A Stabilized Cut Streamline Diffusion Finite Element Method for Convection-Diffusion Problems on Surfaces

Erik Burman[a], Peter Hansbo[b,*], Mats G. Larson[c], André Massing[c], Sara Zahedi[d]

[a]*Department of Mathematics, University College London, London, WC1E 6BT United Kingdom*
[b] *Department of Mechanical Engineering, Jönköping University, SE-551 11 Jönköping, Sweden*
[c]*Department of Mathematics and Mathematical Statistics, Umeå University, SE–901 87 Umeå, Sweden*
[d]*Department of Mathematics, KTH Royal Institute of Technology, SE-100 44 Stockholm, Sweden*

Abstract

We develop a stabilized cut finite element method for the stationary convection diffusion problem on a surface embedded in $\mathbb{R}^d$. The cut finite element method is based on using an embedding of the surface into a three dimensional mesh consisting of tetrahedra and then using the restriction of the standard piecewise linear continuous elements to a piecewise linear approximation of the surface. The stabilization consists of a standard streamline diffusion stabilization term on the discrete surface and a so called normal gradient stabilization term on the full tetrahedral elements in the active mesh. We prove optimal order a priori error estimates in the standard norm associated with the streamline diffusion method and bounds for the condition number of the resulting stiffness matrix. The condition number is of optimal order for a specific choice of method parameters. Numerical examples supporting our theoretical results are also included.

Keywords: cut finite element method, convection–diffusion–reaction, PDEs on surfaces, streamline diffusion, continuous interior penalty

1. Indroduction

Contributions. We develop and analyze cut finite element method for the convection-diffusion problem on surfaces. The cut finite element method is constructed as follows: (i) The surface is embedded into a three dimensional domain equipped with a family of meshes. (ii) A piecewise linear approximation of the surface is computed for instance using an interpolation of the distance function. (iii) The active mesh is defined as the subset of elements that intersect the discrete surface. (iv) A finite element approximation is defined by using a variational formulation together with a restriction of the finite element space to the active mesh as trial and test space.

In order to stabilize the method we add two stabilization terms: (i) A streamline diffusion stabilization, which is added on the discrete surface and stabilizes the method in the convection dominated case. (ii) A normal gradient stabilization term on the full three dimensional tetrahedra in the active mesh, which provides control of the variation of the discrete solution in the direction normal to the surface.

Streamline diffusion or Streamline upwind Petrov–Galerkin methods for stabilization of transport dominated problems in flat domains were introduced in [1] and [17]. In order to show stability of the method and an optimal order bound on the condition number we add the normal gradient stabilization, which enables us to show that the condition number is of optimal order for a specific choice of method parameters. As an alternative to the normal gradient stabilization one may use face stabilization [2] [5], full gradient stabilization [4], or face stabilization in combination

*Corresponding author
Email addresses: e.burman@ucl.ac.uk (Erik Burman), peter.hansbo@ju.se (Peter Hansbo), mats.larson@math.umu.se (Mats G. Larson), andre.massing@umu.se (André Massing), sara.zahedi@math.kth.se (Sara Zahedi)

with normal gradient stabilization on the surface [18]. Full gradient stabilization does not lead to optimal condition number estimates in the convection dominated case and does not extend to higher order approximations. The stabilization proposed in [18] has similar properties as the normal gradient stabilization we analyze here.

Previous Work. CutFEM, or trace FEM, for partial differential equations on surfaces was first introduced for the Laplace-Beltrami operator in [20] without stabilization and is now a rapidly developing technique. In [2] a stabilized version based on so called face stabilization or ghost stabilization, which provides control over the jump in the normal gradient across interior faces in the active mesh was introduced and analyzed. In particular it was shown that the condition number scaled in an optimal way. In [22] a streamline diffusion trace finite element method for the convection-diffusion problem on a surface was studied, here only streamline diffusion stabilization was added and the skew symmetric discretization of the convection term was used. In [5] the pure convection problem on a surface with face stabilization was analyzed. In [4] a full gradient stabilization method for the Laplace-Beltrami operator was developed and analyzed. In [3] an abstract framework for analysis of cut finite element methods on embedded manifolds of arbitrary codimension was developed and, in particular, the normal gradient stabilization term which we employ in this paper was introduced and analyzed (see also [12] for an analysis in the case of high order approximation). Coupled bulk-surface problems were considered in [6] and [13]. Higher order versions of trace fem for the Laplace-Beltrami operator were analyzed in [23, 12]. Finally in, [16], [19], and [21], extensions to time dependent problems were presented.

Several other techniques for solving partial differential equations on surfaces have been proposed. Most notably, the original idea of Dziuk, [8] was to use a triangulation of the surface. We refer to [9] and the references therein for an overview.

Organization of the Paper. In Section 2 we introduce the model problem, in Section 3 we define the cut finite element method, in Section 4 we derive our main theoretical results, in Section 5 we prove a bound on the condition number, and in Section 6 we present numerical results that confirm our theoretical results.

2. The Model Problem

2.1. The Surface

Let Γ be a smooth surface without boundary embedded in $\mathbb{R}^3$ with signed distance function ρ, such that the exterior unit normal to the surface is given by $n = \nabla\rho$. We let $p : \mathbb{R}^3 \to \Gamma$ be the closest point mapping. Then there is a $\delta_0 > 0$ such that p maps each point in $U_{\delta_0}(\Gamma)$ to precisely one point on Γ. Here $U_\delta(\Gamma) = \{x \in \mathbb{R}^3 : |\rho(x)| < \delta\}$ is the open tubular neighborhood of Γ of thickness $\delta > 0$.

2.2. Tangential Calculus

For each function u on Γ we let the extension u^e to the neighborhood $U_{\delta_0}(\Gamma)$ be defined by the pull back $u^e = u \circ p$. For a function $u : \Gamma \to \mathbb{R}$ we define the tangential gradient

$$\nabla_\Gamma u = P_\Gamma \nabla u^e \tag{2.1}$$

where $P_\Gamma = I - n \otimes n$, with $n = n(x)$, is the projection onto the tangent plane $T_x(\Gamma)$, and the surface divergence

$$\mathrm{div}_\Gamma(u) = \mathrm{tr}(u \otimes \nabla_\Gamma) = \mathrm{div}(u^e) - n \cdot (u^e \otimes \nabla) \cdot n \tag{2.2}$$

where $(u \otimes \nabla)_{ij} = \partial_j u_i$. It can be shown that the tangential derivative does not depend on the particular choice of extension.

2.3. The Surface Convection-Diffusion Problem

The strong form of the convection diffusion problem on Γ takes the form: find $u : \Gamma \to \mathbb{R}$ such that

$$Lu = \beta \cdot \nabla_\Gamma u + \alpha u - \epsilon \Delta_\Gamma u = f \qquad \text{on } \Gamma \tag{2.3}$$

where $\beta : \Gamma \to \mathbb{R}^3$ is a given tangential vector field, $\alpha : \Gamma \to \mathbb{R}$, and $f : \Gamma \to \mathbb{R}$ are given functions, and $\epsilon \geq 0$ is a given constant.

We assume that the coefficients α and β are smooth and that there is a constant α_0 such that

$$0 < \alpha_0 \leq \inf_{x \in \Gamma} (\alpha(x) - \frac{1}{2} \text{div}_\Gamma \beta(x)) \tag{2.4}$$

We note that using Green's formula and assumption (2.4) we obtain the estimate

$$(Lv, v)_\Gamma = ((\alpha - \frac{1}{2} \text{div}_\Gamma \beta) v, v)_\Gamma + \epsilon \|\nabla_\Gamma v\|^2_\Gamma \geq \alpha_0 \|v\|^2_\Gamma + \epsilon \|\nabla_\Gamma v\|^2_\Gamma \tag{2.5}$$

The weak formulation of (2.3) takes the form: find $u \in V = H^1(\Gamma)$ such that

$$a(u, v) = (f, v)_\Gamma \qquad \forall v \in V \tag{2.6}$$

where

$$a(v, w) = (\beta \cdot \nabla_\Gamma v, w)_\Gamma + (\alpha v, w)_\Gamma + \epsilon (\nabla_\Gamma v, \nabla_\Gamma w)_\Gamma \tag{2.7}$$

In the case $\epsilon > 0$ we may conclude using Lax–Milgram's lemma that there is a unique solution to (2.6). In the case $\epsilon = 0$ we refer to [5] for an existence result.

We will be interested in the behavior of the numerical method for coefficients that satisfy

$$0 \leq \epsilon \lesssim 1, \quad \alpha \sim 1, \quad \beta \sim 1, \quad \alpha + \frac{1}{2} \text{div}_\Gamma \beta \sim 1 \tag{2.8}$$

with particular focus on the convection dominated case.

3. The Finite Element Method

3.1. The Discrete Surface

Let $\Omega_0 \subset \mathbb{R}^3$ be a polygonal domain that contains $U_{\delta_0}(\Gamma)$ and let $\{\mathcal{T}_{0,h}, h \in (0, h_0]\}$ be a family of quasiuniform partitions of Ω_0 into shape regular tetrahedra with mesh parameter h. Let $\Gamma_h \subset \Omega_0$ be a surface without boundary such that $\Gamma_h \cap \overline{T}$ is either empty or a subset of some hyperplane for each $T \in \mathcal{T}_{0,h}$ and let n_h be the piecewise constant exterior unit normal to Γ_h.

Assumption A. The family $\{\Gamma_h : h \in (0, h_0]\}$ approximates Γ in the following sense:

- $\Gamma_h \subset U_{\delta_0}(\Gamma)$, and the closest point mapping $p : \Gamma_h \to \Gamma$ is a bijection, for all $h \in (0, h_0]$.
- There are constants such that for all $h \in (0, h_0]$,

$$\|\rho\|_{L^\infty(\Gamma_h)} \lesssim h^2, \qquad \|n - n_h\|_{L^\infty(\Gamma_h)} \lesssim h \tag{3.1}$$

Let $\mathcal{T}_h$ be the active mesh in the mesh

$$\mathcal{T}_h = \{T \in \mathcal{T}_{h,0} : \overline{T} \cap \Gamma_h \neq \emptyset\} \tag{3.2}$$

and let $\mathcal{F}_h$ be the set of all interior faces in $\mathcal{T}_h$. Let $\mathcal{K}_h$ be the induced partition of Γ_h, defined by

$$\mathcal{K}_h = \{K = T \cap \Gamma_h : T \in \mathcal{T}_h\} \cup \{F \in \mathcal{F}_h : F \subset \Gamma_h\} \tag{3.3}$$

and let $\mathcal{E}_h$ be the set of edges in the induced partition $\mathcal{K}_h$ of Γ_h.

We shall also use the notation $\omega^l = p(\omega) = \{p(x) \in \Gamma : x \in \omega \subset \Gamma_h\}$, in particular $\mathcal{K}_h^l = \{K^l : K \in \mathcal{K}_h^l\}$ is a partition of Γ, and $\|v\|_\omega$ denotes the L^2 norm over the set ω equipped with the appropriate measure.

3.2. The Stabilized Finite Element Method

Let V_h be the space of continuous piecewise linear functions defined on $\mathcal{T}_h$. To implement a discrete version of the variational form of (2.3) we introduce discrete approximations, α_h and β_h, of the physical parameters α and β. The assumptions made on these approximations will be specified in Assumption B below.

