^{m+1}(m+p+1)^{m+p+1}
\end{aligned}$$

Therefore, there exists $M_1 > 0$ such that

$$|\partial_w^p(\rho_0 b_m)|_{\Omega_w} \leq M_1 C_0^{p+2}C_1^{m+1}(m+p+1)^{m+p+1}. \tag{3.13}$$

By analagous argument, there exists $M_2 > 0$ such that

$$\left|\partial_w^p\int_0^{z(w)}\rho_1(\zeta)b_m(\zeta)d\zeta\right|_{\Omega_w} = \left|\partial_w^{p-1}((\rho_1\partial_w z)b_m)\right|_{\Omega_w} \leq M_2 C_0^{p+1}C_1^{m+1}(m+p)^{m+p}. \tag{3.14}$$

Next, choose $C_1 > C_0 \left(4\max(M_1, M_2, 1)\right)$. Then, combining (3.10), (3.11), (3.13), and (3.14), we have

$$|\partial_w^p b_{m+1}(w)|_{\Omega_w} \leq C_0^{p+1} C_1^{m+2} (m+p+1)^{m+p+1}.$$

Then, since $w \to z$ is a change of variables independent of m which maps $\Omega_w \to \Omega$ and $b_m(w) = d_m(z(w))$, we have

$$|d_m|_\Omega = |b_m|_{\Omega_w} \leq C_0 C_1^{m+1} m^m.$$

Now, choose $\tilde{\Omega} \subset \Omega$ with $\inf\{|z-\zeta| : z \in \tilde{\Omega},\ \zeta \in \partial\Omega\} > \gamma > 0$. Then, since d_m are holomorphic, we apply Cauchy estimates to obtain

$$|\partial^p d_m|_{\tilde{\Omega}} \leq \gamma^{-p} p! |d_m|_\Omega \leq \gamma^{-p} p! C_0 C_1^{m+1} m^m.$$

□

We apply the lemma with $d_m(x) := e^{G(x)} a_m(x)$, $\tau_2 := -\theta$, $\tau_1 := 2\theta G'$, and $\tau_0 := \theta(G'' - (G')^2)$ where θ and G are given above. Analyticity of V implies that a_0, θ, and ψ are holomorphic in a neighbourhood of I.

Then, putting $1/N = eC_1 h$, using Lemma 2 and that ψ is real analytic, we have

$$\begin{aligned} \sup_{x\in[-\delta,\delta]} |h^N \phi_N| &\leq h^N C(C_2 C_1^N (N-1)^{N-1} + 2C_2^2 C_1^N (N-1)^{N-1} + 6C_2^3 C_1^N (N-2)^{N-2}) \\ &\leq Ch^N C_1^N N^N \leq C(h^N C_1^N \left(\frac{1}{eC_1 h}\right)^N) = Ce^{-N} = Ce^{-\frac{1}{eC_1 h}} \end{aligned} \tag{3.15}$$

where C denotes various postive constants that are independent of N. Finally, combining (3.4) with inequalities (3.3), (3.5), (3.6), (3.7), and (3.15) gives $O(e^{-\frac{1}{Ch}})$ quasimodes. We then normalize g to obtain φ. □

Now, applying this result to $P_1(x, hD)$, we obtain

$$\psi = \int_0^x \frac{i}{2} + \left(\xi_0 - \frac{i}{2}\right)\left(1 - \frac{2x_0 t + t^2}{\left(\xi_0 - \frac{i}{2}\right)^2}\right)^{1/2} dt$$

for $x_0 < 0$ and $z = -\xi_0^2 + i\xi_0 + \mu - x_0^2$.

4. Instability

Our approach to obtaining blow-up of (1.1) will follow that used by Sandstede and Scheel in [11]. We will first demonstrate that, from small initial data, we obtain a solution that is ≥ 1 on a deformed ball in time $t_1 = O(1)$. We will then use the fact that the solution is ≥ 1 on this region to demonstrate that after an additional $t_2 = O(h)$, the solution to the equation blows up.

First, we prove that there exists initial data so that the solution to (1.1) is ≥ 1 in time $O(1)$. Recall that φ_t denotes the flow of $i\langle\nabla\rho, D\rangle$.

