Wall's continued-fraction characterization of Hausdorff moment sequences: A conceptual proof

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Abstract

I give an elementary proof of Wall's continued-fraction characterization of Hausdorff moment sequences.

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Let us recall that a sequence $\boldsymbol{a} = (a_n)_{n\ge 0}$ of real numbers is called a Hamburger (resp. Stieltjes, resp. Hausdorff) moment sequence [1, 3, 17, 19, 20] if there exists a positive measure μ on $\mathbb{R}$ (resp. on $[0,\infty)$, resp. on $[0,1]$) such that $a_n = \int x^n \, d\mu(x)$ for all $n \ge 0$. One fundamental characterization of Stieltjes moment sequences was found by Stieltjes [22] in 1894 (see also [26, pp. 327–329]): A sequence $\boldsymbol{a} = (a_n)_{n\ge 0}$ of real numbers is a Stieltjes moment sequence if and only if there exist real numbers $\alpha_0, \alpha_1, \alpha_2, \ldots \ge 0$ such that

$$\sum_{n=0}^{\infty} a_n t^n \;=\; \cfrac{\alpha_0}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}} \tag{1}$$

in the sense of formal power series. (That is, the ordinary generating function $f(t) = \sum\limits_{n=0}^{\infty} a_n t^n$ can be represented as a Stieltjes-type continued fraction with nonnegative coefficients.) Moreover, the coefficients $\boldsymbol{\alpha} = (\alpha_i)_{i\ge 0}$ are unique if we make the convention that $\alpha_i = 0$ implies $\alpha_j = 0$ for all $j > i$; we shall call such a sequence $\boldsymbol{\alpha}$ *standard*.

Since every Hausdorff moment sequence is a Stieltjes moment sequence, its ordinary generating function clearly has a continued-fraction expansion of the form (1) with coefficients $\boldsymbol{\alpha} \ge 0$. But *which* sequences $\boldsymbol{\alpha} \ge 0$ correspond to Hausdorff moment sequences? The answer was given by Wall [24, Theorems 4.1 and 6.1] in 1940: A sequence $\boldsymbol{a} = (a_n)_{n\ge 0}$ of real numbers is a Hausdorff moment sequence if and only if there exist real numbers $c \ge 0$ and $g_1, g_2, g_3, \ldots \in [0,1]$ such that

$$\sum_{n=0}^{\infty} a_n t^n \;=\; \cfrac{c}{1 - \cfrac{g_1 t}{1 - \cfrac{(1-g_1) g_2 t}{1 - \cfrac{(1-g_2) g_3 t}{1 - \cfrac{(1-g_3) g_4 t}{1 - \cdots}}}}} \tag{2}$$

in the sense of formal power series.

Wall's proof of this result was based on an interesting but somewhat mysterious identity for continued fractions [24, Theorem 2.1] together with some complex-analysis arguments.[1] Four years later, Wall [25] gave a new proof, based on Schur's [18] characterization of analytic functions bounded in the unit disc and the Herglotz–Riesz [10,16] integral representation of analytic functions in the unit disc with positive real part.

Here I would like to present an alternate proof of Wall's theorem that is not only very simple but also gives insight into why the coefficients in (2) take the form $\alpha_n = (1 - g_{n-1}) g_n$.

[1] See also [12] for a combinatorial proof of Wall's identity, and see [9, 15, 27] for some interesting applications of it.

This proof requires two well-known elementary facts about moment sequences:

1) $\boldsymbol{a}$ is a Stieltjes moment sequence if and only if the "aerated" sequence $\widehat{\boldsymbol{a}} = (a_0, 0, a_1, 0, a_2, 0, \ldots)$ is a Hamburger moment sequence. Indeed, the even subsequence of a Hamburger moment sequence is always a Stieltjes moment sequence; and conversely, if $\boldsymbol{a}$ is a Stieltjes moment sequence that is represented by a measure μ supported on $[0,\infty)$, then $\widehat{\boldsymbol{a}}$ is represented by the even measure $\widehat{\mu} = (\tau^+ + \tau^-)/2$ on $\mathbb{R}$, where $\tau^\pm$ is the image of μ under the map $x \mapsto \pm\sqrt{x}$. In particular, if μ is supported on $[0,\Lambda]$, then $\widehat{\mu}$ is supported on $[-\sqrt{\Lambda}, \sqrt{\Lambda}]$.