The finite element method takes the form: find $u_h \in V_h$ such that

$$A_h(u_h, v) = L_h(v) \quad \forall v \in V_h \tag{3.4}$$

The forms are defined by

$$A_h(v, w) = a_h(v, w) + s_{h,1}(v, w) + s_{h,2}(v, w) \tag{3.5}$$

with

$$a_h(v, w) = (\beta_h \cdot \nabla_{\Gamma_h} v, w)_{\mathcal{K}_h} + (\alpha_h v, w)_{\mathcal{K}_h} + \epsilon(\nabla_{\Gamma_h} v, \nabla_{\Gamma_h} w)_{\mathcal{K}_h} \tag{3.6}$$

$$s_{h,1}(v, w) = \tau_1 h(\beta_h \cdot \nabla_{\Gamma_h} v + \alpha_h v, \beta_h \cdot \nabla_{\Gamma_h} w)_{\mathcal{K}_h} \tag{3.7}$$

$$s_{h,2}(v, w) = \tau_2 h^\gamma (n_h \cdot \nabla v, n_h \cdot \nabla w)_{\mathcal{T}_h} \tag{3.8}$$

where $\nabla_{\Gamma_h} v = P_{\Gamma_h} \nabla v = (I - n_h \otimes n_h)\nabla v$ is the tangential gradient on Γ_h, and

$$L_h(v) = l_h(v) + l_{s_{h,1}}(v) = (f^e, v)_{\mathcal{K}_h} + \tau_1 h(f^e, \beta_h \cdot \nabla_{\Gamma_h} v)_{\mathcal{K}_h} \tag{3.9}$$

Here the streamline diffusion stabilization term $s_{h,1}$ is added to control the solution in the convection dominated case and the normal gradient stabilization term $s_{h,2}$ is added to control potential instabilities caused by the cut elements. The streamline diffusion stabilization $s_{h,1}$ includes the weighting parameter

$$\tau_1 = \begin{cases} c_{\tau_1} \beta_\infty^{-1} & \text{if } \beta_\infty h \geq \epsilon \text{ (high Peclet number regime)} \\ c_{\tau_1} h \epsilon^{-1} & \text{if } \beta_\infty h \leq \epsilon \text{ (low Peclet number regime)} \end{cases} \tag{3.10}$$

where c_{τ_1} is a positive parameter and $\beta_\infty = \|\beta_h\|_{L^\infty(\Gamma_h)}$. Observe that this can be more compactly written

$$\tau_1 = c_{\tau_1} \min(\beta_\infty^{-1}, h\epsilon^{-1}) \tag{3.11}$$

The normal gradient stabilization term $s_{h,2}$ has two parameters: τ_2 and γ. The parameter γ will be chosen in the interval $\gamma \in [0, 2)$, see the discussion in Remark 3.1 below. The parameter τ_2 should be chosen proportional to τ_1^{-1}, to scale correctly in β_∞ and ϵ, more precisely

$$\tau_2 = c_{\tau_2} \max(\beta_\infty, \epsilon h^{-1}) \tag{3.12}$$

where c_{τ_2} is a positive parameter.

Remark 3.1. *From the stability estimate and consistency we have that the parameter γ in (3.8) must be chosen in the range $0 \leq \gamma < 2$. More precisely, the derivation of the coercivity of A_h provides an upper bound on γ, guaranteeing that the stabilization is strong enough, and the consistency result provides the lower bound, guaranteeing that the stabilization is weak enough not to affect the optimal order of convergence (see the a priori estimate). The condition number estimate, see Theorem 6.1, however shows that the best choice is $\gamma = 1$. This is the largest γ, and thus the weakest stabilization, that gives optimal order condition number.*

Remark 3.2. *In [4] the so called full gradient stabilization was proposed and analyzed for the Laplace-Beltrami operator. Applying the same idea for the convection-diffusion problem we find that a suitable full gradient stabilization term takes the form*

$$s_{h,3}(v, w) = \tau_2 h^2 (\nabla v, \nabla w)_{\mathcal{T}_h} \tag{3.13}$$

where τ_s is defined in (3.12) and the powers of h are given by the a priori error estimate (see comments in Remark 5.3. We note that the normal control provided by the full gradient stabilization term is weaker compared to the normal gradient due to the different h scalings. To prove coercivity, at least using straight forward estimates, we will need to use the antisymmetric formulation of the convection term. See Remark 5.1 below. Note also that the h scaling here is fixed while in the normal gradient stabilization we have some flexibility (the choice of γ).

4. Preliminary Results

4.1. Extension and Lifting of Functions

In this section we summarize basic results concerning extension and liftings of functions. We refer to [2], [7], and [15], for further details.

Extension. Recalling the definition $v^e = v \circ p$ of the extension and using the chain rule we obtain the identity

$$\nabla_{\Gamma_h} v^e = B^T \nabla_\Gamma v \tag{4.1}$$

where

$$B = P_\Gamma (I - \rho\kappa) P_{\Gamma_h} : T_x(K) \to T_{p(x)}(\Gamma) \tag{4.2}$$

and $\kappa = \nabla \otimes \nabla \rho$ is the curvature tensor (or second fundamental form) which may be expressed in the form

$$\kappa(x) = \sum_{i=1}^{2} \frac{\kappa_i^e}{1 + \rho(x)\kappa_i^e} a_i^e \otimes a_i^e \tag{4.3}$$

where κ_i are the principal curvatures with corresponding orthonormal principal curvature vectors a_i, see [11, Lemma 14.7]. We note that there is $\delta > 0$ such that the uniform bound

$$\|\kappa\|_{L^\infty(U_\delta(\Gamma))} \lesssim 1 \tag{4.4}$$

holds. Furthermore, $B : T_x(K) \to T_{p(x)}(\Gamma)$ is invertible for $h \in (0, h_0]$ with h_0 small enough, i.e, there is $B^{-1} : T_{p(x)}(\Gamma) \to T_x(K)$ such that

$$BB^{-1} = P_\Gamma, \qquad B^{-1}B = P_{\Gamma_h} \tag{4.5}$$

See [15] for further details.

Lifting. The lifting w^l of a function w defined on Γ_h to Γ is defined as the push forward

$$(w^l)^e = w^l \circ p = w \quad \text{on } \Gamma_h \tag{4.6}$$

For the derivative it follows that

$$\nabla_{\Gamma_h} w = \nabla_{\Gamma_h} (w^l)^e = B^T \nabla_\Gamma (w^l) \tag{4.7}$$

and thus

$$\nabla_\Gamma (w^l) = B^{-T} \nabla_{\Gamma_h} w \tag{4.8}$$

Estimates Related to B. Using the uniform bound $\|\kappa\|_{U_{\delta_0}(\Gamma)} \lesssim 1$ and the bound $\|\rho\|_{L^\infty(\Gamma_h)} \lesssim h^2$ from the geometry approximation assumption it follows that

$$\|B\|_{L^\infty(\Gamma_h)} \lesssim 1, \qquad \|B^{-1}\|_{L^\infty(\Gamma)} \lesssim 1 \tag{4.9}$$

$$\|P_\Gamma P_{\Gamma_h} - B\|_{L^\infty(\Gamma)} \lesssim h^2, \qquad \|P_{\Gamma_h} P_\Gamma - B^{-1}\|_{L^\infty(\Gamma_h)} \lesssim h^2 \tag{4.10}$$

For the surface measures on Γ and Γ_h we have the identity

$$d\Gamma = |B| d\Gamma_h \tag{4.11}$$

where $|B| = |\det(B)|$ is the absolute value of the determinant of B and we have the following estimates

$$\|1 - |B|\|_{L^\infty(\Gamma_h)} \lesssim h^2, \quad \big\|1 - |B^{-1}|\big\|_{L^\infty(\Gamma_h)} \lesssim h^2, \quad \||B|\|_{L^\infty(\Gamma_h)} \lesssim 1, \quad \big\||B|^{-1}\big\|_{L^\infty(\Gamma_h)} \lesssim 1 \tag{4.12}$$

Norm Equivalences. We have

$$\|v\|_{L^2(\Gamma)} \sim \|v\|_{L^2(\Gamma_h)} \qquad \text{and} \qquad \|\nabla_\Gamma v\|_{L^2(\Gamma)} \sim \|\nabla_{\Gamma_h} v\|_{L^2(\Gamma_h)} \tag{4.13}$$

4.2. Properties of the Discrete Approximations of the Coefficients

We make the following assumptions about the approximations α_h and β_h:

Assumption B. The functions α_h, β_h are approximations of α^e and β^e, such that β_h is tangential $\beta_h = P_{\Gamma_h}\beta_h$, and the following properties hold:

- There are constants such that for all $h \in (0, h_0]$,

$$\|\alpha_h\|_{L^\infty(\Gamma_h)} \lesssim 1, \quad \|\beta_h\|_{L^\infty(\Gamma_h)} \lesssim 1, \quad \|\alpha_h - \frac{1}{2}\mathrm{div}_{\Gamma_h}\beta_h\|_{L^\infty(\Gamma_h)} \lesssim 1 \tag{4.14}$$

- There is a constant $\alpha_{h,0}$ such that for all $h \in (0, h_0]$,

$$0 < \alpha_{h,0} \leq \inf_{x\in\Gamma_h} \alpha_h(x) - \frac{1}{2}\mathrm{div}_{\Gamma_h}\beta_h(x) \tag{4.15}$$

- There are constants (depending on α and β) such that for all $h \in (0, h_0]$,

$$\|\beta - |B|^{-1}B\beta_h^l\|_{L^\infty(\Gamma)} \lesssim C_\beta h^2, \qquad \|\alpha - |B|^{-1}\alpha_h^l\|_{L^\infty(\Gamma)} \lesssim C_\alpha h^2 \tag{4.16}$$

It follows from assumptions A and B that

$$\|[\nu_h \cdot \beta_h]\|_{L^\infty(\mathcal{E}_h)} \lesssim C_\beta h^2 \tag{4.17}$$

for all $h \in (0, h_0]$. Here the jump is defined by

$$[\nu_h \cdot \beta_h] = \nu_{h,K_1} \cdot \beta_h + \nu_{h,K_2} \cdot \beta_h \tag{4.18}$$

where ν_{h,K_i} denotes the unit vector orthogonal to the edge $E \in \mathcal{E}_h$ shared by the elements K_1 and K_2, tangent and exterior to K_i, $i = 1, 2$. See [5] for a proof of (4.17).

Remark 4.1. *Using the bounds (4.12), (4.10) and the identity (4.2) we note that*

$$\|\beta - |B|^{-1}B\beta_h^l\|_{L^\infty(\Gamma)} \lesssim \|\beta - P_\Gamma\beta_h^l\|_{L^\infty(\Gamma)} + O(h^2) \tag{4.19}$$

and using (4.12) we have

$$\|\alpha - |B|^{-1}\alpha_h^l\|_{L^\infty(\Gamma)} \lesssim \|\alpha - \alpha_h^l\|_{L^\infty(\Gamma)} + O(h^2) \tag{4.20}$$

Thus we conclude that (4.16) is equivalent to the simplified assumptions

$$\|\beta - P_\Gamma\beta_h^l\|_{L^\infty(\Gamma)} \lesssim C_\beta h^2, \qquad \|\alpha - \alpha_h^l\|_{L^\infty(\Gamma)} \lesssim C_\alpha h^2 \tag{4.21}$$

Remark 4.2. *A natural choice is to take $\alpha_h = \alpha^e$ and $\beta_h = P_{\Gamma_h}\beta^e$; then we have*

$$C_\alpha = \|\alpha\|_{L^\infty(\Gamma)} = \alpha_\infty, \qquad C_\beta = \|\beta\|_{L^\infty(\Gamma)} = \beta_\infty \tag{4.22}$$

We may then verify (4.16), using the simplified form (4.21), as follows:

$$\|\beta - P_\Gamma\beta_h^l\|_{L^\infty(\Gamma)} = \|P_\Gamma(P_\Gamma\beta^e)^l - P_\Gamma(P_{\Gamma_h}\beta^e)^l\|_{L^\infty(\Gamma)} \tag{4.23}$$

$$= \|P_\Gamma(P_\Gamma\beta^e - P_{\Gamma_h}\beta^e)^l)\|_{L^\infty(\Gamma)} \tag{4.24}$$

$$= \|P_\Gamma(P_\Gamma P_\Gamma\beta^e - P_{\Gamma_h}P_\Gamma\beta^e)^l)\|_{L^\infty(\Gamma)} \tag{4.25}$$

$$= \|(P_\Gamma(P_\Gamma - P_{\Gamma_h})P_\Gamma\beta^e)^l)\|_{L^\infty(\Gamma)} \tag{4.26}$$

$$\leq \|P_\Gamma(P_\Gamma - P_{\Gamma_h})P_\Gamma\|_{L^\infty(\Gamma)}\|\beta\|_{L^\infty(\Gamma)} \tag{4.27}$$

$$= \|P_\Gamma n_h\|^2_{L^\infty(\Gamma)}\|\beta\|_{L^\infty(\Gamma)} \tag{4.28}$$

$$\lesssim h^2\|\beta\|_{L^\infty(\Gamma)} \tag{4.29}$$

where we used the identity $P_\Gamma\beta^e = \beta^e$ and the estimate

$$\|P_\Gamma n_h\|_{L^\infty(\Gamma)} = \|P_\Gamma(n_h - n)\|_{L^\infty(\Gamma)} \leq \|n_h - n\|_{L^\infty(\Gamma)} \lesssim h \tag{4.30}$$

For α we clearly have $\alpha - \alpha_h^l = \alpha - (\alpha^e)^l = 0$.