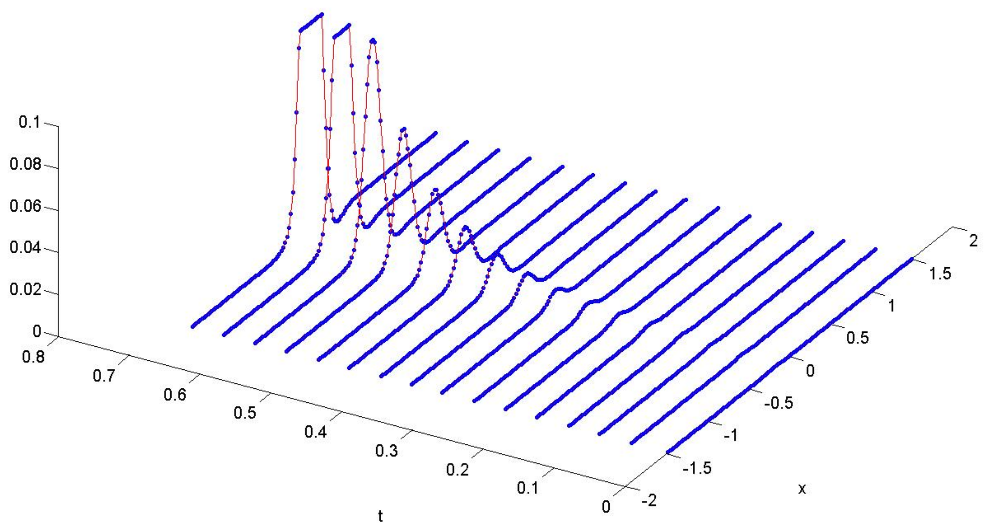

FIGURE 3. We set $h = 10^{-2}$ and see that the difference between the solution to (1.1) with initial data a quasimode with error $O(h^2)$ (red line) and the solution with initial data a quasimode with error $O(h^3)$ (blue dots) is negligible. Thus, by using $O(h^3)$ error quasimodes, we have not introduced large error into our numerical calculations.

Lemma 3. *Fix $\mu > 0$, $\alpha < \mu$, $0 < \epsilon \leq \frac{1}{2}(\mu - \alpha)$, and $(x_0, a, \delta) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^+$ such that both $\varphi_t(B(x_0, 2a)) \subset V^{-1}[0, \mu - \alpha - \epsilon]$ for $t \leq \delta$ and φ_t is defined on $B(x_0, 2a)$ for $0 \leq t < 2\delta$. Then, for each*

$$0 < h < h_0$$

where h_0 is small enough, there exists

$$u_0(x) \geq 0, \quad \|u_0\|_{C^p} \leq \exp\left(-\frac{1}{Ch}\right), \quad p = 0, 1, \dots$$

and $0 < t_1 < \delta$ so that the solution to (1.1) with initial data u_0 satisfies $u(x, t_1) \geq 1$ on $x \in \varphi_{t_1}(B(x_0, a))$.