2) If the Hamburger moment sequence $\boldsymbol{a} = (a_n)_{n\ge 0}$ satisfies $|a_n| \le AB^n$ with $A, B < \infty$, then the representing measure μ is unique and is supported on $[-B, B]$. In particular, a Hausdorff moment sequence always has a unique representing measure. (In fact, the representing measure μ of a Hamburger moment sequence is unique under the vastly weaker hypothesis $|a_n| \le AB^n n!$, or even under the yet weaker hypothesis $\sum\limits_{n=1}^{\infty} a_{2n}^{-1/2n} = \infty$ [17, section 4.2]; but we shall not need these latter results.)

Besides Stieltjes-type continued fractions (1) [henceforth called S-fractions for short], we shall also make use of Jacobi-type continued fractions (J-fractions)

$$f(t) \;=\; \cfrac{1}{1 - \gamma_0 t - \cfrac{\beta_1 t^2}{1 - \gamma_1 t - \cfrac{\beta_2 t^2}{1 - \cdots}}} \tag{3}$$

(always considered as formal power series in the indeterminate t).[2] We shall need three elementary facts about these continued fractions:

1) The contraction formula: We have

$$\cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cfrac{\alpha_3 t}{1 - \cdots}}}} \;=\; \cfrac{1}{1 - \alpha_1 t - \cfrac{\alpha_1 \alpha_2 t^2}{1 - (\alpha_2 + \alpha_3) t - \cfrac{\alpha_3 \alpha_4 t^2}{1 - (\alpha_4 + \alpha_5) t - \cfrac{\alpha_5 \alpha_6 t^2}{1 - \cdots}}}} \tag{4}$$

as an identity between formal power series. In other words, an S-fraction with coefficients $\boldsymbol{\alpha}$ is equal to a J-fraction with coefficients $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$, where

$$\gamma_0 \;=\; \alpha_1 \tag{5a}$$

$$\gamma_n \;=\; \alpha_{2n} + \alpha_{2n+1} \quad \text{for } n \ge 1 \tag{5b}$$

$$\beta_n \;=\; \alpha_{2n-1}\alpha_{2n} \tag{5c}$$

See [26, pp. 20–22] for the classic algebraic proof of the contraction formula (4); see [7, Lemmas 1 and 2] [6, proof of Lemma 1] [5, Lemma 4.5] for a very simple variant

[2] My use of the terms "S-fraction" and "J-fraction" follows the general practice in the combinatorial literature, starting with Flajolet [8]. The classical literature on continued fractions [4,11,13,14,26] generally uses a different terminology. For instance, Jones and Thron [11, pp. 128–129, 386–389] use the term "regular C-fraction" for (a minor variant of) what I have called an S-fraction, and the term "associated continued fraction" for (a minor variant of) what I have called a J-fraction.

algebraic proof; and see [23, pp. V-31–V-32] for an enlightening combinatorial proof, based on Flajolet's [8] combinatorial interpretation of S-fractions (resp. J-fractions) as generating functions for Dyck (resp. Motzkin) paths with height-dependent weights.