4.3. Some Technical Estimates

Here we will recall two inequalities that are useful in the analysis of the stabilized method. The first that was originally introduced in [3] is a Poincaré type inequality showing that the L^2-norm of the finite element solution in the bulk mesh $\mathcal{T}_h$ can be controlled by the L^2-norm over the discrete surface plus the normal component of the bulk gradient, scaled with h. The second is a trace inequality showing that the scaled L^2-norm of the finite element solution over the edges of the tesselation of the discrete surface can be bounded by the L^2-norm over the surface, plus the scaled normal stabilization term.

Lemma 4.1. *There is a constant such that for all $v \in V_h$,*

$$\|v\|^2_{\mathcal{T}_h} \lesssim h\|v\|^2_{\mathcal{K}_h} + h^2\|n_h \cdot \nabla v\|^2_{\mathcal{T}_h} \tag{4.31}$$

Proof. See [3] Proposition 8.8. □

Lemma 4.2. *There is a constant such that for all $v \in V_h$,*

$$h\|v\|^2_{\mathcal{E}_h} \lesssim \|v\|^2_{\mathcal{K}_h} + h\|n_h \cdot \nabla v\|^2_{\mathcal{T}_h} \tag{4.32}$$

There is a constant such that for all $v \in V_h + H^2(\mathcal{T}_h)$,

$$h^2\|v\|^2_{\mathcal{E}_h} \lesssim \|v\|^2_{\mathcal{T}_h} + h^2\|\nabla v\|^2_{\mathcal{T}_h} + h^4\|\nabla^2 v\|^2_{\mathcal{T}_h} \tag{4.33}$$

where $\nabla^2 v = \nabla \otimes \nabla v$ is the Hessian of v.

Proof. To show (4.32) we first use inverse estimates to pass from edges to faces to elements

$$\|v\|^2_E \lesssim h^{-1}\|v\|^2_F \lesssim h^{-2}\|v\|^2_T \tag{4.34}$$

where $E = F \cap \Gamma_h$ and $F \subset \partial T$. Next summing over all $E \in \mathcal{E}_h$, employing (4.31), and using the fact that to each T there is at most four edges E such that $E \subset \partial T$, we obtain the desired estimate

$$h\|v\|^2_{\mathcal{E}_h} \lesssim h^{-1}\|v\|^2_{\mathcal{T}_h} \lesssim \|v\|^2_{\mathcal{K}_h} + h\|n_h \cdot \nabla v\|^2_{\mathcal{T}_h} \tag{4.35}$$

The inequality (4.33) follows by first applying the trace inequality

$$\|v\|^2_E \lesssim h^{-1}\|v\|^2_F + h\|\nabla v\|^2_F \tag{4.36}$$

see [14], to pass from the edge E to the face F such that $E = F \cap \Gamma_h$. Then we apply the standard trace inequality

$$\|w\|^2_F \lesssim h^{-1}\|w\|^2_T + h\|\nabla w\|^2_T \tag{4.37}$$

with $w = \nabla v$ to pass from the face F to an element T such that $F \subset \partial T$. We obtain

$$\begin{aligned}
\|v\|^2_E &\lesssim h^{-1}\|v\|^2_F + h\|\nabla v\|^2_F && (4.38)\\
&\lesssim h^{-1}(h^{-1}\|v\|^2_T + h\|\nabla v\|^2_T) + h(h^{-1}\|\nabla v\|^2_T + h\|\nabla^2 v\|^2_T) && (4.39)\\
&\lesssim h^{-2}\|v\|^2_T + \|\nabla v\|^2_T + h^2\|\nabla^2 v\|^2_T && (4.40)
\end{aligned}$$

Summing over all edges $E \in \mathcal{E}_h$ we obtain (4.33). □

4.4. Interpolation Error Estimates

For the error analysis in the next section we will use the following mesh-dependent norm:

$$|||v|||_h^2 = \|v\|_{\mathcal{K}_h}^2 + \epsilon\|\nabla_{\Gamma_h} v\|_{\mathcal{K}_h}^2 + \tau_1 h\|\beta_h \cdot \nabla_{\Gamma_h} v\|_{\mathcal{K}_h}^2 + \tau_2 h^\gamma \|n_h \cdot \nabla v\|_{\mathcal{T}_h}^2 \tag{4.41}$$

Let $\pi_h : L^2(\mathcal{T}_h) \to V_h$ be the Clément interpolant and recall the standard interpolation error estimate

$$\|w - \pi_h w\|_{H^m(\mathcal{T}_h)} \lesssim h^{s-m}\|w\|_{H^s(\mathcal{T}_h)} \tag{4.42}$$

for $w \in H^s(\mathcal{T}_h)$ and $0 \leq m \leq s \leq 2$. When interpolating functions of the form $w = v^e$, with $v \in H^s(\Gamma)$, the following stability of the extension

$$\|v^e\|_{H^s(U_\delta(\Gamma))} \lesssim \delta^{1/2}\|v\|_{H^s(\Gamma)} \tag{4.43}$$

is used together with a trace inequality to derive the interpolation error estimate

$$\|v - (\pi_h v^e)^l\|_{H^m(\Gamma)} \sim \|v^e - \pi_h v^e\|_{H^m(\Gamma_h)} \lesssim h^{s-m}\|v\|_{H^s(\Gamma)} \quad m \in \{0,1\}, m \leq s \leq 2 \tag{4.44}$$

see [2] and [20] for details. Furthermore, we have the following energy norm estimate of the interpolation error.

Lemma 4.3. *Let τ_1 be defined by (3.11), τ_2 by (3.12), with $0 \leq \gamma$; then there is a constant such that*

$$|||u^e - \pi_h u^e|||_h^2 \lesssim \max(\beta_\infty h^3, \epsilon h^2)\|u\|_{H^2(\Gamma)}^2 \tag{4.45}$$

Proof. Considering the definition of (4.41) we see that the bound for the first two terms is an immediate consequence of (4.44),

$$\|u^e - \pi_h u^e\|_{\mathcal{K}_h}^2 + \epsilon\|\nabla_{\Gamma_h}(u^e - \pi_h u^e)\|_{\mathcal{K}_h}^2 \lesssim h^4\|u\|_{H^2(\Gamma)}^2 + \epsilon h^2\|u\|_{H^2(\Gamma)}^2 \lesssim \max(h^4, \epsilon h^2)\|u\|_{H^2(\Gamma)}^2 \tag{4.46}$$

For the third term of (4.41), that is related to the streamline diffusion stabilization, we have using (4.44) and the definition (3.11) of τ_1,

$$\begin{aligned} \tau_1 h\|\beta_h \cdot \nabla_{\Gamma_h}(u^e - \pi_h u^e)\|_{\mathcal{K}_h}^2 &\lesssim \tau_1 \beta_\infty^2 h^3 \|u\|_{H^2(\Gamma)} \\ &\lesssim \min(\beta_\infty^{-1}, h\epsilon^{-1})\beta_\infty^2 h^3\|u\|_{H^2(\Gamma)}^2 \lesssim \beta_\infty h^3 \|u\|_{H^2(\Gamma)} \end{aligned} \tag{4.47}$$

where for the last inequality we used the fact $\min(\beta_\infty^{-1}, h\epsilon^{-1})\beta_\infty \leq 1$. Finally, for the fourth term we obtain

$$\begin{aligned} \tau_2 h^\gamma \|n_h \cdot \nabla(u^e - \pi_h u^e)\|_{\mathcal{T}_h}^2 &\lesssim \tau_2 h^{\gamma+2}\|u^e\|_{H^2(\mathcal{T}_h)}^2 \\ &\lesssim \tau_2 h^{\gamma+3}\|u\|_{H^2(\Gamma)}^2 \lesssim \max(\beta_\infty h^3, \epsilon h^2)h^\gamma \|u\|_{H^2(\Gamma)}^2 \end{aligned} \tag{4.48}$$

where we used the assumption (3.12) on τ_2 to conclude that

$$\tau_2 h^{\gamma+3} = \max(\beta_\infty, \epsilon h^{-1})h^{\gamma+3} = \max(\beta_\infty h^3, \epsilon h^2)h^\gamma \tag{4.49}$$

Collecting the bounds we obtain

$$|||u^e - \pi_h u^e|||_h^2 \lesssim \max(h^4, \epsilon h^2, \beta_\infty h^3, \epsilon h^2, \beta_\infty h^{3+\gamma}, \epsilon h^{2+\gamma})\|u\|_{H^2(\Gamma)}^2 \tag{4.50}$$

and the desired result follows for $\gamma \geq 0$. □

5. A Priori Error Estimates

In this section we will prove the main result of this paper: an optimal error estimate in the streamline derivative norm and an estimate that is suboptimal with $O(h^{\frac{1}{2}})$ for the error in the L^2-norm. To give some structure to this result we first prove coercivity, which also establishes the existence of the discrete solution, then continuity and finally estimates of the geometrical error and consistency.

5.1. Coercivity

Compared to a standard coercivity result for a problem set in the flat domain we must here control the terms appearing due to jumps in the discrete approximation of β over element faces. To obtain this control we need to use equation (4.17) of Assumption B and the normal grandient stabilization.

Lemma 5.1. *Let $\gamma < 2$ and h_0 be small small enough, there is a constant $c_{coer} > 0$ such that for all $v \in V_h$ and $h \in (0, h_0]$,*

$$c_{coer} |||v|||_h^2 \lesssim A_h(v, v) \qquad \forall v \in V_h \tag{5.1}$$

The constant c_{coer} takes the form

$$c_{coer} = \min\Big(1, \alpha_{h,0} - C_1 h_0, 1 - C_2 h_0^{2-\gamma}\Big) \tag{5.2}$$

where C_1 and C_2 are positive constants.