Proof. Let v solve

$$(h\partial_t - P(x, hD))v = 0, v(x, 0) = v_0. \tag{4.1}$$

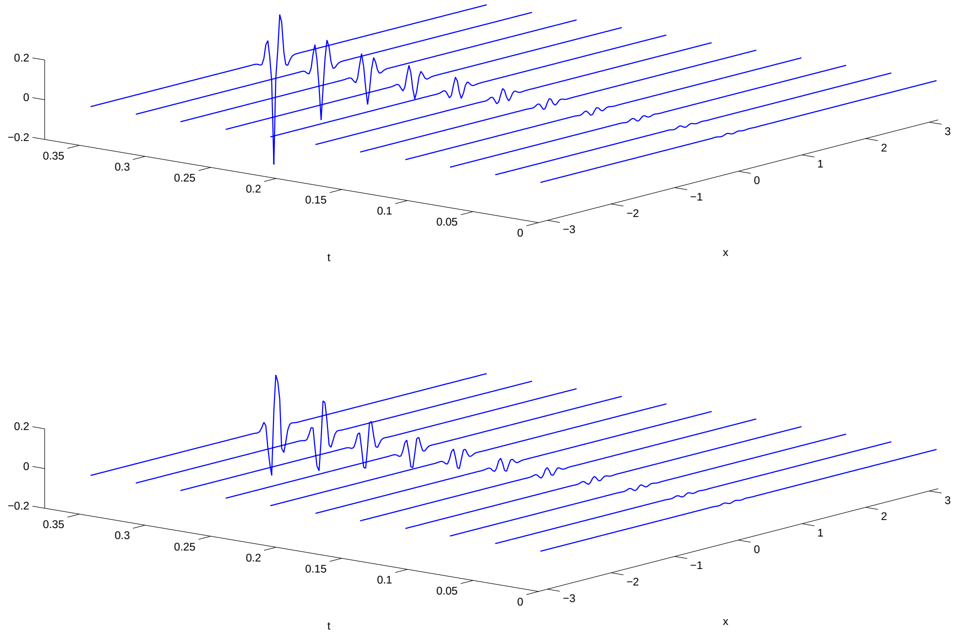

FIGURE 4. We show a numerical simulation $u(t)$ of the evolution of (1.1) with a quasimode at imaginary energy as initial data. Specifically, we set $h = 1/193$ and use a quasimode with eigenvalue $z = \frac{1}{16} + \frac{i}{4\sqrt{2}}$. The real part is shown in the top graph and the imaginary part in the bottom graph. We see that although subsolution methods do not apply to these quasimondes, blow up still occurs in time ≈ 0.35.

Let $w_0 : \mathbb{R}^d \to \mathbb{R}$ and define $O := \{x : w_0 > 0\}$. We make the following assumptions on w_0,

$$w_0 \geq 0, \quad \|w_0\|_{C^p} \leq \exp(-\frac{1}{Ch}), \quad w_0 \in C(\mathbb{R}^d) \tag{4.2}$$

$$w_0 \in \overline{C^\infty}(O), \quad \text{supp } w_0 \subset B(x_0, 2a), \quad w_0 > \exp\left(-\frac{\delta}{2h}\right) \text{ on } B(x_0, a), \tag{4.3}$$

$$\partial O \text{ is smooth}, \quad -\Delta w_0(x) \leq C w_0(x) - \beta \text{ for } x \in O \text{ and } 0 < h < h_0. \tag{4.4}$$

where $\overline{C^\infty}(O)$ are smoothly extendible functions on O. We will construct such a function at the end of the proof.

Define $w : [0, 2\delta) \times \mathbb{R}^d \to \mathbb{R}$ by

$$w := \begin{cases} \exp\left(\frac{\alpha}{h}t\right) w_0(\varphi_t(x)) & \text{where } \varphi_t \text{ is defined} \\ 0 & \text{else.} \end{cases}$$

Since supp $w \subset B(0, 2a)$ and φ_t is defined on $B(0, 2a) \times [0, 2\delta)$, w is continuous. We proceed by showing that w is a viscosity subsolution of (4.1) in the sense of Crandall, Ishi, and Lions [2].

First, we show that w is a subsolution on $O_t := \varphi_t(O)$ for $t < \delta$.

$$\begin{aligned} hw_t - P(x, hD)w &= hw_t - h^2\Delta w - ih\langle \nabla\rho, D\rangle w - \mu w + V(x)w \\ &= (\alpha - \mu + V(x))w - h^2\Delta w \\ &\leq \exp\left(\frac{\alpha}{h}t\right)((\alpha - \mu + V(x))w_0) - h^2\Delta w \end{aligned}$$

Now, by Taylor's formula, $\varphi_t(x) = x + O(t)$. Hence $-\Delta\left[w_0(\varphi_t(x))\right] = -\Delta w_0(x) + O(t)$. We have $t < \delta$, and $-\Delta w_0 \leq Cw_0 - \beta$ on O. Therefore, for δ small enough, $-\Delta w \leq Cw_0$. Hence, for h small enough independent of $0 < \delta < \delta_0$,

$$hw_t - P(x, hD)w \leq \exp\left(\frac{\alpha}{h}t\right)\left(\alpha - \mu + Ch^2 + V(x)\right)w_0$$