2) Binomial transform: Fix a real number ξ, and let $\boldsymbol{a} = (a_n)_{n\ge 0}$ be a sequence of real numbers. Then the ξ-binomial transform of $\boldsymbol{a}$ is defined to be the sequence $\boldsymbol{b} = (b_n)_{n\ge 0}$ given by

$$b_n \;=\; \sum_{k=0}^{n} \binom{n}{k} a_k\, \xi^{n-k} \,. \tag{6}$$

Note that if $a_n = \int x^n \, d\mu(x)$, then $b_n = \int (x+\xi)^n \, d\mu(x)$. In other words, if $\boldsymbol{a}$ is a Hamburger moment sequence with representing measure μ, then $\boldsymbol{b}$ is a Hamburger moment sequence with representing measure $T_\xi \mu$ (the ξ-translate of μ).

Now suppose that the ordinary generating function of $\boldsymbol{a}$ is given by a J-fraction:

$$\sum_{n=0}^{\infty} a_n t^n \;=\; \cfrac{1}{1 - \gamma_0 t - \cfrac{\beta_1 t^2}{1 - \gamma_1 t - \cfrac{\beta_2 t^2}{1 - \cdots}}} \,. \tag{7}$$

Then the ξ-binomial transform $\boldsymbol{b}$ of $\boldsymbol{a}$ is given by a J-fraction in which we make the replacement $\gamma_i \to \gamma_i + \xi$:

$$\sum_{n=0}^{\infty} b_n t^n \;=\; \cfrac{1}{1 - (\gamma_0 + \xi) t - \cfrac{\beta_1 t^2}{1 - (\gamma_1 + \xi) t - \cfrac{\beta_2 t^2}{1 - \cdots}}} \,. \tag{8}$$

See [2, Proposition 4] for an algebraic proof of (8); or see [21] for a simple combinatorial proof based on Flajolet's [8] theory.

3) An upper bound: If $\boldsymbol{a}$ is given by the S-fraction (1) with $0 \le \alpha_i \le 1$ for all i, then $0 \le a_n \le C_n \le 4^n$, where $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$ is the nth Catalan number. The proof is simple: If we consider the coefficients $\boldsymbol{\alpha}$ in (1) to be indeterminates, then it is easy to see that a_n is a polynomial in $\alpha_0, \ldots, \alpha_n$ with nonnegative integer coefficients (namely, α_0 times the Stieltjes–Rogers polynomial $S_n(\boldsymbol{\alpha})$ [8]); so a_n is an increasing function of $\boldsymbol{\alpha}$ on the set $\boldsymbol{\alpha} \ge 0$. On the other hand, if $\alpha_i = 1$ for all i, then (1) represents a series $f(t) = \sum_{n=0}^{\infty} a_n t^n$ satisfying $f(t) = 1/[1 - tf(t)]$, from which it follows that $f(t) = [1 - \sqrt{1-4t}]/(2t)$ and hence (by binomial expansion) that $a_n = C_n$.

Proof of Wall's theorem. Let $\boldsymbol{a} = (a_n)_{n\ge 0}$ be a Hausdorff moment sequence; we can assume without loss of generality that $a_0 = 1$. Then $\boldsymbol{a}$ has a (unique) representing measure μ supported on $[0,1]$, and its ordinary generating function is given by a unique S-fraction (1) with $\alpha_0 = 1$ and standard coefficients $\boldsymbol{\alpha} \ge 0$. Now let $\widehat{\boldsymbol{a}} = (a_0, 0, a_1, 0, a_2, 0, \ldots)$ be the aerated sequence; it is a Hamburger moment sequence with a (unique) even representing measure $\widehat{\mu}$ supported on $[-1,1]$, and its ordinary generating function is given by the J-fraction with coefficients $\boldsymbol{\gamma} = 0$ and $\boldsymbol{\beta} = \boldsymbol{\alpha}$:

$$\sum_{n=0}^{\infty} \widehat{a}_n t^n \;=\; \sum_{n=0}^{\infty} a_n t^{2n} \;=\; \cfrac{1}{1 - \cfrac{\alpha_1 t^2}{1 - \cfrac{\alpha_2 t^2}{1 - \cdots}}} \,. \tag{9}$$