Proof. We have

$$A_h(v, v) = a_h(v, v) + s_{h,1}(v, v) + s_{h,2}(v, v) = I + II + III \tag{5.3}$$

Term I. Using Assumptions (4.15) and (4.17) we obtain

$$a_h(v, v) = (\beta_h \cdot \nabla_{\Gamma_h} v, v)_{\mathcal{K}_h} + (\alpha_h v, v)_{\mathcal{K}_h} + \epsilon(\nabla v, \nabla w)_{\mathcal{K}_h} \tag{5.4}$$

$$= ((\alpha_h - \frac{1}{2}\mathrm{div}_{\Gamma_h} \beta_h) v, v)_{\mathcal{K}_h} + ([\nu_h \cdot \beta_h] v, v)_{\mathcal{E}_h} + \epsilon(\nabla v, \nabla w)_{\mathcal{K}_h} \tag{5.5}$$

$$\geq \alpha_{h,0} \|v\|_{\mathcal{K}_h}^2 - C_* C_\beta (h\|v\|_{\mathcal{K}_h}^2 + h^2 \|n_h \cdot \nabla_{\Gamma_h} v\|_{\mathcal{T}_h}^2) + \epsilon \|\nabla_{\Gamma_h} v\|_{\mathcal{K}_h}^2 \tag{5.6}$$

where we used Green's formula in (5.5), and in (5.6) we used assumption (4.15) for the first term, and for the second term we employed the estimate

$$([\nu_h \cdot \beta_h] v, v)_{\mathcal{E}_h} \lesssim C_\beta (h\|v\|_{\mathcal{K}_h}^2 + h^2 \|n_h \cdot \nabla v\|_{\mathcal{T}_h}^2) \tag{5.7}$$

with hidden constant denoted by C_*. To verify (5.7) we use (4.17) and the inverse estimate (4.32),

$$([\nu_h \cdot \beta_h] v, v)_{\mathcal{E}_h} \lesssim \|[\nu_h \cdot \beta_h]\|_{L^\infty(\mathcal{E}_h)} \|v\|_{\mathcal{E}_h}^2 \tag{5.8}$$

$$\lesssim C_\beta h^2 \|v\|_{\mathcal{E}_h}^2 \tag{5.9}$$

$$\lesssim C_\beta (h\|v\|_{\mathcal{K}_h}^2 + h^2 \|n_h \cdot \nabla v\|_{\mathcal{T}_h}^2) \tag{5.10}$$

Term II. Using the Cauchy-Schwarz inequality and the bound $2ab \leq a^2 + b^2$ we obtain

$$s_{h,1}(v, v) = \tau_1 h(\beta_h \cdot \nabla_{\Gamma_h} v + \alpha_h v, \beta_h \cdot \nabla_{\Gamma_h} v)_{\mathcal{K}_h} \tag{5.11}$$

$$\geq \tau_1 h \|\beta_h \cdot \nabla_{\Gamma_h} v\|_{\mathcal{K}_h}^2 - \tau_1 h \|\alpha_h v\|_{\mathcal{K}_h} \|\beta_h \cdot \nabla_{\Gamma_h} v\|_{\mathcal{K}_h} \tag{5.12}$$

$$\geq \frac{\tau_1}{2} h \|\beta_h \cdot \nabla_{\Gamma_h} v\|_{\mathcal{K}_h}^2 - \frac{\tau_1}{2} h \|\alpha_h\|_{L^\infty(\mathcal{K}_h)}^2 \|v\|_{\mathcal{K}_h}^2 \tag{5.13}$$

Term III. We directly have

$$s_{h,2}(v, v) = \tau_2 h^\gamma \|n_h \cdot \nabla v\|_{\mathcal{T}_h}^2 \tag{5.14}$$

Conclusion. Collecting the estimates we obtain

$$A_h(v,v) \geq \Big(\alpha_{h,0} - C_* C_\beta h - h\frac{\tau_1}{2}\|\alpha_h\|^2_{L^\infty(\mathcal{K}_h)}\Big)\|v\|^2_{\mathcal{K}_h} + \epsilon\|\nabla_{\Gamma_h} v\|^2_{\mathcal{K}_h} + \frac{\tau_1}{2} h\|\beta_h \cdot \nabla_{\Gamma_h} v\|^2_{\mathcal{K}_h} \tag{5.15}$$

$$+ \Big(1 - C_* C_\beta \tau_2^{-1} h^{2-\gamma}\Big)\tau_2 h^\gamma \|n_h \cdot \nabla v\|^2_{\mathcal{T}_h} \tag{5.16}$$

$$\gtrsim \min\Big(1, \alpha_{h,0} - C_1 h_0, 1 - C_2 h_0^{2-\gamma}\Big)|||v|||^2_h \tag{5.17}$$

$$\gtrsim c_{\text{coer}}|||v|||^2_h \tag{5.18}$$

for $h \in (0, h_0]$ with h_0 small enough. The constants C_1 and C_2 satisfy

$$C_1 = C_* C_\beta + \frac{\tau_1}{2}\|\alpha_h\|^2_{L^\infty(\mathcal{K}_h)} \leq C_* C_\beta + \frac{1}{2}\min(\beta_\infty^{-1}, h_0\epsilon^{-1})\alpha_\infty^2 \lesssim 1 \tag{5.19}$$

$$C_2 = C_* C_\beta \tau_2^{-1} \lesssim C_\beta \tau_1 \lesssim C_\beta \min(\beta_\infty^{-1}, h_0\epsilon^{-1}) \lesssim 1 \tag{5.20}$$

where we used the identity $\tau_2 = \tau_1^{-1}$. Here $\alpha_\infty = \|\alpha_h\|_{L^\infty(\Gamma_h)}$. □

Remark 5.1. *Note that if we instead start from the skew symmetric discretization of the convection term*

$$a_\beta(v,w) = \frac{1}{2}\Big((\beta_h \cdot \nabla_{\Gamma_h} v, w)_{\mathcal{K}_h} - (v, \beta_h \cdot \nabla_{\Gamma_h} w)_{\mathcal{K}_h}\Big) - \frac{1}{2}((\text{div}_{\Gamma_h}\beta_h)v, w)_{\mathcal{K}_h} \tag{5.21}$$

we do not have to use partial integration in Term I since $a_\beta(v,v) = 0$. This simplifies the argument since we immediately obtain

$$A_h(v,v) = ((\alpha_h - \frac{1}{2}\text{div}_{\Gamma_h}\beta_h)v, v)_{\mathcal{K}_h} + \epsilon(\nabla_{\Gamma_h} v, \nabla_{\Gamma_h} v)_{\mathcal{K}_h} + s_{h,1}(v,v) + s_{h,s}(v,v) \tag{5.22}$$

We note that [22] uses the skew symmetric form and may thus establish coercivity (with respect to a weaker norm) without using the stabilization term $s_{h,2}$. With normal gradient stabilization we find that we may use the standard or the antisymmetric formulation of the convection term. However, partial integration must still be used to prove an optimal a priori error estimate, see the proof of the continuity result in the next section.

5.2. Continuity

We now prove a continuity result.

Lemma 5.2. *There is a constant such that for all $\eta \in V_h + H^2(\mathcal{T}_h)$, $v \in V_h$,*

$$A_h(\eta, v) \lesssim \Big(\max(\beta_\infty, \epsilon h^{-1})h^{-1}\|\eta\|^2_{\mathcal{K}_h} + |||\eta|||^2_h + C_\beta^2 \max(h^2, \beta_\infty^{-1} h^{3-\gamma})|||\eta|||^2_{h,*}\Big)^{1/2} |||v|||_h \tag{5.23}$$

where

$$|||\eta|||^2_{h,*} = h^{-1}\|\eta\|^2_{\mathcal{T}_h} + h\|\nabla\eta\|^2_{\mathcal{T}_h} + h^3\|\nabla^2\eta\|^2_{\mathcal{T}_h} \tag{5.24}$$

Proof. We have

$$A_h(\eta, v) = a_h(\eta, v) + s_{h,1}(\eta, v) + s_{h,2}(\eta, v) = I + II + III \tag{5.25}$$

Term I. Using partial integration on the discrete surface followed by the Cauchy-Schwarz inequality we obtain

$$
\begin{aligned}
I &= (\alpha_h \eta, v)_{\mathcal{K}_h} + (\beta_h \cdot \nabla_{\Gamma_h} \eta, v)_{\mathcal{K}_h} + \epsilon(\nabla_{\Gamma_h}\eta, \nabla_{\Gamma_h} v)_{\mathcal{K}_h} && (5.26)\\
&= ((\alpha_h - \operatorname{div}_{\Gamma_h}\beta_h)\eta, v)_{\mathcal{K}_h} - (\eta, \beta_h \cdot \nabla_{\Gamma_h} v)_{\mathcal{K}_h} + ([\nu_h \cdot \beta_h]\eta, v)_{\mathcal{E}_h} + \epsilon(\nabla_{\Gamma_h}\eta, \nabla_{\Gamma_h} v)_{\mathcal{K}_h} && (5.27)\\
&\leq \underbrace{(\|\alpha_h - \operatorname{div}_{\Gamma_h}\beta_h\|_{L^\infty(\mathcal{K}_h)})}_{\lesssim 1} \|\eta\|_{\mathcal{K}_h} \underbrace{\|v\|_{\mathcal{K}_h}}_{\leq |||v|||_h} && (5.28)\\
&\quad + \tau_1^{-\frac{1}{2}} h^{-1/2} \|\eta\|_{\mathcal{K}_h} \underbrace{\tau_1^{\frac{1}{2}} h^{1/2} \|\beta_h \cdot \nabla_{\Gamma_h} v\|_{\mathcal{K}_h}}_{\lesssim |||v|||_h}\\
&\quad + \underbrace{\|[\nu_h \cdot \beta_h]\|_{L^\infty(\mathcal{E}_h)} \|\eta\|_{\mathcal{E}_h} \|v\|_{\mathcal{E}_h}}_{\bigstar \lesssim C_\beta \max(h, \tau_2^{-1/2} h^{\frac{3-\gamma}{2}}) |||\eta|||_{h,*} |||v|||_h} + \underbrace{\epsilon^{1/2}\|\nabla_{\Gamma_h}\eta\|_{\mathcal{K}_h}}_{\leq |||\eta|||_h} \underbrace{\epsilon^{1/2}\|\nabla_{\Gamma_h} v\|_{\mathcal{K}_h}}_{\leq |||v|||_h}\\
&\lesssim \tau_1^{-1/2} h^{-1/2} \|\eta\|_{\mathcal{K}_h} |||v|||_h + |||\eta|||_h |||v|||_h + C_\beta \max(h, \tau_2^{-1/2} h^{\frac{3-\gamma}{2}}) |||\eta|||_{h,*} |||v|||_h && (5.29)
\end{aligned}
$$

We thus obtain

$$
\begin{aligned}
I &\lesssim \left(\tau_1^{-1} h^{-1} \|\eta\|^2_{\mathcal{K}_h} + |||\eta|||^2_h + C_\beta^2 \max(h^2, \tau_2^{-1} h^{3-\gamma}) |||\eta|||^2_{h,*}\right)^{1/2} |||v|||_h && (5.30)\\
&\lesssim \left(\max(\beta_\infty, \epsilon h^{-1}) h^{-1} \|\eta\|^2_{\mathcal{K}_h} + |||\eta|||^2_h + C_\beta^2 \max(h^2, \beta_\infty^{-1} h^{3-\gamma}) |||\eta|||^2_{h,*}\right)^{1/2} |||v|||_h && (5.31)
\end{aligned}
$$

where we used the estimates

$$
\tau_1^{-1/2} h^{-1/2} \lesssim \max(\beta_\infty^{1/2}, \epsilon^{1/2} h^{-1/2}), h^{-1/2}) \qquad C_\beta \tau_2^{-1/2} \lesssim C_\beta \beta_\infty^{-1/2} \tag{5.32}
$$

which follows directly from the definitions (3.10) and (3.12), of parameters τ_1 and τ_2.