Now, since for some $\epsilon > 0$ and $t < \delta$, supp $w \subset V^{-1}[0, \mu - \alpha - \epsilon]$ we have that

$$hw_t - P(x, hD)w \leq \exp\left(\frac{\alpha}{h}t\right)(Ch^2 - \epsilon)w \leq 0$$

for h small enough. Thus, w is a subsolution on O_t for $t < \delta$. Next, we observe that on $(\mathbb{R}^d \setminus \overline{O_t})$, $w \equiv 0$ and hence is a subsolution of (4.1).

Finally, we consider $\partial O_t := \varphi_t(\partial O)$. We have that ∂O_t is smooth. If $y_0 \in \partial O_t$ and w is twice differentiable at y_0, then $w_t = (\Delta w)(y_0) = (Dw)(y_0) = w(y_0) = 0$ and w is clearly a subsolution to (4.1) at y_0. Let $y_0 \in \partial O_t$ be a point where w is not twice differentiable. Suppose that $\phi \in C^2$ such that $w - \phi$ has a maximum at y_0.

We take paths through y_0 to reduce to a one dimensional problem. For any path $\gamma : I \to \mathbb{R} \times \mathbb{R}^d$ with $\gamma(0) = (t, y_0)$, define $h_\gamma(s) := w(\gamma(s))$ and $\phi_\gamma(s) := \phi(\gamma(s))$. Since w is nonnegative, continuous, smooth on O_t, and extends smoothly from O_t to a function on $\mathbb{R}^d$ for all $t < \delta$, $h'_{\gamma+} := \lim_{s\to 0^+} h'_\gamma(s)$ and $h'_{\gamma-} := \lim_{s\to 0^-} h'_\gamma(s)$ exist. Simlarly, $h''_{\gamma+}$ and $h''_{\gamma-}$ exist. Therefore, since $w - \phi$ is maximized at y_0, $h'_{\gamma+} \leq \phi'_\gamma(0) \leq h'_{\gamma-}$. Now, since w is not twice differentiable at y_0, either there exists γ such that h_γ is not differentiable at 0 or w is differentiable at y_0, but not twice differentiable.

Case 1: $\gamma(s)$ is a path through y_0 for which h_γ is not differentiable.
Then $h'_- < h'_+$ and there exists no such ϕ.

Case 2: w is differentiable at x_0 but not twice differentiable.
Then, for all γ through y_0, $\varphi'_\gamma(0) = h'_\gamma(0)$ and $\varphi''_\gamma(0) \geq \max(h''_{\gamma+}, h''_{\gamma-})$. Now, let γ_i be the coordinate paths through x_0 with $w(\gamma_i(t)) > 0$ for $0 < t < \delta$. Then, since on $w > 0$, w is a

subsolution of the linearized problem (4.1), we have

$$\begin{aligned}(h\partial_t\phi - P(x,hD)\phi)(x_0) &= h\partial_t w - h^2\Delta\phi - ih\langle\nabla\rho, D\rangle w - \mu w + V(x)w \\ &\leq h\partial_t w - h^2\left(\sum_i \lim_{t\to 0^+} h''_{\gamma_i+}\right) - ih\langle\nabla\rho, D\rangle w - \mu w + V(x)w \leq 0.\end{aligned}$$

Thus, we have that w is a subsolution on ∂O_t. Putting this together with the arguments above, we have that w is a viscosity subsolution for (4.1) on $t<\delta$.

Now, by an adaptation of the maximum principle found in [2, Section 3] to parabolic equations, any solution, v to (4.1) with initial data $v_0 > w_0$ will have $v \geq w$ for $t<\delta$. But, since $v\geq 0$, $v^3 \geq 0$ and hence the solution u to (1.1) with initial data v_0 will have $u \geq v \geq w$ for $t<\delta$. Now, since for $t > \frac{\delta}{2}$, $w(x,t)\geq 1$ on $\varphi_t(B(x_0,a))$, we have the result.