Now let $\widetilde{\boldsymbol{a}}$ be the 1-binomial transform of $\widehat{\boldsymbol{a}}$; it is a Stieltjes moment sequence with a (unique) representing measure $\widetilde{\mu} = T_1 \widehat{\mu}$ supported on $[0,2]$, and its ordinary generating function is given by a J-fraction with coefficients $\boldsymbol{\gamma} = \mathbf{1}$ and $\boldsymbol{\beta} = \boldsymbol{\alpha}$:

$$\sum_{n=0}^{\infty} \widetilde{a}_n t^n \;=\; \cfrac{1}{1 - t - \cfrac{\alpha_1 t^2}{1 - t - \cfrac{\alpha_2 t^2}{1 - \cdots}}} \,. \tag{10}$$

But since $\widetilde{\boldsymbol{a}}$ is a Stieltjes moment sequence, its ordinary generating function is also given by an S-fraction with nonnegative coefficients, call them $\boldsymbol{\alpha}'$. Comparing the J-fraction and the S-fraction using the contraction formula (4)/(5), we see that

$$1 \;=\; \alpha'_1 \;=\; \alpha'_2 + \alpha'_3 \;=\; \alpha'_4 + \alpha'_5 \;=\; \ldots \tag{11a}$$

$$\alpha_1 \;=\; \alpha'_1 \alpha'_2 \,, \quad \alpha_2 \;=\; \alpha'_3 \alpha'_4 \,, \quad \ldots \tag{11b}$$

It follows from (11a) that $\alpha'_i \in [0,1]$ for all i. Setting $g_n = \alpha'_{2n}$ shows that $\alpha_1 = g_1$ and $\alpha_n = (1 - g_{n-1}) g_n$ for $n \ge 2$, which is precisely the representation (2).

Conversely, suppose that $\boldsymbol{a}$ is given by an S-fraction (2) with coefficients $c = 1$ and $g_i \in [0,1]$. Then $\boldsymbol{a}$ is a Stieltjes moment sequence satisfying $a_n \le 4^n$, so that the representing measure μ is unique and is supported on $[0,4]$. (Of course, we will soon see that μ is actually supported on $[0,1]$.) Then the aerated sequence $\widehat{\boldsymbol{a}} = (a_0, 0, a_1, 0, a_2, 0, \ldots)$ is a Hamburger moment sequence with a unique representing measure $\widehat{\mu}$ that is even and supported on $[-2,2]$, and its ordinary generating function is given by a J-fraction with coefficients $\boldsymbol{\gamma} = 0$, $\beta_1 = g_1$ and $\beta_n = (1 - g_{n-1}) g_n$ for $n \ge 2$. Now let $\widetilde{\boldsymbol{a}}$ be the 1-binomial transform of $\widehat{\boldsymbol{a}}$: it is a Hamburger moment sequence with a unique representing measure $\widetilde{\mu} = T_1 \widehat{\mu}$ supported on $[-1,3]$, and its ordinary generating function is given by a J-fraction with coefficients $\boldsymbol{\gamma} = \mathbf{1}$, $\beta_1 = g_1$ and $\beta_n = (1 - g_{n-1}) g_n$ for $n \ge 2$. But the contraction formula (4)/(5) shows that this J-fraction is equivalent to an S-fraction with coefficients $\alpha'_1 = 1$ and $\alpha'_{2n} = g_n$, $\alpha'_{2n+1} = 1 - g_n$ for $n \ge 1$. Since all these coefficients are nonnegative, it follows that $\widetilde{\boldsymbol{a}}$ is a Stieltjes moment sequence. Therefore $\widetilde{\mu}$ is supported on $[0,3]$, so that $\widehat{\mu} = T_{-1} \widetilde{\mu}$ is supported on $[-1,2]$. But since $\widehat{\mu}$ is even, it must actually be supported on $[-1,1]$. Hence μ is supported on $[0,1]$, which shows that $\boldsymbol{a}$ is a Hausdorff moment sequence. □

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