For $\bigstar$ we used (4.17) followed by (4.33) and (4.32) to obtain the bound

$$
\begin{aligned}
\bigstar &= \|[\nu_h \cdot \beta_h]\|_{L^\infty(\mathcal{E}_h)} \|\eta\|_{\mathcal{E}_h} \|v\|_{\mathcal{E}_h} && (5.33)\\
&\lesssim C_\beta h h^{1/2} \|\eta\|_{\mathcal{E}_h} h^{1/2} \|v\|_{\mathcal{E}_h} && (5.34)\\
&\lesssim C_\beta h \underbrace{\left(h^{-1}\|\eta\|^2_{\mathcal{T}_h} + h^1 \|\nabla \eta\|^2_{\mathcal{T}_h} + h^3 \|\nabla^2 \eta\|^2_{\mathcal{T}_h}\right)^{1/2}}_{= |||\eta|||_{h,*}} \underbrace{\left(\|v\|^2_{\mathcal{K}_h} + h\|n_h \cdot \nabla v\|^2_{\mathcal{T}_h}\right)^{1/2}}_{\lesssim \max(1, \tau_2^{-\frac{1}{2}} h^{\frac{1-\gamma}{2}}) |||v|||_h} && (5.35)\\
&\lesssim C_\beta \max(h, \tau_2^{-\frac{1}{2}} h^{\frac{3-\gamma}{2}}) |||\eta|||_{h,*} |||v|||_h && (5.36)
\end{aligned}
$$

where we used the estimate

$$
\begin{aligned}
\|v\|^2_{\mathcal{K}_h} + h\|n_h \cdot \nabla v\|^2_{\mathcal{T}_h} &\leq \|v\|^2_{\mathcal{K}_h} + \tau_2^{-1} h^{1-\gamma} \tau_2 h^\gamma \|n_h \cdot \nabla v\|^2_{\mathcal{T}_h} && (5.37)\\
&\leq \max(1, \tau_2^{-1} h^{1-\gamma})(\|v\|^2_{\mathcal{K}_h} + \tau_2 h^\gamma \|n_h \cdot \nabla v\|^2_{\mathcal{T}_h}) && (5.38)\\
&\leq \max(1, \tau_2^{-1} h^{1-\gamma}) |||v|||^2_h && (5.39)
\end{aligned}
$$

Term II. Using the Cauchy-Schwarz inequality,

$$
\begin{aligned}
II &= \tau_1 h(\alpha_h \eta + \beta_h \cdot \nabla_{\Gamma_h} \eta, \beta_h \cdot \nabla_{\Gamma_h} v)_{\mathcal{K}_h} && (5.40)\\
&\lesssim (\tau_1^{1/2} h^{1/2} \|\alpha_h\|_{L^\infty(\Gamma_h)} \|\eta\|_{\mathcal{K}_h} + \tau_1^{1/2} h^{1/2} \|\beta_h \cdot \nabla_{\Gamma_h} \eta\|_{\mathcal{K}_h}) \tau_1^{1/2} h^{1/2} \|\beta_h \cdot \nabla_{\Gamma_h} v\|_{\mathcal{K}_h} && (5.41)\\
&\lesssim |||\eta|||_h |||v|||_h && (5.42)
\end{aligned}
$$

Term III. Using the Cauchy-Schwarz inequality,

$$
III = \tau_2 h^\gamma (n_h \cdot \nabla \eta, n_h \cdot \nabla v)_{\mathcal{T}_h} \leq \tau_2^{1/2} h^{\gamma/2} \|n_h \cdot \nabla \eta\|_{\mathcal{T}_h} \tau_2^{1/2} h^{\gamma/2} \|n_h \cdot \nabla v\|_{\mathcal{T}_h} \lesssim |||\eta|||_h |||v|||_h \tag{5.43}
$$

Collecting the estimates of terms *I*-*III* we directly obtain the desired estimate. □

5.3. Geometric Error Estimates

We have the following estimates: for $v \in H^1(\Gamma)$ and $w \in V_h$,

$$|a(v, w^l) - a_h(v^e, w)| \lesssim (\epsilon^{1/2} + C_\alpha + C_\beta) h^2 \|v\|_{H^1(\Gamma)} (\|w\|^2_{\mathcal{K}_h} + \epsilon \|\nabla_{\Gamma_h} w\|^2_{\mathcal{K}_h})^{1/2} \tag{5.44}$$

and for $w \in V_h$,

$$|l(w^l) - l_h(w)| \lesssim h^2 \|f\|_\Gamma \|w\|_{\mathcal{K}_h} \tag{5.45}$$

Verification of (5.44). We have

$$|\epsilon(\nabla_\Gamma v, \nabla_\Gamma w^l)_\Gamma - \epsilon(\nabla_{\Gamma_h} v^e, \nabla_{\Gamma_h} w)_{\mathcal{K}_h}| \lesssim \epsilon h^2 \|\nabla_\Gamma v\|_\Gamma \|\nabla_{\Gamma_h} w\|_{\mathcal{K}_h} \tag{5.46}$$

see [2] for details. Using (4.1) and changing domain of integration from Γ_h to Γ we obtain

$$(\beta_h \cdot \nabla_{\Gamma_h} v^e, w)_{\mathcal{K}_h} - (\beta \cdot \nabla_\Gamma v, w^l)_\Gamma = (|B|^{-1}(B\beta_h^l \cdot \nabla_\Gamma v), w^l)_\Gamma - (\beta \cdot \nabla_\Gamma v, w^l)_\Gamma \tag{5.47}$$

$$= ((|B|^{-1} B\beta_h^l - \beta) \cdot \nabla_\Gamma v), w^l)_\Gamma \lesssim C_\beta h^2 \|\nabla_\Gamma v\|_\Gamma \|w\|_{\mathcal{K}_h} \tag{5.48}$$

where we used Assumption B and (4.13) in the last step. Using the same approach we obtain

$$(\alpha_h v^e, w)_{\mathcal{K}_h} - (\alpha v, w^l)_\Gamma = ((\alpha - |B|^{-1}\alpha_h^l) v, w^l)_\Gamma \lesssim C_\alpha h^2 \|v\|_\Gamma \|w\|_{\mathcal{K}_h} \tag{5.49}$$

Finally (5.45) follows in the same way.

5.4. Consistency

We now estimate the consistency error which depends on the geometric error.

Lemma 5.3. *For $\gamma > 0$, there is a constant such that for all $v \in V_h$ and $u \in H^2(\Gamma)$,*

$$A_h(u^e, v) - L_h(v) \lesssim (\max(\beta_\infty h^3, \epsilon h^2))^{1/2} \|u\|_{H^2(\Gamma)} |||v|||_h + h^2 \|f\|_\Gamma |||v|||_h \tag{5.50}$$

Proof. We have the identity

$$A_h(u^e, v) - L_h(v) = \underbrace{a_h(u^e, v) - l_h(v)}_{I} + \underbrace{s_{h,1}(u^e, v) - l_{s_{h,1}}(v)}_{II} + \underbrace{s_{h,2}(u^e, v)}_{III} \tag{5.51}$$

Term I. Using the geometry error estimates (5.44) and (5.45) we directly obtain

$$I = a_h(u^e, v) - a(u, v^l) + l(v^l) - l_h(v) \tag{5.52}$$

$$\lesssim h^2 \|u\|_{H^1(\Gamma)} (\|v\|^2_{\mathcal{K}_h} + \epsilon \|\nabla_{\Gamma_h} v\|^2_{\mathcal{K}_h})^{1/2} + h^2 \|f\|_\Gamma \|v\|_{\mathcal{K}_h} \tag{5.53}$$

$$\lesssim h^2 \|u\|_{H^1(\Gamma)} |||v|||_h + \|f\|_\Gamma \Big) |||v|||_h \tag{5.54}$$

$$\lesssim (\max(\beta_\infty h^3, \epsilon h^2)^{1/2} \|u\|_{H^1(\Gamma)} |||v|||_h + \|f\|_\Gamma |||v|||_h \tag{5.55}$$

where we used the estimate

$$h^2 \lesssim (\max(\beta_\infty h^3, \epsilon h^2))^{1/2} \tag{5.56}$$

which holds for all $h \in (0, h_0]$ with h_0 small enough

Term II. Subtracting the quantity

$$(\beta \cdot \nabla_\Gamma u + \alpha u - \epsilon \Delta_\Gamma u - f)^e = 0 \tag{5.57}$$

and estimating the resulting terms using (5.48) and (5.49) we obtain

$$\begin{aligned}
II &= \tau_1 h(\beta_h \cdot \nabla_{\Gamma_h} u^e + \alpha_h u^e - f_h, \beta_h \cdot \nabla_{\Gamma_h} v)_{\mathcal{K}_h} && (5.58)\\
&= \tau_1 h((\beta_h \cdot \nabla_{\Gamma_h} u^e + \alpha_h u^e - f^e) - (\beta \cdot \nabla_\Gamma u + \alpha u - \epsilon\Delta_\Gamma u - f)^e, \beta_h \cdot \nabla_{\Gamma_h} v)_{\mathcal{K}_h} && (5.59)\\
&\leq \tau_1^{1/2} h^{1/2}\Big(\|\beta_h \cdot \nabla_{\Gamma_h} u^e - (\beta \cdot \nabla_\Gamma u)^e\|_{\mathcal{K}_h} + \|\alpha_h u^e - (\alpha u)^e\|_{\mathcal{K}_h} && (5.60)\\
&\qquad + \epsilon\|(\Delta_\Gamma u)^e\|_{\mathcal{K}_h}\Big) \tau_1^{1/2} h^{1/2} \|\beta_h \cdot \nabla_{\Gamma_h} v\|_{\mathcal{K}_h}\\
&\leq \tau_1^{1/2}(C_\beta h^{5/2} + C_\alpha h^{5/2} + \epsilon h^{1/2})\|u\|_{H^2(\Gamma)} |||v|||_h && (5.61)\\
&\lesssim (\beta_\infty^{-1} C_\beta^2 h^5 + \beta_\infty^{-1} C_\alpha^2 h^5 + \epsilon h^2)^{1/2} \|u\|_{H^2(\Gamma)} |||v|||_h && (5.62)\\
&\lesssim (\max(\beta_\infty h^3, \epsilon h^2))^{1/2} \|u\|_{H^2(\Gamma)} |||v|||_h && (5.63)
\end{aligned}$$

where we used the definition (3.10) of τ_1 and finally the fact that

$$\beta_\infty^{-1} C_\beta^2 h^5 + \beta_\infty^{-1} C_\alpha^2 h^5 \lesssim \beta_\infty h^3 \tag{5.64}$$

for all $h \in (0, h_0]$ with h_0 small enough.

Term III. Using the fact that $n \cdot \nabla u^e = 0$ we directly obtain

$$\begin{aligned}
III &\lesssim \tau_2 h^\gamma \|n_h \cdot \nabla u^e\|_{\mathcal{T}_h} \|n_h \cdot \nabla v\|_{\mathcal{T}_h} && (5.65)\\
&\lesssim \tau_2 h^\gamma \|(n_h - n) \cdot \nabla u^e\|_{\mathcal{T}_h} \|n_h \cdot \nabla v\|_{\mathcal{T}_h} && (5.66)\\
&\lesssim \tau_2^{1/2} h^{\gamma/2} \|(n_h - n)\|_{L^\infty(\Gamma_h)} \|\nabla u^e\|_{\mathcal{T}_h} \tau_2^{1/2} h^{\gamma/2} \|n_h \cdot \nabla v\|_{\mathcal{T}_h} && (5.67)\\
&\lesssim \tau_2^{1/2} h^{(\gamma+3)/2} \|u\|_{H^1(\Gamma)} |||v|||_h && (5.68)\\
&\lesssim (\max(\beta_\infty h^{\gamma+3}, \epsilon h^{\gamma+2})^{1/2} \|u\|_{H^1(\Gamma)} |||v|||_h && (5.69)\\
&\lesssim (\max(\beta_\infty h^3, \epsilon h^2)^{1/2} \|u\|_{H^1(\Gamma)} |||v|||_h && (5.70)
\end{aligned}$$

where we used the definition (3.12) of τ_2 and at last the restriction $\gamma \geq 0$.