We now construct a function v_0 satsifying the assumptions, (4.2),(4.3), and (4.4). Let v_1 be the ground state solution of the Dirichlet Laplacian on $B(0,1)\subset\mathbb{R}^d$ i.e.

$$-\Delta v_1 = \lambda v_1 \text{ on } B(0,1) \quad v_1|_{\partial_{B(0,1)}} = 0.$$

Then, v_1 extends smoothly off of $B(0,1)$ and has $v_1 > 0$ inside $B(0,1)$ (see for instance [7, Section 6.5]).

Let $\chi \in C^\infty(B(0,1))$, $0\leq\chi\leq 1$ with $\chi\equiv 1$ on $B(0,1)\setminus B(0,1-\epsilon)$ and $\operatorname{supp}\chi\subset B(0,1)\setminus B(0,1-2\epsilon)$. Then, define $v_2 := Mv_1 + [\chi(x)](|x|^2-1)$ where M is large enough so that $v_2>0$. There exists such an M since $v_1>0$ in $B(0,1)$ and $\lambda>0$ imply that $-\Delta v_1 = \lambda v_1 > 0$ and hence, by Hopf's Lemma, $\partial_\nu v_1 < 0$ on $\partial B(0,1)$ where ν is the outward normal vector to $\partial B(0,1)$ (see for instance [7, Section 6.4]).

For $|x|\leq 1-\epsilon$, there exists $C>0$ such that,

$$-\Delta v_2 = \lambda M v_1 - [\Delta\chi(x)](|x|^2-1) - 4\langle\partial\chi, x\rangle - 2\chi d \leq \lambda M v_1 + C.$$

For $|x|\leq 1-\epsilon$, $v_1>\delta$. Thus, by increasing M if necessary, we obtain $\beta>0$ such that

$$-\Delta v_2 \leq \lambda M v_1 + C \leq 2\lambda v_2 - \beta.$$

Now, for $1-\epsilon<|x|<1$,

$$-\Delta v_2 = \lambda M v_1 - 2d = \lambda(v_2 - |x|^2 + 1) - 2d \leq \lambda v_2 + \lambda(2\epsilon - \epsilon^2) - 2d.$$

Thus, for $\epsilon>0$ small enough, there exists $\beta>0$ such that

$$-\Delta v_2 \leq \lambda v_2 - \beta.$$

Finally, $\exists\, a\in\mathbb{R}$, $x_0\in\mathbb{R}^d$, and $C_1, C_2 > 0$ constants so that

$$v_0 = \begin{cases} C_1 e^{-\frac{1}{C_2 h}} v_2(a^{-1}(x-x_0)) & x\in B(x_0,a) \\ 0 & \text{else}\end{cases}$$

satisfies the conditions on w_0.

□

Remark 1. If a shorter time is desired, one may use initial data of $O(h^n)$ to obtain the same result in time $O(h|\log h|)$.

Remark 2. Notice that to obtain a growing subsolution it was critical that $\mu > 0$. This corresponds precisely with the movement of the pseudospectrum of $P(x, hD)$ into the right half plane.

Now, we will demonstrate finite time blow-up using the fact that in time $O(1)$ the solution to (1.1) is ≥ 1 on an open region. The proof of theorem 1 follows