Collecting the estimates for the terms *I-III* directly completes the proof. □

Remark 5.2. *Note that from the proof we see that γ must be larger or equal to zero. This lower bound on γ guarantees that the stabilization is weak enough not to affect the optimal order of convergence, see Theorem 5.1 in the next section.*

Remark 5.3. *For the full gradient stabilization (3.13) we note that we get the coarser estimate*

$$\begin{aligned}
III &\lesssim \tau_2 h^\gamma \|\nabla u^e\|_{\mathcal{T}_h} \|\nabla v\|_{\mathcal{T}_h} && (5.71)\\
&\lesssim \tau_2^{1/2} h^{\gamma/2} \|\nabla u^e\|_{\mathcal{T}_h} \tau_2^{1/2} h^{\gamma/2} \|\nabla v\|_{\mathcal{T}_h} && (5.72)\\
&\lesssim \tau_2^{1/2} h^{(\gamma+1)/2} \|u\|_{H^1(\Gamma)} |||v|||_h && (5.73)\\
&\lesssim (\max(\beta_\infty h^{\gamma+1}, \epsilon h^\gamma)^{1/2} \|u\|_{H^1(\Gamma)} |||v|||_h && (5.74)
\end{aligned}$$

and thus we must take $\gamma = 2$ to get optimal order. We note that the normal gradient stabilization is more refined with a smaller consistency error, which allows us to take a larger γ and thus obtain stronger control compared to full gradient stabilization.

5.5. A Priori Error Estimate

In this section we will prove an a priori error estimates that is optimal for both convection and diffusion dominated flows. In the convection dominated regime the error measured in the streamline derivative norm is optimal, $O(h)$, whereas the error in the L^2-norm is $O(h^{3/2})$, which is suboptimal with a factor $O(h^{1/2})$. In the diffusion dominated regime, we show that the error in the H^1-norm is optimal $O(h)$. In the latter case it is also possible to prove optimal error estimates in the L^2-norm following [20, 2], we leave the details of this estimate to the reader.

Theorem 5.1. *Let u be the solution to (2.3) and u_h the finite element approximation defined by (3.4) with $0 \leq \gamma < 2$. If assumptions A and B hold, then there is a constant such that for all $h \in (0, h_0]$, with h_0 small enough,*

$$|||u^e - u_h|||_h^2 \lesssim \max(\beta_\infty h^3, \epsilon h^2)\|u\|_{H^2(\Gamma)}^2 |||v|||_h + h^4\|f\|_\Gamma^2 \tag{5.75}$$

Proof. Adding and subtracting an interpolant and using the triangle inequality

$$|||u^e - u_h|||_h \leq |||u^e - \pi_h u^e|||_h + |||\pi_h u^e - u_h|||_h \tag{5.76}$$

$$\lesssim \min(\beta_\infty h^3, \epsilon h^2)^{1/2}\|u\|_{H^2(\Gamma)} + |||\pi_h u^e - u_h|||_h \tag{5.77}$$

where we used the energy norm interpolation estimate (4.45) for the first term. For the second term we obtain using coercivity, Lemma 5.1 with $\gamma < 2$,

$$|||\pi_h u^e - u_h|||_h \lesssim \sup_{v \in V_h \setminus \{0\}} \frac{A_h(\pi_h u^e - u_h, v)}{|||v|||_h} \tag{5.78}$$

Here we have the identity

$$A_h(\pi_h u^e - u_h, v) = A_h(\pi_h u^e - u^e, v) + A_h(u^e - u_h, v) \tag{5.79}$$

$$= \underbrace{A_h(\pi_h u^e - u^e, v)}_{I} + \underbrace{A_h(u^e, v) - L_h(v)}_{II} \tag{5.80}$$

Term I. Employing the continuity result in Lemma 5.2, with $\eta = \pi_h u^e - u^e$, followed by the interpolation error estimates (4.42) and (4.45), we obtain

$$A_h(\eta, v) \lesssim \Big(\max(\beta_\infty, \epsilon h^{-1})h^{-1}\|\eta\|_{\mathcal{K}_h}^2 + |||\eta|||_h^2 + C_\beta^2 \max(h^2, \beta_\infty^{-1}h^{3-\gamma})|||\eta|||_{h,*}^2 \Big)^{1/2} |||v|||_h \tag{5.81}$$

$$\lesssim \Big(\max(\beta_\infty, \epsilon h^{-1})h^3 + \max(\beta_\infty h^3, \epsilon h^2) + C_\beta^2 \max(h^2, \beta_\infty^{-1}h^{3-\gamma})h^4 \Big)^{1/2} \|u\|_{H^2(\Gamma}^2 |||v|||_h \tag{5.82}$$

$$\lesssim (\max(\beta_\infty h^3, \epsilon h^2))^{1/2}\|u\|_{H^2(\Gamma}^2 |||v|||_h \tag{5.83}$$

where we finally used the estimate

$$C_\beta^2 \max(h^6, \beta_\infty^{-1}h^{7-\gamma}) \lesssim \beta_\infty^2 h^3 \tag{5.84}$$

which holds for all $h \in (0, h_0]$ with h_0 small enough since $\gamma < 2 \leq 4$. Thus we conclude that

$$A_h(\eta, v) \lesssim (\max(\beta_\infty h^3, \epsilon h^2))^{1/2}\|u\|_{H^2(\Gamma}^2 |||v|||_h \tag{5.85}$$

Term II. Using the the consistency error estimate (5.50) we directly obtain

$$A_h(u^e, v) - L_h(v) \lesssim (\max(\beta_\infty h^3, \epsilon h^2))^{1/2}\|u\|_{H^2(\Gamma)} |||v|||_h + h^2\|f\|_\Gamma |||v|||_h \tag{5.86}$$

Combining (5.78), (5.85), and (5.86), we obtain

$$|||\pi_h u^e - u_h|||_h \lesssim (\max(\beta_\infty h^3, \epsilon h^2))^{1/2}\|u\|_{H^2(\Gamma)} |||v|||_h + h^2\|f\|_\Gamma |||v|||_h \tag{5.87}$$

which together with (5.77) completes the proof. □

6. Condition Number Estimate

Let $\{\varphi_i\}_{i=1}^N$ be the standard piecewise linear basis functions associated with the nodes in $\mathcal{T}_h$ and let $\mathcal{A}$ be the stiffness matrix with elements $a_{ij} = A_h(\varphi_i, \varphi_j)$. The condition number is defined by

$$\kappa_h(\mathcal{A}) := |\mathcal{A}|_{\mathbb{R}^N} |\mathcal{A}^{-1}|_{\mathbb{R}^N} \tag{6.1}$$

Using the approach in [2], see also [10], we may prove the following bound on the condition number of the matrix.

Theorem 6.1. *The condition number of the stiffness matrix $\mathcal{A}$ satisfies the estimate*

$$\kappa_h(\mathcal{A}) \lesssim \max(\beta_\infty h^{-1}, \epsilon h^{-2}, h^{-\gamma}) \tag{6.2}$$

for all $h \in (0, h_0]$ with h_0 small enough and $0 \leq \gamma < 2$. In particular, for $\gamma = 1$, we obtain the optimal estimate

$$\kappa_h(\mathcal{A}) \lesssim \max(\beta_\infty h^{-1}, \epsilon h^{-2}) \tag{6.3}$$

Proof. First we note that if $v = \sum_{i=1}^N V_i \varphi_i$ and $\{\varphi_i\}_{i=1}^N$ is the usual nodal basis on $\mathcal{T}_h$ then the following well known estimates hold

$$h^{-d/2} \|v\|_{\mathcal{T}_h} \lesssim |V|_{\mathbb{R}^N} \lesssim h^{-d/2} \|v\|_{\mathcal{T}_h} \tag{6.4}$$

It follows from the definition (6.1) of the condition number that we need to estimate $|\mathcal{A}|_{\mathbb{R}^N}$ and $|\mathcal{A}^{-1}|_{\mathbb{R}^N}$.

Estimate of $|\mathcal{A}|_{\mathbb{R}^N}$. We have

$$|\mathcal{A}V|_{\mathbb{R}^N} = \sup_{W \in \mathbb{R}^N \setminus 0} \frac{(W, \mathcal{A}V)_{\mathbb{R}^N}}{|W|_{\mathbb{R}^N}} \tag{6.5}$$

$$= \sup_{w \in V_h \setminus 0} \frac{A_h(v, w)}{|W|_{\mathbb{R}^N}} \tag{6.6}$$

$$\lesssim \max(\beta_\infty, \epsilon h^{-1}) h^{d-2} |V|_{\mathbb{R}^N} \tag{6.7}$$

where we used the continuity

$$A_h(v, w) \lesssim \max(\beta_\infty, \epsilon h^{-1}) h^{d-2} |V|_{\mathbb{R}^N} |W|_{\mathbb{R}^N} \tag{6.8}$$

To verify (6.8) we use inverse estimates to derive bounds in terms of $\|v\|_{\mathcal{T}_h}$ and $\|w\|_{\mathcal{T}_h}$ and then we employ (6.4) to pass over to the $|\cdot|_{\mathbb{R}^N}$ norms. More precisely, we use the inverse estimate $\|w\|_K \lesssim h^{-\frac{1}{2}} \|w\|_T$, where $K = T \cap \Gamma_h$ to pass from $\mathcal{K}_h$ to $\mathcal{T}_h$, and the standard inverse estimate $\|\nabla w\|_T \lesssim h^{-1} \|w\|_T$ to remove the gradient. We follow the coefficients recalling the notation $\alpha_\infty = \|\alpha_h\|_{L^\infty(\Gamma_h)}$ and $\beta_\infty = \|\beta_h\|_{L^\infty(\Gamma_h)}$, employ the definitions (3.10) and (3.12) of the parameters τ_1 and τ_2, and the estimate $\alpha_\infty \beta_\infty^{-1} h \lesssim \alpha_\infty \beta_\infty^{-1} h_0 \lesssim 1$. The bounds are as follows,

$$\begin{aligned}
a_h(v, w) &\lesssim \|\beta_h \cdot \nabla_{\Gamma_h} v\|_{\mathcal{K}_h} \|w\|_{\mathcal{K}_h} + \|\alpha_h v\|_{\mathcal{K}_h} \|w\|_{\mathcal{K}_h} + \epsilon \|\nabla_{\Gamma_h} v\|_{\mathcal{K}_h} \|\nabla_{\Gamma_h} w\|_{\mathcal{K}_h} && (6.9)\\
&\lesssim \beta_\infty h^{-1} \|\nabla v\|_{\mathcal{T}_h} \|w\|_{\mathcal{T}_h} + \alpha_\infty h^{-1} \|v\|_{\mathcal{T}_h} \|w\|_{\mathcal{T}_h} + \epsilon h^{-1} \|\nabla v\|_{\mathcal{T}_h} \|\nabla w\|_{\mathcal{T}_h} && (6.10)\\
&\lesssim (\beta_\infty h^{-2} + \alpha_\infty h^{-1} + \epsilon h^{-3}) \|v\|_{\mathcal{T}_h} \|w\|_{\mathcal{T}_h} && (6.11)\\
&\lesssim h^{d-2} (\beta_\infty (1 + \alpha_\infty \beta_\infty^{-1} h) + \epsilon h^{-1}) |V|_{\mathbb{R}^N} |W|_{\mathbb{R}^N} && (6.12)\\
&\lesssim \max(\beta_\infty, \epsilon h^{-1}) h^{d-2} |V|_{\mathbb{R}^N} |W|_{\mathbb{R}^N} && (6.13)
\end{aligned}$$

$$
\begin{aligned}
s_{h,1}(v,w) &= \tau_1 h(\beta_h \cdot \nabla_{\Gamma_h} v + \alpha_h v, \beta_h \cdot \nabla_{\Gamma_h} w)_{\mathcal{K}_h} && (6.14)\\
&\lesssim \tau_1 (h\|\beta_h \cdot \nabla_{\Gamma_h} v\|_{\mathcal{K}_h} \|\beta_h \cdot \nabla_{\Gamma_h} w\|_{\mathcal{K}_h} + h\|\alpha_h v\|_{\mathcal{K}_h} \|\beta_h \cdot \nabla_{\Gamma_h} w\|_{\mathcal{K}_h}) && (6.15)\\
&\lesssim \tau_1 (\|\nabla v\|_{\mathcal{T}_h} \|\nabla w\|_{\mathcal{T}_h} + \|v\|_{\mathcal{T}_h} \|\nabla w\|_{\mathcal{T}_h}) && (6.16)\\
&\lesssim \tau_1 (\beta_\infty^2 h^{-2} + \alpha_\infty \beta_\infty h^{-1}) \|v\|_{\mathcal{T}_h} \|w\|_{\mathcal{T}_h} && (6.17)\\
&\lesssim \tau_1 \beta_\infty^2 (1 + \alpha_\infty \beta_\infty^{-1} h) h^{d-2} |V|_{\mathbb{R}^N} |W|_{\mathbb{R}^N} && (6.18)\\
&\lesssim \tau_1 \beta_\infty^2 (1 + \alpha_\infty \beta_\infty^{-1} h) h^{d-2} |V|_{\mathbb{R}^N} |W|_{\mathbb{R}^N} && (6.19)\\
&\lesssim \beta_\infty h^{d-2} |V|_{\mathbb{R}^N} |W|_{\mathbb{R}^N} && (6.20)
\end{aligned}
$$

$$
\begin{aligned}
s_{h,2}(v,w) &\lesssim \tau_2 h^\gamma \|n_h \cdot \nabla v\|_{\mathcal{T}_h} \|n_h \cdot \nabla w\|_{\mathcal{T}_h} && (6.21)\\
&\lesssim \tau_2 h^\gamma \|\nabla v\|_{\mathcal{T}_h} \|\nabla w\|_{\mathcal{T}_h} && (6.22)\\
&\lesssim \tau_2 h^{\gamma-2} \|v\|_{\mathcal{T}_h} \|w\|_{\mathcal{T}_h} && (6.23)\\
&\lesssim \tau_2 h^{d+\gamma-2} |V|_{\mathbb{R}^N} |W|_{\mathbb{R}^N} && (6.24)\\
&\lesssim \max(\beta_\infty, \epsilon h^{-1}) h^{d-2} |V|_{\mathbb{R}^N} |W|_{\mathbb{R}^N} && (6.25)
\end{aligned}
$$

We conclude that

$$
|\mathcal{A}|_{\mathbb{R}^N} \lesssim \underbrace{\max(\beta_\infty, \epsilon h^{-1})}_{\tau_2} h^{d-2} \tag{6.26}
$$

Estimate of $|\mathcal{A}^{-1}|_{\mathbb{R}^N}$*.* We note that using (6.4) and Lemma 4.1 we have

$$
\begin{aligned}
h^d |V|_{\mathbb{R}^N}^2 &\lesssim \|v\|_{\mathcal{T}_h}^2 && (6.27)\\
&\lesssim h\|v\|_{\mathcal{K}_h}^2 + h^2 \|n_h \cdot \nabla v\|\|_{\mathcal{T}_h}^2 && (6.28)\\
&\lesssim h\|v\|_{\mathcal{K}_h}^2 + \tau_2^{-1} h^{2-\gamma} \tau_2 h^\gamma \|n_h \cdot \nabla v\|\|_{\mathcal{T}_h}^2 && (6.29)\\
&\lesssim \max(h, \tau_1 h^{2-\gamma}) |||v|||_h^2 && (6.30)
\end{aligned}
$$

where we used the fact that $\tau_2^{-1} \sim \tau_1$ as follows

$$
\max(h, \tau_2^{-1} h^{2-\gamma}) \lesssim \max(h, \tau_1 h^{2-\gamma}) \tag{6.31}
$$

Thus we obtain

$$
|V|_{\mathbb{R}^N}^2 \lesssim \underbrace{h^{-d} \max(h, \tau_1 h^{2-\gamma})}_{g(h)} |||v|||_h^2 \tag{6.32}
$$

where we introduced the notation $g(h)$ for convenience. Starting from (6.32) and using the coercivity (5.1) we obtain

$$
\begin{aligned}
|V|_{\mathbb{R}^N} \lesssim g(h)^{1/2} |||v|||_h &\lesssim \sup_{w \in V_h \setminus \{0\}} g(h)^{1/2} \frac{A_h(v,w)}{|||w|||_h} && (6.33)\\
&\lesssim \sup_{W \in \mathbb{R}^N \setminus \{0\}} g(h)^{1/2} \frac{|\mathcal{A}V|_{\mathbb{R}^N} |W|_{\mathbb{R}^N}}{g(h)^{-1/2} |W|_{\mathbb{R}^N}} \lesssim g(h) |\mathcal{A}V|_{\mathbb{R}^N} && (6.34)
\end{aligned}
$$

where we used (6.32), $g(h)^{-1/2}|W|_{\mathbb{R}^N} \lesssim |||w|||_h$, to replace $|||w|||_h$ by $g(h)^{-1/2}|W|_{\mathbb{R}^N}$ in the denominator. Setting $V = \mathcal{A}^{-1} X$, $X \in \mathbb{R}^N$, we obtain

$$
|\mathcal{A}^{-1}|_{\mathbb{R}^N} \lesssim g(h) = \max(h, \tau_1 h^{2-\gamma}) \tag{6.35}
$$

Conclusion. Combining the estimates (6.26) and (6.35) we obtain

$$
\begin{aligned}
|\mathcal{A}|_{\mathbb{R}^N} |\mathcal{A}^{-1}|_{\mathbb{R}^N} &(\lesssim h^{d-2} \tau_2)(h^{-d} \max(h, \tau_1 h^{2-\gamma})) && (6.36)\\
&\lesssim \max(\tau_2 h^{-1}, \tau_2 \tau_1 h^{-\gamma}) && (6.37)\\
&\lesssim \max(\beta_\infty h^{-1}, \epsilon h^{-2}, h^{-\gamma}) && (6.38)
\end{aligned}
$$

where we used the estimate $\tau_2 \tau_1 \lesssim 1$. Thus the proof is complete. □

7. Numerical Examples

7.1. Convection–Diffusion

We consider convection–diffusion on the spheroid defined by

$$\frac{(x-1/2)^2+(y-1/2)^2}{r_{\max}^2}+\frac{(z-1/2)^2}{r_{\min}^2}=1$$

with $r_{\max}=1/2$ and $r_{\min}=1/4$. The convective velocity was chosen as

$$\beta=(1/2-y, x-1/2, 0)$$

and it is easily verified that $\beta_\Gamma = P_\Gamma\beta=\beta$, and that $\nabla_\Gamma\cdot\beta_\Gamma=0$. We set $\alpha=0$, $c_{\tau_1}=1/2$ in (3.10), $c_{\tau_2}=0$ in (3.12). The right-hand f was chosen by applying the differential operator to the fabricated solution

$$u(x,y,z)=100(x-1/2)(y-1/2)(z-1/2)$$

In Fig. 1 we show an isoplot of the solution using $\epsilon=10^{-3}$ on a given mesh in a sequence of refinements, and in Fig. 2 we show the velocity field plotted on the same mesh. Finally, in Fig. 3 we present the convergence in $L_2(\Gamma_h)$ obtained by our method, close to second order.

7.2. Convection–Reaction with a Layer

We consider convection–diffusion on the spheroid defined by

$$\frac{(x-1/2)^2+(y-1/2)^2}{r_{\max}^2}+\frac{(z-1/2)^2}{r_{\min}^2}=1$$

with $r_{\max}=0.5$ and $r_{\min}=0.45$. The convective velocity was chosen as

$$\beta=(5-10y, 10x-5, 0)$$

and parameters $\alpha=1$, $\epsilon=0$. The right-hand f was chosen as

$$f=\begin{cases}1 \text{ if } z>0.55\\ 0 \text{ if } z\le 0.55\end{cases}$$

creating a discontinuity at $z=0.55$. In Fig. 4 we show isoplots of the solution using $c_{\tau_1}=0$, $\tau_2=10^{-4}$ (top), and $\tau_2=10^3$ (bottom) on a given mesh with $\gamma=1$. Notice the instability for small τ_2 and excessive diffusivity for large τ_2. In Fig. 5 we show the corresponding isoplot for $\tau_2=1$, $c_{\tau_1}=0$, and in Fig. 6 we used $c_{\tau_1}=1/2$, $c_{\tau_2}=0$. In both cases there are, as expected, slight over- and undershoots close to the discontinuity.

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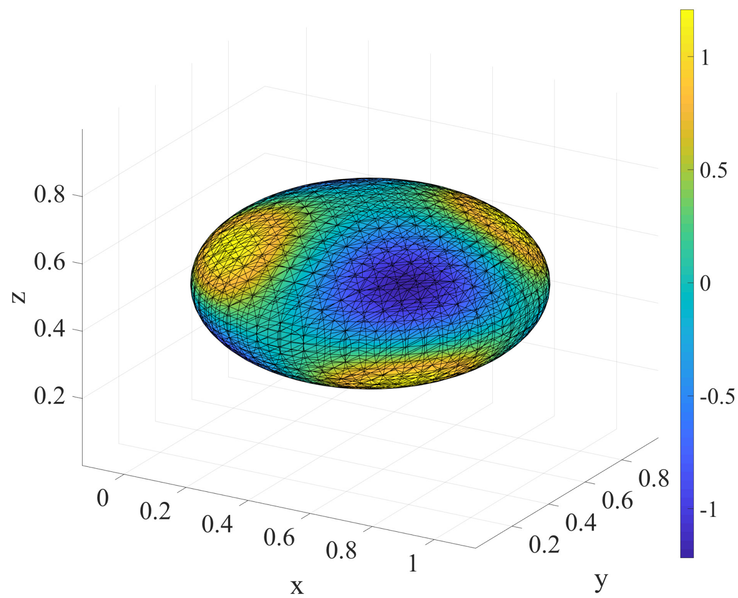

Figure 1: Isoplot of the solution on a given mesh.

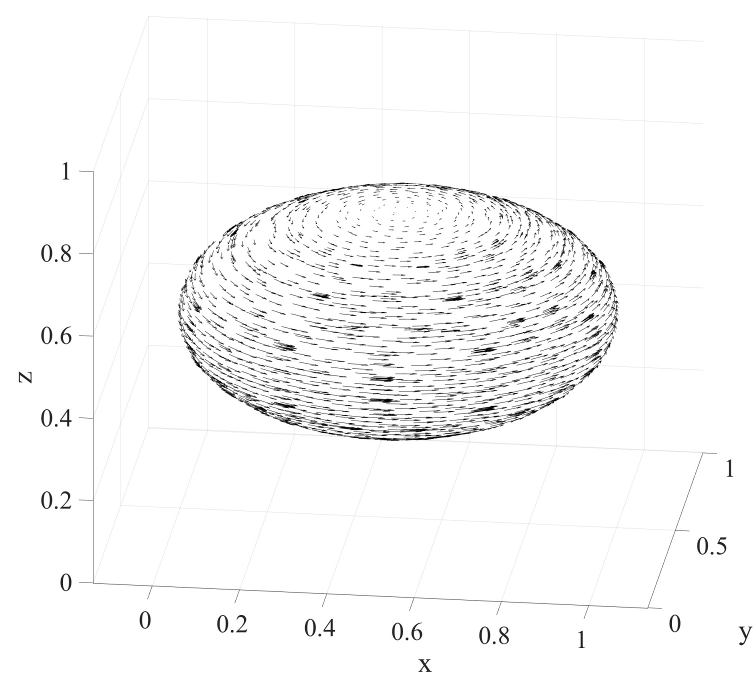

Figure 2: Velocity field on a given mesh.