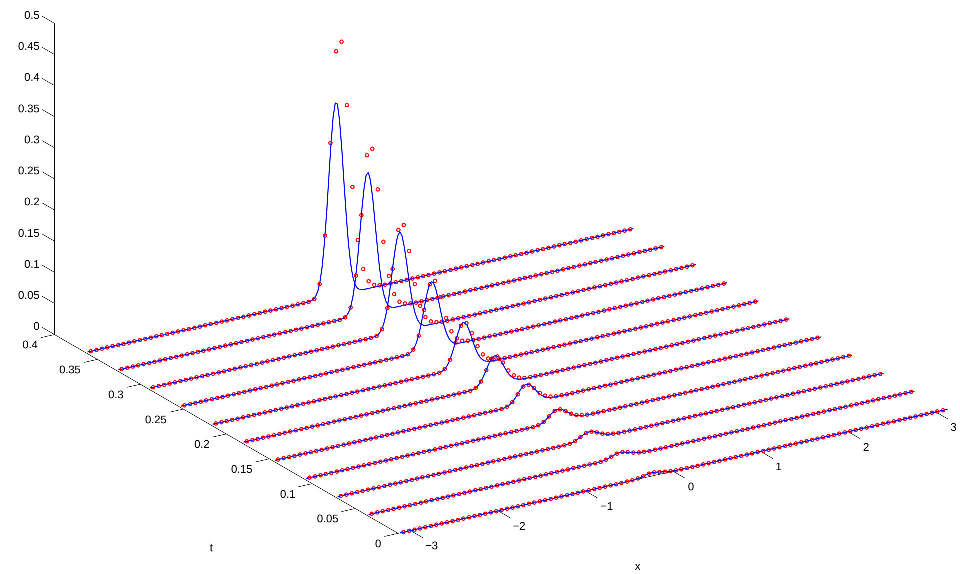

FIGURE 5. We show simulations of solutions to the equation $hu_t = P_1(x, hD)u$ with $h = 1/193$. The solution using a quasimode u_0 with eigenvalue $z = \frac{1}{16}$ and error $O(h^3)$ as initial data is shown in the blue solid linse. The red dotted lines show $u(x, t) = u_0(x) \exp(zt/h)$. We see that the solution to the linearized problem (4.1) with quasimode initial data closely approximates the exponential until $t \approx 0.3$

Proof. Let $u_0(x)$ and t_1 be the initial data and time found in Lemma 3 with (a, x_0, δ) such that φ_t is defined on $B(x_0, a)$, $\varphi_t(B(x_0, a)) \subset V^{-1}[0, \frac{\mu}{2}]$ for $t \in [0, \delta]$, and $t_1 < \delta$. Then, $u(x, t_1) \geq 1$ on $\varphi_t(B(x_0, a))$.

Now, let $\Phi \in C_0^\infty(\mathbb{R})$ be a smooth bump function with $\Phi(y) = 1$ on $|y| \leq 1$, $0 \leq \Phi \leq 1$, supp $\Phi \subset (-2, 2)$, and $\Phi'' \leq C\Phi^{1/3}$ (one such function is given by $e^{-1/x}$ for $\epsilon > x > 0$). Define $\chi : \mathbb{R}^d \to \mathbb{R}$ by $\chi(y) := \Phi\left(2a^{-1}|y|\right)$.

Next, let $y' = \varphi_t(x_0 + y)$ and let

$$v(y,t) := \chi(y)u(y',t).$$

Then, we have that

$$hv_t = h^2\Delta v + \mu v + v^3 - 2h^2\langle\nabla\chi, \nabla u\rangle - h^2\Delta\chi u + (\chi - \chi^3)u^3 - V(y')v.$$

Finally, define the operations, $[f]$ and $[f,g]$ by

$$[f] := \fint_{B(0,a)} f(y)dy \quad \text{and} \quad [f,g] := \fint_{B(0,a)} \langle f(y), g(y)\rangle dy.$$

Then, we have that

$$\begin{aligned} h[v]_t &= h^2[\Delta v] + \mu[v] + [v^3] - h^2[\Delta\chi, u] - 2h^2[\nabla\chi, \nabla u] + \left[\chi - \chi^3, u^3\right] - [V(y'), v] \\ &\geq \mu[v] + [v^3] + h^2[\Delta\chi, u] + \left[\chi - \chi^3, u^3\right] - [V(y'), v] \qquad (4.5) \end{aligned}$$

Here, (4.5) follows from integration by parts, the fact that $\nabla\chi = 0$ at $|y| = a$ and that

$$\fint_{B(0,a)} \Delta v = \frac{c_d}{a^d}\int_{\partial B(0,a)} \nabla v\cdot\nu = 0.$$