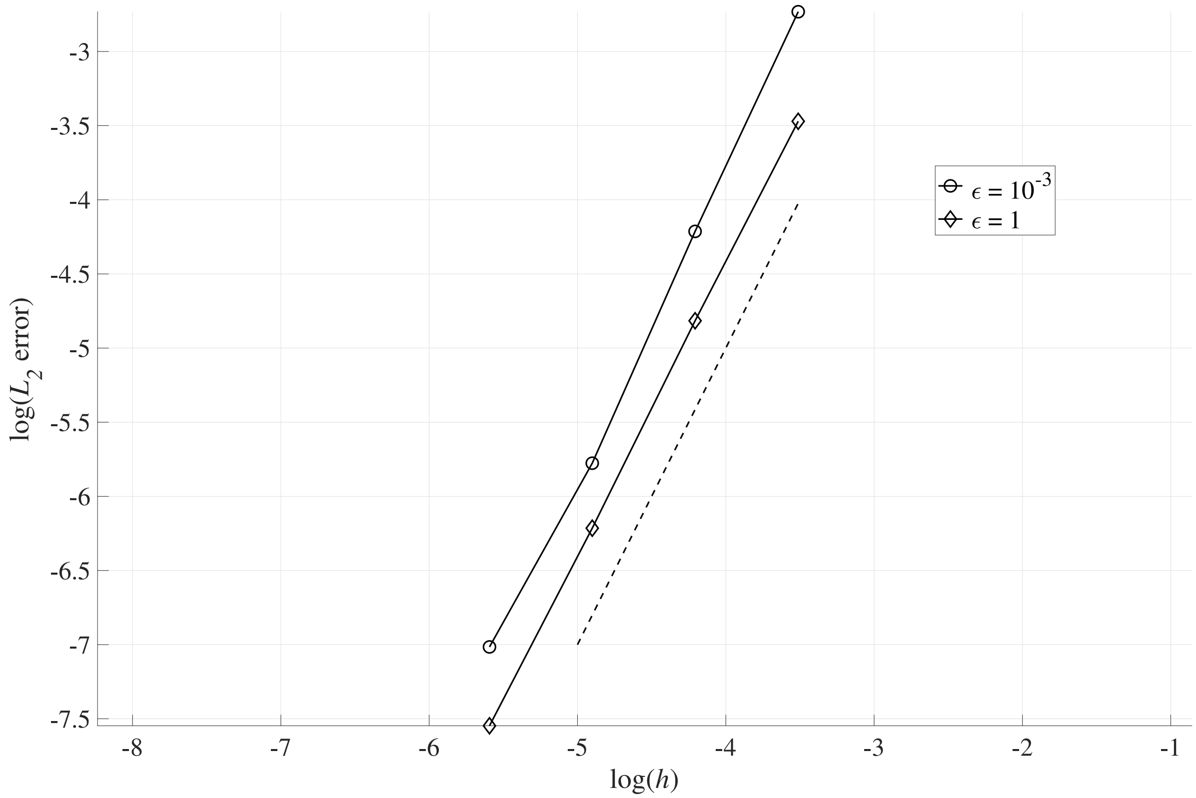

Figure 3: $L_2(\Gamma_h)$–convergence of the discrete solution. Dotted line indicates second order convergence.

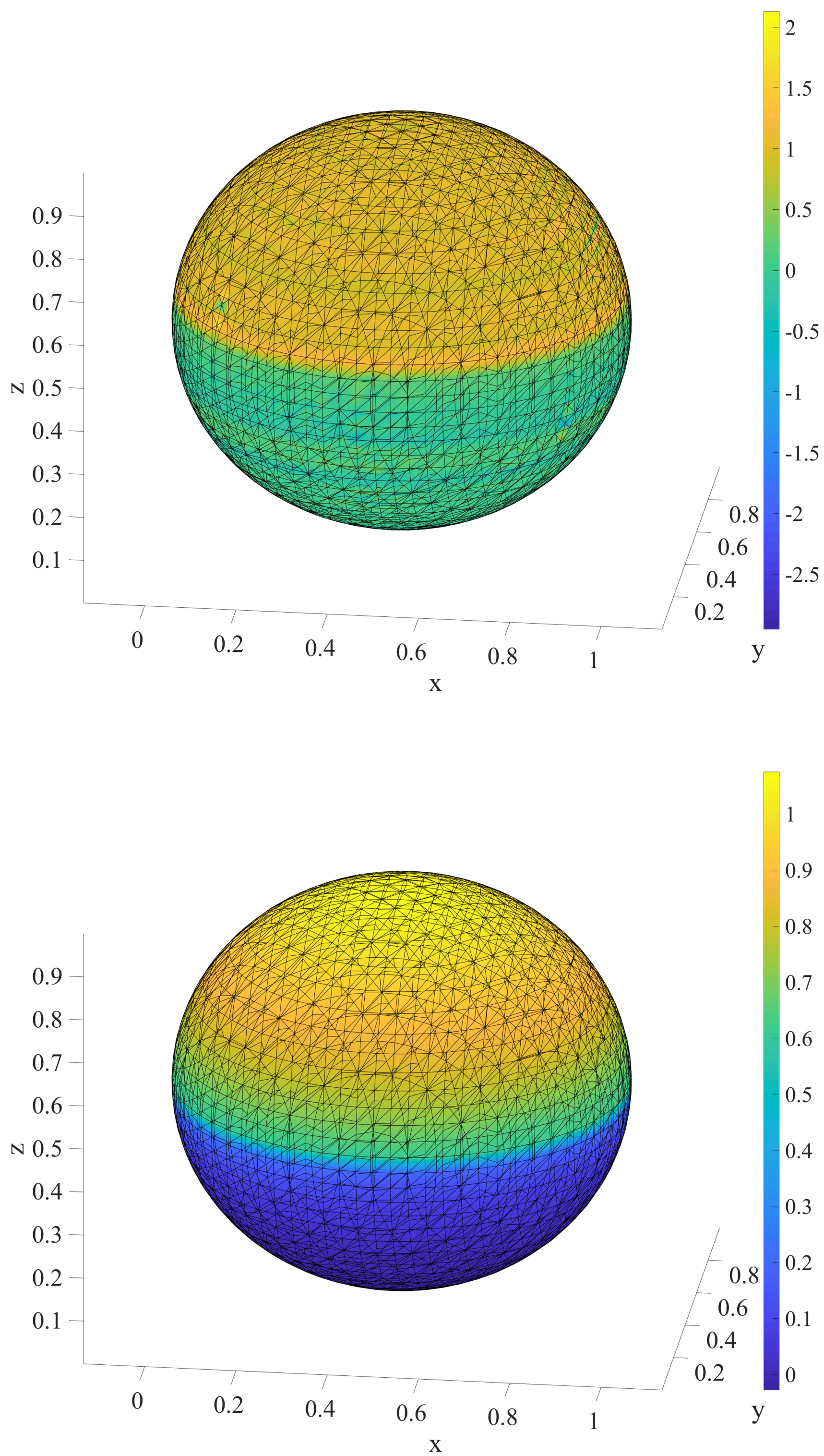

Figure 4: Isoplot of the solution on a given mesh for $c_\tau = 0$. Top: $\tau_2 = 10^{-4}$, bottom: $\tau_2 = 10^3$

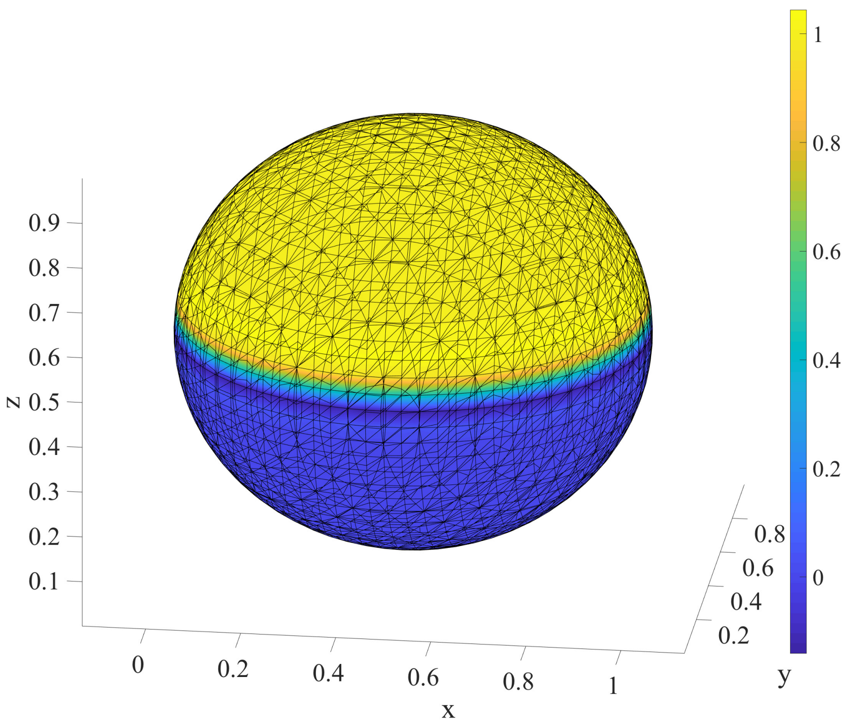

Figure 5: Isoplot of the solution for $c_\tau = 0$, $\tau_2 = 1$

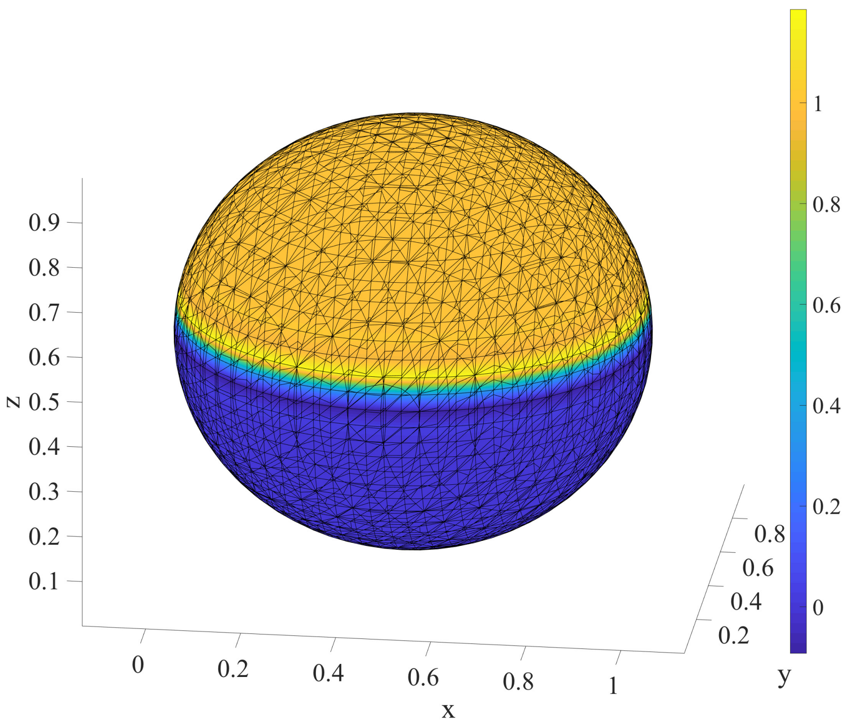

Figure 6: Isoplot of the solution for $\tau_2 = 0$, $c_\tau = 1/2$