We will later need that $[v^3] \geq [v]^3$. To see this use Hölder's inequality with $f = wc_d^{\frac{1}{3}}(a)^{-\frac{d}{3}}$, $g = c_d^{\frac{2}{3}}(a)^{-\frac{2d}{3}}$, $p = 3$, and $q = \frac{3}{2}$ to obtain

$$[v^3] = \int_{B(0,a)} \frac{v^3 c_d}{a^d}d\xi \geq \left(\int_{B(0,a)} \frac{vc_d}{a^d}d\xi\right)^3 = [v]^3.$$

Next, we will estimate $[\Delta\chi, u]$. Observe that

$$\begin{aligned} [\Delta\chi, u] = \left|\fint \Delta\chi u\right| &\leq \left|\frac{1}{C_d}\int_0^a \left(\frac{2a^{-1}(d-1)}{r}\Phi'(2a^{-1}r) + 4a^{-2}\Phi''(2a^{-1}r)\right)\int_{\partial B(0,r)} u(r,\phi)dSdr\right| \\ &\leq C\int_0^a \left(\Phi^{1/2} + \Phi^{1/3}\right)\int_{\partial B(0,a)} u(r,\phi)dSdr \qquad (4.6) \\ &\leq C\fint \chi^{1/3}u \leq C\fint (1 + \chi u^3) \leq C' + C\fint \chi u^3 \qquad (4.7) \end{aligned}$$

where C' and C do not depend on h. Here (4.6) follows from the fact that for any function $\Phi \geq 0$, $\Phi' \leq \Phi^{1/2}$ and that $\Phi' = 0$ near $r = 0$. (4.7) follows from $0 \leq \Phi \leq 1$.

Now, we have

$$\begin{aligned} h[v]_t &= \mu[v] + [v^3] + h^2[\Delta\chi, u] + \left[\chi - \chi^3, u^3\right] - [V(y'), v] \\ &\geq \mu[v] + [v^3] - O(h^2) + \left[(1 - O(h^2))\chi - \chi^3, u^3\right] - [V(y'), v] \\ &\geq \mu[v] + [v^3] - O(h^2) - O(h^2)[v^3] - [V(y'), v] \qquad (4.8) \\ &\geq \mu[v] + (1 - O(h^2))[v]^3 - O(h^2) - [V(y'), v] \qquad (4.9) \end{aligned}$$

Here, (4.8) follows from the fact that $\chi \leq 1$ and (4.9) follows for $h < 1$ since $[v^3] \geq [v]^3$.

Now, on $t < \delta$, we have $V(y') \leq \frac{\mu}{2}$. Thus, for $0 < t < \delta$,

$$h[v]_t \geq \frac{\mu}{2}[v] + (1 - O(h^2))[v]^3 - O(h^2).$$

We have that $[v](t_1) \geq 1/4$ and $\mu > 0$. Therefore there exists $\gamma > 0$ such that, for h small enough and $t_1 \leq t \leq t_1 + \gamma$,

$$h[v]_t \geq \frac{\mu}{4}[v] + \frac{1}{2}[v]^3.$$

But, the solution to this equation with initial data $[v](0) \geq 1/4$ blows up in time $t_2 = O(h)$. Hence, so long as $t_1 + t_2 < \min(\delta, t_1 + \gamma)$ and h is small enough, $[v]$ blows up in time $t_1 + t_2$. Observe that since $t_1 < \delta$, $0 \leq t_1 + t_2 = t_1 + O(h) < \min(\delta, t_1 + \gamma)$ for h small enough. Thus, the solution to 1.1 blows up in time $O(1)$. □

Remark. A similar result holds for polynomially small data with blow up in time $O(h|\log h|)$.

5. Numerical Simulations

We expect that the instability of (1.1) is related to the presence of pseudospectrum in the right half plane. In fact, using numerical simulations for (4.1) based on code from [8] with P as in (1.5), (see Figure 4) we are able to demonstrate that the the solution with a quasimode for a positive eigenvalue as initial data closely approximates an exponential. Based on these results we expect that a proof of blow-up using quasimodes will allow the results of Theorem 1 to be extended to complex energies and wider classes of operators.

All simulations were run in 1 dimension with $\mu = \frac{1}{8}$. Unless otherwise stated, all quasimodes are constructed with errors of $O(h^3)$.

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